Labor Problem Set 2
Example Solutions

Maximilian Blesch & Maxi Schaller

Advanced applied econometrics

July 2025

Organization

Graded Problem Set 4 - Labor:

- Due date: **August 15**
- Additional tutorial: coordinate date!

Exam:

- Date: **July 18, 9am**; Elinor Ostrom Hall; duration 2hrs
- All material covered in the course is relevant for the exam.
- 1/2 Part I (FW), 1/2 Part II: IO (HU) & labor (PH)
- On Part II: There will be no explicit questions on Python programming, however, you need to be able discuss the conceptual implementation of the models and methods discussed in the problem sets.
- Materials allowed:
  - Hand-written or printed notes: 10 pages, double-sided, A4. **No** lecture slides/papers etc.
  - **No** electronic devices
  - Non-programmable calculator
- **Master students** receive the same exam questions. Their final grade will only based on the exam. Passing the graded problem sets is sufficient.

Exercise 2: Dynamic discrete choice labor supply

- Recap: last week we discussed the static discrete choice model of labor supply; ML estimation
- Today: Dynamic discrete choice of working hours
  - Key: introduction of dynamic incentives via endogenous human capital accumulation
  - Finite horizon dynamic programming model; State-space implementation; backwards induction
  - One person 'Robinson' economy, 3 period-decision problem
  - No stochastic components

Setup

Time horizon and agent:

- Time horizon: 3 periods (days)
- Agent: Robinson, a single individual making decisions about labor supply
- Discrete choice: $h \in \{8, 12, 16\}$ hours of work per day

Production and human capital:

- Day 1: Robinson starts with zero experience and produces 1 unit of consumption goods regardless of hours worked
- Human capital accumulation based on hours worked: For every additional 4 hours worked (beyond 8hrs), Robinson gains 1 unit of experience.

$$c_t = (1 + 0.25e_t)^2$$

Utility function:

- Cobb-Douglas utility function

$$u(c_t, l_t) = c_t^{\alpha} l_t^{\beta} \quad \text{with } \alpha = \beta = 0.5$$
$$l_t = 24 - h_t \quad \text{(leisure time)}$$

A: Question 1

1. If Robinson had to survive only for 1 day, how many hours would he work? How much utility would he get? What incentive does Robinson have to make a different choice on Day 1 if he has to survive for 3 days?

$$\max_{c,l} u(c_t, h_t) = c_t^{\alpha}(T - h_t)^{\beta}$$

$$\max_{h_1} u(c_1 = 1, h_1) = 1(24 - h_1)^{0.5}$$

- Production/consumption is fixed to 1 unit (experience is zero)
- First derivative: $-0.5 \cdot (24 - h_1)^{-0.5} < 0$. Utility is maximised when hours of work are minimised.
- Computing the utility for different choices of h_1:
  - $h_1 = 8$: $u = 16^{0.5} = 4$
  - $h_1 = 12$: $u = 12^{0.5} = 3.4641$
  - $h_1 = 16$: $u = 8^{0.5} = 2.8284$
- Optimal choice: $h_1 = 8$ hours, $u(c_1, h_1) = 4$.

If Robinson has to survive for 3 days, he may choose to work longer hours on the first day to gain experience and increase his future production returns. Depending on the parameters of the model, it may well be optimal to work longer hours on the first day(s) and reap the returns to experience gathered in the following day(s).

A: Question 2

2. Is Robinson risk averse? The curvature of the utility function measures the consumers attitude towards risk. The Cobb-Douglas utility is a concave function (can be verified by showing that the Hessian is negative semidefinite), implying that the utility of the expected consumption is higher than the expected utility of consumption. Therefore Robinson is risk averse.

