Order of convergence

$\lambda = |g'(x)|$, $\lambda = 0 \rightarrow$ Super Linear Convergent

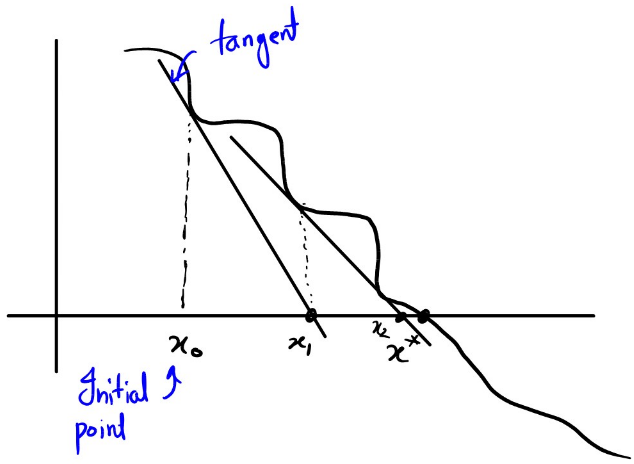
$0 < \lambda < 1 \rightarrow$ Linear Convergent

$\lambda \geq 1 \rightarrow$ Divergent

Newton's Method / Newton Raphson Method

→ Root Finding Algorithm → Iterative Approach

→ Super Linear Convergent $(\lambda = 0)$



We keep iterating until we are reaching the root or reaching close enough (ϵ)

formula:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Example: Given, $f(x) = x^2 - 2x e^{-x} + e^{-2x}$, Initial point $x_0 = 1$ and Error bound $= 0.01$. Find the root of $f(x)$ using Newton's Method.

Solution:

$$f(x) = x^2 - \underbrace{2x}_{u}\,\underbrace{e^{-x}}_{v} + e^{-2x}$$

$\frac{d}{dx} uv = u'v + uv'$

$$f'(x) = 2x - \left[2e^{-x} + 2x e^{-x}(-1)\right] + e^{-2x}(-2)$$

$$= 2x - 2e^{-x} + 2x e^{-x} - 2e^{-2x}$$

k	x_k	$f(x_k)$	$f(x_k) <$ Error Bound (0.01)	$f'(x_k)$	$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$
0	1	0.3995	No	1.729	$x_1 = 0.7689$
1	0.7689	0.0933	No	0.8939	$x_2 = 0.6645$
2	0.6645	0.0225	No	0.4542	$x_3 = 0.6149$
3	0.6149	0.005506	Yes		

Root x^*

$\therefore x^* = 0.6149$

Example: $f(x) = \frac{1}{x} - 0.5$ $[x^*$ is at $2]$, Given, initial point $x_0 = 1$, find the root of $f(x)$ with relative error/ Differences at each iteration until less than 0.01.

$$f'(x) = (-1)x^{-2} = -\frac{1}{x^2}$$

k	x_k	$f(x_k)$	$f'(x_k)$	$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$	Relative Error $\lvert x_{k+1} - x_k \rvert$
0	1	0.5	-1	1.5	0.5
1	1.5	0.1667	-0.4444	1.875	0.375
2	1.875	0.0333	-0.284	1.99225	0.11725
3	1.99225	0.00194	-0.25	2.00	0.00775
4	2.00	0	-0.25	2.00	0

< 0.05 iteration stop.

Root

Root $= 1.99225$

** Important **

Prove that newton raphson method is a super linear convergent.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\downarrow$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

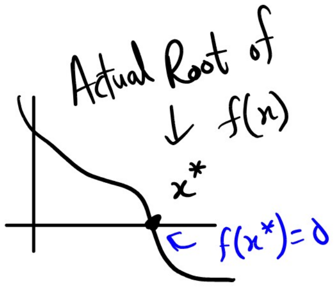
We know,

Convergence Rate $\lambda = |g'(x)|$

$$= \left| \frac{d}{dx}\left(x - \frac{\overbrace{f(x)}^{u}}{\underbrace{f'(x)}_{v}}\right) \right| \qquad \frac{d}{dx}\frac{u}{v} = \frac{vu' - uv'}{v^2}$$

$$= \left| 1 - \frac{\left(f'(x)\, f'(x) - f(x)\, f''(x)\right)}{[f'(x)]^2} \right|$$

$$= \left| \frac{\cancel{[f'(x)]^2} - \cancel{[f'(x)]^2} + f(x)\, f''(x)}{[f'(x)]^2} \right|$$



$$\lambda = \left| \frac{f(x)\, f''(x)}{[f'(x)]^2} \right|$$

$$\lambda = |g'(x^*)| = \left| \frac{f(x^*)\, f''(x^*)}{[f'(x^*)]^2} \right|$$

$$= \left| \frac{0 * f''(0)}{(f'(0))^2} \right|$$

$$\lambda = 0 \leftarrow \text{Super Linear Convergent}$$

$\therefore$ Newton Raphson method is super linear convergent.

