

# Fundamentals of Linear Algebra and Optimization

Jean Gallier

## Homework 2

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Francine Leech, Chen Xiang, Reffat Manzur

**Problem B1 (10 pts).**

Suppose  $A = (a_{i,j})_{m \times n}$ ,  $B = (b_{i,j})_{n \times p}$ ,  $C = AB = (c_{i,j})_{m \times p}$ . It is easy to write  $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$ ,  $\forall i = 1, 2, \dots, m$ .  $j = 1, 2, \dots, p$  and then consider  $(A^1B_1 + \dots + A^nB_n)_{ij}$ , we have  $(A^1B_1 + \dots + A^nB_n)_{ij} = (a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}) = \sum_{k=1}^n a_{ik}b_{kj}$ . So  $AB = A^1B_1 + \dots + A^nB_n$ .

**Problem B2 (10 pts).**

Because  $f$  is a linear map, thus we have

$$f(x + y) = f(x) + f(y)$$

$$f(\lambda x) = \lambda f(x)$$

So write the inverse function  $g = f^{-1}$  and  $x_1, x_2 \in E$ , we have

$$\begin{aligned} \exists x \in E, x = g(f(x_1) + f(x_2)) &\Rightarrow f(x) = f(x_1) + f(x_2) = f(x_1 + x_2) \\ &\Rightarrow x = x_1 + x_2 \\ &\Rightarrow g(f(x)) = g(f(x_1 + x_2)) = g(f(x_1)) + g(f(x_2)) \\ \exists x^* \in E, x^* = g(f(\lambda x)) &\Rightarrow f(x^*) = f(\lambda x) \\ &\Rightarrow x^* = \lambda x \\ &\Rightarrow g(\lambda f(x)) = g(f(\lambda x)) = \lambda g(f(x)) \end{aligned}$$

**Problem B3 (10 pts).**

For  $x \in E$ , it can be written as  $x_1u_1 + x_2u_2 + \dots + x_nu_n$  as  $(u_i)_{i \in I}$  is a basis of  $E$  and  $(u_i)_{i \in I}$  spans  $E$ . Then,  $f(x) = x_1f(u_1) + x_2f(u_2) + \dots + x_nf(u_n)$ . Since  $f(u_i) = v_i$ , this can be rewritten as:  $f(x) = x_1v_1 + x_2v_2 + \dots + x_nv_n$ . If the linear map  $f$  is surjective, then every element in  $F$  has a corresponding element in  $E$  and  $f(x) = x_1v_1 + x_2v_2 + \dots + x_nv_n$  indicates that  $(v_i)_{i \in I}$  must span  $F$ .

**Problem B4 (10 pts).**

Let  $f: E \rightarrow F$  be a linear map with  $\dim(E) = n$  and  $\dim(F) = m$ . Prove that  $f$  has rank 1

iff  $f$  is represented by an  $m \times n$  matrix of the form

$$A = uv^\top$$

with  $u$  a nonzero column vector of dimension  $m$  and  $v$  a nonzero column vector of dimension  $n$ .  $\dim(E) = n$  indicates that the basis of  $E$  consists of  $n$  vectors and  $\dim(F) = m$  indicates the basis of  $F$  consists of  $m$  vectors. From class notes, we know that  $M(f) =$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = A, \text{ an } m \times n \text{ matrix.}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} \text{ and } v^T = (v_1 \ v_2 \ v_3 \ \cdots \ v_n) \text{ so } uv^T \text{ is an } m \times n \text{ matrix of the form:}$$

$$\begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & u_2v_3 & \cdots & u_2v_n \\ u_3v_1 & u_3v_2 & u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ u_mv_1 & u_mv_2 & u_mv_3 & \cdots & u_mv_n \end{pmatrix} = v_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + v_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + \cdots v_n \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix}. \text{ Since every}$$

column in the matrix of  $f$  is a multiple of one column, the rank is 1 for  $f$  and the matrix is always of this form by definition for an  $m \times n$  matrix.

