Hermite Interpolation

node = $(n+1)$

degree $: (n+1)-1$
$= n$

x	$f(x)$	$f'(x)$
x_0	$f(x_0)$	$f'(x_0)$
x_1	$f(x_1)$	$f'(x_1)$
$\vdots$	$\vdots$	$\vdots$
x_n	$f(x_n)$	$f(x_n)$
	$n+1$	$n+1$

$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)$ total, $(n+1)$ node/conditions

$(x_0, f'(x_0)), (x_1, f'(x_1)), \ldots, (x_n, f'(x_n))$ total, $(n+1)$ " / "

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total $= 2(n+1)$ " / "

$= 2n+2$ nodes/conditions

$\therefore$ hermite polynomial degree $= (2n+2)-1$
$= 2n+1$

Hermite interpolation,

$$P_{2n+1}(x) = \sum_{k=0}^{n} \underbrace{h_k(x)} \cdot f(x_k) + \underbrace{\hat{h}(x)} \cdot f'(x_k)$$

$$\boxed{h_k(x) = [1 - 2(x - x_k)\{l_k'(x_k)\}]\{l_k(x)\}^2}$$

$$\boxed{\hat{h}_k(x) = (x - x_k)\{l_k(x)\}^2}$$

Example:

Given function, $f(x) = \sin(x)$ and nodes $\{0, \frac{\pi}{2}\}$, find out the hermite polynomial:

node=2
$n=1$

	x	$f(x)$	$f'(x) = \cos(x)$
x_0	0	$f(x_0)$ 0	1 $f'(x_0)$
x_1	$\pi/2$	$f(x_1)$ 1	0 $f'(x_1)$

$$P_{2n+1}(x) = \sum_{k=0}^{n} \underbrace{h_k(x)} \cdot f(x_k) + \underbrace{\hat{h}(x)} \cdot f'(x_k)$$

$k=0$
$k=1$

$$P_{2n+1} = P_{2\cdot 1+1} = P_3(x) = \cancel{h_0(x) \overbrace{f(x_0)}^{0}} + \hat{h}_0(x) f'(x_0)$$
$$+ h_1(x) f(x_1) + \cancel{\hat{h}(x) \underbrace{f'(x_1)}_{0}}$$

$$= \hat{h}_0(x) f'(x_0) + h_1(x) f(x_1)$$

$$\underset{k=0}{\hat{h}_0(x)} = (x - x_k) \{l_k(x)\}^2$$
$$= (x - x_0) \{l_0(x)\}^2$$
$$= (x - 0) \left\{\frac{x - \frac{\pi}{2}}{-\frac{\pi}{2}}\right\}^2$$

$$\hat{h}_0(x) = x \left\{\frac{x - \frac{\pi}{2}}{-\frac{\pi}{2}}\right\}^2$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - \frac{\pi}{2}}{0 - \frac{\pi}{2}}$$

$$\underset{k=1}{h_1(x)} = [1 - 2(x - x_k) \{l_k'(x_k)\}] \{l_k(x)\}^2$$
$$= [1 - 2(x - x_1) \{\underline{l_1'(x_1)}\}] \{\underline{l_1(x)}\}^2$$

$\frac{\pi}{2}$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{\pi/2 - 0} = \frac{x}{\pi/2} = \frac{2x}{\pi}$$

$$l_1'(x) = \frac{2}{\pi}$$

$$l_1'\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$h_1(x) = \left[1 - 2\left(x - \frac{\pi}{2}\right)\left(\frac{2}{\pi}\right)\right]\left(\frac{2x}{\pi}\right)^2$$

$$P_3(x) = \hat{h}_0(x)\, f'(x_0) + h_1(x) f(x_1)$$

$$P_3(x) = x\left\{\frac{x - \pi/2}{-\pi/2}\right\}^2 + \left[1 - 2\left(x - \frac{\pi}{2}\right)\left(\frac{2}{\pi}\right)\right]\left(\frac{2x}{\pi}\right)^2$$

Example:

node = 3

degree, n = 2

	x	$f(x)$	$f'(x)$	
x_0	-1	$f(x_0)$ 1	2	$f'(x_0)$
x_1	0	$f(x_1)$ 0	2	$f'(x_1)$
x_2	1	$f(x_2)$ 1	0	$f'(x_2)$

$$P_{2n+1}(x) = \sum_{k=0}^{n} \underbrace{h_k(x)} \cdot f(x_k) + \underbrace{\hat{h}(x)} \cdot f'(x_k)$$

