

# Fundamentals of Linear Algebra and Optimization

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## Homework 2

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**Problem B1 (10 pts).**

Suppose  $A = (a_{i,j})_{m \times n}$ ,  $B = (b_{i,j})_{n \times p}$ ,  $C = AB = (c_{i,j})_{m \times p}$ . It is easy to write  $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$ ,  $\forall i = 1, 2, \dots, m$ .  $j = 1, 2, \dots, p$  and then consider  $(A^1B_1 + \dots + A^nB_n)_{ij}$ , we have  $(A^1B_1 + \dots + A^nB_n)_{ij} = (a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}) = \sum_{k=1}^n a_{ik}b_{kj}$ . So  $AB = A^1B_1 + \dots + A^nB_n$ .

**Problem B2 (10 pts).**

Because  $f$  is a linear map, thus we have

$$f(x + y) = f(x) + f(y)$$

$$f(\lambda x) = \lambda f(x)$$

So write the inverse function  $g = f^{-1}$  and  $x_1, x_2 \in E$ , we have

$$\begin{aligned} \exists x \in E, x = g(f(x_1) + f(x_2)) &\Rightarrow f(x) = f(x_1) + f(x_2) = f(x_1 + x_2) \\ &\Rightarrow x = x_1 + x_2 \\ &\Rightarrow g(f(x)) = g(f(x_1 + x_2)) = g(f(x_1)) + g(f(x_2)) \\ \exists x^* \in E, x^* = g(f(\lambda x)) &\Rightarrow f(x^*) = f(\lambda x) \\ &\Rightarrow x^* = \lambda x \\ &\Rightarrow g(\lambda f(x)) = g(f(\lambda x)) = \lambda g(f(x)) \end{aligned}$$

**Problem B3 (10 pts).** Given two vectors spaces  $E$  and  $F$ , let  $(u_i)_{i \in I}$  be any basis of  $E$  and let  $(v_i)_{i \in I}$  be any family of vectors in  $F$ . Prove that the unique linear map  $f: E \rightarrow F$  such that  $f(u_i) = v_i$  for all  $i \in I$  is surjective iff  $(v_i)_{i \in I}$  spans  $F$ .

For  $x \in E$ , it can be written as  $x_1u_1 + x_2u_2 + \dots + x_nu_n$  as  $(u_i)_{i \in I}$  is a basis of  $E$  and  $(u_i)_{i \in I}$  spans  $E$ . Then,  $f(x) = x_1f(u_1) + x_2f(u_2) + \dots + x_nf(u_n)$ . Since  $f(u_i) = v_i$ , this can be rewritten as:  $f(x) = x_1v_1 + x_2v_2 + \dots + x_nv_n$ . If the linear map  $f$  is surjective, then every element in  $F$  has a corresponding element in  $E$  and  $f(x) = x_1v_1 + x_2v_2 + \dots + x_nv_n$  indicates that  $(v_i)_{i \in I}$  must span  $F$ .

**Problem B4 (10 pts).** Let  $f: E \rightarrow F$  be a linear map with  $\dim(E) = n$  and  $\dim(F) = m$ . Prove that  $f$  has rank 1 iff  $f$  is represented by an  $m \times n$  matrix of the form

$$A = uv^\top$$

with  $u$  a nonzero column vector of dimension  $m$  and  $v$  a nonzero column vector of dimension  $n$ .  $\dim(E) = n$  indicates that the basis of  $E$  consists of  $n$  vectors and  $\dim(F) = m$  indicates the basis of  $F$  consists of  $m$  vectors. From class notes, we know that  $M(f) =$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = A, \text{ an } m \times n \text{ matrix.}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} \text{ and } v^T = (v_1 \ v_2 \ v_3 \ \cdots \ v_n) \text{ so } uv^T \text{ is an } m \times n \text{ matrix of the form:}$$

$$\begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & u_2v_3 & \cdots & u_2v_n \\ u_3v_1 & u_3v_2 & u_3v_3 & \cdots & u_3v_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ u_mv_1 & u_mv_2 & u_mv_3 & \cdots & u_mv_n \end{pmatrix} = v_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + v_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} + \cdots + v_n \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix}. \text{ Since every}$$

column in the matrix of  $f$  is a multiple of one column, the rank is 1 for  $f$  and the matrix is always of this form by definition for an  $m \times n$  matrix.