### Problem B5 (120 pts).

(1)

$$\begin{aligned} W_{3,3}c &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot c_1 + 1 \cdot c_5 \\ 1 \cdot c_1 + -1 \cdot c_5 \\ 1 \cdot c_2 + 1 \cdot c_6 \\ 1 \cdot c_2 + -1 \cdot c_6 \\ 1 \cdot c_3 + 1 \cdot c_7 \\ 1 \cdot c_3 + -1 \cdot c_7 \\ 1 \cdot c_4 + 1 \cdot c_8 \\ 1 \cdot c_4 + -1 \cdot c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_5 \\ c_1 - c_5 \\ c_2 + c_6 \\ c_2 - c_6 \\ c_3 + c_7 \\ c_3 - c_7 \\ c_4 + c_8 \\ c_4 - c_8 \end{pmatrix} \end{aligned}$$

(2)

If the inverse of  $W_{3,3}$  is  $(1/2)W_{3,3}^\top$ , then the product of the two matrices is the identity matrix.

$$\begin{aligned}
W_{3,3} \cdot (1/2)W_{3,3}^\top &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix} \\
&= \begin{pmatrix} 0.5+0.5 & 0.5+-0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5+-0.5 & 0.5--0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5+0.5 & 0.5+-0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5+-0.5 & 0.5--0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5+0.5 & 0.5+-0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5+-0.5 & 0.5--0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5+0.5 & 0.5+-0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5+-0.5 & 0.5--0.5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

(3)

**First:**  $W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8)$

$$\begin{aligned}
W_{3,2}c &= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ c_1 + 0 - c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + c_2 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + c_2 + 0 - c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_3 \\ c_1 - c_3 \\ c_2 + c_4 \\ c_2 - c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}
\end{aligned}$$

Second:  $W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8)$

$$W_{3,1}c = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ c_1 - c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

Now check  $W_{3,2} * W_{3,1}$

$$W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},$$

Conclude  $W_{3,3} * W_{3,2} * W_{3,1} = W_3$

$$W_{3,3} * W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We see that in the first row of the  $W_{3,3}$  matrix, there is at least 1 1 in the columns 1 through 4 in each of the 8 rows. We are guaranteed that the first column in the  $W_3$  matrix is all 1's because, every 1 in  $W_{3,3}$  will be multiplied by at least 1's located in the first 4 rows in the first column of  $W_{3,2} * W_{3,1}$ . We can apply the same theory to get the second column of  $W_3$ . The change here is that there are 2  $-1$ 's located in the 3rd and 4th row in the second column of  $W_{3,2} * W_{3,1}$ . Now, the last 4 rows in the second column of  $W_3$  will be  $-1$ . Column 3 in  $W_3$  will have a 1 in the first 2 rows and a  $-1$  in the next two rows because only the first 4 rows in  $W_{3,3}$  have 1's in the first two columns. Using the same idea, rows 5 and 6 will have 1 and the last two rows will have  $-1$  in  $W_3$ . Finally, the last 4 columns in  $W_3$  will have a 1 and  $-1$ . The 1 and  $-1$  in  $W_{3,3}$  is in the 5th, 6th, 7th, and 8th columns, and the last 4 columns in  $W_{3,2} * W_{3,1}$  are 0s in the first 4 rows and an  $4 \times 4$  identity matrix in the last 4 rows. With the row and column multiplication, we are guaranteed the shifting 1 and  $-1$  in the last 4 columns of  $W_3$ .

(4)

We have  $W_{3,3} = (\alpha_1, \alpha_2, \dots, \alpha_8)$ , then write  $\alpha_i^T \alpha_j, \forall i, j = 1, 2, \dots, 8 \ i \neq j$  there are two conditions:

1.  $\alpha_i$  and  $\alpha_j$  have elements on the same positions.
2.  $\alpha_i$  and  $\alpha_j$  have elements on different postions.

For the first condition, we have  $\alpha_i^T \alpha_j = 1 - 1 = 0$  and for the second condition, we have  $\alpha_i^T \alpha_j = 0$ . Also,  $\alpha_i^T \alpha_i = 1 + 1 = 2$ , so we can prove that the columns are orthogonal. It is the same to prove the rows are orthogonal.

### Inverse of $W_{3,2}$

We see that in  $W_{3,2}$ , the last 4 rows already have a diagonal with all 1's. The only thing left is to manipulate the first 4 rows. Similar to  $W_{3,3}$ , the transpose of  $W_{3,2}$  is  $(1/2)W_{3,2}^T$  of the first 4 rows and 4 columns.

$$\text{inv}(W_{3,2}) = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

### Inverse of $W_{3,1}$

6 rows of  $W_{3,1}$  is already the identity matrix. Thus only the first  $2 \times 2$  submatrix in  $W_{3,1}$  has to be manipulated to find its inverse. We can use a similar method as above, to find that we only work that has to be done is taking the  $\frac{1}{2}W_{3,1}^\top$  of the  $2 \times 2$  submatrix.

$$\text{inv}(W_{3,1}) = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) We can find that

$$W_{n,n} = \begin{pmatrix} 1 & \overbrace{0 \dots 0}^{2^{n-1}-1} & 1 & \overbrace{0 \dots 0}^{2^{n-1}-1} \\ 1 & \overbrace{0 \dots 0}^{2^{n-1}-1} & -1 & \overbrace{0 \dots 0}^{2^{n-1}-1} \\ 0 & 1 & \overbrace{\dots 0}^{2^{n-1}-1} & 1 & \overbrace{0 \dots 0}^{2^{n-1}-1} \\ 0 & 1 & \overbrace{\dots 0}^{2^{n-1}-1} & -1 & \overbrace{0 \dots 0}^{2^{n-1}-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \overbrace{\dots 0}^{2^{n-1}-1} & 1 & \overbrace{\dots 0}^{2^{n-1}-1} & 1 \\ \dots & \overbrace{\dots 0}^{2^{n-1}-1} & 1 & \overbrace{\dots 0}^{2^{n-1}-1} & -1 \end{pmatrix}$$

So

$$W_{n,n}c = \begin{pmatrix} c_1 + c_{2^{n-1}+1} \\ c_1 - c_{2^{n-1}+1} \\ c_2 + c_{2^{n-1}+2} \\ c_2 - c_{2^{n-1}+2} \\ c_3 + c_{2^{n-1}+3} \\ c_3 - c_{2^{n-1}+3} \\ \vdots \\ c_{2^{n-1}} + c_{2^n} \\ c_{2^{n-1}} - c_{2^n} \end{pmatrix}$$

It is the last step in process of reconstructing a vector from its Haar coefficients  $c$ .

Write  $W_{n,n} = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_{2^n}^T \end{pmatrix}$ , So  $W_{n,n}^T W_{n,n} = \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_{2^n} \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_{2^n}^T \end{pmatrix} = \sum_{i=1}^{2^n} v_i v_i^T$ . Be-

cause we can calculate that  $v_i v_i^T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$  there are two 1 in the diagonal. Because we have pairs  $u_i = (\underbrace{0, 0, \dots, 1, \dots, 0, 0}_{2^{n-1}}, \underbrace{0, 0, \dots, 1, \dots, 0, 0}_{2^{n-1}})$  and  $v_i =$

$(\underbrace{0, 0, \dots, 1, \dots, 0, 0}_{2^{n-1}}, \underbrace{0, 0, \dots, -1, \dots, 0, 0}_{2^{n-1}})$ , also there is  $uu^T + vv^T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$ .

When  $i \neq j$ ,  $(u_i u_i^T) \cdot (u_j u_j^T) = \mathbf{0}$ ,  $(v_i v_i^T) \cdot (v_j v_j^T) = \mathbf{0}$ , thus  $W_{n,n}^T W_{n,n} = 2I \Rightarrow W_{n,n}^{-1} = \frac{1}{2} W_{n,n}^T$ .

When  $W_{n,n} = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_{2^n}^T \end{pmatrix}$ , it is easy to calculate  $v_i^T \cdot v_i = 2$ ,  $v_i^T \cdot v_j = 0$  or  $v_i^T \cdot v_j = 1 - 1 = 0$ ,  $i \neq j$ .

So rows are orthogonal, it is the same to prove columns are orthogonal.

**Extra credit (30 pts.)**

We can find

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \in M_{2^n, 2^n}$$

and it is easy to get

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in M_{2^n, 2^{n-1}},$$

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \in M_{2^n, 2^{n-1}},$$

so

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 1 & 0 & \cdots & \cdots & -1 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 1 & \vdots \\ 0 & 1 & \cdots & \cdots & 0 & -1 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -1 \end{pmatrix} = W_{n,n}.$$

$$\begin{aligned} W_{n,n} W_{n,n}^\top &= \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}^\top \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}]^\top + I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^\top \end{aligned}$$



according to the facts above, we have

$$\begin{aligned}
W_{n,n}W_{n,n}^\top &= [I_{2^{n-1}} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}][I_{2^{n-1}} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}^\top] + [I_{2^{n-1}} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}][I_{2^{n-1}} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^\top] \\
&= I_{2^{n-1}} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + I_{2^{n-1}} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
&= I_{2^{n-1}} \otimes \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_{2^n}
\end{aligned}$$

Use Induction:

When  $n = 1$  we have  $W_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ , So  $W_1^T \cdot W_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = B_1$ .

Suppose it is true when  $n = k$ , we have  $W_k^T W_k = B_k$ , then when  $n = k + 1$

$$\begin{aligned}
W_{k+1}^T W_{k+1} &= \begin{pmatrix} [W_k \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}]^T \\ [I_{2^k} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}]^T \end{pmatrix} \cdot \begin{pmatrix} [W_k \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}][I_{2^k} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}] \\ \end{pmatrix} \\
&= \begin{pmatrix} [W_k^T \otimes (1 \quad 1)][W_k \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}] & [W_k^T \otimes (1 \quad 1)][I_{2^k} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}] \\ [I_{2^k} \otimes (1 \quad -1)][W_k \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}] & [I_{2^k} \otimes (1 \quad -1)][I_{2^k} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}] \end{pmatrix} \\
&= \begin{pmatrix} W_k^T W_k \otimes I & \mathbf{0} \\ \mathbf{0} & I_{2^k} \otimes 2 \end{pmatrix} \\
&= \begin{pmatrix} 2B_k \otimes I & \mathbf{0} \\ \mathbf{0} & 2I_{2^k} \end{pmatrix} = 2 \begin{pmatrix} B_k & \mathbf{0} \\ \mathbf{0} & I_{2^k} \end{pmatrix} \\
&= B_{k+1}
\end{aligned}$$

So  $W_n^T W_n = B_n$ .

(6) When  $k = 1$ , it is easy to prove.

Suppose it is true when  $n = k$ , To understand what  $W_{n,i}$  does to a vector  $c = (c_1, c_2, \dots, c_{2^n})$ , let's define  $(c_A + c_B), (c_A - c_B)$  to be a pair. When  $W_{n,i}$  acts on  $c$ , it reconstructs  $c$  so that it still contains  $2^n$  elements but among the elements, there are  $2^{i-1}$  pairs going from left to right. For example,  $W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8)$  shows  $2^{2-1} = 2$  pairs consisting of  $c_1, c_3$  and  $c_2, c_4$ . Furthermore, for every pair consisting of  $c_A, c_B$ ,  $B = A + 2^{i-1}$ . It appears that as the difference between  $n$  and  $i$  get smaller, more of the elements from the original vector  $c$  get reconstructed as a sum and difference with another element from  $c$ .

(2) We can prove that  $W_{n,n}W_{n,n-1} \cdots W_{n,1} = W_n$

We are given that  $W_{n,k} = \begin{pmatrix} W_{k,k} & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & I_{2^{n-2^k}} \end{pmatrix}$ ;

When  $k = 1$ ,  $W_{n,1} = \begin{pmatrix} W_{1,1} & 0_{2^k, 2^{n-2}} \\ 0_{2^{n-2}, 2} & I_{2^{n-2}}(6) \end{pmatrix}$  The matrix  $W_{n,i}$  is obtained from the matrix  $W_{i,i}$  ( $1 \leq i \leq n-1$ ) as follows:

$$W_{n,k} W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & 2^n - 2^k \end{pmatrix}.$$

Then when  $n = k+1$ ,

$$W_{n,k+1} \cdot W_{n,k} W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_{k+1,k+1} & 0_{2^{k+1}, 2^{n-2^{k+1}}} \\ 0_{2^{n-2^{k+1}}, 2^i} & I_{2^{n-2^{k+1}}} \end{pmatrix} \begin{pmatrix} W_k & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & 2^n - 2^k \end{pmatrix}.$$

From (5) we can easily find that

$$W_{k+1,k+1} C = \begin{pmatrix} c_1 + c_{2^{k+1}} \\ c_1 - c_{2^{k+1}} \\ c_2 + c_{2^{k+2}} \\ c_2 - c_{2^{k+2}} \\ c_3 + c_{2^{k+3}} \\ c_3 - c_{2^{k+3}} \\ \vdots \\ c_{2^k} + c_{2^{k+1}} \\ c_{2^k} - c_{2^{k+1}} \end{pmatrix}$$

So

$$\begin{aligned} & \begin{pmatrix} W_{k+1,k+1} & 0_{2^{k+1}, 2^{n-2^{k+1}}} \\ 0_{2^{n-2^{k+1}}, 2^i} & I_{2^{n-2^{k+1}}} \end{pmatrix} \begin{pmatrix} W_k & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & 2^n - 2^k \end{pmatrix} \\ &= \begin{pmatrix} W_{k+1,k+1} & 0_{2^{k+1}, 2^{n-2^{k+1}}} \\ 0_{2^{n-2^{k+1}}, 2^i} & I_{2^{n-2^{k+1}}} \end{pmatrix} \begin{pmatrix} W_k^* & 0_{2^{k+1}, 2^{n-2^{k+1}}} \\ 0_{2^{n-2^{k+1}}, 2^{k+1}} & I_{2^{n-2^{k+1}}} \end{pmatrix} \\ &= \begin{pmatrix} W_{k+1,k+1} \cdot W_k^* & 0_{2^{k+1}, 2^{n-2^{k+1}}} \\ 0_{2^{n-2^{k+1}}, 2^{k+1}} & I_{2^{n-2^{k+1}}} \end{pmatrix} \end{aligned}$$

Because

$$\begin{aligned} W_{k+1,k+1} \cdot W_k^* &= \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \cdots & \vdots \\ a_{2^{k+1}1} & a_{2^{k+1}2} & a_{2^{k+1}3} \cdots & a_{2^{k+1}2^{k+1}} \\ a_{2^{k+1}1} & a_{2^{k+1}2} & a_{2^{k+1}3} \cdots & a_{2^{k+1}2^{k+1}} \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \cdots & 0 \\ -1 & 0 & 0 \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & -1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \cdots & \vdots \\ 0 & 0 & 0 \cdots & 1 \\ 0 & 0 & 0 \cdots & -1 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} W_k \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^k} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \\ &= W_{k+1} \end{aligned}$$

So it is true when  $n = k + 1$ , thus

$$W_{n,k}W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & 2^n - 2^k \end{pmatrix}.$$

From above, Because  $W_{n,k}^T \cdot W_{n,k} = \begin{pmatrix} W_{k,k}^T \cdot W_{k,k} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} = \begin{pmatrix} 2I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$  and  $W_{n,k} \cdot W_{n,k}^T = \begin{pmatrix} W_{k,k} \cdot W_{k,k}^T & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} = \begin{pmatrix} 2I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$  thus rows and columns of  $W_{n,k}$  are orthogonal. And  $W_{n,k} \cdot \begin{pmatrix} \frac{1}{2}W_{k,k}^T & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & I_{2^{n-2^k}} \end{pmatrix} = \begin{pmatrix} W_{k,k} \cdot \frac{1}{2}W_{k,k}^T & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} = I$ , so

$$W_{n,k}^{-1} = \begin{pmatrix} \frac{1}{2}W_{k,k}^T & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & I_{2^{n-2^k}} \end{pmatrix}.$$

**Problem B6 (20 pts).** We can use Proposition 3.1 to prove that  $E = \text{Ker}(f) \oplus \text{Im}(f)$ . According to the proposition, given vector space  $E$ ,  $\text{Ker}(f) + \text{Im}(f)$  is a direct sum iff  $\text{Ker}(f) \cap \text{Im}(f) = (0)$ .

