$\boxed{\textbf{Fall 2016} \quad \textbf{CIS 515}}$

Fundamentals of Linear Algebra and Optimization
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Homework 2

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Problem B1 (10 pts).
Suppose $A = (a_{i,j})_{m\times n}$, $B = (b_{i,j})_{n\times p}$, C $= AB = (c_{i,j})_{m\times p}$. It is easy to write $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, $\forall i = 1, 2, \cdots m$. $j = 1, 2, \cdots p$ and then consider $(A^1B_1 + \cdots + A^nB_n)_{ij}$, we have $(A^1B_1+\cdots+A^nB_n)_{ij} = (a_{i1}b_{1j}+a_{i2}b_{2j}\cdots a_{in}b_{nj}) = \sum_{k=1}^{n} a_{ik}b_{kj}$. So $AB = A^1B_1+\cdots+A^nB_n$.

Problem B2 (10 pts).
Because f is a linear map, thus we have

$$f(x + y) = f(x) + f(y)$$

$$f(\lambda x) = \lambda f(x)$$

So write the inverse function $g = f^{-1}$ and $x_1, x_2 \in E$, we have

$$\begin{aligned}
\exists x \in E,\ x = g(f(x_1) + f(x_2)) &\Rightarrow f(x) = f(x_1) + f(x_2) = f(x_1 + x_2)\\
&\Rightarrow x = x_1 + x_2\\
&\Rightarrow g(f(x)) = g(f(x_1 + x_2)) = g(f(x_1)) + g(f(x_2))\\
\exists x^* \in E,\ x^* = g(f(\lambda x)) &\Rightarrow f(x^*) = f(\lambda x)\\
&\Rightarrow x^* = \lambda x\\
&\Rightarrow g(\lambda f(x)) = g(f(\lambda x)) = \lambda g(f(x))
\end{aligned}$$

Problem B3 (10 pts). Given two vectors spaces E and F, let $(u_i)_{i\in I}$ be any basis of E and let $(v_i)_{i\in I}$ be any family of vectors in F. Prove that the unique linear map $f\colon E \to F$ such that $f(u_i) = v_i$ for all $i \in I$ is surjective iff $(v_i)_{i\in I}$ spans F.

For $x\epsilon E$, it can be written as $x_1u_1 + x_2u_2 + \cdots + x_nu_n$as $(u_i)_{i\epsilon I}$is a basis of E and $(u_i)_{i\epsilon I}$spans E . Then, $f(x) = x_1f(u_1) + x_2f(u_2) + \cdots + x_nf(u_n)$. Since $f(u_i) = v_i$, this can be rewritten as: $f(x) = x_1v_1 + x_2v_2 + \cdots + x_nv_n$. If the linear map f is surjective, then every element in F has a corresponding element in E and $f(x) = x_1v_1 + x_2v_2 + \cdots + x_nv_n$ indicates that $(v_i)_{i\epsilon I}$ must span F.

Problem B4 (10 pts). Let $f \colon E \to F$ be a linear map with $\dim(E) = n$ and $\dim(F) = m$. Prove that f has rank 1 iff f is represented by an $m \times n$ matrix of the form

$$A = uv^\top$$

with u a nonzero column vector of dimension m and v a nonzero column vector of dimension n. $dim(E) = n$ indicates that the basis of E consists of n vectors and $dim(F) = m$ indicates the basis of F consists of m vectors. From class notes, we know that $M(f) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = A$, an mxn matrix.

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix}$ and $v^T = \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{pmatrix}$ so uv^T is an mxn matrix of the form: $\begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & u_2v_3 & \cdots & u_2v_n \\ u_3v_1 & u_3v_2 & u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ u_mv_1 & u_mv_2 & u_mv_3 & \cdots & u_mv_n \end{pmatrix} = v_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + v_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + \cdots v_n \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix}$. Since every column in the matrix of f is a multiple of one column, the rank is 1 for f and the matrix is always of this form by definition for an mxn matrix.

Problem B5 (120 pts). (Haar extravaganza) Consider the matrix

$$W_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

(1)

$$W_{3,3}c = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot c_1 + 1 \cdot c_5 \\ 1 \cdot c_1 + -1 \cdot c_5 \\ 1 \cdot c_2 + 1 \cdot c_6 \\ 1 \cdot c_2 + -1 \cdot c_6 \\ 1 \cdot c_3 + 1 \cdot c_7 \\ 1 \cdot c_3 + -1 \cdot c_7 \\ 1 \cdot c_4 + 1 \cdot c_8 \\ 1 \cdot c_4 + -1 \cdot c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_5 \\ c_1 - c_5 \\ c_2 + c_6 \\ c_2 - c_6 \\ c_3 + c_7 \\ c_3 - c_7 \\ c_4 + c_8 \\ c_4 - c_8 \end{pmatrix}$$

(2)
If the inverse of $W_{3,3}$ is $(1/2)W_{3,3}^{\top}$, then the product of the two matrices is the identity matrix.

