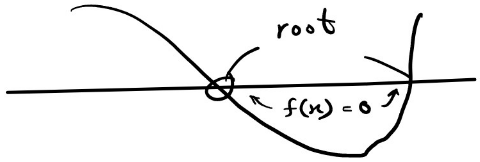
Fixed Point Iteration:

- Root Finding Algorithm
- Iterative method.



$f(x) = 0$ → We try to manipulate this function

Mathematically Manipulate

$\boxed{g(x) = x}$ ← Condition → x is the root of $f(x)$

1. Finding $g(x)$

Given $f(x) = x^2 - 2x - 3$. Construct 3 $g(x)$ from $f(x)$

$g(x) = x$ ← subject

$f(x) = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow x^2 = 2x + 3$

$\Rightarrow \underbrace{x}_{x} = \underbrace{\sqrt{2x+3}}_{g(x)}$

$$\boxed{\Rightarrow g_1(x) = \sqrt{2x+3}}$$

$x^2 - 2x - 3 = 0$

$\Rightarrow x^2 - x - x - 3 = 0$

$\Rightarrow x^2 - x - 3 = x$

$\Rightarrow x = x^2 - x - 3$

$$\Rightarrow \boxed{g_2(x) = x^2 - x - 3}$$

$x^2 - 2x - 3 = 0$

$\Rightarrow 2x^2 - x^2 - 2x - 3 = 0$

$\Rightarrow 2x^2 - 2x = x^2 + 3$

$\Rightarrow x(2x - 2) = x^2 + 3$

$\Rightarrow x = \dfrac{x^2+3}{2x-2}$

$$\boxed{g_3(x) = \frac{x^2+3}{2x-2}}$$

2. Given the initial point $x_0 = 0$, we find the root iteratively.

Find the root using fixed point iteration for $f(x) = x^2 - 2x - 3$ using the initial point $x_0 = 0$. [use upto 3 significant figure]

$\underbrace{2.25}_{\text{3 significant figure}}112$

$$f(x) = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow x^2 = 2x + 3$$

$$\Rightarrow x = \sqrt{2x+3}$$

$$\Rightarrow g_1(x) = \sqrt{2x+3}$$

Now,

$x_0 = 0$ $g_1(x_0) = g_1(0) = \sqrt{2.0+3} = \sqrt{3} = 1.73$

Iteration 1 : $g_1(0) = 1.73$

" 2: $g_1(1.73) = 2.54$

" 3: $g_1(2.54) = 2.84$

" 4: $g_1(2.84) = 2.95$

" 5: $g_1(2.95) = 2.98$

" 6: $g_1(2.98) = 3.00$

" 7: $g_1(\underbrace{3.00}_{x}) = \underbrace{3.00}_{x}$

we continue this until we get $g_1(x) = x$.

$\therefore g_1(x) = x$

[Fixed point Reach]

Convergence [we reached to the root]

$\therefore$ Root, $x^* = 3$

$g_2(x) = x^2 - x - 3$

$x_0 = 0$

$g_2(0) = -3.00$

$g_2(-3) = 9.00$

$g_2(9.00) = 69.0$

$g_2(69.0) = 4.69 \times 10^3$

Divergence

(As we did not reach to the root)

$$g_3(x) = \frac{x^2+3}{2x-2}$$

$x_0 = 0$

$g_3(0) = -1.50$

$g_3(-1.50) = -1.05$

$g_3(-1.05) = -1.00$

$g_3(-1.00) = -1.00$

Convergence

[we have reached the root]

$\therefore\ g_3(x) = x$

$\therefore$ Root $x^* = -1$

$\therefore$ Actual Roots:

$x^2 - 2x - 3 = 0$

$\Rightarrow x^2 - 3x + x - 3 = 0$

$\Rightarrow x(x-3) + 1(x-3) = 0$

$\Rightarrow (x-3)(x+1) = 0$

$x = 3, -1$

Let, $x_0 = 42$,

$$g_3(x) = \frac{x^2+3}{2x-2}$$

$$g_3(42) = 21.6$$

$$g_3(21.6) = 11.4$$

$$g_3(11.4) = 6.39$$

$$g_3(6.39) = 4.07$$

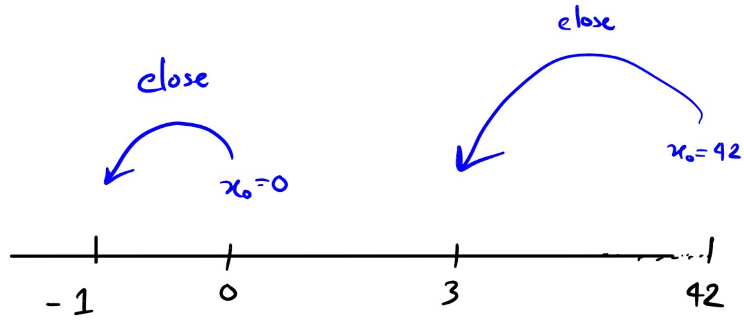
$$g_3(4.07) = 3.19$$

$$g_3(3.19) = 3.01$$

$$g_3(3.01) = 3.00$$

$$g_3(3.00) = 3.00$$

$$g_3(x) = x$$

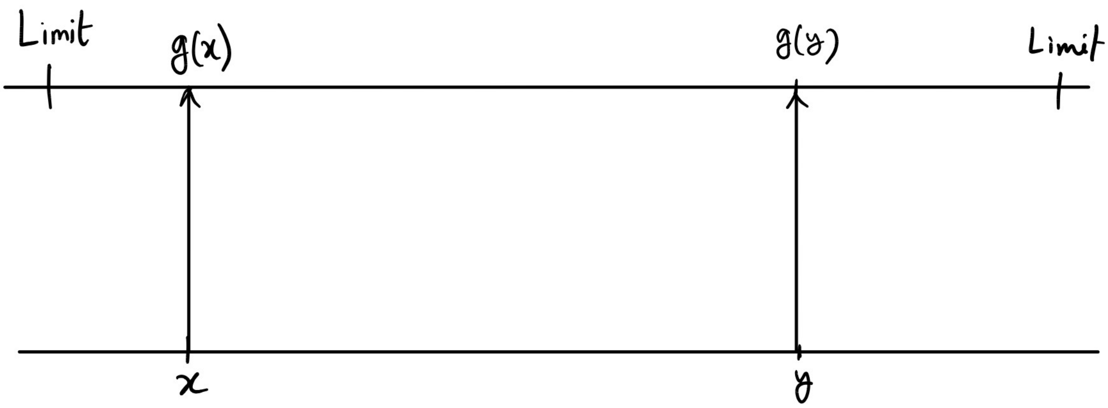


$\therefore\ x^* = 3$

The root we converge to depends on the choice of the initial point (x_0) and the chosen $g(x)$.

How do we know which $g(x)$ is convergent?

Contraction Mapping Theorem



$$\lambda = \left| \frac{g(y) - g(x)}{y - x} \right|$$

Gradient of $g(x)$

Converging Rate, $\lambda = |g'(x)|$

$\lambda = 0 \longrightarrow$ Super Linear Convergent.

↳ [less number of iteration to find the root]

$0 < \lambda < 1 \longrightarrow$ Linear Convergent

↳ [It will find the root but requires more iteration]

$\lambda \geq 1 \longrightarrow$ Divergent

↳ [we will not find the root]

$f(x) = x^3 - 2x^2 - x + 2$

(a) Find the actual roots of this equation.

(b) Construct three $g(x)$ from $f(x)$

(c) Determine which $g(x)$ are convergent & which are divergent? or Find the converging $g(x)$ and which root it will converge to?

(a)

$$x^3 - 2x^2 - x + 2 = 0$$

$$\Rightarrow x^2(x-2) - 1(x-2) = 0$$

$$\Rightarrow (x^2-1)(x-2) = 0$$

$$\Rightarrow (x-1)(x+1)(x-2) = 0$$

$$\boxed{\therefore\ x = 1, -1, 2}$$

(b)

$$x^3 - 2x^2 - x + 2 = 0$$

$$2x^2 = x^3 - x + 2$$

$$\Rightarrow x^2 = \frac{1}{2}(x^3 - x + 2)$$

$$\Rightarrow x = \sqrt{\frac{1}{2}(x^3 - x + 2)}$$

$$\Rightarrow g_1(x) = \sqrt{\frac{1}{2}(x^3 - x + 2)}$$

$$= \frac{1}{\sqrt{2}}(x^3 - x + 2)^{\frac{1}{2}}$$

$$x^3 - 2x^2 - x + 2 = 0$$

$$x = x^3 - 2x^2 + 2$$

$$g_2(x) = x^3 - 2x^2 + 2$$

$$x^3 - 2x^2 - x + 2 = 0$$

$$x(x^2 - 2x - 1) = -2$$

$$\Rightarrow x = \frac{-2}{x^2 - 2x - 1}$$

$$g_3(x) = \frac{-2}{x^3 - 2x^2 + 3}$$

(c)

$$\lambda_1 = |g_1(x)| = \left| \frac{d}{dx} \frac{1}{\sqrt{2}} (x^3 - x + 2)^{\frac{1}{2}} \right|$$

$$= \left| \frac{1}{\sqrt{2}} \frac{1}{2} (x^3 - x + 2)^{-\frac{1}{2}} (3x^2 - 1) \right|$$

From 'a',

$x = -1$, $\lambda_1 = 0.6$ Linear Convergent

$x = 1$, $\lambda_1 = 0.5$ " "

$x = 2$, $\lambda_1 = 3.75$ Divergent

$$\lambda_2 = |g_2'(x)| = \left| \frac{d}{dx} (x^3 - 2x^2 + 2) \right|$$

$$= |3x^2 - 4x|$$

$x = 1$, $\lambda_2 = 1$ divergent

$x = -1$, $\lambda_2 = 7$ "

$x = 2$, $\lambda_2 = 4$ "

$$\lambda_3 = |g_3'(x)| = \left| \frac{d}{dx} \left(\frac{-2}{x^2 - 2x - 1} \right) \right|$$

$$= \left| \frac{d}{dx} -2 (x^2 - 2x - 1)^{-1} \right|$$

$$= \left| -2 \cdot (-1) (x^2 - 2x - 1)^{-2} (2x - 2) \right|$$

$$= \left| 2 \frac{2x-2}{(x^2-2x-1)^2} \right|$$

$x = 1$, $\lambda_3 = 0$ Super Linear Convergent

$x = -1$, $\lambda_3 = 2$ Divergent

$x = 2$, $\lambda_3 = 4$ Divergent

Advantage & Disadvantage:

\# Convergence very fast if $\lambda = 0$ or $g(x)$ is super linear convergent

\# Not always guaranteed — as it depends on $g'(x)$ & initial point.

Graph Representation: