

Fundamentals of Linear Algebra and Optimization

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Homework 3

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Problem B1 (20 pts).

Let us prove this first for two subspaces. $E = U_1 \oplus U_2$ if and only if

$$(1) \ E = U_1 + U_2$$

and

$$(2) \ U_1 \cap U_2 = \{0\}$$

For (1), every vector $e \in E$ can be uniquely written as $e = u_{11} + u_{21}$ with $u_{11} \in U_1$ and $u_{21} \in U_2$.

For (2), let $e \in U_1 \cap U_2$. Since $e \in U_1$ and $e \in U_2$, then we can write,

$$(1) \ e = e + 0 \text{ where } e \in U_1 \text{ and } 0 \in U_2 \text{ and}$$

$$(2) \ e = 0 + 0 \text{ where } 0 \in U_1 \text{ and } e \in U_2.$$

But $e = u_{11} + u_{21}$ is unique, so $e = 0$. Since $E = U_1 + U_2$, we will check uniqueness. Suppose $e = u_{11} + u_{21}$ and $e = u_{12} + u_{22}$ where $u_{11}, u_{12} \in U_1$ and $u_{21}, u_{22} \in U_2$. Then $u_{11} + u_{21} = u_{12} + u_{22}$, so $u_{11} - u_{12} = u_{22} - u_{21}$. Let x be a vector such that $x = u_{11} - u_{12} = u_{22} - u_{21}$. Then $x \in U_1$ and $x \in U_2$, and $u_{11} = u_{12}$ and $u_{22} = u_{21}$, so $x \in U_1 \cap U_2 = \{0\}$.

Now by induction we can extend our logic for any number of $p \geq 2$ subspaces of some vector space E .

Problem B2 (50 pts).

$$(1)$$

By definition an involution is a function f that is its own inverse.

$f : E \rightarrow E$ is in an involution when $\forall x \in E : f(f(x)) = x$. Since $f^{-1} = f$, we just have to check that $f(f(x)) = x$ for all x in the domain of f .

Let us find the inverse of $f(x) = a - x$ where a is any real number. Let,

$$f(x) = a - x$$

$$y = a - x$$

$$f^{-1} = a - y$$

$$x = a - y$$

$$y = a - x$$

so $f(x) = f^{-1}(x)$. So it's an inverse of itself.

(2)

$$\text{Let } u_1 = \frac{u+f(u)}{2} \text{ and } u_{-1} = \frac{u-f(u)}{2}.$$

$$\begin{aligned} \text{Next, we can find that we have something in both the spaces. } f(u_1) &= f\left(\frac{u+f(u)}{2}\right) = \\ \frac{f(u)+f(f(u))}{2} &= \frac{f(u)+u}{2} = u_1 \\ f(u_{-1}) &= f\left(\frac{u-f(u)}{2}\right) = \frac{f(u)-f(f(u))}{2} = \frac{f(u)-u}{2} = u_{-1}. \end{aligned}$$

Next we need to look at uniqueness. Let $v_1 \in E_1$ and $v_{-1} \in E_{-1}$.

$$f(v_1) = v_1$$

$$f(v_1) = -v_1$$

$v_1 = -v_1$ can only have this if $v_1 = -v_1 = 0$.

$$(3) \text{ Let the basis of } E \text{ be } (\xi_i), i = 1, 2, \dots, n, \text{ where } \xi_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \text{ Define}$$

$$f(\xi_i) = \begin{cases} \xi_i & i = 1, 2, \dots, k \\ -\xi_i & i = k+1, k+2, \dots, n \end{cases}. \text{ Then } \forall u \in E, \text{ we can write as } u = \sum_i \lambda_i \xi_i \text{ and}$$

$$f(u) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \\ -\xi_{k+1} \\ -\xi_{k+2} \\ \vdots \\ -\xi_n \end{pmatrix} = I_{k,n-k} \cdot u. \text{ And } (f \circ f)(u) = I_{k,n-k} I_{k,n-k} I_{k,n-k} u = I u = u. \text{ Thus } f \text{ is an}$$

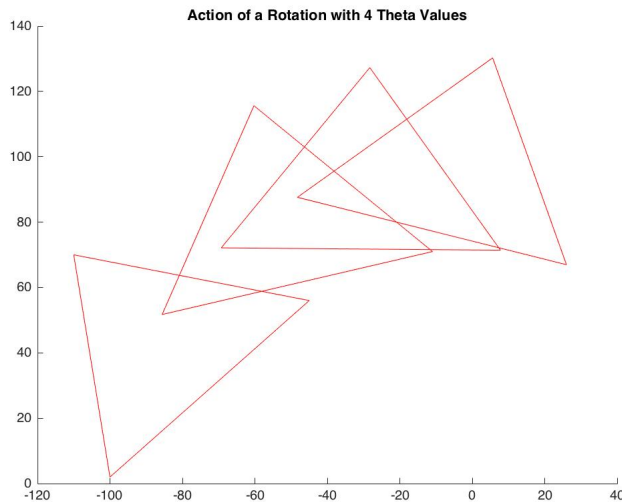
involution.

