Lecture 4.1

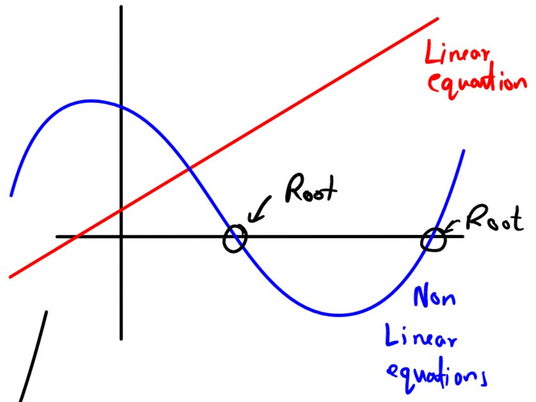
Root finding for non linear equations:

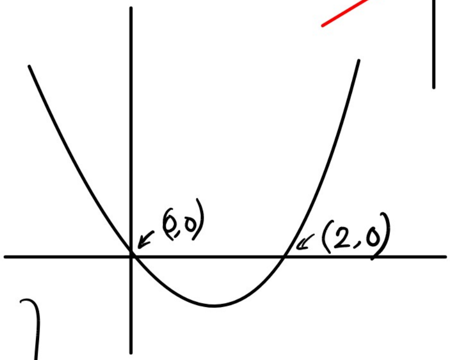
$$f(x) = x^2 - 2x = 0$$

$$\Rightarrow x(x-2) = 0$$

$$\Rightarrow x = 0, 2$$

Roots $\rightarrow x^* = 0, 2$





- # Polynomials degree ≥ 2
- # Trigono... functions (sin, cos)
- # exponential " (e^x)
- # $\left(\frac{1}{x} - 2\right)$

} Non linear function

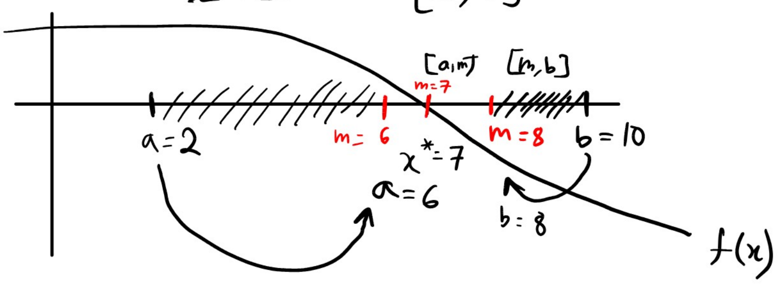
Bisection Root Finding Algorithm

Interval Bisection method

Given,

$f(x) = \sim$, $[a, b]$, find x^*

~~[a, m]~~ [m, b]



$$m = \frac{a+b}{2} = \frac{2+10}{2} = 6$$

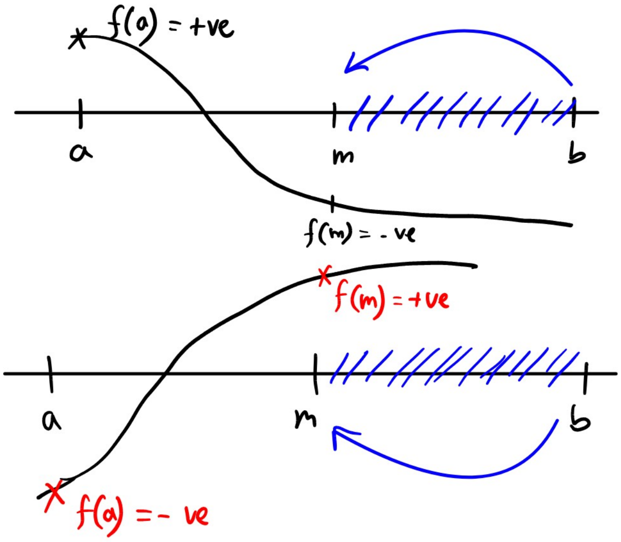
$$m = \frac{a+b}{2} = \frac{6+10}{2} = 8$$

$$m = \frac{a+b}{2} = \frac{6+8}{2} = 7$$

$f(m) = f(7) = 0$

$m = 7$ is the root.

