$$\therefore \quad D_h^{(1)} = \frac{4 D_{\frac{h}{2}} - D_h}{3} \Rightarrow \text{removes } h^2 \text{ term}$$

$$\therefore \quad D_h^{(2)} = \frac{16 D_{\frac{h}{2}}^{(1)} - D_h^{(1)}}{15} \Rightarrow \text{remove } h^4 \text{ term}$$

t $\qquad$ P

$$\boxed{D_h^t = \frac{2^p D_{\frac{h}{2}}^{t-1} - D_h^{t-1}}{2^p - 1}}$$

$$\underbrace{h^2}_{1} + \underbrace{h^4}_{2} + \underbrace{h^6}_{3} + \cdots$$

$p=4 \qquad p=6$

$$D_h^2 = \frac{2^4 D_{\frac{h}{2}}^1 - D_h^1}{2^4 - 1} = \frac{16 D_{\frac{h}{2}}^1 - D_h^1}{15}$$

$$D_h^3 = \frac{2^6 D_{\frac{h}{2}}^2 - D_h^2}{2^6 - 1} = \frac{64 D_{\frac{h}{2}}^2 - D_h}{63}$$

Example: $f(x) = \boxed{e^x \sin(x)}$, find $D_h^{(1)}$ using Richardson Extrapolation at $x = 1$ for $h = 0.5$.

Soln.:

$f'(1) = ?$ $\qquad h = 0.5 \qquad x = 1$

$$D_h^1 = \frac{4 D_{\frac{h}{2}} - D_h}{3}$$

$$\boxed{D_h} = \frac{f(x+h) - f(x-h)}{2h} = \frac{e^{1.5}\sin(1.5) - e^{0.5}\sin(0.5)}{2 * 0.5} = 3.68$$

$h = 0.5 \qquad \frac{h}{2} = 0.25$

$$\boxed{D_{\frac{h}{2}}} = \frac{f(1+0.25) - f(1-0.25)}{2 * 0.25} = \frac{e^{1.25}\sin(1.25) - e^{0.75}\sin(0.75)}{2 * 0.25} = 3.7385$$

$$D_h^1 = \frac{4 * 3.7385 - 3.68}{3} = 3.757$$

(Ans)

<u>Example:</u>

$f(x) = e^{2x} + 3x$, find $f'(2)$ using Richardson Extrapolation for $h = 1.2$ & 0.6.

$$D_h = \frac{f(2+1.2) - f(2-1.2)}{2 * 1.2} = \frac{e^{2.(3.2)} + 3*3.2 - (e^{2*0.8} + 3*0.8)}{2*3.2}$$

$$= 251.705$$

$$D_{\frac{h}{2}} = \frac{f(2+0.6) - f(2-0.6)}{2 * 0.6} = \frac{e^{2*(2.6)} + 3*(2.6) - (e^{2*1.4} + 3*1.4)}{2*0.6}$$

$$= 140.356$$

$$D_h^1 = \frac{4 * D_{\frac{h}{2}} - D_h}{3} = \frac{4 * 140.356 - 251.705}{3}$$

$$= 103.23$$

Example: $f'(1) = ?$ $h = 0.4$ using $D_h^{(2)}$ RE.

x	$f(x)$
0.6	0.707178
0.8	0.8559892
0.9	0.925863
1	0.984007
1.1	1.033743
1.2	1.074575
1.4	1.127986

$$D_h^{(2)} = \frac{16 D'_{h/2} - D'_h}{15}$$

$$D'_h = \frac{4D_{h/2} - D_h}{3} \qquad D'_{h/2} = \frac{4\,D_{h/4} - D_{h/2}}{3}$$

$$D_h = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(1+0.4) - f(1-0.4)}{2 * 0.4} = \frac{1.127986 - 0.707178}{0.8}$$

$$= 0.52601$$

$$D_{\frac{h}{2}} = D_{0.2} = \frac{f(1+0.2) - f(1-0.2)}{2 * 0.2} = \frac{1.074575 - 0.8559892}{0.4}$$

$$= 0.5464$$

$$D_{\frac{h}{4}} = D_{0.1} = \frac{f(1+0.1) - f(1-0.1)}{2 * 0.1} = \frac{1.033743 - 0.925863}{0.2}$$

$$= 0.5394$$

$$D'_h = \frac{4D_{h/2} - D_h}{3}$$

$$= \frac{4 * 0.5464 - 0.52601}{3} = 0.553$$

$$D'_{h/2} = \frac{4\,D_{h/4} - D_{h/2}}{3} = \frac{4 * 0.5394 - 0.5464}{3}$$

$$= 0.537$$

$$D_h^{(2)} = \frac{16\,\underbrace{D'_{h/2}} - \underbrace{D'_h}}{15}$$

$$= \frac{16 * 0.537 - 0.553}{15} = \boxed{0.535933}$$

Ans.

