

Fundamentals of Linear Algebra and Optimization

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Homework 2

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Problem B1 (10 pts).

Suppose $A = (a_{i,j})_{m \times n}$, $B = (b_{i,j})_{n \times p}$, $C = AB = (c_{i,j})_{m \times p}$. It is easy to write $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$, $\forall i = 1, 2, \dots, m$. $j = 1, 2, \dots, p$ and then consider $(A^1B_1 + \dots + A^nB_n)_{ij}$, we have $(A^1B_1 + \dots + A^nB_n)_{ij} = (a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}) = \sum_{k=1}^n a_{ik}b_{kj}$. So $AB = A^1B_1 + \dots + A^nB_n$.

Problem B2 (10 pts).

Because f is a linear map, thus we have

$$f(x + y) = f(x) + f(y)$$

$$f(\lambda x) = \lambda f(x)$$

So write the inverse function $g = f^{-1}$ and $x_1, x_2 \in E$, we have

$$\begin{aligned} \exists x \in E, x = g(f(x_1) + f(x_2)) &\Rightarrow f(x) = f(x_1) + f(x_2) = f(x_1 + x_2) \\ &\Rightarrow x = x_1 + x_2 \\ &\Rightarrow g(f(x)) = g(f(x_1 + x_2)) = g(f(x_1)) + g(f(x_2)) \\ \exists x^* \in E, x^* = g(f(\lambda x)) &\Rightarrow f(x^*) = f(\lambda x) \\ &\Rightarrow x^* = \lambda x \\ &\Rightarrow g(\lambda f(x)) = g(f(\lambda x)) = \lambda g(f(x)) \end{aligned}$$

Problem B3 (10 pts).

For $x \in E$, it can be written as $x_1u_1 + x_2u_2 + \dots + x_nu_n$ as $(u_i)_{i \in I}$ is a basis of E and $(u_i)_{i \in I}$ spans E . Then, $f(x) = x_1f(u_1) + x_2f(u_2) + \dots + x_nf(u_n)$. Since $f(u_i) = v_i$, this can be rewritten as: $f(x) = x_1v_1 + x_2v_2 + \dots + x_nv_n$. If the linear map f is surjective, then every element in F has a corresponding element in E and $f(x) = x_1v_1 + x_2v_2 + \dots + x_nv_n$ indicates that $(v_i)_{i \in I}$ must span F .

Problem B4 (10 pts).

Let $f: E \rightarrow F$ be a linear map with $\dim(E) = n$ and $\dim(F) = m$. Prove that f has rank 1

iff f is represented by an $m \times n$ matrix of the form

$$A = uv^\top$$

with u a nonzero column vector of dimension m and v a nonzero column vector of dimension n . $\dim(E) = n$ indicates that the basis of E consists of n vectors and $\dim(F) = m$ indicates the basis of F consists of m vectors. From class notes, we know that $M(f) =$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = A, \text{ an } m \times n \text{ matrix.}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} \text{ and } v^T = (v_1 \ v_2 \ v_3 \ \cdots \ v_n) \text{ so } uv^T \text{ is an } m \times n \text{ matrix of the form:}$$

$$\begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & u_2v_3 & \cdots & u_2v_n \\ u_3v_1 & u_3v_2 & u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ u_mv_1 & u_mv_2 & u_mv_3 & \cdots & u_mv_n \end{pmatrix} = v_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + v_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + \cdots + v_n \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix}.$$

Since every column in the matrix of f is a multiple of one column, the rank is 1 for f and the matrix is always of this form by definition for an $m \times n$ matrix.

Problem B5 (120 pts).

