QR Decomposition

$$\underset{[m\times n]}{A} = QR = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix} \begin{bmatrix} u_1^T q_1 & u_2^T q_1 & \dots & u_n^T q_1 \\ 0 & u_2^T q_2 & \dots & u_n^T q_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_n^T q_n \end{bmatrix}$$

Here, Q is an orthonormal set of vector which we can find by using Gram - Schmidt Process.

$$Q = (q_1 \quad q_2 \quad \dots \quad q_n)$$

* Suppose we have a co efficient matrix

$$A = \begin{array}{c} \left[\begin{matrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{matrix}\right] \\ \begin{matrix} u_1 & u_2 \end{matrix} \end{array}$$

Let each column as $u_1, u_2 \dots u_n$

Now, to find q, we need to know two formulas:

$$p_k = u_k - \sum_{i=1}^{k-1} \underbrace{\left(u_k^T q_i\right)}_{\text{Scaler Dot product. (Will give a value)}} q_i$$

$$q_k = \frac{p_k}{|p_k|}$$

Gram - schmidt Process

Using these formulas, we will convert u into q

Step by step solution of the problem

step 1: $k = 1$

Note, the formula of $P_k = u_k - \underbrace{\sum_{k=1}^{k-1} (u_k^T \cdot q_i) q_i}$

if $k = 1$ then this part will run 0 times as $k-1 = 0$ and $k=1$

$$P_1 = u_1 - 0$$
$$= u_1 \quad = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$q_1 = \frac{P_K}{|P_K|} = \frac{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}{\sqrt{3^2+6^2+0^2}} = \frac{1}{\sqrt{45}} \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

step 2: $k = 2$

$$P_2 = u_2 - \sum_{i=1}^{2-1} (u_2^T q_i) q_i$$
$$= u_2 - (u_2^T q_1) q_i$$
$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \frac{1}{\sqrt{45}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{45}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{\sqrt{45}} \cdot \frac{1}{\sqrt{45}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{\cancel{15}}{3\,\cancel{45}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}{\sqrt{0^2+0^2+2^2}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since, number of unknown $n = 2$, we will stop at $k = 2$.

$$\therefore Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 3/\sqrt{45} & 0 \\ 6/\sqrt{45} & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we need to solve the system.

QR Decomposition formula Derivation:

We know, $Ax = B$

From least square method, $A^T A x = A^T b$

Since we defined $A = QR$, Replacing $A = QR$ we get,

$$(QR)^T QR x = (QR)^T b$$

$$\Rightarrow Q^T R^T QR x = Q^T R^T b$$

$$\Rightarrow \underbrace{Q^T Q}_{1} \cancel{R^T} R x = Q^T \cancel{R^T} b$$

As Q is orthonormal, $QQ^T = 1$

$$\therefore \boxed{Rx = Q^T b}$$

This is the final formula which we will use to solve the system by QR decomposition.

Example:

$$f(-3) = 0$$
$$f(0) = 0$$
$$f(6) = 2$$

Fit a straight line using QR Decomposition.

$$P_1(x) = a_0 + a_1 x = f(x)$$

$$P_1(-3) = a_0 + a_1(-3) = 0$$

$$P_1(0) = a_0 + a_1(0) = 0$$

$$P_1(6) = a_0 + a_1(6) = 2$$

Coefficient Matrix $A = \begin{vmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{vmatrix}$ with columns $\underbrace{}_{u_1}$ $\underbrace{}_{u_2}$

Now,

We have two column so, there will be two steps.

Step 1

$$P_1 = u_1 - 0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{1^2+1^2+1^2}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}}$$

step 2

$$P_2 = u_2 - \sum_{i=1}^{2-1} (u_2^T q_i) q_i$$

$$= u_2 - (u_2^T q_1) q_1$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - \left\{ \begin{bmatrix} -3 & 0 & 6 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} q_1$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - \left(\frac{3}{\sqrt{3}} \right) q_1$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - \cancel{\sqrt{3}} \frac{1}{\cancel{\sqrt{3}}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$q_2 = \frac{\begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}}{\sqrt{(-4)^2 + (-1)^2 + 5^2}} = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix}$$

$$R = \begin{bmatrix} u_1^T q_1 & u_2^T q_1 \\ 0 & u_2^T q_2 \end{bmatrix}$$

$$u_1^T q_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \sqrt{3}$$

$$u_2^T q_1 = \begin{bmatrix} -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \frac{-3}{\sqrt{3}} + \frac{6}{\sqrt{3}} = \sqrt{3}$$

$$u_2^T q_2 = \begin{bmatrix} -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{bmatrix} = \sqrt{42}$$

$$\therefore \; R = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ 10/\sqrt{42} \end{bmatrix}$$

$$\sqrt{42}\, a_1 = 10/\sqrt{42}$$

$$a_1 = \frac{10}{42} = \frac{5}{21}$$

$$\sqrt{3}\, a_0 + \sqrt{3} \times \frac{5}{21} = \frac{2}{\sqrt{3}} \qquad a_0 = \frac{3}{7}$$