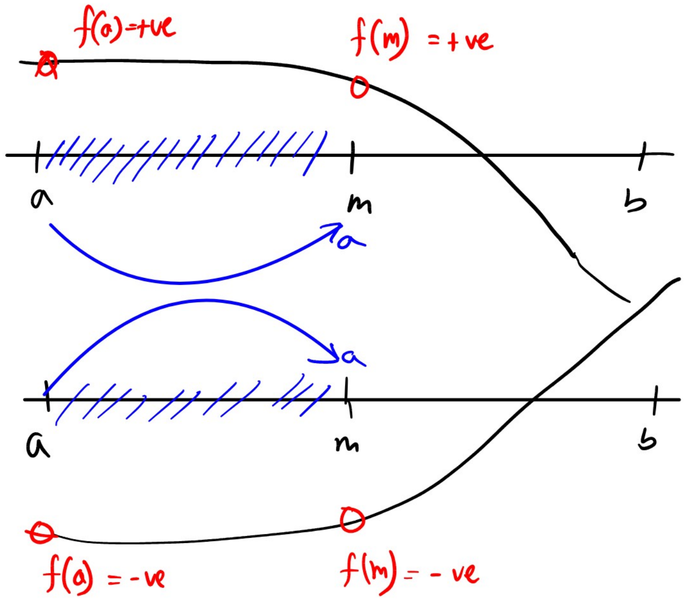
Case 1: Root exists in [a, m]



f(a) and f(m) has different sign.

$f(a) * f(m) = -ve$ means root exist in [a, m], $b = m$

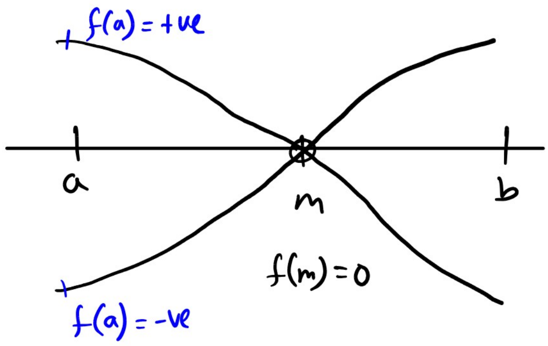
Case 2: Root doesn't exist in [a, m]



f(a) & f(m) have similar sign.

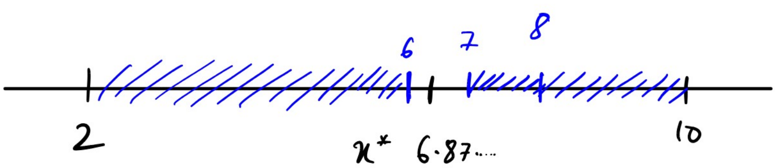
$f(a) * f(m) = +ve$ root doesn't exist in [a, m], $a = m$

Case 3:



if $f(a) * f(m) = 0$, then $f(m) = 0$

$\therefore$ m is the root.



