Fall 2016 CIS 515

Fundamentals of Linear Algebra and Optimization
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Homework 2

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Problem B1 (10 pts).
Suppose $A = (a_{i,j})_{m\times n}$, $B = (b_{i,j})_{n\times p}$, C $= AB = (c_{i,j})_{m\times p}$. It is easy to write $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, $\forall i = 1, 2, \cdots m$. $j = 1, 2, \cdots p$ and then consider $(A^1B_1 + \cdots + A^nB_n)_{ij}$, we have $(A^1B_1 + \cdots + A^nB_n)_{ij} = (a_{i1}b_{1j} + a_{i2}b_{2j} \cdots a_{in}b_{nj}) = \sum_{k=1}^{n} a_{ik}b_{kj}$. So $AB = A^1B_1 + \cdots + A^nB_n$.

Problem B2 (10 pts).
Because f is a linear map, thus we have

$$f(x + y) = f(x) + f(y)$$

$$f(\lambda x) = \lambda f(x)$$

So write the inverse function $g = f^{-1}$ and $x_1, x_2 \in E$, we have

$$\begin{aligned}
\exists x \in E,\ x = g(f(x_1) + f(x_2)) &\Rightarrow f(x) = f(x_1) + f(x_2) = f(x_1 + x_2)\\
&\Rightarrow x = x_1 + x_2\\
&\Rightarrow g(f(x)) = g(f(x_1 + x_2)) = g(f(x_1)) + g(f(x_2))\\
\exists x^* \in E,\ x^* = g(f(\lambda x)) &\Rightarrow f(x^*) = f(\lambda x)\\
&\Rightarrow x^* = \lambda x\\
&\Rightarrow g(\lambda f(x)) = g(f(\lambda x)) = \lambda g(f(x))
\end{aligned}$$

Problem B3 (10 pts).
For $x \epsilon E$, it can be written as $x_1u_1 + x_2u_2 + \cdots + x_nu_n$as $(u_i)_{i\epsilon I}$is a basis of E and $(u_i)_{i\epsilon I}$spans E . Then, $f(x) = x_1f(u_1) + x_2f(u_2) + \cdots + x_nf(u_n)$. Since $f(u_i) = v_i$, this can be rewritten as: $f(x) = x_1v_1 + x_2v_2 + \cdots + x_nv_n$. If the linear map f is surjective, then every element in F has a corresponding element in E and $f(x) = x_1v_1 + x_2v_2 + \cdots + x_nv_n$ indicates that $(v_i)_{i\epsilon I}$ must span F.

Problem B4 (10 pts).
Let $f\colon E \to F$ be a linear map with $\dim(E) = n$ and $\dim(F) = m$. Prove that f has rank 1

iff f is represented by an $m \times n$ matrix of the form

$$A = uv^\top$$

with u a nonzero column vector of dimension m and v a nonzero column vector of dimension n. $dim(E) = n$ indicates that the basis of E consists of n vectors and $dim(F) = m$ indicates the basis of F consists of m vectors. From class notes, we know that $M(f) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = A$, an mxn matrix.

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix}$ and $v^T = \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{pmatrix}$ so uv^T is an mxn matrix of the form: $\begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & u_2v_3 & \cdots & u_2v_n \\ u_3v_1 & u_3v_2 & u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ u_mv_1 & u_mv_2 & u_mv_3 & \cdots & u_mv_n \end{pmatrix} = v_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + v_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + \cdots v_n \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix}$. Since every column in the matrix of f is a multiple of one column, the rank is 1 for f and the matrix is always of this form by definition for an mxn matrix.

