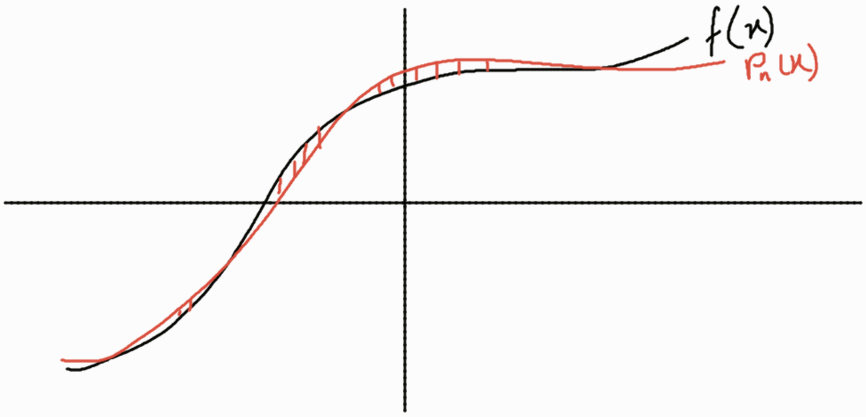
Convergence, Runge Phenomena and Chebyshev Nodes



$$\text{Error} = |f(x) - P_n(x)| \qquad n\uparrow \quad \text{error}\downarrow$$

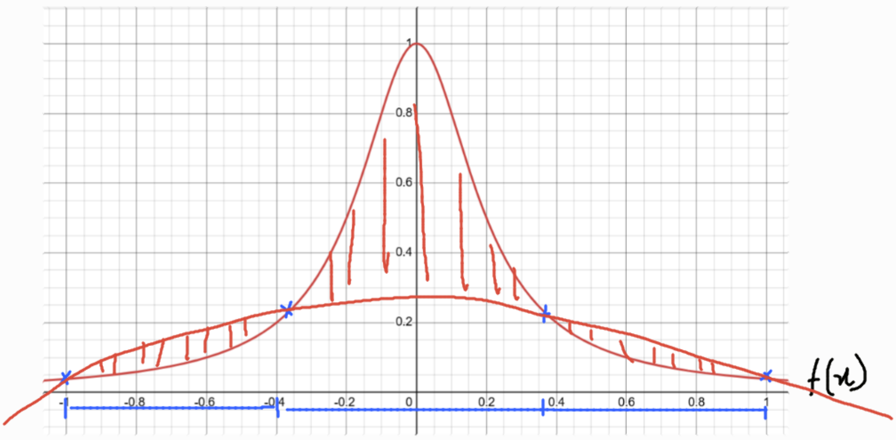
$$n \to \infty \qquad \text{error} \to 0$$

But it's not always true.

