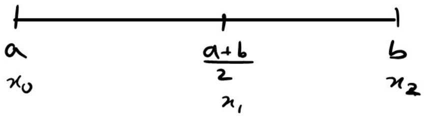
# Closed Newton Cote's formula for $n = 2$ [Simpson Rules]

$$I_2(f) = \int_a^b P_2(x)\, dx$$

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

Now, as $n = 2$, we need three nodes.




$$x_0 = a \qquad x_1 = \frac{a+b}{2} = m \qquad x_2 = b$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} = \frac{x - m}{a - m} \cdot \frac{x - b}{a - b}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} = \frac{x - a}{m - a} \cdot \frac{x - b}{m - b}$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{x - a}{b - a} \cdot \frac{x - m}{b - m}$$

Now, weight function,

$$\sigma_0 = \int_a^b l_0(x)\, dx$$

$$= \int_a^b \frac{(x-m)}{(a-m)} \frac{(x-b)}{(a-b)}\, dx$$

$$= \frac{1}{(a-m)(a-b)} \int_a^b (x-m)(x-b)\, dx$$

$$= \frac{1}{6}(b-a)$$

This is a long procedure. You are not required to write step by step in the question is about the derivation of Simson rule.

The detailed step by step procedure is given below at the end of the lecture note.

Similarly,

$$\sigma_1 = \int_a^b l_1(x)\,dx$$

$$\int_a^b \frac{(x-a)\,(x-b)}{(m-a)\,(m-b)}\,dx$$

$$\sigma_1 = \frac{1}{(m-a)(m-b)} \int_a^b (x-a)\,(x-b)\,dx$$

$$= \frac{2}{3}\,(b-a)$$

$$\sigma_2 = \int_a^b l_2(x)\,dx$$

$$= \int_a^b \frac{(x-a)\,(x-m)}{(b-a)\,(b-m)}\,dx$$

$$= \frac{1}{(b-a)\,(b-m)} \int_a^b (x-a)\,(x-m)\,dx$$

$$= \frac{1}{6}\,(b-a)$$

$$I_2(f) = \sum_{K=0}^{2} \sigma_K\, f(x_K)$$

$$= \sigma_0 f(x_0) + \sigma_1 f(x_1) + \sigma_2 f(x_2)$$

$$= \frac{b-a}{6} f(x_0) + \frac{2(b-a)}{3} f(x_1) + \frac{b-a}{6} f(x_2)$$

$$= \frac{b-a}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\boxed{I_2(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}$$

← Simpson Rule formula

**Example:** Find $\int_0^2 e^{0.5x} + \sin x \, dx$ using closed newton cotes formula for $n=2$.

Here, $a=0$, $b=2$, $\frac{a+b}{2}=1$, $f(x)=e^{0.5x}+\sin x$

Simpson rule,

$$I_2(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$f(0) = e^{0.5*0} + \sin(0) = 1$$

$$f(1) = e^{0.5*1} + \sin(1) = 2.4902$$

$$f(2) = e^{0.5*2} + \sin(2) = 3.6276$$

$$I_2(F) = \frac{2-0}{6}\left(1 + 4*(2.4902) + 3.6276\right)$$

$$= 4.8628$$

The detailed integration steps:

$$\sigma_0(x) = \int_a^b l_0(x)\, dx$$

$$= \int_a^b \frac{(x-m)}{(a-m)} \frac{(x-b)}{(a-b)}\, dx$$

$$= \frac{1}{(a-m)(a-b)} \int_a^b (x-m)(x-b)\, dx$$

$$= \frac{1}{\left(a-\frac{a+b}{2}\right)(a-b)} \int_a^b \left(x^2 - mx - bx + bm\right) dx$$

$$= \frac{1}{\left(\frac{2a-a-b}{2}\right)(a-b)} \left[\frac{x^3}{3} - \frac{mx^2}{2} - \frac{bx^2}{2} + bmx\right]_a^b$$

$$= \frac{1}{\frac{(a-b)}{2}(a-b)} \left[\frac{b^3}{3} - \frac{(a+b)}{2}\frac{b^2}{2} - \frac{b^3}{2} + \frac{(a+b)}{2} b^2 - \frac{a^3}{3} + \frac{(a+b)}{2}\frac{a^2}{2} + \frac{ba^2}{2} - \frac{(a+b)}{2} ab\right]$$

$$= \frac{2}{(a-b)^2}\left[\frac{4b^3 - 3b^2(a+b) - 6b^3 + 6b^2(a+b) - 4a^3 + 3a^2(a+b) + 6a^2b - 6ab(a+b)}{12}\right]$$

$$= \frac{2}{(a-b)^2}\left[\frac{-2b^3 + 3b^2(a+b) - 4a^3 + 3a^3 + 3a^2b + \cancel{6a^2b} - \cancel{6a^2b} - 6ab^2}{12}\right]$$

$$= \frac{2}{(a-b)^2}\left[\frac{-2b^3 + 3ab^2 + 3b^3 - 4a^3 + 3a^3 + 3a^2b - 6ab^2}{12}\right]$$

$$= \frac{2}{(a-b)^2}\left[\frac{b^3 - 3ab^2 + 3a^2b - a^3}{12}\right]$$

$$= \frac{2}{(a-b)^2} \frac{(b-a)^3}{12} = \frac{\cancel{2}}{\cancel{(b-a)^2}} \frac{\cancel{(b-a)^3}(b-a)}{\cancel{12}6} = \frac{b-a}{6}$$