$$W_{3,3} \cdot (1/2)W_{3,3}^{\top} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 \end{pmatrix}$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

We have $W_{3,3} = (\alpha_1, \alpha_2, \cdots, \alpha_8)$, then write $\alpha_i^T \alpha_j, \forall i, j = 1, 2, \cdots 8 \ i \neq j$ there are two conditions:
1. α_i and α_j have elements on the same positions.
2. α_i and α_j have elements on different postions.
For the first condition, we have $\alpha_i^T \alpha_j = 1 - 1 = 0$ and for the second condition, we have $\alpha_i^T \alpha_j = 0$. Also, $\alpha_i^T \alpha_i = 1 + 1 = 2$, so we can prove that the columns are orthogonal. It is the same to prove the rows are orthogonal.

(3) Let $W_{3,2}$ and $W_{3,1}$ be the following matrices:

$$
W_{3,2} = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
W_{3,1} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Show that given any vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$, the result $W_{3,2}c$ of applying $W_{3,2}$ to c is

$$
W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8),
$$

the second step in reconstructing a vector from its Haar coefficients, and the result $W_{3,1}c$ of applying $W_{3,1}$ to c is

$$
W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8),
$$

the first step in reconstructing a vector from its Haar coefficients.

Conclude that

$$
W_{3,3}W_{3,2}W_{3,1} = W_3,
$$

the Haar matrix

$$W_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Hint. First, check that

$$W_{3,2}W_{3,1} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},$$

where

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

First: $W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8)$

$$W_{3,2}c = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ c_1 + 0 - c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + c_2 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + c_2 + 0 - c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_3 \\ c_1 - c_3 \\ c_2 + c_4 \\ c_2 - c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

Second: $W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8)$

$$W_{3,1}c = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ c_1 - c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

Now check $W_{3,2} * W_{3,1}$

$$
W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

=)(

(4) Prove that the columns and the rows of $W_{3,2}$ and $W_{3,1}$ are orthogonal. Deduce from this that the columns of W_3 are orthogonal, and the rows of W_3^{-1} are orthogonal. Are the rows of W_3 orthogonal? Are the columns of W_3^{-1} orthogonal? Find the inverse of $W_{3,2}$ and the inverse of $W_{3,1}$.

(5) For any $n \geq 2$, the $2^n \times 2^n$ matrix $W_{n,n}$ is obtained form the two rows

$$
\underbrace{1, 0, \ldots, 0}_{2^{n-1}}, \underbrace{1, 0, \ldots, 0}_{2^{n-1}}
$$
$$
\underbrace{1, 0, \ldots, 0}_{2^{n-1}}, \underbrace{-1, 0, \ldots, 0}_{2^{n-1}}
$$

by shifting them $2^{n-1} - 1$ times over to the right by inserting a zero on the left each time.

Given any vector $c = (c_1, c_2, \ldots, c_{2^n})$, show that $W_{n,n}c$ is the result of the last step in the process of reconstructing a vector from its Haar coefficients c. Prove that $W_{n,n}^{-1} = (1/2)W_{n,n}^{\top}$, and that the columns and the rows of $W_{n,n}$ are orthogonal.

Extra credit (30 pts.)

Given a $m \times n$ matrix $A = (a_{ij})$ and a $p \times q$ matrix $B = (b_{ij})$, the *Kronecker product* (or *tensor product*) $A \otimes B$ of A and B is the $mp \times nq$ matrix

$$
A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.
$$

It can be shown (and you may use these facts without proof) that $\otimes$ is associative and that

$$
\begin{aligned}
(A \otimes B)(C \otimes D) &= AC \otimes BD \\
(A \otimes B)^{\top} &= A^{\top} \otimes B^{\top},
\end{aligned}
$$

whenever AC and BD are well defined.

Check that

$$W_{n,n} = \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix},$$

and that

$$W_n = \begin{pmatrix} W_{n-1} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}.$$

Use the above to reprove that

$$W_{n,n} W_{n,n}^\top = 2I_{2^n}.$$

Let

$$B_1 = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and for $n \geq 1$,

$$B_{n+1} = 2 \begin{pmatrix} B_n & 0 \\ 0 & I_{2^n} \end{pmatrix}.$$

Prove that

$$W_n^\top W_n = B_n, \quad \text{for all } n \geq 1.$$

We can find

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \in M_{2^n, 2^n}$$

and it is easy to get

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in M_{2^n, 2^{n-1}},$$

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \in M_{2^n, 2^{n-1}},$$

so

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 1 & 0 & \cdots & \cdots & -1 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 1 & \vdots \\ 0 & 1 & \cdots & \cdots & 0 & -1 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -1 \end{pmatrix} = W_{n,n}.$$

$$\begin{aligned} W_{n,n}W_{n,n}^\top &= \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}^\top \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}]^\top + I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^\top \end{aligned}$$

according to the facts above, we have

$$\begin{aligned} W_{n,n}W_{n,n}^\top &= [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}][I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top] + [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}][I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}^\top] \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + I_{2^{n-1}} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_{2^n} \end{aligned}$$