**Problem B5 (120 pts).** (Haar extravaganza) Consider the matrix

$$W_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

(1)

$$\begin{aligned}
W_{3,3}c &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot c_1 + 1 \cdot c_5 \\ 1 \cdot c_1 + -1 \cdot c_5 \\ 1 \cdot c_2 + 1 \cdot c_6 \\ 1 \cdot c_2 + -1 \cdot c_6 \\ 1 \cdot c_3 + 1 \cdot c_7 \\ 1 \cdot c_3 + -1 \cdot c_7 \\ 1 \cdot c_4 + 1 \cdot c_8 \\ 1 \cdot c_4 + -1 \cdot c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_5 \\ c_1 - c_5 \\ c_2 + c_6 \\ c_2 - c_6 \\ c_3 + c_7 \\ c_3 - c_7 \\ c_4 + c_8 \\ c_4 - c_8 \end{pmatrix}
\end{aligned}$$

(2)

If the inverse of  $W_{3,3}$  is  $(1/2)W_{3,3}^\top$ , then the product of the two matrices is the identity matrix.

$$\begin{aligned}
W_{3,3} \cdot (1/2)W_{3,3}^\top &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix} \\
&= \begin{pmatrix} 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 + 0.5 & 0.5 + -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 + -0.5 & 0.5 - -0.5 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) Let  $W_{3,2}$  and  $W_{3,1}$  be the following matrices:

$$W_{3,2} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad W_{3,1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Show that given any vector  $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$ , the result  $W_{3,2}c$  of applying  $W_{3,2}$  to  $c$  is

$$W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8),$$

the second step in reconstructing a vector from its Haar coefficients, and the result  $W_{3,1}c$  of applying  $W_{3,1}$  to  $c$  is

$$W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8),$$

the first step in reconstructing a vector from its Haar coefficients.

Conclude that

$$W_{3,3}W_{3,2}W_{3,1} = W_3,$$

the Haar matrix

$$W_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

*Hint.* First, check that

$$W_{3,2}W_{3,1} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},$$

where

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

**First:**  $W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8)$

$$W_{3,2}c = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ c_1 + 0 - c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + c_2 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + c_2 + 0 - c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_3 \\ c_1 - c_3 \\ c_2 + c_4 \\ c_2 - c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

**Second:**  $W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8)$

$$W_{3,1}c = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ c_1 - c_2 + 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + c_3 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + c_4 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + c_5 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + c_6 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + c_7 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 + 0 + c_8 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

**Now check  $W_{3,2} * W_{3,1}$**

$$W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},$$

Conclude  $W_{3,3} * W_{3,2} * W_{3,1} = W_3$

$$W_{3,3} * W_{3,2} * W_{3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

=

(4) Prove that the columns and the rows of  $W_{3,2}$  and  $W_{3,1}$  are orthogonal. Deduce from this that the columns of  $W_3$  are orthogonal, and the rows of  $W_3^{-1}$  are orthogonal. Are the rows of  $W_3$  orthogonal? Are the columns of  $W_3^{-1}$  orthogonal? Find the inverse of  $W_{3,2}$  and the inverse of  $W_{3,1}$ .

We have  $W_{3,3} = (\alpha_1, \alpha_2, \dots, \alpha_8)$ , then write  $\alpha_i^T \alpha_j, \forall i, j = 1, 2, \dots, 8 \ i \neq j$  there are two conditions:

1.  $\alpha_i$  and  $\alpha_j$  have elements on the same positions.
2.  $\alpha_i$  and  $\alpha_j$  have elements on different positions.

For the first condition, we have  $\alpha_i^T \alpha_j = 1 - 1 = 0$  and for the second condition, we have  $\alpha_i^T \alpha_j = 0$ . Also,  $\alpha_i^T \alpha_i = 1 + 1 = 2$ , so we can prove that the columns are orthogonal. It is the same to prove the rows are orthogonal.