2b]

$x_0 \quad\quad x_1 \quad\quad x_2$

$a \quad\quad \frac{a+b}{2} = m \quad\quad b$

$$\sigma_1 = \int_a^b l_1(x)\, dx.$$

$$= \int_a^b \frac{x-x_0}{x_1-x_0} \times \frac{x-x_2}{x_1-x_2} \, dx.$$

$$= \int_a^b \frac{x-a}{m-a} \times \frac{x-b}{m-b} \, dx.$$

$$= \frac{1}{(m-a)(m-b)} \int_a^b (x-a)(x-b)\, dx$$

$$= \frac{1}{(m-a)(m-b)} \int_a^b x^2 - ax - bx + ab \, dx.$$

$$= \frac{1}{(m-a)(m-b)} \left[ \frac{x^3}{3} - \frac{ax^2}{2} - \frac{bx^2}{2} + abx \right]_a^b$$

$$= \frac{1}{(m-a)(m-b)} \left( \frac{b^3}{3} - \frac{ab^2}{2} - \frac{b^3}{2} + ab^2 - \frac{a^3}{3} + \frac{a^3}{2} + \frac{a^2b}{2} - a^2b \right)$$

$$= \frac{1}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} \left( \frac{2b^3 - 3ab^2 - 3b^3 + 6ab^2 - 2a^3 + 3a^3 + 3a^2b - 6a^2b}{6} \right)$$

$$= \frac{1}{\left(\frac{a+b-2a}{2}\right)\left(\frac{a+b-2b}{2}\right)} \left( \frac{-b^3 + 3ab^2 + a^3 - 3a^2b}{6} \right)$$

$$= \frac{1}{\left(\frac{b-a}{2}\right)\left(\frac{a-b}{2}\right)} \left\{ \frac{-(b^3 - 3ab^2 - a^3 + 3a^2b)}{6} \right\}$$

$$= \frac{4}{(b-a)(b-a)} \times \left\{ \frac{b^3 - a^3 - 3ab(b-a)}{6} \right\}$$

$$= \frac{4(b-a)^3}{6(b-a)^2}$$

$$= \frac{2}{3}(b-a)$$

b) $\sigma_k = \int_a^b l_k(x)\, dx$

$[a, b]$

$\swarrow \quad \searrow$

$x_0 \quad x_2$

$\sigma_2 = \int_a^b l_2(x)\, dx$

$x_1 = \frac{a+b}{2} = m$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$l_2(x) = \frac{(x-a)(x-m)}{(b-a)(b-m)}$$

$$\sigma_2 = \int_a^b \frac{(x-a)(x-m)}{(b-a)(b-m)}\, dx$$

$$= \frac{1}{(b-a)(b-m)} \int_a^b (x-a)(x-m)\, dx$$

$$= \frac{1}{(b-a)(b-m)} \int_a^b (x^2 - xm - ax + am)\, dx$$

$$= \frac{1}{(b-a)(b-m)} \left[ \frac{x^3}{3} - \frac{x^2 m}{2} - \frac{ax^2}{2} + amx \right]_a^b$$

$$= \frac{1}{(b-a)(b-m)} \left[ \left( \frac{b^3}{3} - \frac{b^2 m}{2} - \frac{ab^2}{2} + amb \right) - \left( \frac{a^3}{3} - \frac{a^2 m}{2} - \frac{a^3}{2} + a^2 m \right) \right]$$

$$= \frac{1}{(b-a)(b-m)}\left(\frac{b^3}{3} - \frac{b^2 m}{2} - \frac{ab^2}{2} + amb - \frac{a^3}{3} + \frac{a^2 m}{2} + \frac{a^3}{2} - a^2 m\right)$$

$$= \frac{1}{(b-a)\left(b - \frac{a+b}{2}\right)}\left\{\frac{b^3}{3} - \frac{b^2}{2}\times\frac{(a+b)}{2} - \frac{ab^2}{2} + ab\times\frac{(a+b)}{2} - \frac{a^3}{3} + \frac{a^2}{2}\times\frac{(a+b)}{2} + \frac{a^3}{2} - a^2\left(\frac{a+b}{2}\right)\right\}$$

$$= \frac{1}{(b-a)\left(\frac{b-a}{2}\right)}\left(\frac{b^3}{3} - \frac{ab^2}{4} - \frac{b^3}{4} - \frac{ab^2}{2} + \frac{a^2 b}{2} + \frac{ab^2}{2} - \frac{a^3}{3} + \frac{a^3}{4} + \frac{a^2 b}{4} + \frac{a^3}{2} - \frac{a^3}{2} - \frac{a^2 b}{2}\right)$$

$$= \frac{2}{(b-a)^2}\left(\frac{b^3}{12} - \frac{3ab^2}{4} + \frac{3a^2 b}{4} + \frac{ab^2}{2} - \frac{a^3}{12} - \frac{a^2 b}{2}\right)$$

$$= \frac{2}{(b-a)^2}\left(\frac{b^3}{12} - \frac{ab^2}{4} + \frac{a^2 b}{4} - \frac{a^3}{12}\right)$$

$$= \frac{2}{(b-a)^2} \times \frac{b^3 - 3ab^2 + 3a^2 b - a^3}{12}$$

$$= \frac{2}{(b-a)^2} \times \frac{(b-a)^3}{12}$$

$$= \frac{b-a}{6}$$

$$\therefore\ \sigma_2 = \frac{1}{6}(b-a) \quad \text{[verified]}$$