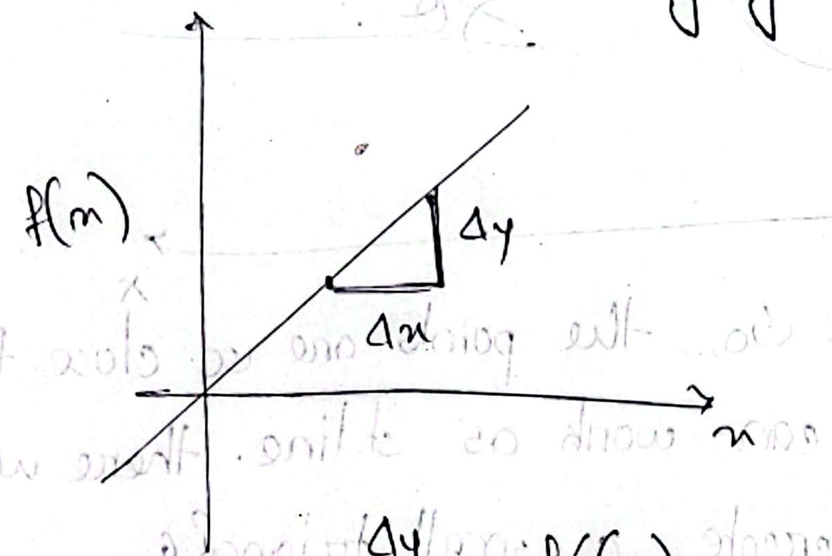
Chapter 03

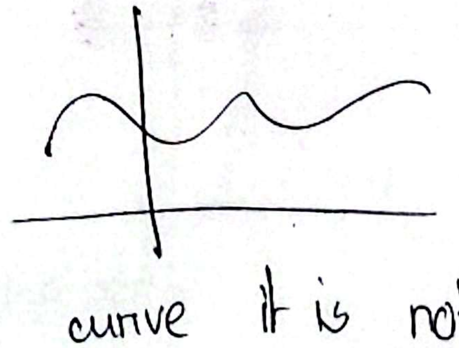
we want to find differenciation numerically (how fast a function changing?).



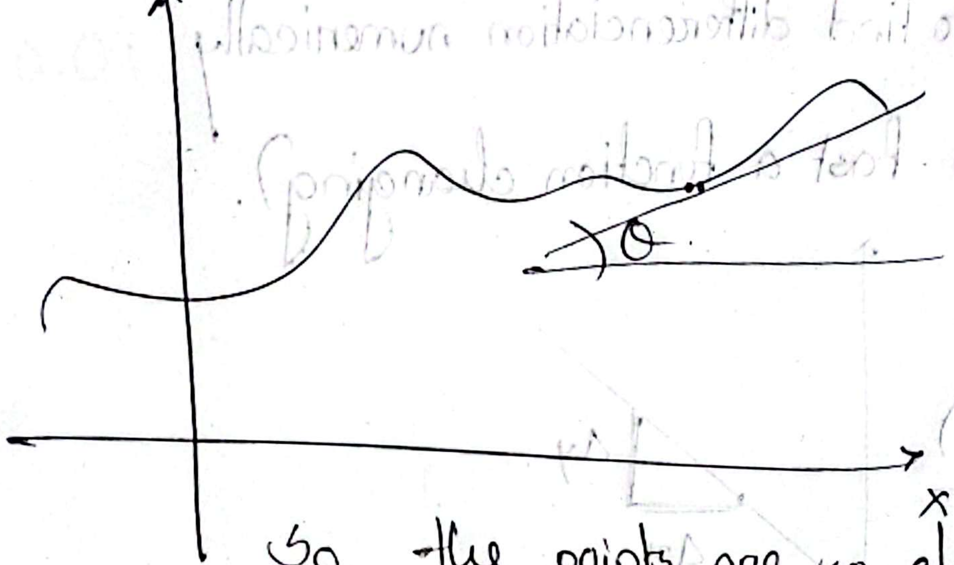
$$\frac{\Delta y}{\Delta x} = f'(x)$$

$$\tan\theta = f'(x)$$

For st. line taking 2 points give same slope.