Geometric interpretation of the action of f is that $f(u)$ is the reflection of u in some axes. When $k = n-1$, we can easily find that $f(u)$ differs from u only in the last entry $f(u)_i = -u_i$, so it is the reflection across the n^{th} axis.

Problem B3 (50 pts). A rotation R_θ in the plane \mathbb{R}^2 is given by the matrix

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

(1) Use **Matlab** to show the action of a rotation R_θ on a simple figure such as a triangle or a rectangle, for various values of θ , including $\theta = \pi/6, \pi/4, \pi/3, \pi/2$.



(2) Prove that R_θ is invertible and that its inverse is $R_{-\theta}$.

$$\begin{aligned} & \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix} \cdot \begin{pmatrix} \cos(-1) & -\sin(-1) \\ \sin(-1) & \cos(-1) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} (\cos \theta \cdot \cos(-\theta)) \cdot (-\sin \theta \sin(-\theta)) & -\cos(\theta)0 \\ \sin(\theta)\cos(-\theta) + \cos\theta \sin(-\theta) & 1 \cos(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(3) Because $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, we have

$$\begin{aligned} R_\alpha \circ R_\beta &= \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \cdot \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \\ &= R_\beta \circ R_\alpha = R_{\alpha+\beta} \end{aligned}$$

We can get

$$\begin{aligned} 1. & (R_\alpha \circ R_\beta) \circ R_\gamma = R_{\alpha+\beta+\gamma} = R_\alpha \circ (R_\beta \circ R_\gamma) \\ 2. & R_\alpha \circ I = I \circ R_\alpha = R_\alpha \\ 3. & R_\alpha \circ R_{-\alpha} = I \\ 4. & R_\alpha \circ R_\beta = R_{\alpha+\beta} = R_\beta \circ R_\alpha \end{aligned}$$

So rotations in the plane form a commutative group.

Problem B4 (110 pts). Consider the affine map $R_{\theta, (a_1, a_2)}$ in \mathbb{R}^2 given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

(1) Prove that if $\theta \neq k2\pi$, with $k \in \mathbb{Z}$, then $R_{\theta, (a_1, a_2)}$ has a unique fixed point (c_1, c_2) , that is, there is a unique point (c_1, c_2) such that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = R_{\theta, (a_1, a_2)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and this fixed point is given by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2 \sin(\theta/2)} \begin{pmatrix} \cos(\pi/2 - \theta/2) & -\sin(\pi/2 - \theta/2) \\ \sin(\pi/2 - \theta/2) & \cos(\pi/2 - \theta/2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

(2) In this question, we still assume that $\theta \neq k2\pi$, with $k \in \mathbb{Z}$. By translating the coordinate system with origin $(0, 0)$ to the new coordinate system with origin (c_1, c_2) , which means that if (x_1, x_2) are the coordinates with respect to the standard origin $(0, 0)$ and if (x'_1, x'_2) are the coordinates with respect to the new origin (c_1, c_2) , we have

$$\begin{aligned} x_1 &= x'_1 + c_1 \\ x_2 &= x'_2 + c_2 \end{aligned}$$

and similarly for (y_1, y_2) and (y'_1, y'_2) , then show that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = R_{\theta, (a_1, a_2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

becomes

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = R_{\theta} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}.$$

Conclude that with respect to the new origin (c_1, c_2) , the affine map $R_{\theta, (a_1, a_2)}$ becomes the rotation R_{θ} . We say that $R_{\theta, (a_1, a_2)}$ is a *rotation of center* (c_1, c_2) .

(3) Use **Matlab** to show the action of the affine map $R_{\theta, (a_1, a_2)}$ on a simple figure such as a triangle or a rectangle, for $\theta = \pi/3$ and various values of (a_1, a_2) . Display the center (c_1, c_2) of the rotation.

What kind of transformations correspond to $\theta = k2\pi$, with $k \in \mathbb{Z}$?

