Oppg. 1

a) $X_A \sim Pois(\lambda_A \cdot 365)$, $X_B \sim Pois(\lambda_B \cdot 365)$

i) $E[X_A] = \lambda_A \cdot 365 = \underline{\underline{15}}$

ii) Tabell:

$P(X_A) \leq 15 = \underline{\underline{0,5681}}$

iii) $P(X_A \leq 15, X_B \leq 10) = P(X_A \leq 15) P(X_B \leq 10)$

Tabellverdier Poissonfordeling:

$= 0,5681 \cdot 0,9863 = \underline{\underline{0,5603}}$

iiii) Uavhengige stokastiske variabler:

$P(X_A \leq 15 \mid X_B \leq 10) = P(X_A \leq 15) = \underline{\underline{0,5681}}$

b) $Z = X_A + X_B$

$M_{X+Y} = M_X \cdot M_Y = e^{\mu_A(e^t-1)} e^{\mu_B(e^t-1)}$

$= e^{(\mu_A+\mu_B)(e^t-1)}$

Vi ser at X+Y får en momentgenererende funksjon på formen $M_A = e^{\mu(e^t-1)}$ hvor parameteren blir $\mu = \mu_A + \mu_B$

c) $P(X_A = x_A \mid Z = z)$

$$= \frac{P(X_A = x_A \cap X_B = z - x_A)}{P(Z = z)}$$

$\mu_{X_A} = 15$, $\mu_{X_B} = 5$, $\mu_Z = \mu_A + \mu_B = 20$

$$\Rightarrow \frac{f_{X_A}(x_A) \cdot f_{X_B}(z - x_A)}{f_Z(z)}$$

$$\frac{\ldots x_A \ldots \quad x_B \ldots}{f_Z(z)}$$

$$= \frac{\left(\dfrac{\mu_{x_A}^{x_A} e^{-\mu_A}}{x_A!}\right)\left(\dfrac{\mu_{x_B}^{z-x_A} e^{-\mu_B}}{(z-x_A)!}\right)}{\left(\dfrac{\mu_z^{z} e^{-\mu_z}}{z!}\right)}$$

$$= \frac{\mu_{x_A}^{x_A} \cdot \mu_{x_B}^{z-x_A}}{\mu_z^{z}} \cdot \frac{z!}{(z-x_A)!} = \frac{\mu_{x_A}^{x_A} \cdot \mu_{x_B}^{z-x_A}}{(\mu_A+\mu_B)^{z}} \cdot \binom{z}{x_A}$$

Hvis $\mu_A + \mu_B = 1$, så får vi

$\mu_{x_A}^{x_A} \cdot \mu_{x_B}^{z-x_A} \cdot \binom{z}{x_A}$ som er binomisk fordelt.

Oppg. 2

a) $E[X] = 3500$, $SD[X] = 570$

$$P(X \geq 3000) = P\left(Z \geq \frac{3000-3500}{570}\right) = P(Z \geq -0{,}877)$$
$$= 1 - P(Z \leq -0{,}877) = \underline{\underline{0{,}8106}}$$

$P(3200 \leq X < 4000) = ?$

$P(X < 4000) = P(X \geq 3000) = 0{,}8106$

$P(X \leq 3200) = P(Z \leq -0{,}526) = 0{,}2981$

$P(X < 4000) - P(X \leq 3200) = 0{,}8106 - 0{,}2981$
$= \underline{\underline{0{,}5125}}$

$$P(X \geq 3500 \mid X \geq 3000) = \frac{0{,}5}{P(X \geq 3000)} = \underline{\underline{0{,}617}}$$

$$P(X \leq 4000 \mid X \geq 3200) = \frac{P(X \leq 4000) - P(X \leq 3200)}{P(X \geq 3200)}$$