Problem B5 (120 pts).
(1)

$$W_{3,3}c = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot c_1 + 1 \cdot c_5 \\ 1 \cdot c_1 + -1 \cdot c_5 \\ 1 \cdot c_2 + 1 \cdot c_6 \\ 1 \cdot c_2 + -1 \cdot c_6 \\ 1 \cdot c_3 + 1 \cdot c_7 \\ 1 \cdot c_3 + -1 \cdot c_7 \\ 1 \cdot c_4 + 1 \cdot c_8 \\ 1 \cdot c_4 + -1 \cdot c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_5 \\ c_1 - c_5 \\ c_2 + c_6 \\ c_2 - c_6 \\ c_3 + c_7 \\ c_3 - c_7 \\ c_4 + c_8 \\ c_4 - c_8 \end{pmatrix}$$

(2)
If the inverse of $W_{3,3}$ is $(1/2)W_{3,3}^{\top}$, then the product of the two matrices is the identity matrix.

$$W_{3,3}\cdot(1/2)W_{3,3}^{\top} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5+0.5 & 0.5+-0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5+-0.5 & 0.5--0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5+0.5 & 0.5+-0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5+-0.5 & 0.5--0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5+0.5 & 0.5+-0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5+-0.5 & 0.5--0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5+0.5 & 0.5+-0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5+-0.5 & 0.5--0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(3)
First: $W_{3,2}c = (c_1+c_3, c_1-c_3, c_2+c_4, c_2-c_4, c_5, c_6, c_7, c_8)$

$$W_{3,2}c = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1+0+c_3+0+0+0+0+0 \\ c_1+0-c_3+0+0+0+0+0 \\ 0+c_2+0+c_4+0+0+0+0 \\ 0+c_2+0-c_4+0+0+0+0 \\ 0+0+0+0+c_5+0+0+0 \\ 0+0+0+0+0+c_6+0+0 \\ 0+0+0+0+0+0+c_7+0 \\ 0+0+0+0+0+0+0+c_8 \end{pmatrix} = \begin{pmatrix} c_1+c_3 \\ c_1-c_3 \\ c_2+c_4 \\ c_2-c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

Second: $W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8)$

$$W_{3,1}c = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ c_1 - c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

Now check $W_{3,2} * W_{3,1}$

$$W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},$$

Conclude $W_{3,3} * W_{3,2} * W_{3,1} = W_3$

$$W_{3,3} * W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
= \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

We see that in the first row of the $W_{3,3}$ matrix, there is at least 1 1 in the columns 1 through 4 in each of the 8 rows. We are guaranteed that the first column in the W_3 matrix is all 1's because, every 1 in $W_{3,3}$ will be multiplied by at least 1's located in the first 4 rows in the first column of $W_{3,2} * W_{3,1}$. We can apply the same theory to get the second column of W_3. The change here is that there are 2 -1's located in the 3rd and 4th row in the second column of $W_{3,2} * W_{3,1}$. Now, the last 4 rows in the second column of W_3 will be -1. Column 3 in W_3 will have a 1 in the first 2 rows and a -1 in the next two rows because only the first 4 rows in $W_{3,3}$ have 1's in the first two columns. Using the same idea, rows 5 and 6 will have 1 and the last two rows will have -1 in W_3. Finally, the last 4 columns in W_3 will have a 1 and -1. The 1 and -1 in $W_{3,3}$ is in the 5th, 6th, 7th, and 8th columns, and the last 4 columns in $W_{3,2} * W_{3,1}$ are $0s$ in the first 4 rows and an $4x4$ identity matrix in the last 4 rows. With the row and column multiplication, we are guaranteed the shifting 1 and -1 in the last 4 columns of W_3.

(4)

We have $W_{3,3} = (\alpha_1, \alpha_2, \cdots, \alpha_8)$, then write $\alpha_i^T \alpha_j, \forall i, j = 1, 2, \cdots 8 \ i \neq j$ there are two conditions:
1. α_i and α_j have elements on the same positions.
2. α_i and α_j have elements on different postions.
For the first condition, we have $\alpha_i^T \alpha_j = 1 - 1 = 0$ and for the second condition, we have $\alpha_i^T \alpha_j = 0$. Also, $\alpha_i^T \alpha_i = 1 + 1 = 2$, so we can prove that the columns are orthogonal. It is the same to prove the rows are orthogonal.