$$f(x) = \frac{1}{1+25x^2} \qquad \text{interval } [-1, 1]$$

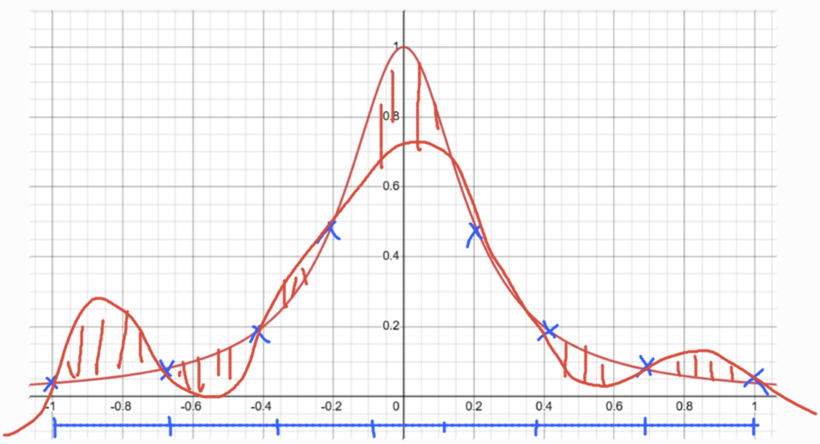
$n = 3$

$P_3(x)$
↑
node = 4



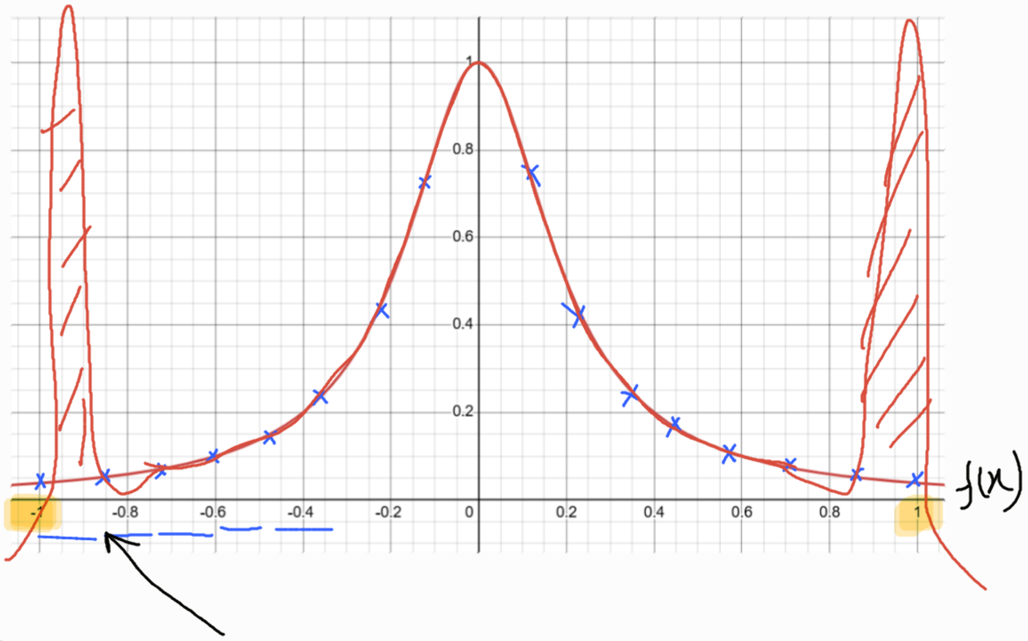
$n = 7$

node = 8



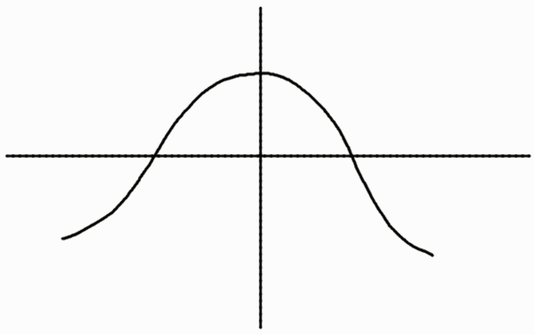
$n = 15$

node = 16



Runge Phenomenon

→ Runge functions.



→ Depends on 2 things:

① Depends on the function (Symmetric function)

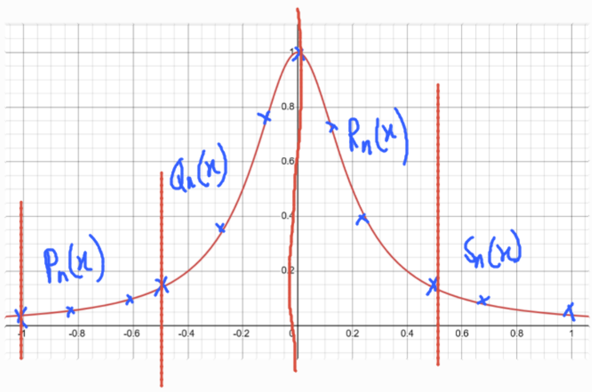
② Depends on the node (Equally spaced)

Solution:

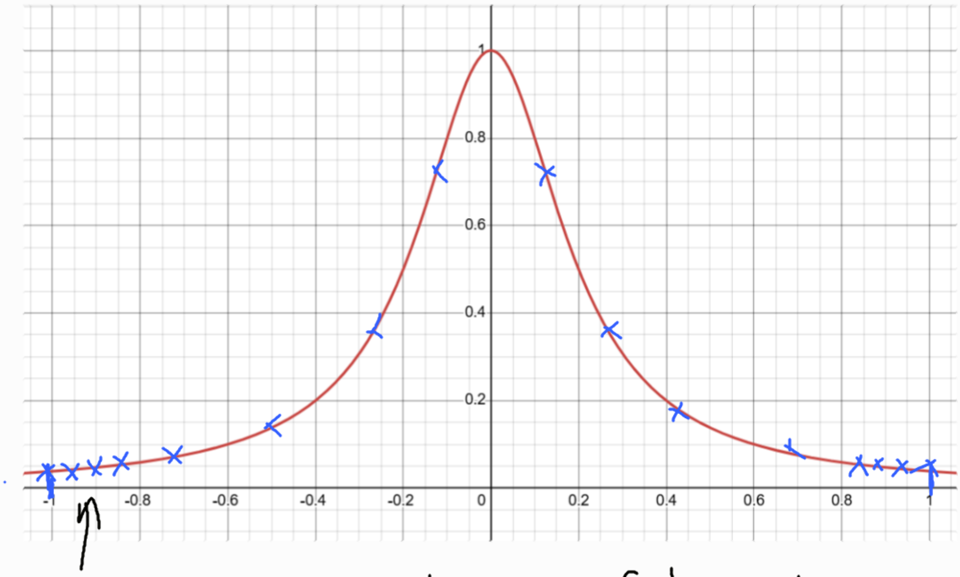
① Piece wise interpolation.

→ take smaller intervals and find the interpolating polynomial

→ Add them up / Merge them up



(ii) No equidistant Node:



how do we find these perfect nodes such that we get least amount of error and reduce the effect of runge phenomenon ?