(1)

$$\begin{aligned} W_{3,3}c &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot c_1 + 1 \cdot c_5 \\ 1 \cdot c_1 + -1 \cdot c_5 \\ 1 \cdot c_2 + 1 \cdot c_6 \\ 1 \cdot c_2 + -1 \cdot c_6 \\ 1 \cdot c_3 + 1 \cdot c_7 \\ 1 \cdot c_3 + -1 \cdot c_7 \\ 1 \cdot c_4 + 1 \cdot c_8 \\ 1 \cdot c_4 + -1 \cdot c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_5 \\ c_1 - c_5 \\ c_2 + c_6 \\ c_2 - c_6 \\ c_3 + c_7 \\ c_3 - c_7 \\ c_4 + c_8 \\ c_4 - c_8 \end{pmatrix} \end{aligned}$$

(2)

If the inverse of $W_{3,3}$ is $(1/2)W_{3,3}^\top$, then the product of the two matrices is the identity matrix.

$$\begin{aligned}
W_{3,3} \cdot (1/2)W_{3,3}^\top &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix} \\
&= \begin{pmatrix} 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

(3)

First: $W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8)$

$$\begin{aligned}
W_{3,2}c &= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ c_1 + 0 - c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + c_2 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + c_2 + 0 - c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_3 \\ c_1 - c_3 \\ c_2 + c_4 \\ c_2 - c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}
\end{aligned}$$

Second: $W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8)$

$$W_{3,1}c = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ c_1 - c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

Now check $W_{3,2} * W_{3,1}$

$$W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},$$

Conclude $W_{3,3} * W_{3,2} * W_{3,1} = W_3$

$$W_{3,3} * W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We see that in the first row of the $W_{3,3}$ matrix, there is at least 1 1 in the columns 1 through 4 in each of the 8 rows. We are guaranteed that the first column in the W_3 matrix is all 1's because, every 1 in $W_{3,3}$ will be multiplied by at least 1's located in the first 4 rows in the first column of $W_{3,2} * W_{3,1}$. We can apply the same theory to get the second column of W_3 . The change here is that there are 2 -1 's located in the 3rd and 4th row in the second column of $W_{3,2} * W_{3,1}$. Now, the last 4 rows in the second column of W_3 will be -1 . Column 3 in W_3 will have a 1 in the first 2 rows and a -1 in the next two rows because only the first 4 rows in $W_{3,3}$ have 1's in the first two columns. Using the same idea, rows 5 and 6 will have 1 and the last two rows will have -1 in W_3 . Finally, the last 4 columns in W_3 will have a 1 and -1 . The 1 and -1 in $W_{3,3}$ is in the 5th, 6th, 7th, and 8th columns, and the last 4 columns in $W_{3,2} * W_{3,1}$ are 0s in the first 4 rows and an 4×4 identity matrix in the last 4 rows. With the row and column multiplication, we are guaranteed the shifting 1 and -1 in the last 4 columns of W_3 .

(4)

We have $W_{3,3} = (\alpha_1, \alpha_2, \dots, \alpha_8)$, then write $\alpha_i^T \alpha_j, \forall i, j = 1, 2, \dots, 8 \ i \neq j$ there are two conditions:

1. α_i and α_j have elements on the same positions.
2. α_i and α_j have elements on different positions.

For the first condition, we have $\alpha_i^T \alpha_j = 1 - 1 = 0$ and for the second condition, we have $\alpha_i^T \alpha_j = 0$. Also, $\alpha_i^T \alpha_i = 1 + 1 = 2$, so we can prove that the columns are orthogonal. It is the same to prove the rows are orthogonal.

Inverse of $W_{3,2}$

We see that in $W_{3,2}$, the last 4 rows already have a diagonal with all 1's. The only thing left is to manipulate the first 4 rows. Similar to $W_{3,3}$, the transpose of $W_{3,2}$ is $(1/2)W_{3,2}^T$ of the first 4 rows and 4 columns.

$$\text{inv}(W_{3,2}) = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse of $W_{3,1}$

6 rows of $W_{3,1}$ is already the identity matrix. Thus only the first 2×2 submatrix in $W_{3,1}$ has to be manipulated to find its inverse. We can use a similar method as above, to find that we only work that has to be done is taking the $\frac{1}{2}W_{3,1}^\top$ of the 2×2 submatrix.

$$\text{inv}(W_{3,1}) = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) For any $n \geq 2$, the $2^n \times 2^n$ matrix $W_{n,n}$ is obtained from the two rows

$$\underbrace{1, 0, \dots, 0}_{2^{n-1}}, \underbrace{1, 0, \dots, 0}_{2^{n-1}} \\ \underbrace{1, 0, \dots, 0}_{2^{n-1}}, \underbrace{-1, 0, \dots, 0}_{2^{n-1}}$$

by shifting them $2^{n-1} - 1$ times over to the right by inserting a zero on the left each time.