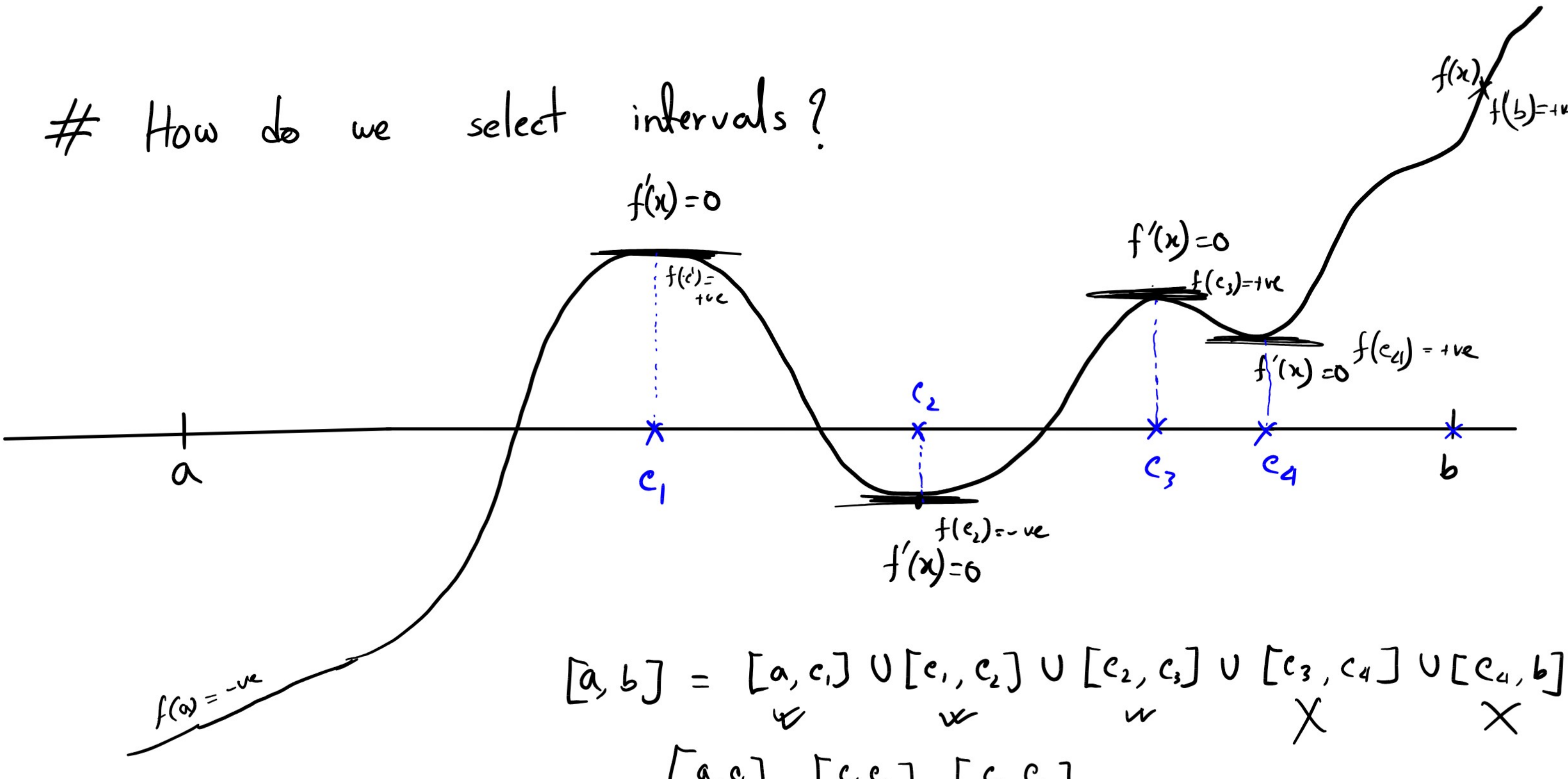
There is a limit for iteration. We do this until we find actual root, or we reach $\underbrace{\text{machine epsilon } \epsilon.}$ → Stopping Criteria.

Using Bisection method, find the root of $\underbrace{f(x) = x^3 - 7x^2 + 14x - 6}$ in interval $\underbrace{[1, 3.2]}_{a \quad b}$. Your solution must be accurate within $\underbrace{0.05}_{\epsilon}$

Iteration	a	b	$m = \frac{a+b}{2}$	f(a)	f(m)	Root exist in [a,m]	new Interval
0	1	3.2	2.1	2	1.79	Doesn't exist	[m, b] = [2.1, 3.2]
1	2.1	3.2	2.65	1.79	0.55	Doesn't exist	[m,b] = [2.65, 3.2]
2	2.65	3.2	2.925	0.55	0.086	Doesn't exist	[m,b] = [2.925, 3.2]
3	2.925	3.2	3.0625	0.086	-0.054	Exists	[a,m] = [2.925, 3.0625]
4	2.925	3.0625	$\underbrace{2.99375}$ → root.	0.086	$\underbrace{0.006}$ → $< \epsilon$ 0.05		

$\therefore$ Root $x^* = 2.99375$

How do we select intervals ?



$[a, b] = [a, c_1] \cup [c_1, c_2] \cup [c_2, c_3] \cup [c_3, c_4] \cup [c_4, b]$

✓ ✓ ✓ X X

$\underline{[a, c_1]}$ $\underline{[c_1, c_2]}$ $\underline{[c_2, c_3]}$

how many iteration is required for given intervals & epsilon values to reach the root ?

$[a, b]$, $\epsilon = 10^{-2}$

Don't include if accuracy is specified.

number of iteration required $\Big\}$ $k \geq \dfrac{\log(|b_0 - a_0|) - \log(\epsilon)}{\log(2)} \boxed{-1}$

Find the number of iteration required to get the root with error bound of machine epsilon, $\epsilon = 1.1 \times 10^{-6}$ in interval $[\underset{a_0}{1.5}, \underset{b_0}{3}]$

$$k \geq \frac{\log(|b_0 - a_0|) - \log(\epsilon)}{\log(2)} - 1$$

$$k \geq \frac{\log(|3 - 1.5|) - \log(1.1 \times 10^{-6})}{\log(2)} - 1$$

$$k \geq 19.379$$

$$k \geq 20$$

$\therefore$ number of iteration required is 20.