(5) For any  $n \geq 2$ , the  $2^n \times 2^n$  matrix  $W_{n,n}$  is obtained from the two rows

$$\underbrace{1, 0, \dots, 0}_{2^{n-1}}, \underbrace{1, 0, \dots, 0}_{2^{n-1}}$$

$$\underbrace{1, 0, \dots, 0}_{2^{n-1}}, \underbrace{-1, 0, \dots, 0}_{2^{n-1}}$$

by shifting them  $2^{n-1} - 1$  times over to the right by inserting a zero on the left each time.

Given any vector  $c = (c_1, c_2, \dots, c_{2^n})$ , show that  $W_{n,n}c$  is the result of the last step in the process of reconstructing a vector from its Haar coefficients  $c$ . Prove that  $W_{n,n}^{-1} = (1/2)W_{n,n}^\top$ , and that the columns and the rows of  $W_{n,n}$  are orthogonal.

**Extra credit (30 pts.)**

Given a  $m \times n$  matrix  $A = (a_{ij})$  and a  $p \times q$  matrix  $B = (b_{ij})$ , the *Kronecker product* (or *tensor product*)  $A \otimes B$  of  $A$  and  $B$  is the  $mp \times nq$  matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

It can be shown (and you may use these facts without proof) that  $\otimes$  is associative and that

$$\begin{aligned} (A \otimes B)(C \otimes D) &= AC \otimes BD \\ (A \otimes B)^\top &= A^\top \otimes B^\top, \end{aligned}$$

whenever  $AC$  and  $BD$  are well defined.

Check that

$$W_{n,n} = \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix},$$

and that

$$W_n = \begin{pmatrix} W_{n-1} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}.$$

Use the above to reprove that

$$W_{n,n}W_{n,n}^\top = 2I_{2^n}.$$

Let

$$B_1 = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and for  $n \geq 1$ ,

$$B_{n+1} = 2 \begin{pmatrix} B_n & 0 \\ 0 & I_{2^n} \end{pmatrix}.$$

Prove that

$$W_n^\top W_n = B_n, \quad \text{for all } n \geq 1.$$

We can find

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \in M_{2^n, 2^n}$$

and it is easy to get

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in M_{2^n, 2^{n-1}},$$

$$I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \in M_{2^n, 2^{n-1}},$$

so

$$\begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 1 & 0 & \cdots & \cdots & -1 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 1 & \vdots \\ 0 & 1 & \cdots & \cdots & 0 & -1 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -1 \end{pmatrix} = W_{n,n}.$$

$$\begin{aligned} W_{n,n} W_{n,n}^\top &= \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}^\top \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}]^\top + I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^\top \end{aligned}$$

according to the facts above, we have

$$\begin{aligned} W_{n,n} W_{n,n}^\top &= [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}] [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}]^\top + [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}] [I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^\top \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + I_{2^{n-1}} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= I_{2^{n-1}} \otimes \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_{2^n} \end{aligned}$$



(6) The matrix  $W_{n,i}$  is obtained from the matrix  $W_{i,i}$  ( $1 \leq i \leq n-1$ ) as follows:

$$W_{n,i} = \begin{pmatrix} W_{i,i} & 0_{2^i, 2^{n-2^i}} \\ 0_{2^{n-2^i}, 2^i} & I_{2^{n-2^i}} \end{pmatrix}.$$

It consists of four blocks, where  $0_{2^i, 2^{n-2^i}}$  and  $0_{2^{n-2^i}, 2^i}$  are matrices of zeros and  $I_{2^{n-2^i}}$  is the identity matrix of dimension  $2^n - 2^i$ .

Explain what  $W_{n,i}$  does to  $c$  and prove that

$$W_{n,n} W_{n,n-1} \cdots W_{n,1} = W_n,$$

where  $W_n$  is the Haar matrix of dimension  $2^n$ .