Let  $u \in E$  then calculate  $v = u - f(u)$ , because  $f$  is an idempotent linear map  $f \circ f = f$  which means  $f(v) = f(u - f(u)) = f(u) - f^2(u) = 0$ . Thus  $v \in \text{Ker}(f)$ , So  $E = \text{Ker}(f) + \text{Im}(f)$ . Let  $u^* \in \text{Ker}(f) \cap \text{Im}(f)$ , then  $u^* = f(s)$ ,  $\exists s \in E$ . Because  $s = f(u^*) = f \circ f(u^*) = 0$  and  $f$  is a linear map,  $u^* = f(s) = 0$ , which means  $\text{Ker}(f) \cap \text{Im}(f) = (0)$ , then  $E = \text{Ker}(f) \oplus \text{Im}(f)$ .

**Problem B7 (20 pts).** Let  $U_1, \dots, U_p$  be any  $p \geq 2$  subspaces of some vector space  $E$  and recall that the linear map

$$a: U_1 \times \cdots \times U_p \rightarrow E$$

is given by

$$a(u_1, \dots, u_p) = u_1 + \cdots + u_p,$$

with  $u_i \in U_i$  for  $i = 1, \dots, p$ .

(1) If we let  $Z_i \subseteq U_1 \times \cdots \times U_p$  be given by

$$Z_i = \left\{ \left( u_1, \dots, u_{i-1}, - \sum_{j=1, j \neq i}^p u_j, u_{i+1}, \dots, u_p \right) \mid \sum_{j=1, j \neq i}^p u_j \in U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right) \right\},$$

for  $i = 1, \dots, p$ , then prove that

$$\text{Ker } a = Z_1 = \cdots = Z_p.$$

In general, for any given  $i$ , the condition  $U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right) = (0)$  does not necessarily imply that  $Z_i = (0)$ . Thus, let

$$Z = \left\{ \left( u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_p \right) \mid u_i = - \sum_{j=1, j \neq i}^p u_j, u_i \in U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right), 1 \leq i \leq p \right\}.$$

Since  $\text{Ker } a = Z_1 = \cdots = Z_p$ , we have  $Z = \text{Ker } a$ . Prove that if

$$U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right) = (0) \quad 1 \leq i \leq p,$$

then  $Z = \text{Ker } a = (0)$ .

(2) Prove that  $U_1 + \cdots + U_p$  is a direct sum iff

$$U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right) = (0) \quad 1 \leq i \leq p.$$

We can use the definition of a vector space to prove this. The vector space  $E$  is the direct sum of 2 more subspaces,  $U_1 + \cdots + U_p$ . The subspaces span the whole space  $E$  and have the 0 intersection.

(3) **Extra credit (40 pts)**. Assume that  $E$  is finite-dimensional, and let  $f_i: E \rightarrow E$  be any  $p \geq 2$  linear maps such that

$$f_1 + \cdots + f_p = \text{id}_E.$$

Prove that the following properties are equivalent:

(1)  $f_i^2 = f_i, 1 \leq i \leq p$ .

(2)  $f_j \circ f_i = 0$ , for all  $i \neq j, 1 \leq i, j \leq p$ .

(1)  $\Rightarrow$  (2)

We multiply  $f_i$  on each side of the equation, then get

$$f_i^2 + \sum_{j \neq i} f_j \circ f_i = f_i$$

$$\sum_{j \neq i} f_j \circ f_i = 0$$

Because  $E$  is finite-dimensional, then for all  $y \in E$  we can write  $y = \sum \lambda_i x_i$ , where  $(x_i)$  is the base. Then for all  $y \in E$ , we have  $\sum_{j \neq i} f_j \circ f_i(y) = 0$ , suppose there is a  $k$  which makes  $f_k \circ f_i \neq 0$  then  $\text{ker}(f_k \circ f_i) = E$ , thus  $f_k \circ f_i = 0$ , which is a contradiction.

(2)  $\Rightarrow$  (1)

We multiply  $f_i, \forall i = 1, 2, \dots, p$  on each side of the equation, then get

$$f_i^2 + \sum_{j \neq i} f_j \circ f_i = f_i$$

$$f_i^2 = f_i$$

**TOTAL: 200 + 70 points.**