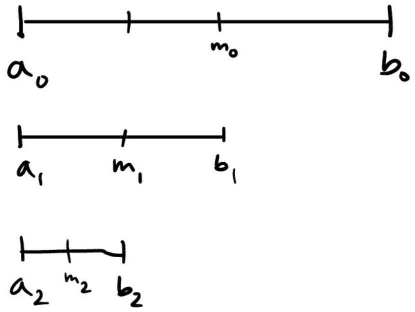
Find the number of iteration required to get the root with the accuracy of 1.1×10^{-6} in interval $[\underset{a_0}{1.5}, \underset{b_0}{3}]$

$$k \geq \frac{\log(|b_0 - a_0|) - \log(1.1 \times 10^{-6})}{\log(2)}$$

$$k \geq 20.379$$

$$k \geq 21$$

Bisection number of iteration formula derivation:



For interval, a_0, b_0:

$$|b_1 - a_1| = \frac{|b_0 - a_0|}{2}$$

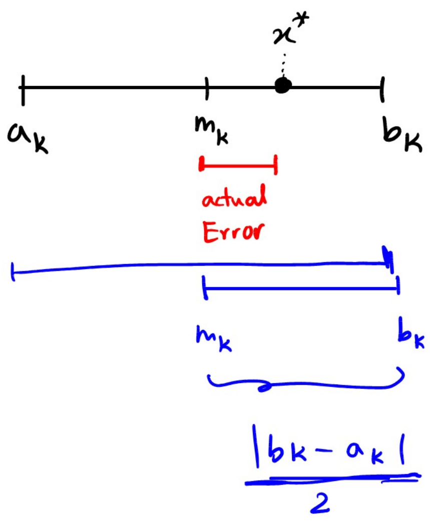
$$|b_2 - a_2| = \frac{|b_1 - a_1|}{2} = \frac{|b_0 - a_0|}{2^2}$$

$$|b_3 - a_3| = \frac{|b_2 - a_2|}{2} = \frac{|b_0 - a_0|}{2^3}$$

$$\vdots$$

$$|b_k - a_k| = \frac{|b_0 - a_0|}{2^k} \quad \text{——————} \quad (1)$$

After k iteration,



$$\text{Actual Error} = |m_k - x^*|$$

Intuitively,

$$\text{Actual Error} \leq \frac{|b_k - a_k|}{2}$$

$$\Rightarrow |m_k - x^*| \leq \frac{|b_k - a_k|}{2}$$

$$\Rightarrow |m_k - x^*| \leq \frac{|b_0 - a_0|}{2^k \cdot 2} \quad \text{[from equation 1]}$$

$$\Rightarrow |m_k - x^*| \leq \frac{|b_0 - a_0|}{2^{k+1}}$$, for maximum possible error, $$|m_k - x^*|_{max} = \frac{|b_0 - a_0|}{2^{k+1}} \quad \text{—} (1)$$

Since, we stop our iteration when actual error is less than given epsilon,

Actual Error $\leq \epsilon$

$$\Rightarrow |m_k - x^*| \leq \epsilon$$

$$\Rightarrow \frac{|b_0 - a_0|}{2^{k+1}} \leq \epsilon \qquad [\text{from equation } 2]$$

$$\Rightarrow \frac{|b_0 - a_0|}{\epsilon} \leq 2^{k+1}$$

$$\Rightarrow \log\left(\frac{|b_0 - a_0|}{\epsilon}\right) \leq \log\left(2^{k+1}\right) \qquad \left[\log\left(\frac{x}{y}\right) = \log(x) - \log(y)\right]$$

$$\left[\log(x^m) = m\log(x)\right]$$

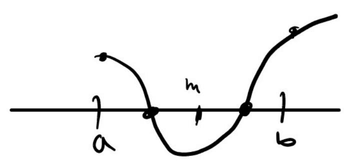
$$\Rightarrow \log(|b_0 - a_0|) - \log(\epsilon) \leq (k+1)\log(2)$$

$$\Rightarrow k+1 \geq \frac{\log(|b_0 - a_0|) - \log(\epsilon)}{\log(2)}$$

$$\therefore k \geq \frac{\log(|b_0 - a_0|) - \log(\epsilon)}{\log(2)} - 1$$

[showed]

<u>Disadvantages of Bisection:</u>



Slow convergence.

Can't find Multiple roots in single interval.

<u>Advantages:</u> Even though it is slow, it is more robust and guaranteed.