Practice Problems

Chapter-6: Least-Squares Approximations

1. Let's say the following matrix below is an orthonormal matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

   Compute the value of $(A^TA)^{-1}$

2. For $f(2) = 3,\ f(4) = 5,\ f(5) = 8$, prepare an overdetermined system and then solve the system using,
   a. Least Square Method
   b. QR Decomposition

3. For $f(18) = 22,\ f(19) = 35,\ f(20) = 60,\ f(25) = 80$, prepare an overdetermined system and then solve the system using,
   a. Least Square Method
   b. QR Decomposition

4. Consider a set of four data points:
$f(0) = 3,\ f(2) = -2,\ f(-1) = 2,\ f(1) = 1$. Find the best-fit polynomial of degree two, $p_2(x)$, for the above data points using the least-squares method by answering the following:
   a. Write down the matrices, A and b, for the above data.
   b. Compute the normal matrix A^TA and A^Tb.
   c. Use the results of the previous part to compute the column matrix $x = (a_0\ a_1\ a_2)^T$, where $a_0,\ a_1, a_2$ are the coefficients of the polynomial p_2, and then write the expression of the polynomial p_2.

5. Consider a set of four data points:
$f(0) = 1,\ f(2) =- 2,\ f(-1) = 2,\ f(1) = 1$. Find the best-fit polynomial of degree two, $p_2(x)$, for the above data points using the least-squares method by answering the following:
   a. Write down the matrices, A and b, for the above data.

b. Compute the normal matrix A^TA and A^Tb.

c. Use the results of the previous part to compute the column matrix, x and write the expression of the polynomial p_2.

6. Consider a set of four data points:

$f(0) = 0$, $f(2) = -1$, $f(-1) = 2$, $f(1) = 2$. We now find the solution by the QR-decomposition method using these four data points by answering the following:

a. Write down the matrices A and b. Also identify the linearly independent column vectors u_1, u_2, u_3 from the matrix A.

b. Using the Gram-Schmidt process, construct the orthonormal column matrices (or vectors) q_1, q_2, q_3 from the linearly independent column vectors u_1, u_2, u_3 obtained in the previous part, and then write down the Q matrix.

c. Calculate the matrix elements of R and write down the matrix R.

d. Compute Rx and Q^Tb, where $x = (a_0\ a_1\ a_2)^T$ is a column vector with a_0, a_1, a_2 which are the coefficients of the polynomial p_2.

e. Using the above result, find the values of a_0, a_1, a_2 and write the expression of the polynomial $p_2(x)$.

7. Consider a set of four data points:

$f(0) = 1$, $f(0.5) = 1.4$, $f(1) = 1.7$, $f(1.5) = 2$. We now find the solution by the QR-decomposition method using these four data points by answering the following:

a. Write down the matrices A and b. Also identify the linearly independent column vectors u_1, u_2, u_3 from the matrix A.

b. Using the Gram-Schmidt process, construct the orthonormal column matrices (or vectors) q_1, q_2, q_3 from the linearly independent column vectors u_1, u_2, u_3 obtained in the previous part, and then write down the Q matrix.

c. Calculate the matrix elements of R and write down the matrix R.

d. Compute Rx and Q^Tb, where $x = (a_0\ a_1\ a_2)^T$ is a column vector with a_0, a_1, a_2 which are the coefficients of the polynomial p_2.

e. Using the above result, find the values of a_0, a_1, a_2 and write the expression of the polynomial $p_2(x)$.