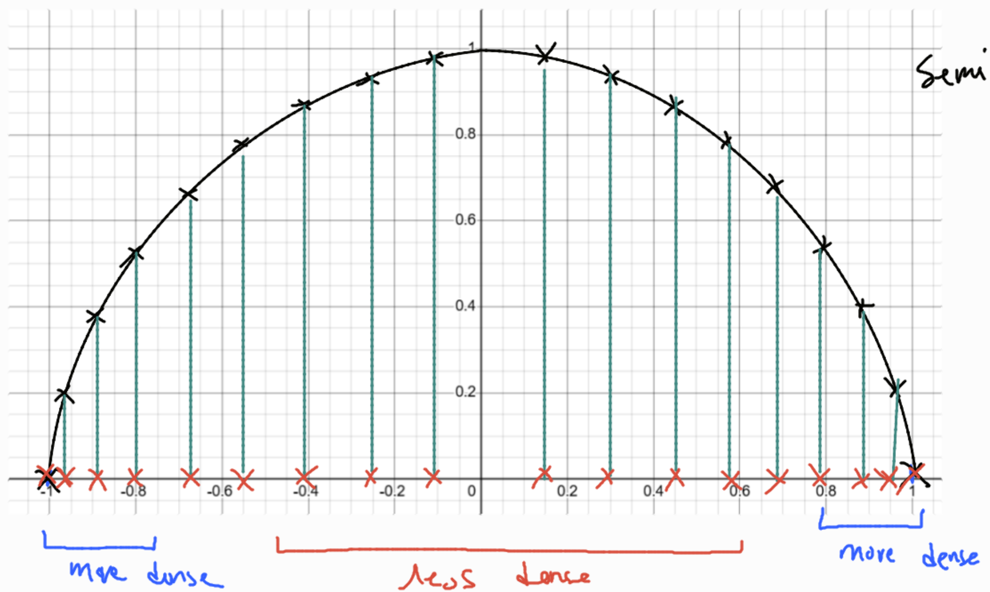
↓

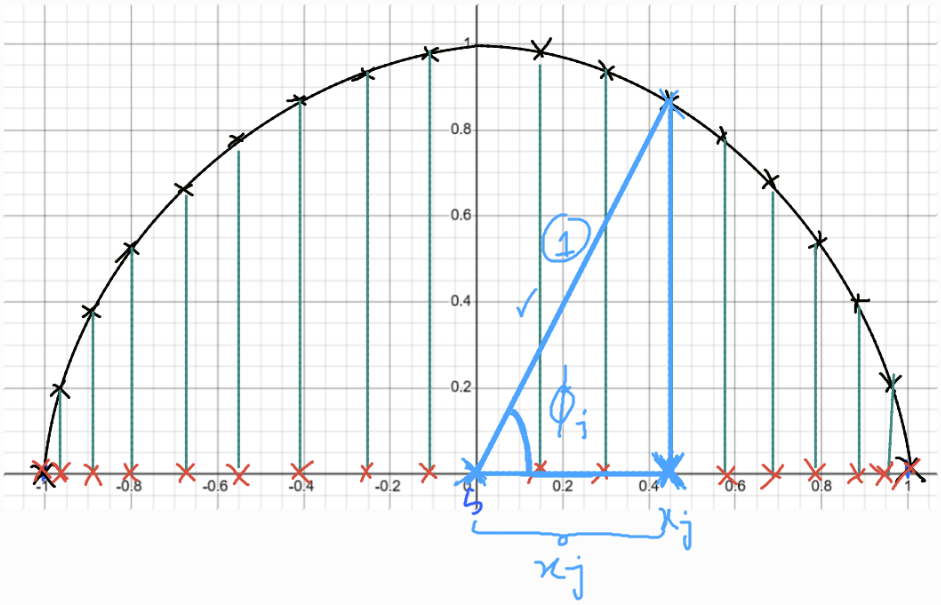
Chebyshev's node

* Takes more node at the end point rather than taking equidistance Node

[-1, 1]

node : 18





[-1, 1]

j = j th node

$$\phi_j = \frac{(2j+1)\pi}{2(n+1)}$$

$\uparrow$ degree

$$\cos \phi_j = \frac{x_j}{[\text{distance between intervals}]/2}$$

$$\cos \phi_j = \frac{x_j}{1}$$

$$\Rightarrow x_j = \cos \phi_j$$

$$\boxed{\Rightarrow x_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)}$$

for interval [-1, 1] only.

for other intervals,

[2, 6]

$\frac{4}{2} = 2$

2 4 6

$$\boxed{x_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right) * \underbrace{\text{radius}}_{\frac{\text{interval distance}}{2}} + \text{center}}$$

* $f(x) = \dfrac{1}{1+25x^2}$, $\underbrace{[-1,1]}_{\text{interval}}$, $n = 3$. Find the
$\quad \hookrightarrow$ degree

Chebyshev's node.

$n=3$ means, nodes $= 4$ $\quad \therefore j = 0, 1, 2, 3$

x_0, x_1, x_2, x_3

$x_j = \cos\left(\dfrac{(2j+1)\pi}{2(n+1)}\right)$

$$x_0 = \cos\left(\frac{(2*0+1)\pi}{2(3+1)}\right) = \cos\frac{\pi}{8}$$

$$x_1 = \cos\left(\frac{(2*1+1)\pi}{2(3+1)}\right) = \cos\frac{3\pi}{8}$$

$$x_2 = \cos\left(\frac{(2*2+1)\pi}{8}\right) = \cos\frac{5\pi}{8}$$

$$x_3 = \cos\left(\frac{(2*3+1)\pi}{8}\right) = \cos\frac{7\pi}{8}$$

(b) Using the nodes, find $P_3(x)$ using lagrange.