A: Question 3

3. Are consumption and leisure substitutes for Robinson?
Sketch of dynamic problem for intuition:

- Robinson faces a trade-off between consumption and leisure over time. If he works more hours today, he can consume more tomorrow, but at the cost of leisure. The decision depends on the intertemporal elasticity of substitution.
- Consider a two-period setting:

$$
\begin{aligned}
U(\mathrm{c},\mathrm{l}) = u(c_t, l_t) &+ \delta u(c_{t+1}, l_{t+1}) \\
\text{s.t.} \quad & l_t = T - h_t; \\
& c_{t+1} = f(h_t), \text{ with } f'(h_t) > 0 \ (f'(l_t) < 0)
\end{aligned}
$$

A: Question 3

Consider the total derivative of the utility function with respect to the hours worked today, h_t:

$$\frac{dU}{dh_t} = \frac{\partial u_t}{\partial l_t}\frac{dl_t}{dh_t} + \delta \frac{\partial u_{t+1}}{\partial c_{t+1}}\frac{dc_{t+1}}{dh_t}$$

$$\text{where} \quad \frac{dl_t}{dh_t} = -1 \quad \text{and} \quad \frac{dc_{t+1}}{dh_t} = f'(h_t) > 0$$

At the optimum:

$$\frac{\partial u_t}{\partial l_t} = \delta f'(h_t)\frac{\partial u_{t+1}}{\partial c_{t+1}}$$

- Marginal utility of leisure in period t is balanced against the (discounted) marginal utility of consumption in period $t+1$.

A: Question 4

4. What are the state variables of the model? How many states are there? Write down a state space representation as a matrix with dimensions *number of states* by *number of state variables*.

One obvious state variable in the model is experience. Given that this is a finite discrete time model, there is a second state variable: period.
There are 9 states.

period	experience
1	0
2	0
2	1
2	2
3	0
3	1
3	2
3	3
3	4

A: Question 5

5. In dynamic programming problems like this one, the agent makes choices based on the choice specific value functions. These are given by the flow utility plus the continuation value. Assume that continuation values on the last day are zero. What would be optimal number of hours of work for Robinson on Day 3?

A: Question 5

- Choice-specific value functions $V^h(h_t, x_t, \theta)$ depend on choice alternative of hours worked h_t and the current state x_t (here only experience in period t), and parameters θ.
- Combination of current flow utility from choice and the continuation value in the next period (which depends on the choice via experience accumulation).

$$\begin{aligned} V^h(h_t, x_t, \theta) &= u(c_t(x_t), l_t(h_t); \theta) + \delta V(x_{t+1}, \theta | x_t, h_t) \\ &= c_t(x_t)^\alpha (T - h_t)^\beta + \delta V(x_{t+1}, \theta | x_t, h_t) \\ &= \left[(1 + 0.25x_t)^2\right]^\alpha (T - h_t)^\beta + \delta V(x_{t+1}, \theta | x_t, h_t) \end{aligned}$$

- $V(x_{t+1}, \theta | x_t, h_t)$ is the maximum value function in the next period, i.e. the utility derived from behaving optimally from the next period onwards, discounted by δ.

- This follows from the principles introduced in lecture 3 (HU) in a infinite horizon setting. Cp. the choice specific value function on slide 33:

$$V^i(x_t, \theta) = u(x_t, i_t, \theta) + \delta E\Big[V(x_{t+1}, \epsilon_{t+1}, \theta) | x_t, i_t, \theta\Big]$$

A: Question 5

- In this simple setting, the optimal choice in any period t, given some state x_t, is given by the maximum of the choice specific value functions.

$$\underset{h_t}{\operatorname{argmax}}\, V^h(h_t, x_t, \theta)$$

- Finite horizon setting: the dynamic programming problem can be solved by backwards induction, starting from the last period.
- Last period: suppose the continuation value is zero. (Alternatives?)

$$V^h(h_3, x_3, \theta) = \left[(1 + 0.25x_3)^2\right]^\alpha (T - h_3)^\beta + \delta \cdot 0$$

- Example derivation for some state: in the last period (day 3) Robinson has not previously accumulated any experience, i.e. $x_3 = 0$.