For curve it is not true. rate of change of a

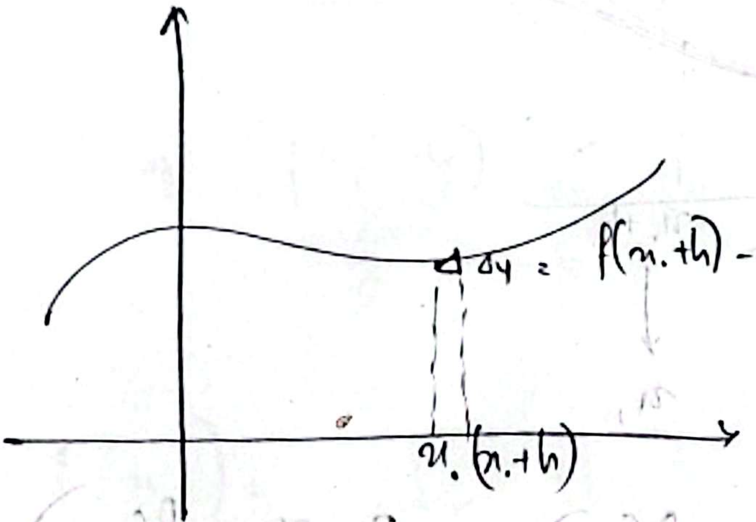


So, the points are so close that, they can work as st. line. there we can create a small triangle.

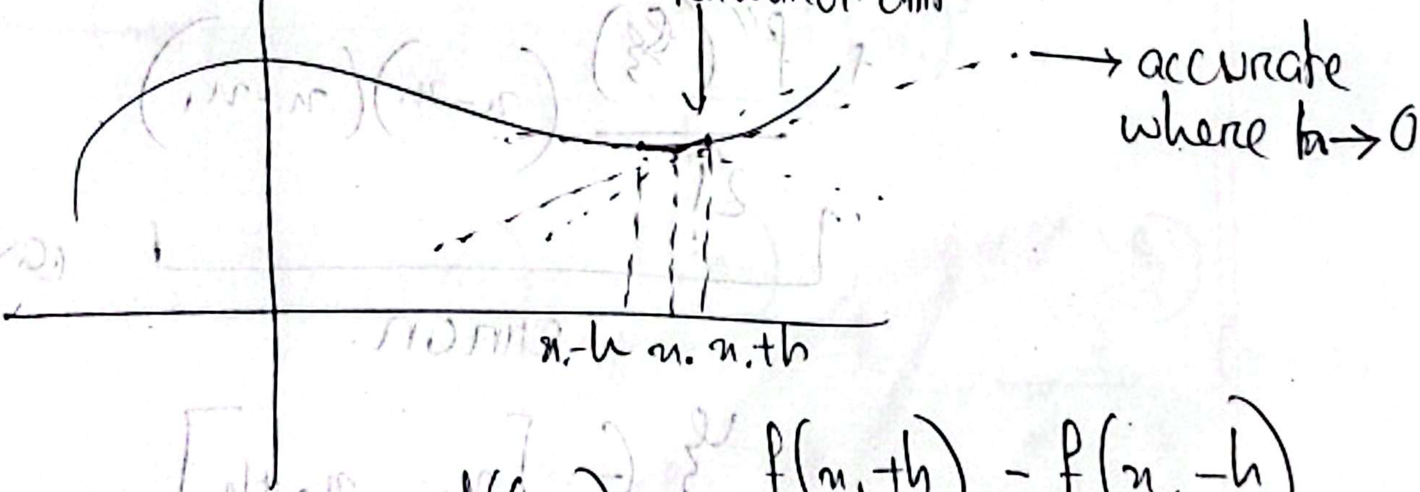
So we find a tangent at that point.

$$\tan\theta = \frac{\Delta y}{\Delta x}.$$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{——— (i)}$$

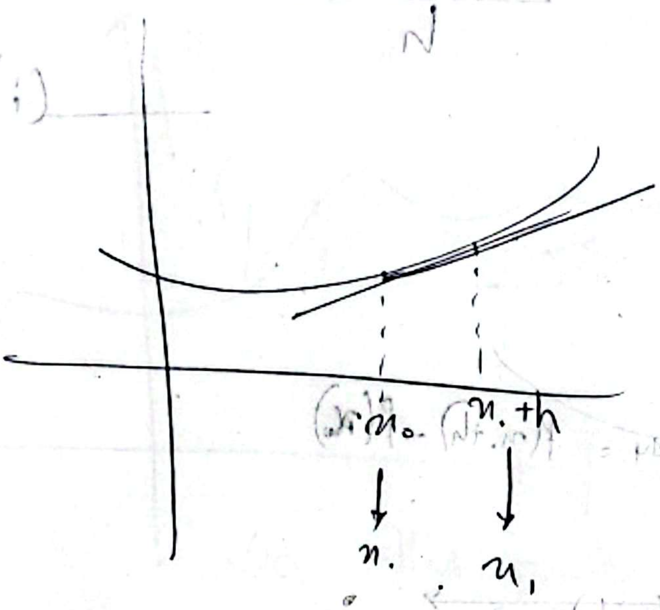


in (i), if $h>0$, forward diff.
if $h<0$, backward diff



central diff, $$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

FD:



$$P_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$f(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1) + \underbrace{\frac{f''(\xi)}{2!}(x-x_0)(x-x_1)}_{\text{error.}}$$

cauchy's theorem

$$\xi \in [x_0, x_0+h]$$

since $x - x_1 \rightarrow$ is the only part not constant

$\frac{d}{dx} uv = u\frac{dv}{dx} + v\frac{du}{dx}$

$$f'(x) = \frac{1}{x_0 - x_1} f(x_0) + \frac{1}{x_1 - x_0} f(x_1)$$

$$+ \frac{f'''(\xi)}{2}(x-x_0)(x-x_1)\frac{d\xi}{dx}$$

$$+ \frac{f''(\xi)}{2}(2x - x_0 - x_1)$$

$\frac{dy}{dx} \sin(x^2)$

$\frac{d\sin(x^2)}{dx^2} \cdot \frac{dx^2}{dx}$

$$f'(x_0) = \frac{1}{x_0 - x_1} f(x_0) + \frac{1}{x_1 - x_0} f(x_1)$$

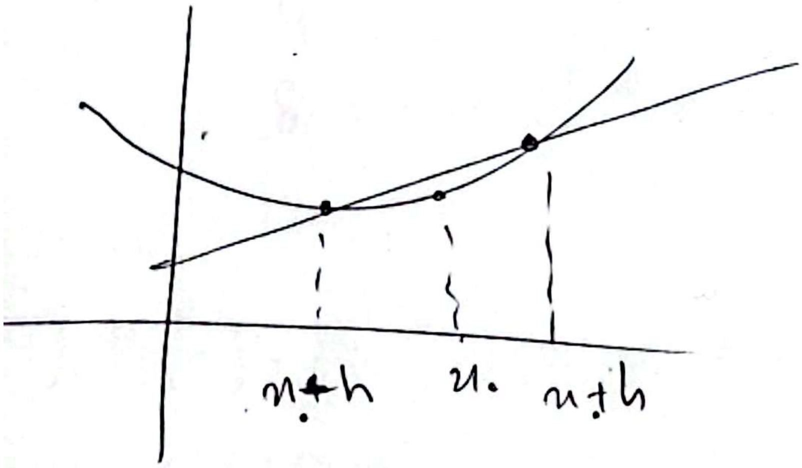
$$+ \frac{f''(\xi)}{2}(x_0 - x_1)$$

$$\cancel{f'(x_1) = \frac{f(x_0+h) - f(x_0)}{h}}$$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{f''(\xi)}{2}(x_0 - x_1)$$

$$f'(x_1) = \frac{f(x_0+h) - f(x_0)}{h} + \frac{f''(\xi)}{2}(-h)$$

Central Difference.



x_0, $x_1 = x_0+h$, $x_2 = \cancel{x_0 + 2h}\; x_0+2h$.

$$f(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) + \frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$f'(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} f(x_2) + \frac{f^3(\xi)}{3!} \Big[(x - x_1)(x - x_2)$$

$$+ (x - x_0)(x - x_2) + (x - x_1)(x - x_2) \Big]$$

$$+ \frac{f^4(\xi)}{3!} (x - x_0)(x - x_1)(x - x_2)$$

$$f'(x_1) = \frac{x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x_1 - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{x_1 - x_0}{(x_2 - x_0)(x_2 - x_1)} f(x_2) + \frac{f^3(\xi)}{3!} (x_1 - x_0)(x_1 - x_2)$$

$$= \frac{x_0 + h - x_0 - 2h}{(-h)(-2h)} f(x_0) + \frac{2x_0 + 2h - x_0 - x_0 - 2h}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{h}{(2h)(h)} f(x_2) + \frac{f^3(\xi)}{3!} (h)(-h)$$

$$= \frac{-\cancel{h}}{2h} f(x_1) + \frac{1}{2h} f(x_1+2h) - \frac{f^3(\xi)}{3!} h^2$$

$$= \frac{f(x_0+2h) - f(x_1)}{2h} - \frac{f^3(\xi)}{3!} h^2$$

Rounding Error

FD, $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$.

CD, $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h}$.

During chap-1, we saw if we subtract two close number then there is a big rounding error.