*Hint.* Use induction on  $k$ , with the induction hypothesis

$$W_{n,k} W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & I_{2^{n-2^k}} \end{pmatrix}.$$

Prove that the columns and rows of  $W_{n,k}$  are orthogonal, and use this to prove that the columns of  $W_n$  and the rows of  $W_n^{-1}$  are orthogonal. Are the rows of  $W_n$  orthogonal? Are the columns of  $W_n^{-1}$  orthogonal? Prove that

$$W_{n,k}^{-1} = \begin{pmatrix} \frac{1}{2} W_{k,k}^\top & 0_{2^k, 2^{n-2^k}} \\ 0_{2^{n-2^k}, 2^k} & I_{2^{n-2^k}} \end{pmatrix}.$$

**Problem B6 (20 pts).** Prove that for every vector space  $E$ , if  $f: E \rightarrow E$  is an idempotent linear map, i.e.,  $f \circ f = f$ , then we have a direct sum

$$E = \text{Ker } f \oplus \text{Im } f,$$

so that  $f$  is the projection onto its image  $\text{Im } f$ .

**Problem B7 (20 pts).** Let  $U_1, \dots, U_p$  be any  $p \geq 2$  subspaces of some vector space  $E$  and recall that the linear map

$$a: U_1 \times \cdots \times U_p \rightarrow E$$

is given by

$$a(u_1, \dots, u_p) = u_1 + \cdots + u_p,$$

with  $u_i \in U_i$  for  $i = 1, \dots, p$ .

(1) If we let  $Z_i \subseteq U_1 \times \cdots \times U_p$  be given by

$$Z_i = \left\{ \left( u_1, \dots, u_{i-1}, - \sum_{j=1, j \neq i}^p u_j, u_{i+1}, \dots, u_p \right) \mid \sum_{j=1, j \neq i}^p u_j \in U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right) \right\},$$

for  $i = 1, \dots, p$ , then prove that

$$\text{Ker } a = Z_1 = \dots = Z_p.$$

In general, for any given  $i$ , the condition  $U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right) = (0)$  does not necessarily imply that  $Z_i = (0)$ . Thus, let

$$Z = \left\{ \left( u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_p \right) \mid u_i = - \sum_{j=1, j \neq i}^p u_j, u_i \in U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right), 1 \leq i \leq p \right\}.$$

Since  $\text{Ker } a = Z_1 = \dots = Z_p$ , we have  $Z = \text{Ker } a$ . Prove that if

$$U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right) = (0) \quad 1 \leq i \leq p,$$

then  $Z = \text{Ker } a = (0)$ .

(2) Prove that  $U_1 + \dots + U_p$  is a direct sum iff

$$U_i \cap \left( \sum_{j=1, j \neq i}^p U_j \right) = (0) \quad 1 \leq i \leq p.$$

(3) **Extra credit (40 pts)**. Assume that  $E$  is finite-dimensional, and let  $f_i: E \rightarrow E$  be any  $p \geq 2$  linear maps such that

$$f_1 + \dots + f_p = \text{id}_E.$$

Prove that the following properties are equivalent:

$$(1) \quad f_i^2 = f_i, \quad 1 \leq i \leq p.$$

$$(2) \quad f_j \circ f_i = 0, \text{ for all } i \neq j, \quad 1 \leq i, j \leq p.$$

$$(1) \Rightarrow (2)$$

We multiply  $f_i$  on each side of the equation, then get

$$f_i^2 + \sum_{j \neq i} f_j \circ f_i = f_i$$

$$\sum_{j \neq i} f_j \circ f_i = 0$$

Because  $E$  is finite-dimensional, then for all  $y \in E$  we can write  $y = \sum \lambda_i x_i$ , where  $(x_i)$  is the base. Then for all  $y \in E$ , we have  $\sum_{j \neq i} f_j \circ f_i(y) = 0$ , suppose there is a  $k$  which

makes  $f_k \circ f_i \neq 0$  then  $\ker(f_k \circ f_i) = E$ , thus  $f_k \circ f_i = 0$ , which is a contradiction.

(2)  $\Rightarrow$  (1)

We multiply  $f_i$ ,  $\forall i = 1, 2, \dots, p$  on each side of the equation, then get

$$f_i^2 + \sum_{j \neq i} f_j \circ f_i = f_i$$

$$f_i^2 = f_i$$

**TOTAL: 200 + 70 points.**