Inverse of $W_{3,2}$

We see that in $W_{3,2}$, the last 4 rows already have a diagonal with all 1's. The only thing left is to manipulate the first 4 rows. Similar to $W_{3,3}$, the transpose of $W_{3,2}$ is $(1/2)W_{3,2}^\top$ of the first 4 rows and 4 columns.

$$inv(W_{3,2}) = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse of $W_{3,1}$
6 rows of $W_{3,1}$ is already the identity matrix. Thus only the first $2x2$ submatrix in $W_{3,1}$ has to be manipulated to find it's inverse. We can use a similar method as above, to find that we only work that has to be done is taking the $\frac{1}{2}W_{3,1}^{\top}$ of the $2x2$ submatrix.

$$inv(W_{3,1}) = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) We can find that

$$W_{n,n} = \begin{pmatrix} 1 & \overbrace{0\cdots0}^{2^{n-1}-1} & 1 & \overbrace{0\cdots0}^{2^{n-1}-1} & \\ 1 & \overbrace{0\cdots0}^{2^{n-1}-1} & -1 & \overbrace{0\cdots0}^{2^{n-1}-1} & \\ 0 & 1 & \overbrace{\cdots0}^{2^{n-1}-1} & 1 & \overbrace{0\cdots0}^{2^{n-1}-1} \\ 0 & 1 & \overbrace{\cdots0}^{2^{n-1}-1} & -1 & \overbrace{0\cdots0}^{2^{n-1}-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \overbrace{\cdots0}^{2^{n-1}-1} & 1 & \overbrace{\cdots0}^{2^{n-1}-1} & 1 \\ & \overbrace{\cdots0}^{2^{n-1}-1} & 1 & \overbrace{\cdots0}^{2^{n-1}-1} & -1 \end{pmatrix}$$

So

$$W_{n,n}c = \begin{pmatrix} c_1 + c_{2^{n-1}+1} \\ c_1 - c_{2^{n-1}+1} \\ c_2 + c_{2^{n-1}+2} \\ c_2 - c_{2^{n-1}+2} \\ c_3 + c_{2^{n-1}+3} \\ c_3 - c_{2^{n-1}+3} \\ \vdots \\ c_{2^{n-1}} + c_{2^n} \\ c_{2^{n-1}} - c_{2^n} \end{pmatrix}$$

It is the last step in process of reconstructing a vector from its Haar coefficients c.

Write $W_{n,n} = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_{2^n}^T \end{pmatrix}$, So $W_{n,n}^T W_{n,n} = \begin{pmatrix} v_1 & v_2 & v_3 \cdots & v_{2^n} \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_{2^n}^T \end{pmatrix} = \sum_{i=1}^{2^n} v_i v_i^T$. Because we can calculate that $v_i v_i^T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$ there are two 1 in the diagonal. Because we have pairs $u_i = (\underbrace{0, 0, \cdots, 1, \cdots, 0, 0,}_{2^{n-1}} \underbrace{0, 0, \cdots, 1, \cdots, 0, 0}_{2^{n-1}})$ and $v_i = (\underbrace{0, 0, \cdots, 1, \cdots, 0, 0,}_{2^{n-1}} \underbrace{0, 0, \cdots, -1, \cdots, 0, 0}_{2^{n-1}})$, also there is $uu^T + vv^T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$.

When $i \neq j$, $(u_i u_i^T) \cdot (u_j u_j^T) = \mathbf{0}$, $(v_i v_i^T) \cdot (v_j v_j^T) = \mathbf{0}$, thus $W_{n,n}^T W_{n,n} = 2I \Rightarrow W_{n,n}^{-1} = \frac{1}{2} W_{n,n}^T$.

When $W_{n,n} = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_{2^n}^T \end{pmatrix}$, it is easy to calculate $v_i^T \cdot v_i = 2,$, $v_i^T \cdot v_j = 0$ or $v_i^T \cdot v_j = 1 - 1 = 0,\ i \neq j$.

So rows are orthogonal, it is the same to prove columns are orthogonal.

Extra credit (30 pts.)