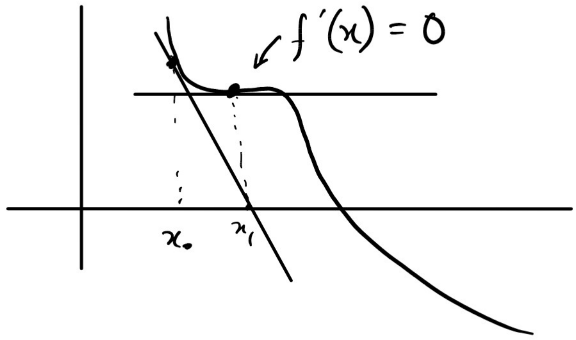
[Proved]

Advantage:

Since it is super linear convergent, this approach is much faster.

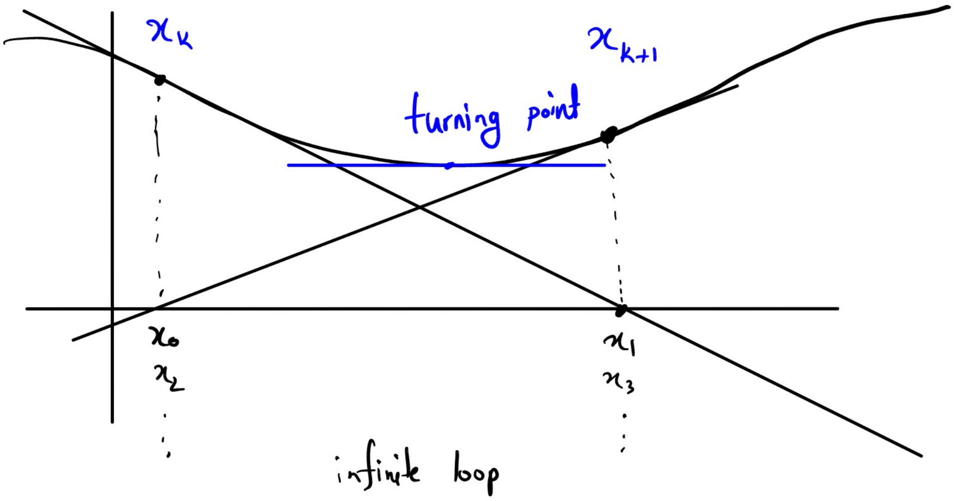
Disadvantages:

① formula, $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \leftarrow 0$



→ if the iteration leads to a turning point ($f'(x)=0$) then we can't find the root as x_{k+1} becomes infinity.

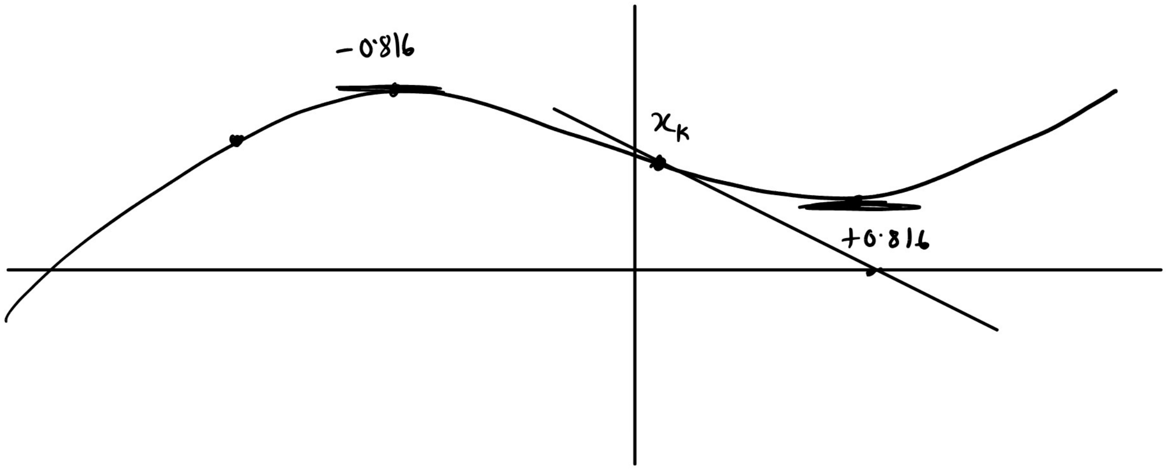
②



If the initial point is chosen such that it lies near a turning point, the iteration may enter a loop and making it impossible to find the root.

Newton's method doesn't work if there is a turning point between x_k and x_{k+1}.

Let,



$f(x) = x^3 - 2x^2 + 2$

Turning point, $f'(x) = 3x^2 - 2 = 0$

$$\Rightarrow x^2 = \frac{2}{3}$$

$$\therefore x = \sqrt{\frac{2}{3}} = \pm 0.816$$

We need to make sure $x_0 < -0.816$