Given any vector $c = (c_1, c_2, \dots, c_{2^n})$, show that $W_{n,n}c$ is the result of the last step in the process of reconstructing a vector from its Haar coefficients c . Prove that $W_{n,n}^{-1} = (1/2)W_{n,n}^\top$, and that the columns and the rows of $W_{n,n}$ are orthogonal.

Extra credit (30 pts.)

We can find

$$\left(I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \in M_{2^n, 2^n}$$

and it is easy to get

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in M_{2^n, 2^{n-1}},$$

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \in M_{2^n, 2^{n-1}},$$

so

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 1 & 0 & \cdots & \cdots & -1 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 1 & \vdots \\ 0 & 1 & \cdots & \cdots & 0 & -1 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -1 \end{pmatrix} = W_{n,n}.$$

$$\begin{aligned} W_{n,n} W_{n,n}^\top &= \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}^\top \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}]^\top + I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^\top \end{aligned}$$

according to the facts above, we have

$$\begin{aligned} W_{n,n} W_{n,n}^\top &= [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}] [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}]^\top + [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}] [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^\top \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + I_{2^{n-1}} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_{2^n} \end{aligned}$$

Use Induction:

When $n = 1$ we have $W_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, So $W_1^T \cdot W_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = B_1$.

Suppose it is true when $n = k$, we have $W_k^T W_k = B_k$, then when $n = k + 1$

$$\begin{aligned}
W_{k+1}^T W_{k+1} &= \begin{pmatrix} [W_k \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}]^T \\ [I_{2^k} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^T \end{pmatrix} \cdot \begin{pmatrix} [W_k \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}][I_{2^k} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}] \\ [I_{2^k} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}][W_k \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}] \end{pmatrix} \\
&= \begin{pmatrix} [W_k^T \otimes (1 \ 1)][W_k \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}] & [W_k^T \otimes (1 \ 1)][I_{2^k} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}] \\ [I_{2^k} \otimes (1 \ -1)][W_k \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}] & [I_{2^k} \otimes (1 \ -1)][I_{2^k} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}] \end{pmatrix} \\
&= \begin{pmatrix} W_k^T W_k \otimes I & \mathbf{0} \\ \mathbf{0} & I_{2^k} \otimes 2 \end{pmatrix} \\
&= \begin{pmatrix} 2B_k \otimes I & \mathbf{0} \\ \mathbf{0} & 2I_{2^k} \end{pmatrix} = 2 \begin{pmatrix} B_k & \mathbf{0} \\ \mathbf{0} & I_{2^k} \end{pmatrix} \\
&= B_{k+1}
\end{aligned}$$

(6) The matrix $W_{n,i}$ is obtained from the matrix $W_{i,i}$ ($1 \leq i \leq n-1$) as follows:

$$W_{n,i} = \begin{pmatrix} W_{i,i} & 0_{2^i, 2^{n-2^i}} \\ 0_{2^{n-2^i}, 2^i} & I_{2^{n-2^i}} \end{pmatrix}.$$

It consists of four blocks, where $0_{2^i, 2^{n-2^i}}$ and $0_{2^{n-2^i}, 2^i}$ are matrices of zeros and $I_{2^{n-2^i}}$ is the identity matrix of dimension $2^n - 2^i$.

Explain what $W_{n,i}$ does to c and prove that

$$W_{n,n} W_{n,n-1} \cdots W_{n,1} = W_n,$$

where W_n is the Haar matrix of dimension 2^n .