So in computer $f(x+h) - f(x-h)$ for $h \to 0$ gives rounding error.

$$fl[f(x_1+h)] = (1+\delta_1) f(x_1+h)$$

$$= f(x_1+h) + \underbrace{\delta_1 f(x_1+h)}_{\text{total error.}}$$

$$fl[f(x_1-h)] = (1+\delta_2) f(x_1-h)$$

$$|\delta_1|, |\delta_2| \leq \epsilon_M,$$

$$f'(x_1) \approx \frac{fl[f(x_1+h)] - fl[f(x_1-h)]}{2h}$$

$$|5 + (-1)| \leq |5| + |-1|$$

$$|4| \leq 5+1 \qquad |4| \leq 6.$$

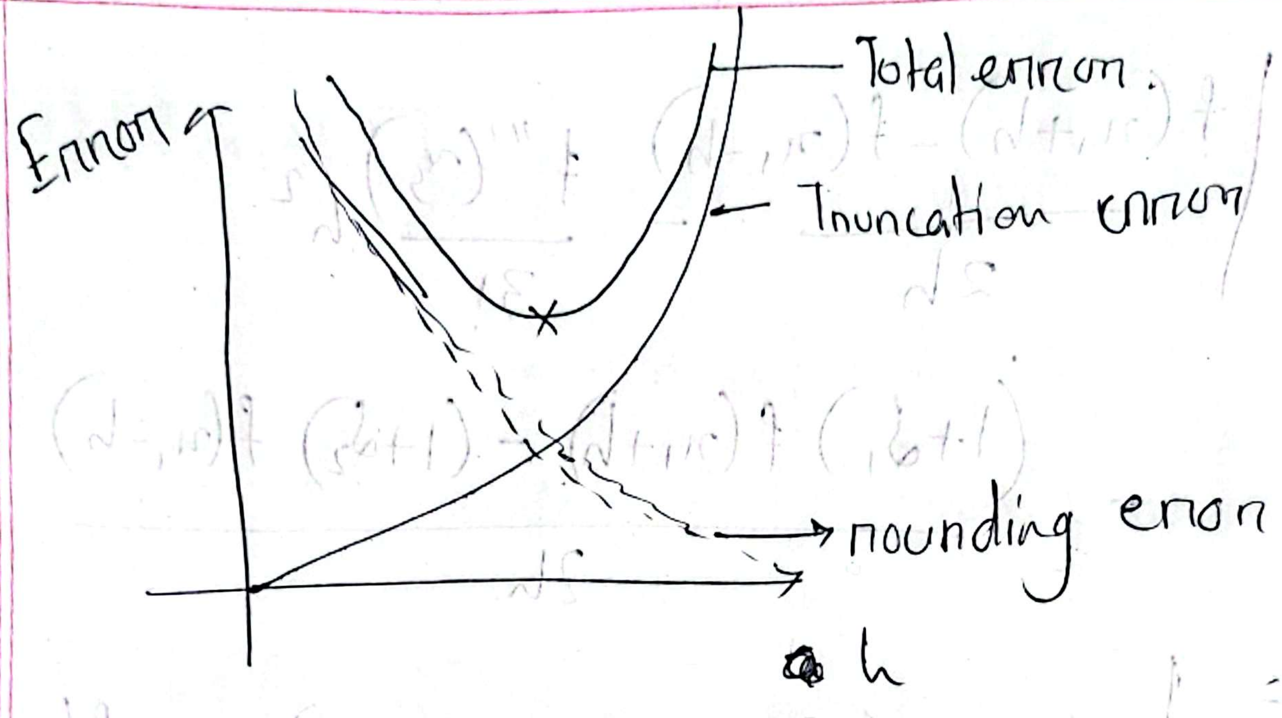
$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{f'''(\xi)}{3!} h^2 - \frac{(1+\delta_1) f(x_1+h) - (1+\delta_2) f(x_1-h)}{2h} \right|$$

$$= \left| -\frac{f'''(\xi)}{3!} h^2 - \frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h} \right|$$

$$= \left| \frac{f'''(\xi)}{3!} \right| h^2 - \left| \frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h} \right| \qquad [\,|a+b| \leq |a| + |b|$$

$$\leq \left| \frac{f'''(\xi)}{3!} \right| h^2 + \left| \frac{\delta_1 f(x_1+h) - \delta_2 f(x_2-h)}{2h} \right|$$

$$\leq \underbrace{\frac{f'''(\xi)}{3!} h^2}_{\text{error}} + \underbrace{\epsilon_m \frac{f(x_1+h) + f(x_1-h)}{2h}}_{\text{rounding error}}.$$



Error
Total error.
Truncation error
rounding error.
h

Richardson Extrapolation.

$$D_u = \frac{f(x_1+h) - f(x_1-h)}{2h}$$

~~cf~~ Taylor series:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^2(x_0)}{2!}(x-x_0)^2$$

$$+ \ldots$$

$$\cancel{f(x_1) = f(x_1) + f'(x_1)(x - x_1) + f^2(x_1)(x-x_1}$$

$$\cancel{f((x_1+h) = f(x_1) + f'(x_1)(x_1+h-x}$$

centering at x_1:

$$f(x) = f(x_1) + f'(x_1)(x-x_1) + \frac{f^2(x_1)}{2!}(x-x_1)^2$$

$$+$$

$$\Rightarrow f(x_1+h) = f(x_1) + f'(x_1)(x_1+h-x_1) + \frac{f^2}{2!}(x_1)$$

$$(x_1+h-x_1)^2 + \frac{f^3(x_1)}{3!}(x_1+h-x_1)^3 +$$

$$\frac{f^4(x_1)(x_1+h-x_1)^4}{4!} + \frac{f^5(x_1)(x_1+h-x_1)^5}{5!} + O(h^6)$$

$$= f(x_1) + f^1(x_1)h + \frac{f^2(x_1)}{2!}h^2 + \frac{f^3(x_1)}{3!}h^3$$

$$+ \frac{f^4(x_1)}{4!}h^4 + \frac{f^5(x_1)}{5!}h^5 + O(h^6)$$

$$\text{---}(i)$$

$$\therefore f(x_1 - h) = f(x_1) + f^1(x_1)(x_1 - h - x_1)$$

$$+ \frac{f^2(x_1)(x_1 - h - x_1)^2}{2!} + \frac{f^3(x_1)(x_1 - h - x_1)^3}{3!}$$

$$+ \frac{f^4(x_1)(x_1 - h - x_1)^4}{4!} + \frac{f^5(x_1)(x_1 - h - x_1)^5}{5!}$$

$$+ O(h^6)$$

$$= f(x_1) - f^1(x_1)(h) + \frac{f^2(x_1)}{2!}h^2$$

$$- \frac{f^3(x_1)}{3!}h^3 + \frac{f^4(x_1)}{4!}h^4 - \frac{f^5(x_1)}{5!}h^5$$

$$+ O(h^6) \quad (ii)$$

$$D_h^{\varphi} = \frac{1}{2h}(i - ii)$$

$$= \frac{1}{2h}\left\{2f'(x_1)h + 2\frac{f^3(x_1)}{3!}h^3 + 2\frac{f^5(x_1)}{5!}h^5 + O(7)\right\}$$

$$= \underbrace{f'(x_1)}_{\text{exact value}} + \underbrace{\frac{f^3(x_1)h^2}{3!} + \frac{f^5(x_1)}{5!}h^4 + O(h^6)}_{\text{error.}}$$

error $\propto h^2$

how to make it better?

Take combination in such a way that h^2 goes away.

Now,

$$D_h = f'(x_1) + \frac{f^3(x_1)}{3!}h^2 + \frac{f^5(x_1)}{5!}h^4 + O(h$$

$$D_{h/2} = f'(x_1) + \frac{f^3(x_1)}{3!}\left(\frac{h}{2}\right)^2 + \frac{f^5(x_1)}{5!}(h/2)^4 + O($$

Now, we can remove h^2 term iff.

$$2^2 D_{h/2} - D_h = 2^2 f'(x_1) - f'(x_1)$$

$$+ 2^2 \frac{f^5(x_1)}{5!}\left(h/2\right)^4 - \frac{f^5(x_1)}{5!}h^4$$

$$+ 2^2 O(h/2)^6 - O(h^6)$$

$$= \cancel{f(x)} \left(2^2-1\right) f'(x_1) + \left(\frac{1}{2^2}-1\right) \frac{f^5(x_1)}{5!} h^4$$

$$+ O(h^6)$$

$$\frac{2^2 D_{h/2} - D_h}{2^2-1} = f'(x_1) + \frac{\left(\frac{1}{2^2}-1\right)}{\left(2^2-1\right)} \frac{f^5(x_1)}{5!} h^4$$

$$+ O(h^6)$$

$$D_h = f'(x_1) + \underbrace{c h^n}_{\text{can be anything}} + O(h^{n+1})$$

$$D_{h/2} = f'(x_1) + c\left(\frac{h}{2}\right)^n + O(h^{n+1})$$

multiply by $D_{h/2}$ with 2^n

$$D_h^{(1)} = \frac{2^n (D_{h/2}) - D_h}{2^n - 1}$$