Example: Deduce an expression for D_h^1 from D_h by replacing h with $\left(\frac{4h}{3}\right)$ using RE Method.

$$D_h = f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}h^4 + O(h^6) \rightarrow (I)$$

$$D_{\frac{4h}{3}} = f'(x) + \frac{f^3(x)}{3!}\left(\frac{4h}{3}\right)^2 + \frac{f^5(x)}{5!}\left(\frac{4h}{3}\right)^4 + O(h^6)$$

$$= f'(x) + \frac{f^3(x)}{3!}\frac{16}{9}h^2 + \frac{f^5(x)}{5!}\left(\frac{4h}{3}\right)^4 + O(h^6) \rightarrow (II)$$

$\frac{9}{16}$ (II) $-$ (I) $\Rightarrow$

$$\frac{9}{16}D_{\frac{4h}{3}} - D_h = \left(\frac{9}{16}-1\right)f'(x) + \left(\frac{9}{16}-1\right)\frac{f^5(x)}{5!}\left(\frac{4h}{3}\right)^4 + O(h^6)$$

$$\Rightarrow \frac{\frac{9}{16}D_{4h/3} - D_h}{\frac{9}{16}-1} = f'(x) + \frac{f^5(x)}{5!}\left(\frac{4h}{3}\right)^4 + O(h^6)$$

$$D_h^1 = \frac{\frac{9}{16}D_{\frac{4h}{3}} - D_h}{\frac{9}{16}-1}$$

Example:

x	1	1.2	1.4
$f(x)$	1.2712	1.3642	1.5496

① Calculate $f'(1.2)$ using C.D.

⑪ Let, $f(x) = x\sin x + x^2\cos x$, compute the upper bound error/ maximum truncation error using CD, $\xi = [1, 1.4]$

Soln:

①
$$h = 0.2$$

$$f'(1.2) = \frac{f(1.2+0.2) - f(1.2-0.2)}{2 * 0.2} = \frac{1.5496 - 1.2712}{2*0.2}$$

$$= 0.69475$$

⑪ Using CD,

$$T.\ Error = \left| \frac{f^3(\xi)}{3!} h^2 \right|$$

$\frac{d}{dx} uv = u'v + uv'$

$$f(x) = \underbrace{x}_{u}\ \underbrace{\sin x}_{v} + \underbrace{x^2}_{u}\ \underbrace{\cos x}_{v}$$

$$f'(x) = \sin x + x\cos x + 2x\cos x - x^2\sin x$$

$$= \sin x + \underbrace{3x}_{u}\ \underbrace{\cos x}_{v} - \underbrace{x^2}_{u}\ \underbrace{\sin x}_{v}$$

$$f''(x) = \cos x + 3\cos x - 3x\sin x - 2x\sin x - x^2\cos x$$

$$= 4\cos x - \underbrace{5x}_{u}\ \underbrace{\sin x}_{v} - \underline{x^2}\ \underline{\cos x}$$

$$f^3(x) = -4\sin x - 5\sin x - 5x\cos x - 2x\cos x + x^2\sin x$$

$$= -9\sin x - 7x\cos x + x^2 \sin x$$

$$\left|\frac{f^3(\xi)}{3!}h^2\right| = \left|\frac{(0.2)^2}{3!}\right| \left| f^3(\xi) \right| \qquad \xi = [1, 1.4]$$

$$= \left|\frac{(0.2)^2}{3!}\right| \left| +9\sin(\xi) + 7(\xi)\cos(\xi) + (\xi)^2 \sin(\xi) \right|$$

$$= \left|\frac{(0.2)^2}{3!}\right| \left| 9\sin(1.4) + 7(1.4)\cos(1) + (1.4)^2 \sin(1.4) \right|$$

$$= \sim \qquad [\text{Ans}]$$

Upper Bound Error