Cauchy's Theorem

x_0	$f(x_0)$
x_1	$f(x_1)$
$\vdots$	$\vdots$
x_n	$f(x_n)$

$$\underbrace{\left| f(x) - P_n(x) \right|}_{\text{Error}} = \left| \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n) \right|$$

To find the upper bound error / maximum possible Error

Example: $f(x) = \cos(x)$, interval $\xi \in [-1, 1]$

node=3, degree=2, n=2

	x	$f(x)$	
$x_0 =$	$-\frac{\pi}{4}$	$\cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$	$f(x_0)$
$x_1 =$	0	$\cos(0) = 1$	$f(x_1)$
$x_2 =$	$\frac{\pi}{4}$	$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$	$f(x_2)$

$$\left| f(x) - P_n(x) \right| = \left| \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n) \right|$$

$$\left| f(x) - P_2(x) \right| = \left| \frac{f^3(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2) \right|$$

$$= \underbrace{\left| \frac{\sin(\xi)}{3!} \right|}_{\sin(x)} \underbrace{\left| (x+\tfrac{\pi}{4})(x-0)(x-\tfrac{\pi}{4}) \right|}_{\omega(x)}$$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

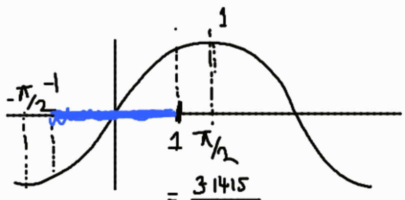
$f^2(x) = -\cos(x)$

$f^3(x) = \sin(x)$

$\sin(x)$ $\begin{cases} \to \min(-1) \\ \to \max(1) \end{cases}$

$\sin(1) = 0.8415$ ✓

$\sin(-1) = -0.8415$



$= \frac{3.1415}{2}$

$= 1.57$

$$\left| \frac{\sin(\xi)}{3!} \right| = \left| \frac{\sin(1)}{3!} \right| = \boxed{\frac{0.8415}{6}}$$

$$\omega(x) = \left| (x+\tfrac{\pi}{4})(x-0)(x-\tfrac{\pi}{4}) \right|$$

$$= (x+\tfrac{\pi}{4})(x-\tfrac{\pi}{4})\, x$$

To get max value of a function,

$$\frac{dy}{dx} = 0$$

$\therefore$ $\omega'(x) = 0$, $x \Rightarrow$ maximum

$$= \left(x^2 - \frac{\pi^2}{4^2}\right) x$$

$$w(x) = x^3 - \frac{\pi^2}{16} x$$

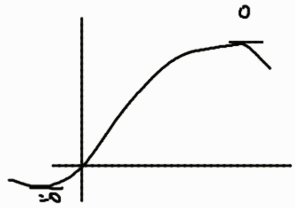
$$w'(x) = 3x^2 - \frac{\pi^2}{16} = 0$$

$$\Rightarrow \quad 3x^2 = \frac{\pi^2}{16}$$

$$\Rightarrow \quad x^2 = \frac{\pi^2}{16 \times 3}$$

$$\therefore \quad x = \pm \frac{\pi}{4\sqrt{3}}$$

$$(a+b)(a-b) = a^2 - b^2$$



x	$w(x) = x^3 - \frac{\pi^2}{16} x$	
$\frac{\pi}{4\sqrt{3}}$	$\left(\frac{\pi}{4\sqrt{3}}\right)^3 - \frac{\pi^2}{16}\left(\frac{\pi}{4\sqrt{3}}\right) = -0.186$	
$-\frac{\pi}{4\sqrt{3}}$	0.186	
-1	-0.3831	
1	0.3831	= maximum

$$\therefore \text{ Max/upper bound Error} = \left| \frac{0.8415}{6} * 0.3831 \right|$$

$$= 0.0537$$

(Ans)