$= \frac{0{,}5125}{\ldots} = 0{,}72$

$P(X \geq 3200)$

$$= \frac{0{,}5125}{1-0{,}2981} = \underline{\underline{0{,}73}}$$

b) i) $P(X < C) = 0{,}1$

$P(Z > z_a) = 0{,}1 \Rightarrow z_a = 2{,}326$

$P(Z < -2{,}326) = 0{,}1$

$$\frac{X - 3500}{570} = -2{,}326$$

$\Rightarrow X = \underline{\underline{2174g}}$

ii) - Vi må anta at babyenes fødselsvekt er uavhengige
- To mulige utfall: over eller under 2174g
- 0,01 sjanse for alle babyer er under 2174g

iii) $Y \sim B(100, 0{,}01)$

$$P(Y \geq 1) = 1 - P(Y=0) = 1 - \binom{100}{0} 0{,}01^0 \cdot 0{,}99^{100}$$
$$= \underline{\underline{0{,}634}}$$

$$P(Y \geq 2 \mid Y \geq 1) = \frac{P(Y \geq 2 \cap Y \geq 1)}{P(Y \geq 1)}$$

$$= \frac{P(Y \geq 2)}{P(Y \geq 1)} = \frac{1 - P(Y=0) - P(Y=1)}{0{,}634}$$

$$= \frac{0{,}634 - \binom{100}{1} 0{,}01^1 \cdot 0{,}99^{99}}{0{,}634} = \underline{\underline{0{,}417}}$$

Oppg. 3 a) $P(X_1 \geq 2)$

$$f(x) \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{ellers} \end{cases}$$

$F(x) = 1 - e^{-\lambda x}$

$P(X_1 \geq 2) = 1 - F(2) = \underline{\underline{e^{-2\lambda}}}$

$P(X_1 + X_2 \geq 4)$

$$= \iint f_1(x) f_2(x)\, dx\, dy$$

$$= \iint_{x+y \geq 4} f_1(x) f_2(x)\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty \lambda^2 e^{-\lambda x} e^{-\lambda y}\, dx\, dy - \int_0^4 \int_0^{4-y} \lambda^2 e^{-\lambda x} e^{-\lambda y}\, dx\, dy$$

$$= 1 - e^{-4\lambda}(-4\lambda + e^{4\lambda} - 1) = \underline{\underline{4\lambda e^{-4\lambda} + e^{-4\lambda}}}$$

$$P(X_1 + X_2 \geq 4 \mid X_1 \geq 2) = \frac{P(X_1 + X_2 \geq 4 \cap X_1 \geq 2)}{P(X_1 \geq 2)}$$

$P(X_1 + X_2 \geq 4 \cap X_1 \geq 2)$

$$= \int_2^\infty \int_0^\infty f_1(x) f_2(y)\, dy\, dx - \int_2^4 \int_0^{4-x} f_1(x) f_2(y)\, dy\, dx$$

$$= \lambda^2 \int_2^\infty \int_0^\infty e^{-\lambda(x+y)}\, dy\, dx - \lambda^2 \int_2^4 \int_0^{4-x} e^{-\lambda(x+y)}\, dy\, dx$$

$$= e^{-2\lambda} + 2\lambda e^{-4\lambda} - e^{2\lambda} + e^{-4\lambda}$$

$$= 2\lambda e^{-4\lambda} + e^{-4\lambda}$$

$$P(X_1 + X_2 \geq 4 \mid X_1 \geq 2) = \underline{\underline{\frac{2\lambda e^{-4\lambda} + e^{-4\lambda}}{e^{-2\lambda}}}}$$

For $\lambda = 0,5$:

$$P(X_1 \geq 2) = e^{-2 \cdot 0,5} = 0,368$$

$$P(X_1 + X_2 \geq 4) = 4 \cdot 0,5 e^{-4 \cdot 0,5} + e^{-4 \cdot 0,5} = 0,406$$

$$P(X_1 + X_2 \geq 4 \mid X_1 \geq 2) = \frac{e^{-2} + e^{-2}}{e^{-1}} = \underline{\underline{0,736}}$$

b) $X \sim \Gamma(\alpha, \beta)$

$$\Rightarrow f(x) = \frac{1}{\beta^\alpha \Gamma(\ )} x^{\alpha-1} e^{-\frac{x}{\beta}}$$

$$\Rightarrow f(x) = \frac{1}{\beta^a \Gamma(a)} x^{a-1} e^{-\frac{x}{\beta}}$$

for $a=1,\ \beta = \frac{1}{\lambda}$:

$$f(x) = \frac{1}{(\frac{1}{\lambda})^1 \Gamma(1)} x^{1-1} e^{-\left(\frac{x}{\frac{1}{\lambda}}\right)} = \lambda e^{-\lambda x}$$

Vi ser at $X \sim Exp(\lambda)$

c) Formelhefte:

$$M_X = \frac{1}{(1-\beta t)^a} \quad \text{for } X \sim \Gamma(a,\beta)$$

$$M_Y = \frac{1}{1-\beta t} \quad \text{for } Y \sim Exp\left(\frac{1}{\beta}\right)$$

Anta at $\frac{\lambda^{n-1}}{\Gamma(n-1)} x^{n-2} e^{-\lambda x} = f_{Z_{n-1}}$

siden Z_{n-1} og X_n er uavhengige

$$Z_n = Z_{n-1} + X_n$$

$$M_{Z_n} = M_{Z_{n-1}} \cdot M_{X_n} = \left(\frac{1}{1-\frac{t}{\lambda}}\right)^{n-1} \left(\frac{1}{1-\frac{t}{\lambda}}\right)$$

$= \left(\frac{1}{1-\frac{t}{\lambda}}\right)^n$ Dette er gammafordelingen sin momentgenererende funksjon med $a=n$ og $\beta = \frac{1}{\lambda}$ $\blacksquare$

Oppg. 4

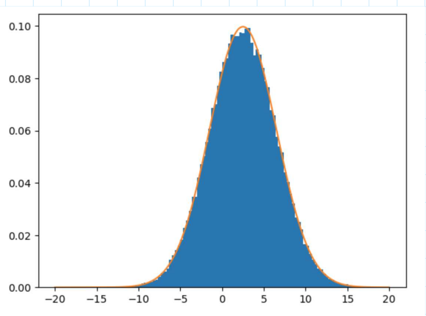
a) $X \sim N(\mu, \sigma^2)$

$Y = aX + b$

$E[Y] = E[aX+b] = a\mu + b$

$Var[Y] = Var[aX+b] = a^2 Var[X] + Var[b]$

$= \underline{\underline{a^2\sigma^2}}$



```python
import numpy as np
import matplotlib.pyplot as plt
n = 100000
my = 1
sigma = 2
a = 2
b = 0.5

def simulateY(n, my, sigma, a, b):
    normal_array = np.random.normal(my, sigma, n)
    y_array = normal_array*a+b
    return y_array

def normalfordeling(x, my, sigma):
    return 1/np.sqrt(2*np.pi)/sigma * np.exp(-0.5*((x-my)/sigma)**2)

data = simulateY(n, my, sigma, a, b)
x = np.linspace(-20,20,400+1)
plt.hist(data,100,density = True)
plt.plot(x, normalfordeling(x,a*my+b,(a*sigma)))
plt.show()
```

b) $Y = u(X)$

$g_Y(y) = f_X(w(y)) \cdot |w'(y)|$

$w(y)$ er $u(x)$ sin invers

$u(x) = ax + b \Rightarrow w(y) = \frac{y-b}{a}$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

$$f_X(w(y)) \cdot |w'(y)| = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{\frac{y-b}{a}-\mu}{\sigma}\right)^2\right\} \frac{1}{a}$$

$$g_Y(y) = \frac{1}{\sqrt{2\pi}a\sigma} \exp\left\{-\frac{1}{2}\left(\frac{y-b-a\mu}{a\sigma}\right)^2\right\}$$

$\sigma_{ny} = a\sigma, \quad \mu_{ny} = b + a\mu$

$\Rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$

Oppg 5. a) $M_{X_i}(t) = E[e^{tX_i}]$

$$E[g(x)] = \sum_x g(x) P(X=x)$$

$$E[e^{tX_i}] = \sum_{x_i} e^{tX_i} P(X_i = x_i)$$

$$= e^{-t} \cdot \frac{1}{2} + e^{t} \cdot \frac{1}{2} = \frac{e^{-t} + e^{t}}{2}$$

