Circulant Matrix

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$$C_x w^{(m)} = \lambda_m w^{(m)} \quad \rightarrow \quad \begin{bmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & & x_2 \\ \vdots & & \ddots & \vdots \\ x_{n-1} & x_{n-2} & \cdots & x_0 \end{bmatrix} \begin{bmatrix} 1 \\ \rho_m \\ \vdots \\ \rho_m^{n-1} \end{bmatrix} = \lambda_m \begin{bmatrix} 1 \\ \rho_m \\ \vdots \\ \rho_m^{n-1} \end{bmatrix}, \qquad \rho_m^k = e^{i\frac{2\pi}{n} mk}$$

habia para $m = 0, \dots, n-1$ (n raices λ_m)

podemos obtener la primera raiz con $m=0$; $\rho_0 = 1$

$$\begin{bmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & & \vdots \\ \vdots & \vdots & & \\ x_{n-1} & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \lambda_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad ; \quad \lambda_0 = x_0 + x_{n-1} + x_{n-2} + \dots x_2 + x_1$$

$$\begin{bmatrix} x_0 & x_{n-1} & x_{n-2} & \dots & x_2 & x_1 \\ x_1 & x_0 & x_{n-1} & & & \\ \vdots & x_1 & x_0 & & & \end{bmatrix}$$

la 2da raiz λ_1 con $m=1$; $\rho_1 = e^{i\frac{2\pi}{n} 1}$

$$\begin{bmatrix} x_0 & x_{n-1} & x_{n-2} & \dots & x_2 & x_1 \\ x_1 & x_0 & x_{n-1} & & & \\ \vdots & x_1 & x_0 & & & \\ x_{n-1} & & & & & x_0 \end{bmatrix} \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{n}1} \\ e^{i\frac{2\pi}{n}2} \\ \vdots \\ e^{i\frac{2\pi}{n}(n-1)} \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ \rho_1 \\ \rho_1^2 \\ \vdots \\ \rho_1^{n-1} \end{bmatrix} \Rightarrow \lambda_1 = \begin{bmatrix} x_0 & x_{n-1} & x_{n-2} & \dots & x_2 & x_1 \end{bmatrix} \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{n}1} \\ e^{i\frac{2\pi}{n}2} \\ \vdots \\ e^{i\frac{2\pi}{2}(n-1)} \end{bmatrix}$$

en este caso se ordenan $\lambda_k = \sum_{\ell=1}^{n} x_{n-\ell}\, e^{i\frac{2\pi}{n}\ell k}$ $\quad \left| \begin{array}{l} \boxed{\ell = n-j} \\ j = n-\ell \\ j(n) = 0 \\ j(1) = n-1 \end{array} \right\} \sum_{j=0}^{n-1} x_j\, e^{i\frac{2\pi}{n}k(n-j)}$

$$\lambda_k = \sum_{j=0}^{n-1} x_j\, e^{i\frac{2\pi}{h}k(-j)} = \sum_{\ell=0}^{n-1} x_\ell\, e^{i\frac{2\pi}{n}\ell(-k)}$$

$$\lambda_1 = x_0 1 + x_1 e^{i\frac{2\pi}{n}(-1)} + x_2 e^{i\frac{2\pi}{2}(-2)} + \dots + x_{n-2} e^{i\frac{2\pi}{n}(2-n)} + x_{n-1} e^{i\frac{2\pi}{n}(1-n)}$$

$\left\{ e^{i\frac{2\pi}{n}(\pm n)} = 1 \right.$

$$\lambda_1 = x_0 1 + x_{n-1} e^{i\frac{2\pi}{n}} + x_{n-2} e^{i\frac{2\pi}{n}2} + \dots + x_2 e^{i\frac{2\pi}{n}} \Rightarrow \underbrace{\sum_{j=0} x_j\, e^{-i\frac{2\pi}{n}kj} = \hat{X}_k}_{\text{DFT}}$$

así teniendo todas las $\boxed{\lambda_k = \hat{X}_k}$ transformada discreta de Fourier