Example 2:

$$\begin{aligned} a_0 + a_1 + a_2 &= 2 \\ a_0 + 2a_1 + 4a_2 &= 3 \\ a_0 + 3a_1 + 9a_2 &= 6 \\ a_0 + 4a_1 + 16a_2 &= 4 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 4 \end{bmatrix}$$

↳ 4X3 Matrix

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \\ \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \end{array}$$

There will be 3 steps.

step 1: $K=1$

$$P_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_1 = \frac{P_1}{|P_1|} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{1^2+1^2+1^2+1^2}} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

step 2: $K=2$

$$P_2 = u_2 - (u_2^T q_1) q_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \left(\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \right) \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \begin{vmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{vmatrix} \times \frac{1}{\sqrt{(-1.5)^2 + (-.5)^2 + (0.5)^2 + (1.5)^2}}$$

$$= \frac{1}{\sqrt{5}} \begin{vmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{vmatrix}$$

step 3: $k = 3,$

$$P_k = u_3 - (u_3^T q_1) q_1 - (u_3^T q_2) q_2$$

$$= u_3 - \left(\begin{bmatrix} 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \right) \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} - \left(\begin{bmatrix} 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix} \frac{1}{\sqrt{5}} \right) \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$= u_3 - (15) \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} - \frac{25}{\sqrt{5}} \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$= \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix} - \begin{bmatrix} 7.5 \\ 7.5 \\ 7.5 \\ 7.5 \end{bmatrix} - 5 \begin{bmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$q_3 = \frac{P_3}{|P_3|} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{1^2 + (-1)^2 + (-1)^2 + 1^2}}$$

$$= \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix}$$

$$\therefore Q = \begin{vmatrix} q_1 & q_2 & q_3 \end{vmatrix} = \begin{vmatrix} 0.5 & -1.5/\sqrt{5} & 0.5 \\ 0.5 & -0.5/\sqrt{5} & -0.5 \\ 0.5 & 0.5/\sqrt{5} & -0.5 \\ 0.5 & 1.5/\sqrt{5} & 0.5 \end{vmatrix}$$

$$\therefore R = \begin{vmatrix} u_1^T q_1 & u_2^T q_1 & u_3^T q_1 \\ 0 & u_2^T q_2 & u_3^T q_2 \\ 0 & & u_3^T q_3 \end{vmatrix}$$

$$u_1^T q_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = 2$$

$$u_2^T q_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = 5$$

$$u_3^T q_1 = \begin{bmatrix} 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = 15$$

$$u_2^T q_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1.5/\sqrt{5} \\ -0.5/\sqrt{5} \\ 0.5/\sqrt{5} \\ 1.5/\sqrt{5} \end{bmatrix} = \sqrt{5}$$

$$u_3^T q_2 = \begin{bmatrix} 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} -1.5/\sqrt{5} \\ -0.5/\sqrt{5} \\ 0.5/\sqrt{5} \\ 1.5/\sqrt{5} \end{bmatrix} = 5\sqrt{5}$$

$$u_3^T q_3 = \begin{bmatrix} 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix} = 2$$

$$\therefore R = \begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ \frac{-1.5}{\sqrt{5}} & \frac{-0.5}{\sqrt{5}} & \frac{0.5}{\sqrt{5}} & \frac{1.5}{\sqrt{5}} \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7.5 \\ 9/2\sqrt{5} \\ -1.5 \end{bmatrix}$$

$$\therefore 2a_2 = -1.5 \qquad a_2 = -\frac{3}{4} = -0.75$$

$$\sqrt{5}\,a_1 + 5\sqrt{5}\,a_2 = 9/2\sqrt{5}$$

$$a_1 = \frac{9/2\sqrt{5} - 5\sqrt{5}(-0.75)}{\sqrt{5}}$$

$$= 4.65$$

$$2a_0 + 5a_1 + 15a_2 = 7.5$$

$$a_0 = \frac{7.5 - 5(4.65) - 15(-0.75)}{2}$$

$$= -2.25$$

$$\therefore \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -2.25 \\ 4.65 \\ -0.75 \end{bmatrix}$$