(4)

$$R_{\theta, (a_1, a_2)}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & a_1 \\ \sin \theta & \cos \theta & a_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$$

$$R_{-\theta, (b_1, b_2)}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) & b_1 \\ \sin(-\theta) & \cos(-\theta) & b_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & b_1 \\ -\sin(\theta) & \cos(\theta) & b_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$$

$$\begin{aligned} R_{\theta, (a_1, a_2)} \cdot R_{-\theta, (b_1, b_2)} &= \begin{pmatrix} \cos \theta & -\sin \theta & a_1 \\ \sin \theta & \cos \theta & a_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) & b_1 \\ -\sin(\theta) & \cos(\theta) & b_2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \cos \theta \sin \theta & \cos \theta b_1 - \sin \theta b_2 + a_1 \\ \cos \theta \sin \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta & \sin \theta b_1 + \cos \theta b_2 + a_2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

So $\cos \theta b_1 - \sin \theta b_2 + a_1 = 0$ and $\sin \theta b_1 + \cos \theta b_2 + a_2 = 0$.

$$\begin{aligned}
&\text{Solve for } b_1, \\
b_1 &= \frac{-a_1 + \sin \theta b_2}{\cos \theta} \\
b_1 &= \frac{-a_1}{\cos \theta} + \frac{\sin \theta b_2}{\cos \theta} \\
b_1 &= -\sec \theta a_1 + \tan \theta b_2
\end{aligned}$$

$$\begin{aligned}
&\text{Solve for } b_2, \\
b_2 &= \frac{-a_2 + \sin \theta b_1}{\cos \theta} \\
b_2 &= \frac{-a_2}{\cos \theta} - \frac{\sin \theta b_1}{\cos \theta} \\
b_2 &= \sec \theta a_2 - \tan \theta b_1
\end{aligned}$$

$$\begin{aligned}
&\text{Now plug in } b_2 \text{ into } b_1. \\
b_1 &= -\sec \theta a_1 + \tan \theta (\sec \theta a_2 - \tan \theta b_1) \\
b_1 &= \sec \theta a_1 - \tan \theta \sec \theta a_2 - \tan^2 \theta b_1 \\
(1 - \tan^2 \theta) b_1 &= \sec \theta a_1 - \tan \theta \sec \theta a_2 \\
b_1 &= \frac{-\tan \theta}{\sec \theta} a_2 - \cos \theta a_1 \\
b_1 &= -\sin \theta a_2 - \cos \theta a_1
\end{aligned}$$

Now plug b_1 into b_2 we get, $b_2 = \sin \theta a_1 - \cos \theta a_1$.

The final answer for b_1 and b_2 in terms of θ , a_1 , and a_2 is,

$$b_1 = -\sin \theta a_2 - \cos \theta a_1$$

$$b_2 = \sin \theta a_1 - \cos \theta a_1$$

(5) Given two affine maps $R_{\alpha, (a_1, a_2)}$ and $R_{\beta, (b_1, b_2)}$, prove that

$$R_{\beta, (b_1, b_2)} \circ R_{\alpha, (a_1, a_2)} = R_{\alpha + \beta, (t_1, t_2)}$$

for some (t_1, t_2) , and find (t_1, t_2) in terms of β , (a_1, a_2) and (b_1, b_2) .

Even in the case where $(a_1, a_2) = (0, 0)$, prove that in general

$$R_{\beta, (b_1, b_2)} \circ R_{\alpha} \neq R_{\alpha} \circ R_{\beta, (b_1, b_2)}.$$

Use (4)-(5) to show that the affine maps of the plane defined in this problem form a nonabelian group denoted $\mathbf{SE}(2)$.

Prove that $R_{\beta, (b_1, b_2)} \circ R_{\alpha, (a_1, a_2)}$ is not a translation (possibly the identity) iff $\alpha + \beta \neq k2\pi$, for all $k \in \mathbb{Z}$. Find its center of rotation when $(a_1, a_2) = (0, 0)$.

If $\alpha + \beta = k2\pi$, then $R_{\beta, (b_1, b_2)} \circ R_{\alpha, (a_1, a_2)}$ is a pure translation. Find the translation vector of $R_{\beta, (b_1, b_2)} \circ R_{\alpha, (a_1, a_2)}$.

Problem B5 (80 pts). A subset \mathcal{A} of \mathbb{R}^n is called an *affine subspace* if either $\mathcal{A} = \emptyset$, or there is some vector $a \in \mathbb{R}^n$ and some subspace U of \mathbb{R}^n such that

$$\mathcal{A} = a + U = \{a + u \mid u \in U\}.$$

We define the dimension $\dim(\mathcal{A})$ of \mathcal{A} as the dimension $\dim(U)$ of U .

(1) Because U is a subspace of \mathbb{R}^n , then $0 \in U$, we have $a + 0 \in \mathcal{A}$

What are affine subspaces of dimension 0? What are affine subspaces of dimension 1 (begin with \mathbb{R}^2)? What are affine subspaces of dimension 2 (begin with \mathbb{R}^3)?

Prove that any nonempty affine subspace is closed under affine combinations.

(2) At least from (1) we know $a \in \mathcal{A}$, \mathcal{A} is not empty. For any $b \in \mathcal{A}$, we can find u , write as $b = a + u$. Then $b - a \in U$, $a + U = a + b - a + U = b + U$.

(3) Choose $x - a \in U_a$, $y - a \in U_a$, $\lambda, \mu \in \mathbb{R}$, then compute $\lambda(x - a) + \mu(y - a) = \frac{1}{\lambda + \mu}(\frac{\lambda}{\lambda + \mu}(x - a) + \frac{\mu}{\lambda + \mu}(y - a)) = \frac{\lambda}{\lambda + \mu}x + \frac{\mu}{\lambda + \mu}y - a$. Because \mathcal{A} is closed under affine combinations, we have $\frac{\lambda}{\lambda + \mu}x + \frac{\mu}{\lambda + \mu}y \in \mathcal{A}$, then $\lambda(x - a) + \mu(y - a) = \frac{\lambda}{\lambda + \mu}x + \frac{\mu}{\lambda + \mu}y - a \in \mathbb{R}^n$, U_a is a subspace of \mathbb{R}^n .

For any $u \in U_a$, we can write as $u = x - a = x + (b - a) - b$. The sum of coefficients of x, b, a is $1 - 1 + 1 = 1$, and \mathcal{A} is closed under affine combinations, then $u = x + (b - a) - b \in \mathcal{A} - b \Rightarrow u \in U_b$. U_a does not depend on the choice of $a \in \mathcal{A}$, $U_a = U_b$ for all $a, b \in \mathcal{A}$. So $U_a = U = \{y - x \in \mathbb{R}^n \mid x, y \in \mathcal{A}\}$ and $\mathcal{A} = a + U$, for any $a \in \mathcal{A}$.

(4) If $\mathcal{A} \cap \mathcal{B} = (x)$, then we can find $x = a + u = b + v$ where $a \in \mathcal{A}, b \in \mathcal{B}, u, v \in U$. Write $b = a + u - v$, thus $b \in \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{B}$, which is against $\mathcal{A} \neq \mathcal{B}$. We have $\mathcal{A} \cap \mathcal{B} = \emptyset$ if $\mathcal{A} \neq \mathcal{B}$ and they are parallel.

Problem B6 (120 pts). (Affine frames and affine maps) For any vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, let $\widehat{v} \in \mathbb{R}^{n+1}$ be the vector $\widehat{v} = (v_1, \dots, v_n, 1)$. Equivalently, $\widehat{v} = (\widehat{v}_1, \dots, \widehat{v}_{n+1}) \in \mathbb{R}^{n+1}$ is the vector defined by

$$\widehat{v}_i = \begin{cases} v_i & \text{if } 1 \leq i \leq n, \\ 1 & \text{if } i = n + 1. \end{cases}$$

(1) For any $m + 1$ vectors (u_0, u_1, \dots, u_m) with $u_i \in \mathbb{R}^n$ and $m \leq n$, prove that if the m vectors $(u_1 - u_0, \dots, u_m - u_0)$ are linearly independent, then the $m + 1$ vectors $(\widehat{u}_0, \dots, \widehat{u}_m)$ are linearly independent.

(2) Prove that if the $m + 1$ vectors $(\widehat{u}_0, \dots, \widehat{u}_m)$ are linearly independent, then for any choice of i , with $0 \leq i \leq m$, the m vectors $u_j - u_i$ for $j \in \{0, \dots, m\}$ with $j - i \neq 0$ are linearly independent.

Any $m + 1$ vectors (u_0, u_1, \dots, u_m) such that the $m + 1$ vectors $(\widehat{u}_0, \dots, \widehat{u}_m)$ are linearly independent are said to be *affinely independent*.