$k=0$
$=1$
$=2$

find out the value for $x = 2$

hermite polynomial:

$$P_{2n+1} = P_5(x) = h_0(x) f(x_0) + \hat{h}_0(x) f'(x_0)$$
$$+ \cancel{h_1(x) f(x_1)} + \hat{h}_1(x) f'(x_1)$$
$$+ h_2(x) f(x_2) + \cancel{\hat{h}_2(x) f'(x_2)}$$

$$= h_0(x) \cdot 1 + \hat{h}_0(x) \cdot 2 + \hat{h}_1(x) \cdot 2 + h_2(x) \cdot 1$$

$$P_5(x) = h_0(x) + 2\hat{h}_0(x) + 2\hat{h}_1(x) + h_2(x)$$

$$h_0(x) = [1 - 2(x - x_0)\{\underbrace{l_0'(x_0)}_{-1}\}] \{l_0(x)\}^2$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2} = \frac{x - 0}{-1 - 0} \times \frac{x - 1}{-1 - 1}$$

$$= \frac{x(x-1)}{-1 \cdot -2}$$

$$= \frac{x^2 - x}{2}$$

$$l_0(x) = \frac{1}{2}(x^2 - x)$$

$$l_0'(x) = \frac{1}{2}(2x - 1)$$

$$l_0'(-1) = \frac{1}{2}(2(-1) - 1) = \frac{1}{2}(-2 - 1) = -\frac{3}{2}$$

$$h_0(x) = [1 - 2(x - (-1)) \cdot (-\tfrac{3}{2})] * \left(\tfrac{1}{2}(x^2 - x)\right)^2$$

$$\hat{h}_0(x) = (x - x_0)\{l_0(x)\}^2 = (x - (-1))\{\tfrac{1}{2}(x^2 - x)\}^2$$

$$= (x+1)\left(\tfrac{1}{2}(x^2 - x)\right)^2$$

$$\hat{h}_1(x) = (x-x_1)\{l_1(x)\}^2$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2} = \frac{x+1}{0+1}\cdot\frac{x-1}{0-1} = \frac{(x+1)(x-1)}{-1}$$

$$= \frac{x^2-1^2}{-1}$$

$$= 1-x^2$$

$$\hat{h}_1(x) = (x-0)(1-x^2)^2$$

$$= x(1-x^2)^2$$

$$h_2(x) = [1-2(x-x_2)(l_2'(\underbrace{x_2}_{1}))]\{l_2(x)\}^2$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1} = \frac{x-(-1)}{1-(-1)}\cdot\frac{x-0}{1-0}$$

$$= \frac{x+1}{2}\cdot x$$

$$l_2(x) = \frac{1}{2}(x^2+x)$$

$$l_2'(x) = \frac{1}{2}(2x+1)$$

$$l_2'(1) = \frac{1}{2}(2\cdot 1+1) = \frac{1}{2}\cdot 3 = \frac{3}{2}$$

$$h_2(x) = \left[1-2(x-1)\cdot\frac{3}{2}\right]\left\{\frac{1}{2}(x^2+x)\right\}^2$$

$$P_5(x) = h_0(x)+2\hat{h}_0(x)+2\hat{h}_1(x)+h_2(x)$$

$$P_5(x) = \left[1-2(x-(-1))\cdot\left(-\frac{3}{2}\right)\right] * \left(\frac{1}{2}(x^2-x)\right)^2$$
$$+ 2(x+1)\left(\frac{1}{2}(x^2-x)\right)^2 + 2x(1-x^2)^2$$
$$+ \left[1-2(x-1)\cdot\frac{3}{2}\right]\left\{\frac{1}{2}(x^2+x)\right\}^2$$

$$P_5(2) = \left[1 - 2\left(2-(-1)\right)\cdot\left(-\tfrac{3}{2}\right)\right] * \left(\tfrac{1}{2}\left(2^2-2\right)\right)^2$$

$$+ 2(2+1)\left(\tfrac{1}{2}(2^2-2)\right)^2 + 2\cdot 2\left(1-2^2\right)^2$$

$$+ \left[1 - 2(2-1)\cdot\tfrac{3}{2}\right]\left\{\tfrac{1}{2}\left(2^2+2\right)\right\}^2$$

$$= \;.\,-- \qquad \text{Ans}$$

<u>Advantage:</u>

Weierstrass theorem,

increase

decrease

$$\left| f(x) - P_n(x) \right| = \text{error}$$

<u>3</u> degree $= 2$

hermite $= 5$