We can find

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \in M_{2^n,2^n}$$

and it is easy to get

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in M_{2^n,2^{n-1}},$$

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \in M_{2^n,2^{n-1}},$$

so

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 1 & 0 & \cdots & \cdots & -1 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 1 & \vdots \\ 0 & 1 & \cdots & \cdots & 0 & -1 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -1 \end{pmatrix} = W_{n,n}.$$

$$\begin{aligned} W_{n,n}W_{n,n}^\top &= \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}^\top \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}]^\top + I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^\top \end{aligned}$$

according to the facts above, we have

$$
\begin{aligned}
W_{n,n}W_{n,n}^{\top} &= [I_{2^{n-1}} \otimes \begin{pmatrix}1\\1\end{pmatrix}][I_{2^{n-1}} \otimes \begin{pmatrix}1\\1\end{pmatrix}^{\top}] + [I_{2^{n-1}} \otimes \begin{pmatrix}1\\-1\end{pmatrix}][I_{2^{n-1}} \otimes \begin{pmatrix}1\\-1\end{pmatrix}^{\top}] \\
&= I_{2^{n-1}} \otimes \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + I_{2^{n-1}} \otimes \begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} \\
&= I_{2^{n-1}} \otimes \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = 2I_{2^n}
\end{aligned}
$$

Use Induction:
When $n = 1$ we have $W_1 = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$, So $W_1^T \cdot W_1 = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = B_1$.
Suppose it is true when $n = k$, we have $W_k^T W_k = B_k$, then when $n = k+1$

$$
\begin{aligned}
W_{k+1}^T W_{k+1} &= \begin{pmatrix}[W_k \otimes \begin{pmatrix}1\\1\end{pmatrix}]^T \\ [I_{2^k} \otimes \begin{pmatrix}1\\-1\end{pmatrix}]^T\end{pmatrix} \cdot \left([W_k \otimes \begin{pmatrix}1\\1\end{pmatrix}][I_{2^k} \otimes \begin{pmatrix}1\\-1\end{pmatrix}]\right) \\
&= \begin{pmatrix}[W_k^T \otimes \begin{pmatrix}1 & 1\end{pmatrix}][W_k \otimes \begin{pmatrix}1\\1\end{pmatrix}] & [W_k^T \otimes \begin{pmatrix}1 & 1\end{pmatrix}][I_{2^k} \otimes \begin{pmatrix}1\\-1\end{pmatrix}] \\ [I_{2^k} \otimes \begin{pmatrix}1 & -1\end{pmatrix}][W_k \otimes \begin{pmatrix}1\\1\end{pmatrix}] & [I_{2^k} \otimes \begin{pmatrix}1 & -1\end{pmatrix}][I_{2^k} \otimes \begin{pmatrix}1\\-1\end{pmatrix}]\end{pmatrix} \\
&= \begin{pmatrix}W_k^T W_k \otimes I & \mathbf{0} \\ \mathbf{0} & I_{2^k} \otimes 2\end{pmatrix} \\
&= \begin{pmatrix}2B_k \otimes I & \mathbf{0} \\ \mathbf{0} & 2I_{2^k}\end{pmatrix} = 2\begin{pmatrix}B_k & \mathbf{0} \\ \mathbf{0} & I_{2^k}\end{pmatrix} \\
&= B_{k+1}
\end{aligned}
$$

So $W_n^T W_n = B_n$.

(6) When $k = 1$, it is easy to prove.
Suppose it is true when $n = k$,

$$
W_{n,k}W_{n,k-1}\cdots W_{n,1} = \begin{pmatrix}W_k & 0_{2^k,2^n-2^k} \\ 0_{2^n-2^k,2^k} & 2^n-2^k\end{pmatrix}.
$$

Then when $n = k+1$,

$$
W_{n,k+1} \cdot W_{n,k}W_{n,k-1}\cdots W_{n,1} = \begin{pmatrix}W_{k+1,k+1} & 0_{2^{k+1},2^n-2^{k+1}} \\ 0_{2^n-2^{k+1},2^i} & I_{2^n-2^{k+1}}\end{pmatrix}\begin{pmatrix}W_k & 0_{2^k,2^n-2^k} \\ 0_{2^n-2^k,2^k} & 2^n-2^k\end{pmatrix}.
$$

