Chapter 5

Linear Equation.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

if all b are 0 → homogenous system.

else → non- " "

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\quad A \qquad\qquad\qquad X \qquad\qquad B$$

$Ax = b$

$x = A^{-1}b$

basic properties:

- A is a square matrix
- A^T is the transpose of A, hence $(A^T)_{ij} = a_{ji}$
- A is symmetric if $A^T = A$
- A is non-singular iff $\exists$ a soln $x \in R^n$, for every $b \in R^n$
- A is non-singular if $\det(A) \neq 0$
- A is non- " iff there exists a unique inverse A^{-1} such that $AA^{-1} = A^{-1}A = I$

Gaussian Elimination Method: $(n^2) \to$ complexity

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 2x_2 + 2x_3 = 04$$

$$2x_1 + 12x_2 - 2x_3 = 4$$

Augmented matrix.

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right)$$

$$R_2' = R_2 - \left(\frac{1}{1}\right)R_1$$

$$R_3' = R_3 - \left(\frac{2}{1}\right)R_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right)$$

$$R_3' = R_3 - \left(\frac{8}{-4}\right) R_2$$
$$= R_3 + 2R_2$$

$-2x_3 = 12$

or, $x_3 = -6$

$-4x_2 + x_3 = 4$

or $-4x_2 - 6 = 4$

or, $-4x_2 = 10$

or, $x_2 = \frac{10}{-4}$

or $x_2 = -2.5$

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2 \times -2.5 + (-6) = 0.$

$x_1 = 11$

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$x_1 = \frac{b_1}{l_{11}}$$

$$x_2 = \frac{b_2 - l_{21}x_1}{l_{22}}$$

$$x_1 + 2x_1 + x_3 = 0$$

$$x_1 - 2x_2 + 2x_3 = 4$$

$$2x_1 + 12x_2 - 2x_3 = 4$$

LU decomposition

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} = A^{(1)} \Longrightarrow \begin{array}{l} R_2 = R_2 - (1/1) R_1 \\ R_3 = R_3 - (2/1) R_1 \end{array} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Frobenius matrix,

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$A^{(2)} = F^{(1)} A^{(1)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1+1+0 & -2-2+0 & -1+2+0 \\ -2+0+2 & -4+0+12 & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix}$$

$$A^2 \xrightarrow{R_3' = R_3 - \left(\frac{8}{-4}\right) R_2}$$

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^{(3)} = F^{(2)} \cdot A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 2+0+0 & 1+0+0 \\ 0+0+0 & 0-4+0 & 0+1+0 \\ 0+0+0 & 0-8+8 & 0+2-4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$L \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$a_1 = 0$

$\Rightarrow a_1 + a_2 = 4$

or, $0 + a_2 = 4$

or, $a_2 = 4$

$2a_1 - 2a_2 + a_3 = 4$

or, $0 - 8 + a_3 = 4$

or, $a_3 = 12$

$$[U]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$$-2x_3 = 12$$

$$x_3 = -6$$

$$-4x_2 + x_3 = 4$$

$$-4x_2 = 4 + 6$$

$$x_2 = -\frac{10}{4}$$

$$x_2 = -5/2$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2(-5/2) - 6 = 0.$$

$$x_1 - 11 = 0.$$

$$x_1 = 11$$

$$2x+y+z=10$$

$$3x+2y+3z=18$$

$$x+4y+9z=16.$$

$$A^{(1)}=\begin{bmatrix}2&1&1\\3&2&3\\1&4&9\end{bmatrix}\xrightarrow[R_3'=R_3-\left(1/2\right)R_1]{R_2'=R_2-\left(3/2\right)R_1}$$

$$F^{(1)}=\begin{bmatrix}1&0&0\\-3/2&1&0\\-1/2&0&1\end{bmatrix}$$

$$A^{(2)}=F^{(1)}A^{(1)}=\begin{bmatrix}1&0&0\\-3/2&1&0\\-1/2&0&1\end{bmatrix}\begin{bmatrix}2&1&1\\3&2&3\\1&4&9\end{bmatrix}=\begin{bmatrix}2&1&1\\0&0.5&1.5\\0&3.5&8.5\end{bmatrix}$$

$$A^{(2)} \xrightarrow{R_3 = R_3 - (3.5/0.5) R_2}$$

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix}$$

$$A^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 3.5 & 8.5 \end{bmatrix}$$

$$U = A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 0.5 & 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}.$$

$$a = 10,\ b = 3,\ c = -10.$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ -10 \end{bmatrix}$$

$z = 5$

$y = 9$

$x = 7$

Pivoting:

if diagonals are 0, then multipliers may become ~~inf~~ undefined.

also, if order of elements in the diagonals are different, then it might cause loss of significance.

pivoting is a techinique to change/swap two row or collumns, so that diagonals elements do not have any zero.

$$\begin{pmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$