From (1) and (2), the vector (u_0, u_1, \dots, u_m) are affinely independent iff for any any choice of i , with $0 \leq i \leq m$, the m vectors $u_j - u_i$ for $j \in \{0, \dots, m\}$ with $j - i \neq 0$ are linearly independent. If $m = n$, we say that $n + 1$ affinely independent vectors (u_0, u_1, \dots, u_n) form an *affine frame* of \mathbb{R}^n .

(3) if (u_0, u_1, \dots, u_n) is an affine frame of \mathbb{R}^n , then prove that for every vector $v \in \mathbb{R}^n$, there is a unique $(n + 1)$ -tuple $(\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n+1}$, with $\lambda_0 + \lambda_1 + \dots + \lambda_n = 1$, such that

$$v = \lambda_0 u_0 + \lambda_1 u_1 + \dots + \lambda_n u_n.$$

The scalars $(\lambda_0, \lambda_1, \dots, \lambda_n)$ are called the *barycentric* (or *affine*) *coordinates* of v w.r.t. the affine frame (u_0, u_1, \dots, u_n) .

If we write $e_i = u_i - u_0$, for $i = 1, \dots, n$, then prove that we have

$$v = u_0 + \lambda_1 e_1 + \dots + \lambda_n e_n,$$

and since (e_1, \dots, e_n) is a basis of \mathbb{R}^n (by (1) & (2)), the n -tuple $(\lambda_1, \dots, \lambda_n)$ consists of the standard coordinates of $v - u_0$ over the basis (e_1, \dots, e_n) .

Conversely, for any vector $u_0 \in \mathbb{R}^n$ and for any basis (e_1, \dots, e_n) of \mathbb{R}^n , let $u_i = u_0 + e_i$ for $i = 1, \dots, n$. Prove that (u_0, u_1, \dots, u_n) is an affine frame of \mathbb{R}^n , and for any $v \in \mathbb{R}^n$, if

$$v = u_0 + x_1 e_1 + \dots + x_n e_n,$$

with $(x_1, \dots, x_n) \in \mathbb{R}^n$ (unique), then

$$v = (1 - (x_1 + \dots + x_n))u_0 + x_1 u_1 + \dots + x_n u_n,$$

so that $(1 - (x_1 + \dots + x_n), x_1, \dots, x_n)$, are the barycentric coordinates of v w.r.t. the affine frame (u_0, u_1, \dots, u_n) .

The above shows that there is a one-to-one correspondence between affine frames (u_0, \dots, u_n) and pairs $(u_0, (e_1, \dots, e_n))$, with (e_1, \dots, e_n) a basis. Given an affine frame (u_0, \dots, u_n) , we obtain the basis (e_1, \dots, e_n) with $e_i = u_i - u_0$, for $i = 1, \dots, n$; given the pair $(u_0, (e_1, \dots, e_n))$ where (e_1, \dots, e_n) is a basis, we obtain the affine frame (u_0, \dots, u_n) , with $u_i = u_0 + e_i$, for $i = 1, \dots, n$. There is also a one-to-one correspondence between barycentric coordinates w.r.t. the affine frame (u_0, \dots, u_n) and standard coordinates w.r.t. the basis (e_1, \dots, e_n) . The barycentric coordinates $(\lambda_0, \lambda_1, \dots, \lambda_n)$ of v (with $\lambda_0 + \lambda_1 + \dots + \lambda_n = 1$) yield the standard coordinates $(\lambda_1, \dots, \lambda_n)$ of $v - u_0$; the standard coordinates (x_1, \dots, x_n) of $v - u_0$ yield the barycentric coordinates $(1 - (x_1 + \dots + x_n), x_1, \dots, x_n)$ of v .

(4) Let (u_0, \dots, u_n) be an affine frame $\in \mathbb{R}^n$, and $(v_0, \dots, v_n) \in \mathbb{R}^m$. There is a unique affine map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $f(u_i) = v_i \forall i = 0, \dots, n$.

Let (u_0, \dots, u_n) be a basis of $E \in \mathbb{R}^n$ and (v_0, \dots, v_n) be a basis for $F \in \mathbb{R}^m$. Let the affine map have the property that $f(u_i) = v_i \forall i = 0, \dots, n$. Because we have affine frame, every vector $u \in \mathbb{R}^n$ can be written uniquely as $u_0 \lambda_0 + u_1 \lambda_1 + \dots + u_n \lambda_n$ where $\sum_i \lambda_i = 1$.