$M^{(r)}_{X_i}(0) = E[X_i^r]$

$M'_{X_i}(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$

$M'_{X_i}(0) = 0 = E[X_i]$

$M''_{X_i}(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$

$$M''_{X_i}(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$

$$E[X_i^2] = M''_{X_i}(0) = 1$$

$$Var[X_i] = E[X_i^2] - E[X]^2 = \underline{\underline{1}}$$

$$E[\bar{X}] = E\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right)$$

$$= E\left[\frac{X_1}{n}\right] + E\left[\frac{X_2}{n}\right] + \dots + E\left[\frac{X_n}{n}\right] = \underline{\underline{0}}$$

$$Var[\bar{X}] = Var\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right]$$

$$= Var\left[\frac{X_1}{n}\right] + Var\left[\frac{X_2}{n}\right] + \dots + Var\left[\frac{X_n}{n}\right]$$

$$= \frac{1}{n^2}\cdot 1 + \frac{1}{n^2}\cdot 1 \dots = \frac{n}{n^2} = \underline{\underline{\frac{1}{n}}}$$

b) $\bar{X} = \frac{X_1}{n} + \frac{X_2}{n} + \dots + \frac{X_n}{n}$

$$M_{\bar{X}}(t) = \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) = \prod_{i=1}^{n} \cosh\frac{t}{n} = \cosh^n \frac{t}{n}$$

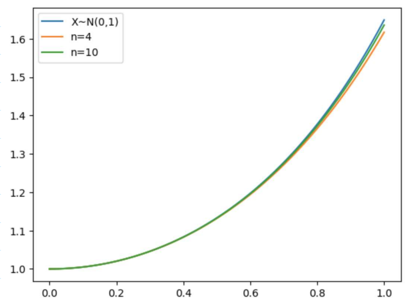
$$M_U(t) = E\left[e^{t\left(\frac{\bar{X} - E[\bar{X}]}{\sqrt{Var[\bar{X}]}}\right)}\right] = E\left[e^{t\left(\frac{\bar{X}}{\sqrt{\frac{1}{n}}}\right)}\right] = E\left[e^{t\sqrt{n}\bar{X}}\right]$$

$$M_U(t) = \cosh^n \frac{t}{n}\sqrt{n} = \cosh^n = \frac{t}{\sqrt{n}}$$

$$\ln M_U(t) = \ln\left(\cosh^n \frac{t}{\sqrt{n}}\right)$$

$$\ln M_U(t) = n \ln \cosh\frac{t}{\sqrt{n}}$$

$$\ln M_U(t) = n \ln \frac{e^{-\frac{t}{\sqrt{n}}} + e^{\frac{t}{\sqrt{n}}}}{2}$$

c) 

Tok ikke med n=100 fordi da ligger den oppå Mx.
Ser ut som at sentralgrenseteoremet stemmer, siden de blir mer og mer identiske med større n

```python
import numpy as np
import matplotlib.pyplot as plt
def M_Z(t):
    return np.exp(t**2/2)
def M_U(t,n):
    return ((np.exp(t/np.sqrt(n))+np.exp(-t/np.sqrt(n)))/2)**n
def lnM_Z(t):
    return t**2/2
def lnM_U(t,n):
    return n * np.log(np.cosh(t/np.sqrt(n)))
T = np.linspace(0,1,101)
plt.plot(T,M_Z(T), label = "X~N(0,1)")
plt.plot(T,M_U(T,4), label = "n=4")
plt.plot(T,M_U(T,10), label = "n=10")
#plt.plot(T,M_U(T,100), label = "n=100")


plt.legend()
plt.show()
```

$$M_Z = e^{\frac{t^2}{2}}$$

$$M_U(t) = \left(\cosh \frac{t}{\sqrt{n}}\right)^n$$

$$\lim_{n\to\infty} \ln M_U(t) = \lim_{n\to\infty} n \ln \cos \frac{t}{\sqrt{n}}$$

Taylorutvikler for å gjøre det lettere:

$$f(t) = n \ln \cosh \frac{t}{\sqrt{n}} \rightarrow f(0) = 0$$

$$f'(t) = \frac{n}{\cosh \frac{t}{\sqrt{n}}} \cdot \sinh \frac{t}{\sqrt{n}} \frac{1}{\sqrt{n}} = \sqrt{n} \tanh \frac{t}{\sqrt{n}} \qquad f'(0) = 0$$

$$f''(t) = 1 - \tanh^2 \frac{t}{\sqrt{n}} \qquad f''(0) = 1$$

$$\ln M_U(t) \sim 0 + 0 \cdot t + \frac{1}{2} t^2 = \frac{1}{2} t^2$$

$$\lim_{n\to\infty} \ln M_U(t) = \frac{t^2}{2}$$