(6) The matrix $W_{n,i}$ is obtained from the matrix $W_{i,i}$ $(1 \leq i \leq n-1)$ as follows:

$$W_{n,i} = \begin{pmatrix} W_{i,i} & 0_{2^i,2^n-2^i} \\ 0_{2^n-2^i,2^i} & I_{2^n-2^i} \end{pmatrix}.$$

It consists of four blocks, where $0_{2^i,2^n-2^i}$ and $0_{2^n-2^i,2^i}$ are matrices of zeros and $I_{2^n-2^i}$ is the identity matrix of dimension $2^n - 2^i$.

Explain what $W_{n,i}$ does to c and prove that

$$W_{n,n}W_{n,n-1} \cdots W_{n,1} = W_n,$$

where W_n is the Haar matrix of dimension 2^n.

Hint. Use induction on k, with the induction hypothesis

$$W_{n,k}W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k,2^n-2^k} \\ 0_{2^n-2^k,2^k} & I_{2^n-2^k} \end{pmatrix}.$$

Prove that the columns and rows of $W_{n,k}$ are orthogonal, and use this to prove that the columns of W_n and the rows of W_n^{-1} are orthogonal. Are the rows of W_n orthogonal? Are the columns of W_n^{-1} orthogonal? Prove that

$$W_{n,k}^{-1} = \begin{pmatrix} \frac{1}{2}W_{k,k}^{\top} & 0_{2^k,2^n-2^k} \\ 0_{2^n-2^k,2^k} & I_{2^n-2^k} \end{pmatrix}.$$

Problem B6 (20 pts). Prove that for every vector space E, if $f\colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \operatorname{Ker} f \oplus \operatorname{Im} f,$$

so that f is the projection onto its image $\operatorname{Im} f$.

Problem B7 (20 pts). Let $U_1, \ldots, U_p$ be any $p \geq 2$ subspaces of some vector space E and recall that the linear map

$$a\colon U_1 \times \cdots \times U_p \to E$$

is given by

$$a(u_1, \ldots, u_p) = u_1 + \cdots + u_p,$$

with $u_i \in U_i$ for $i = 1, \ldots, p$.

(1) If we let $Z_i \subseteq U_1 \times \cdots \times U_p$ be given by

$$Z_i = \left\{ \left(u_1, \ldots, u_{i-1}, -\sum_{j=1, j\neq i}^{p} u_j, u_{i+1}, \ldots, u_p \right) \;\middle|\; \sum_{j=1, j\neq i}^{p} u_j \in U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) \right\},$$

for $i = 1, \ldots, p$, then prove that

$$\operatorname{Ker} a = Z_1 = \cdots = Z_p.$$

In general, for any given i, the condition $U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0)$ does not necessarily imply that $Z_i = (0)$. Thus, let

$$Z = \left\{ \left(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_p \right) \;\middle|\; u_i = -\sum_{j=1, j\neq i}^{p} u_j,\ u_i \in U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right),\ 1 \leq i \leq p \right\}.$$

Since $\operatorname{Ker} a = Z_1 = \cdots = Z_p$, we have $Z = \operatorname{Ker} a$. Prove that if

$$U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p,$$

then $Z = \operatorname{Ker} a = (0)$.

(2) Prove that $U_1 + \cdots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p.$$

(3) **Extra credit (40 pts)**. Assume that E is finite-dimensional, and let $f_i \colon E \to E$ be any $p \geq 2$ linear maps such that

$$f_1 + \cdots + f_p = \mathrm{id}_E.$$

Prove that the following properties are equivalent:

(1) $f_i^2 = f_i$, $1 \leq i \leq p$.

(2) $f_j \circ f_i = 0$, for all $i \neq j$, $1 \leq i, j \leq p$.

(1) $\Rightarrow$ (2)
We multiply f_i on each side of the equation, then get

$$f_i^2 + \sum_{j\neq i} f_j \circ f_i = f_i$$

$$\sum_{j\neq i} f_j \circ f_i = 0$$

Because E is finite-dimensional, then for all $y \in E$ we can write $y = \sum \lambda_i x_i$, where (x_i) is the base. Then for all $y \in E$, we have $\sum_{j\neq i} f_j \circ f_i(y) = 0$, suppose there is a k which makes $f_k \circ f_i \neq 0$ then $ker(f_k \circ f_i) = E$, thus $f_k \circ f_i = 0$, which is a contradiction.

(2) $\Rightarrow$ (1)
We multiply f_i, $\forall i = 1, 2, \cdots, p$ on each side of the equation, then get

$$f_i^2 + \sum_{j\neq i} f_j \circ f_i = f_i$$

$$f_i^2 = f_i$$

TOTAL: 200 + 70 points.