We can write $f(u) = f(u_0 \lambda_0 + u_1 \lambda_1 + \dots + u_n \lambda_n) = \lambda_0 f(u_0) + \lambda_1 f(u_1) + \dots + \lambda_n f(u_n) = \lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_n v_n$. So we can define a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ as above, such that $f(\lambda_0 u_0 + \lambda_1 u_1 + \dots + \lambda_n u_n) = \lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_n v_n$. The function is unique, however we have to check if it is affine.

To check if it is affine, we first pick $w_1, \dots, w_p \in \mathbb{R}^n$ and a set of $\mu_1 + \dots + \mu_p = 1$. We can write linear combination of μ and w , such that, $f(\mu_1 w_1 + \dots + \mu_p w_p) = \mu_1 f(w_1) + \dots + \mu_p f(w_p)$. Now we can rewrite all the w 's as the affine frame, such that,

$$w_1 = \sum_{i=0}^n \lambda_i \mu_i \dots w_p = \sum_{i=0}^n \lambda_p \mu_p$$

Now we can expand and regroup in terms of the frame vectors.

(5) Let (a_0, \dots, a_n) be any affine frame in \mathbb{R}^n and let (b_0, \dots, b_n) be any $n+1$ points in \mathbb{R}^n . Prove that the $(n+1) \times (n+1)$ matrix A corresponding to the unique affine map f such that

$$f(a_i) = b_i, \quad i = 0, \dots, n,$$

is given by

$$A = \begin{pmatrix} \hat{b}_0 & \hat{b}_1 & \dots & \hat{b}_n \end{pmatrix} \begin{pmatrix} \hat{a}_0 & \hat{a}_1 & \dots & \hat{a}_n \end{pmatrix}^{-1}.$$

In the special case where (a_0, \dots, a_n) is the canonical affine frame with $a_i = e_{i+1}$ for $i = 0, \dots, n-1$ and $a_n = (0, \dots, 0)$ (where e_i is the i th canonical basis vector), show that

$$\begin{pmatrix} \hat{a}_0 & \hat{a}_1 & \dots & \hat{a}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \hat{a}_0 & \hat{a}_1 & \dots & \hat{a}_n \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ -1 & -1 & \dots & -1 & 1 \end{pmatrix}.$$

For example, when $n = 2$, if we write $b_i = (x_i, y_i)$, then we have

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 & x_2 - x_3 & x_3 \\ y_1 - y_3 & y_2 - y_3 & y_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

(6) Recall that a nonempty affine subspace \mathcal{A} of \mathbb{R}^n is any nonempty subset of \mathbb{R}^n closed under affine combinations. For any affine map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, for any affine subspace \mathcal{A} of \mathbb{R}^n , and any affine subspace \mathcal{B} of \mathbb{R}^m , prove that $f(\mathcal{A})$ is an affine subspace of \mathbb{R}^m , and that $f^{-1}(\mathcal{B})$ is an affine subspace of \mathbb{R}^n .

Problem B7 (30 pts). Let A be any $n \times k$ matrix

(1) Prove that the $k \times k$ matrix $A^\top A$ and the matrix A have the same nullspace. Use this to prove that $\text{rank}(A^\top A) = \text{rank}(A)$. Similarly, prove that the $n \times n$ matrix AA^\top and the matrix A^\top have the same nullspace, and conclude that $\text{rank}(AA^\top) = \text{rank}(A^\top)$.

We will prove later that $\text{rank}(A^\top) = \text{rank}(A)$.

(2) Let a_1, \dots, a_k be k linearly independent vectors in \mathbb{R}^n ($1 \leq k \leq n$), and let A be the $n \times k$ matrix whose i th column is a_i . Prove that $A^\top A$ has rank k , and that it is invertible. Let $P = A(A^\top A)^{-1}A^\top$ (an $n \times n$ matrix). Prove that

$$P^2 = P$$

$$P^\top = P.$$

What is the matrix P when $k = 1$?

(3) Prove that the image of P is the subspace V spanned by a_1, \dots, a_k , or equivalently the set of all vectors in \mathbb{R}^n of the form Ax , with $x \in \mathbb{R}^k$. Prove that the nullspace U of P is the set of vectors $u \in \mathbb{R}^n$ such that $A^\top u = 0$. Can you give a geometric interpretation of U ?

Conclude that P is a projection of \mathbb{R}^n onto the subspace V spanned by a_1, \dots, a_k , and that

$$\mathbb{R}^n = U \oplus V.$$

Hint. You may use results from HW2.

TOTAL: 460 points.