From (5) we can easily find that

$$W_{k+1,k+1}c = \begin{pmatrix} c_1 + c_{2^k+1} \\ c_1 - c_{2^k+1} \\ c_2 + c_{2^k+2} \\ c_2 - c_{2^k+2} \\ c_3 + c_{2^k+3} \\ c_3 - c_{2^k+3} \\ \vdots \\ c_{2^k} + c_{2^{k+1}} \\ c_{2^k} - c_{2^{k+1}} \end{pmatrix}$$

So

$$\begin{aligned} &\begin{pmatrix} W_{k+1,k+1} & 0_{2^{k+1},2^n-2^{k+1}} \\ 0_{2^n-2^{k+1},2^i} & I_{2^n-2^{k+1}} \end{pmatrix} \begin{pmatrix} W_k & 0_{2^k,2^n-2^k} \\ 0_{2^n-2^k,2^k} & 2^n-2^k \end{pmatrix} \\ &= \begin{pmatrix} W_{k+1,k+1} & 0_{2^{k+1},2^n-2^{k+1}} \\ 0_{2^n-2^{k+1},2^i} & I_{2^n-2^{k+1}} \end{pmatrix} \begin{pmatrix} W_k^* & 0_{2^{k+1},2^n-2^{k+1}} \\ 0_{2^n-2^{k+1},2^{k+1}} & I_{2^n-2^{k+1}} \end{pmatrix} \\ &= \begin{pmatrix} W_{k+1,k+1} \cdot W_k^* & 0_{2^{k+1},2^n-2^{k+1}} \\ 0_{2^n-2^{k+1},2^{k+1}} & I_{2^n-2^{k+1}} \end{pmatrix} \end{aligned}$$

Because

$$\begin{aligned} W_{k+1,k+1} \cdot W_k^* &= \left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \cdots & \vdots \\ a_{2^{k+1}1} & a_{2^{k+1}2} & a_{2^{k+1}3} \cdots & a_{2^{k+1}2^{k+1}} \\ a_{2^{k+1}1} & a_{2^{k+1}2} & a_{2^{k+1}3} \cdots & a_{2^{k+1}2^{k+1}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \cdots & 0 \\ -1 & 0 & 0 \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & -1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \cdots & \vdots \\ 0 & 0 & 0 \cdots & 1 \\ 0 & 0 & 0 \cdots & -1 \end{pmatrix} \right) \\ &= \left(W_k \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad I_{2^k} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \\ &= W_{k+1} \end{aligned}$$

So it is true when $n = k+1$, thus

$$W_{n,k}W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k,2^n-2^k} \\ 0_{2^n-2^k,2^k} & 2^n-2^k \end{pmatrix}.$$

From above, Because $W_{n,k}^T \cdot W_{n,k} = \begin{pmatrix} W_{k,k}^T \cdot W_{k,k} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} = \begin{pmatrix} 2I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$ and $W_{n,k} \cdot W_{n,k}^T = \begin{pmatrix} W_{k,k} \cdot W_{k,k}^T & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} = \begin{pmatrix} 2I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$ thus rows and columns of $W_{n,k}$ are orthogonal.

And $W_{n,k} \cdot \begin{pmatrix} \frac{1}{2}W_{k,k}^{\top} & 0_{2^k,2^n-2^k} \\ 0_{2^n-2^k,2^k} & I_{2^n-2^k} \end{pmatrix} = \begin{pmatrix} W_{k,k} \cdot \frac{1}{2}W_{k,k}^{T} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} = I$, so

$$W_{n,k}^{-1} = \begin{pmatrix} \frac{1}{2}W_{k,k}^{\top} & 0_{2^k,2^n-2^k} \\ 0_{2^n-2^k,2^k} & I_{2^n-2^k} \end{pmatrix}.$$