$$\begin{aligned}
V^h(h_3 = 8, x_3 = 0, \theta) &= \left[(1 + 0.25 * 0)^2\right]^\alpha (24 - 8)^\beta + \delta \cdot 0 &&= 4 \\
V^h(h_3 = 12, x_3 = 0, \theta) &= \left[(1 + 0.25 * 0)^2\right]^\alpha (24 - 12)^\beta + \delta \cdot 0 &&= 3.461 \\
V^h(h_3 = 16, x_3 = 0, \theta) &= \left[(1 + 0.25 * 0)^2\right]^\alpha (24 - 16)^\beta + \delta \cdot 0 &&= 2.8284
\end{aligned}$$

A: Question 5

5. What would be optimal number of hours of work for Robinson on Day 3?

period	experience	$h_t = 8$	$h_t = 12$	$h_t = 16$	V_{max}	choice
3	0	4	3.4641	2.8284	4	8h
3	1	5	4.3301	3.5355	5	8h
3	2	6	5.1962	4.2426	6	8h
3	3	7	6.0621	4.9497	7	8h
3	4	8	6.9282	5.6569	8	8h

- Upon calculating the choice specific value functions in this state it becomes clear that no matter what choices were made on day 1 and day 2, on day 3 it is always optimal for Robinson to work for 8 hours only.
- No further dynamic incentives are at play: working more to obtain additional experience does not yield any additional utility, since the continuation value is zero (Robinson is rescued).

A: Question 6

6. Note that the flow utilities you calculated in order to arrive to the previous answer are the Day 2 continuation values. What would be optimal numbers of hours to work for Robinson on Day 2? And Day 1?

$$V^h(h_t, x_t, \theta) = \left[(1 + 0.25x_t)^2\right]^\alpha (T - h_t)^\beta + \delta V(x_{t+1}, \theta)$$

A: Question 6

- Moving back one period $t = 2$, we can use the continuation values from Day 3 to calculate the choice specific value functions for Day 2.
- Note that the continuation values depend on the state an the choice in Day 2, i.e. the choice determining to which (experience) state Robinson will transition to on Day 3.
- Example: starting from the state $x_2 = 0$ (no experience), depending on the choice of hours Robinson transitions to states with $x_3 = (0, 1, 2)$ on Day 3.

The table below summarises the choice specific value functions on Day 1 and Day 2. Comparing the values one arrives at the *state specific* optimal number of hours to work. One gets to these values by *solving the model backwards*.

period	experience	$h_t = 8$	$h_t = 12$	$h_t = 16$	V_{max}	choice
1	0	12.8284	13.9996	15.0711		
2	0	8	8.4641	8.8284		
2	1	10	10.3301	10.5355		
2	2	12	12.1962	12.2426		

A: Question 6

- Again, the working hours choice is derived by maximising the choice specific value functions.

period	experience	$h_t = 8$	$h_t = 12$	$h_t = 16$	V_{max}	choice
1	0	12.8284	13.9996	15.0711	15.0711	16
2	0	8	8.4641	8.8284	8.8284	16
2	1	10	10.3301	10.5355	10.5355	16
2	2	12	12.1962	12.2426	12.2426	16
3	0	4	3.4641	2.8284	4	8h
3	1	5	4.3301	3.5355	5	8h
3	2	6	5.1962	4.2426	6	8h
3	3	7	6.0621	4.9497	7	8h
3	4	8	6.9282	5.6569	8	8h

Moving backward through the model, we can establish that Robinson will choose to work for 16h on Day 1 and Day 2, and 8 hours on Day 3.

See code for Question B.

Graded Assignment 4

Extension of today's framework:

- Longer time horizon: 10 (60) periods (days)
- Heterogeneous individuals: introduction of differences in preferences over consumption and leisure
- Stochastic shocks to productivity/consumption: introduction of uncertainty in the model
- Discrete choice: occupation - fishing or farming
- Estimation of the model with simulated method of moments

Key questions for model implementation:

- State-space setup: What are the state variables? How many states are there?
- How to compute the model solution at every state in every period?