Hint. Use induction on k , with the induction hypothesis

$$W_{n,k} W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & I_{2^{n-2^k}} \end{pmatrix}.$$

Prove that the columns and rows of $W_{n,k}$ are orthogonal, and use this to prove that the columns of W_n and the rows of W_n^{-1} are orthogonal. Are the rows of W_n orthogonal? Are the columns of W_n^{-1} orthogonal? Prove that

$$W_{n,k}^{-1} = \begin{pmatrix} \frac{1}{2} W_{k,k}^\top & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & I_{2^{n-2^k}} \end{pmatrix}.$$

Problem B6 (20 pts). Prove that for every vector space E , if $f: E \rightarrow E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \text{Ker } f \oplus \text{Im } f,$$

so that f is the projection onto its image $\text{Im } f$.

Problem B7 (20 pts). Let U_1, \dots, U_p be any $p \geq 2$ subspaces of some vector space E and recall that the linear map

$$a: U_1 \times \dots \times U_p \rightarrow E$$

is given by

$$a(u_1, \dots, u_p) = u_1 + \dots + u_p,$$

with $u_i \in U_i$ for $i = 1, \dots, p$.

(1) If we let $Z_i \subseteq U_1 \times \dots \times U_p$ be given by

$$Z_i = \left\{ \left(u_1, \dots, u_{i-1}, - \sum_{j=1, j \neq i}^p u_j, u_{i+1}, \dots, u_p \right) \mid \sum_{j=1, j \neq i}^p u_j \in U_i \cap \left(\sum_{j=1, j \neq i}^p U_j \right) \right\},$$

for $i = 1, \dots, p$, then prove that

$$\text{Ker } a = Z_1 = \dots = Z_p.$$

In general, for any given i , the condition $U_i \cap \left(\sum_{j=1, j \neq i}^p U_j \right) = (0)$ does not necessarily imply that $Z_i = (0)$. Thus, let

$$Z = \left\{ \left(u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_p \right) \mid u_i = - \sum_{j=1, j \neq i}^p u_j, u_i \in U_i \cap \left(\sum_{j=1, j \neq i}^p U_j \right), 1 \leq i \leq p \right\}.$$

Since $\text{Ker } a = Z_1 = \dots = Z_p$, we have $Z = \text{Ker } a$. Prove that if

$$U_i \cap \left(\sum_{j=1, j \neq i}^p U_j \right) = (0) \quad 1 \leq i \leq p,$$

then $Z = \text{Ker } a = (0)$.

(2) Prove that $U_1 + \dots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1, j \neq i}^p U_j \right) = (0) \quad 1 \leq i \leq p.$$

We can use the definition of a vector space to prove this. The vector space E is the direct sum of 2 more subspaces, $U_1 + \cdots + U_p$. The subspaces span the whole space E and have the 0 intersection.

(3) **Extra credit (40 pts)**. Assume that E is finite-dimensional, and let $f_i: E \rightarrow E$ be any $p \geq 2$ linear maps such that

$$f_1 + \cdots + f_p = \text{id}_E.$$

Prove that the following properties are equivalent:

(1) $f_i^2 = f_i, 1 \leq i \leq p$.

(2) $f_j \circ f_i = 0$, for all $i \neq j, 1 \leq i, j \leq p$.

(1) \Rightarrow (2)

We multiply f_i on each side of the equation, then get

$$f_i^2 + \sum_{j \neq i} f_j \circ f_i = f_i$$

$$\sum_{j \neq i} f_j \circ f_i = 0$$

Because E is finite-dimensional, then for all $y \in E$ we can write $y = \sum \lambda_i x_i$, where (x_i) is the base. Then for all $y \in E$, we have $\sum_{j \neq i} f_j \circ f_i(y) = 0$, suppose there is a k which makes $f_k \circ f_i \neq 0$ then $\ker(f_k \circ f_i) = E$, thus $f_k \circ f_i = 0$, which is a contradiction.

(2) \Rightarrow (1)

We multiply $f_i, \forall i = 1, 2, \dots, p$ on each side of the equation, then get

$$f_i^2 + \sum_{j \neq i} f_j \circ f_i = f_i$$

$$f_i^2 = f_i$$

TOTAL: 200 + 70 points.