Problem B6 (20 pts). We can use Proposition 3.1 to prove that $E = Ker(f) \oplus Im(f)$. According to the proposition, given vector space E, $Ker(f) + Im(f)$ is a direct sum iff $Ker(f) \cap Im(f) = (0)$.
Let $u \in E$ then calculate $v = u - f(u)$, because f is an idempotent linear map $f \circ f = f$ which means $f(v) = f(u - f(u)) = f(u) - f^2(u) = 0$. Thus $v \in ker(f)$, So $E = Ker(f) + Im(f)$. Let $u^* \in Ker(f) \cap Im(f)$, then $u^* = f(s), \exists s \in E$.Because $s = f(u^*) = f \circ f(u^*) = 0$ and f is a linear map, $u^* = f(s) = 0$, which means $Ker(f) \cap Im(f) = (0)$, then $E = Ker(f) \oplus Im(f)$

Problem B7 (20 pts). Let $U_1, \ldots, U_p$ be any $p \geq 2$ subspaces of some vector space E and recall that the linear map

$$a\colon U_1 \times \cdots \times U_p \to E$$

is given by

$$a(u_1, \ldots, u_p) = u_1 + \cdots + u_p,$$

with $u_i \in U_i$ for $i = 1, \ldots, p$.

(1) If we let $Z_i \subseteq U_1 \times \cdots \times U_p$ be given by

$$Z_i = \left\{ \left(u_1, \ldots, u_{i-1}, -\sum_{j=1, j\neq i}^{p} u_j, u_{i+1}, \ldots, u_p \right) \,\middle|\, \sum_{j=1, j\neq i}^{p} u_j \in U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) \right\},$$

for $i = 1, \ldots, p$, then prove that

$$\mathrm{Ker}\, a = Z_1 = \cdots = Z_p.$$

In general, for any given i, the condition $U_i \cap \left(\sum_{j=1,j\neq i}^{p} U_j \right) = (0)$ does not necessarily imply that $Z_i = (0)$. Thus, let

$$Z = \left\{ \left(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_p \right) \,\middle|\, u_i = -\sum_{j=1, j\neq i}^{p} u_j,\ u_i \in U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right),\ 1 \leq i \leq p \right\}.$$

Since $\mathrm{Ker}\, a = Z_1 = \cdots = Z_p$, we have $Z = \mathrm{Ker}\, a$. Prove that if

$$U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p,$$

then $Z = \mathrm{Ker}\, a = (0)$.

(2) Prove that $U_1 + \cdots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1, j \neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p.$$

We can use the definition of a vector space to prove this. The vector space E is the direct sum of 2 more subspaces, $U_1 + \cdots + U_p$. The subspaces span the whole space E and have the 0 intersection.

(3) **Extra credit (40 pts)**. Assume that E is finite-dimensional, and let $f_i \colon E \to E$ be any $p \geq 2$ linear maps such that

$$f_1 + \cdots + f_p = \mathrm{id}_E.$$

Prove that the following properties are equivalent:

(1) $f_i^2 = f_i$, $1 \leq i \leq p$.

(2) $f_j \circ f_i = 0$, for all $i \neq j$, $1 \leq i, j \leq p$.

(1) $\Rightarrow$ (2)
We multiply f_i on each side of the equation, then get

$$f_i^2 + \sum_{j \neq i} f_j \circ f_i = f_i$$

$$\sum_{j \neq i} f_j \circ f_i = 0$$

Because E is finite-dimensional, then for all $y \in E$ we can write $y = \sum \lambda_i x_i$, where (x_i) is the base. Then for all $y \in E$, we have $\sum_{j \neq i} f_j \circ f_i(y) = 0$, suppose there is a k which makes $f_k \circ f_i \neq 0$ then $ker(f_k \circ f_i) = E$, thus $f_k \circ f_i = 0$, which is a contradiction.

(2) $\Rightarrow$ (1)
We multiply f_i, $\forall i = 1, 2, \cdots, p$ on each side of the equation, then get

$$f_i^2 + \sum_{j \neq i} f_j \circ f_i = f_i$$

$$f_i^2 = f_i$$

TOTAL: 200 + 70 points.