Diagonalización

$$C_x w^{(m)} = \lambda_{(m)} w^{(m)}$$

$$\underbrace{\begin{bmatrix} x_0 & x_{n-1} & x_{n-2} & \dots & x_2 & x_1 \\ x_1 & x_0 & x_{n-1} & & & \\ \vdots & x_1 & x_0 & & & \\ x_{n-1} & & & & & x_0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \\ 1 & \rho_1 & \rho_2 & \\ 1 & \rho_1^2 & \rho_2^2 & \dots \\ 1 & \vdots & \vdots & \\ 1 & \rho_1^{(n-1)} & \rho_2^{(n-1)} & \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \\ 1 & \rho_1 & \rho_2 & \\ 1 & \rho_1^2 & \rho_2^2 & \dots \\ 1 & \vdots & \vdots & \\ 1 & \rho_1^{(n-1)} & \rho_2^{(n-1)} & \end{bmatrix} \begin{bmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \lambda_2 \\ & & & \ddots \end{bmatrix}}_{\text{RHS}}$$

$$\left[\begin{bmatrix} 1 \end{bmatrix} \quad 1 \right]$$

$\rho_1 \qquad \rho_2$

$$x_0 1 + x_{n-1}\rho_1 + x_{n-2}\rho_1^2 + \dots + x_1 \rho_1^{(n-1)} = \lambda_1 1$$

que contiene la ecuación

RHS

$$\lambda_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}, \lambda_1 \begin{bmatrix} 1 \\ \rho_1 \\ \rho_1^2 \\ \vdots \end{bmatrix}, \dots$$

$$\lambda_k = \sum_{j=0}^{n-1} x_j e^{i\frac{2\pi}{n}(n-j)k} \overset{k=1}{\Rightarrow} x_0 1 + x_1 e^{i\frac{2\pi}{n}(n-1)} + \dots + x_{n-1} e^{i\frac{2\pi}{n}1} + \dots \quad // = \hat{x}_k$$

esto $C_x W = W(\text{diag}\,\lambda_k) = W(\text{diag}\,\hat{x}_k) \qquad /(\ )\cdot W^{-1}$

$$C_x W W^{-1} = C_x = W(\text{diag}\,\hat{x}_k) W^{-1} = W(\text{diag}\,\hat{x}_k)\frac{W^*}{n}$$

$$C_x = \underbrace{\frac{W}{n}}_{\text{IDFT}} \text{diag}\,\hat{x} \underbrace{W^*}_{\text{DFT}}$$

entonces C_x puede actuar sobre un vector

$$C_x y = \frac{W}{n}\,\text{diag}\,\hat{x}\,\underbrace{W^* y}_{\hat{y}} \Rightarrow C_x y = z \qquad / W^* \cdot (\ )$$

$$\underbrace{\frac{W^* W}{n}}_{W^{-1}W} \text{diag}\,\hat{x}\, W^* y = W^* z$$

se simplifica

$$\begin{bmatrix} x_0 & x_{n-1} & \\ x_1 & x_0 & \\ x_2 & x_1 & \\ \vdots & & \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} z_0 \\ \vdots \\ z_{n-1} \end{bmatrix} \rightarrow \boxed{(\text{diag}\,\hat{x})\,\hat{y} = \hat{z}}$$

$$\begin{bmatrix} \hat{x}_0 & & & \\ & \hat{x}_1 & & \\ & & \ddots & \\ & & & \hat{x}_{n-1} \end{bmatrix}\begin{bmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \vdots \\ \hat{y}_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{z}_0 \\ \hat{z}_1 \\ \vdots \\ \hat{z}_{n-1} \end{bmatrix}$$

#Construyendo W

otra cosa

eigenvector $1\ \lambda_0$ eigenvector λ_1

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -i & 1 & -1 \\ 1 & -1 & -1 & i \end{bmatrix}$$

$e^{i\frac{2\pi}{n}k} = e^{i\frac{\pi}{2}k} = \{1, e^{i\frac{\pi}{2}}, e^{i\frac{\pi}{2}3}, e^{i\pi}\} = \{1, i, -i, -1\} = \rho_1$

(caso $n=4$)

$\rho_2^k = e^{i\frac{\pi}{2}\cdot 2k} = e^{i\pi k} = \{1, e^{i\pi}, e^{i2\pi}, e^{i\pi}\} = \{1, -1, 1, -1\}$

$\rho_3^k = e^{i\frac{\pi}{2}3k} = \{1, e^{i\frac{3}{2}\pi}, e^{i3\pi}, e^{i\frac{9}{2}\pi}\} = \{1, -i, -1, i\}$

$e^{i\frac{8+1}{2}\pi} = e^{i(8\pi + \frac{\pi}{2})} = e^{i\frac{\pi}{2}} = i$

$4 = e^{i\frac{9\pi}{2}} = i$

Bamieh, B. (2018). Discovering the Fourier Transform: A Tutorial on Circulant Matrices, Circular Convolution, and the DFT. *ArXiv.org*. https://doi.org/10.48550/arXiv.1805.05533