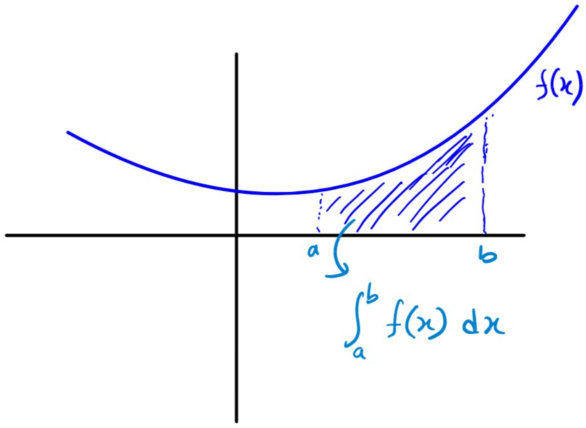
# Chapter 7

## Numerical Integration:




We know, The definite integral,

$$I(f) = \int_a^b f(x)\,dx$$

This means the exact area under the curve $f(x)$ from a to b.

But in reality, there is no exact formula for integral, so we need an approximation.

So, to evaluate $I(f) = \int_a^b f(x)\,dx$

Since $f(x)$ can be too complex as it is a realworld function, We approximate $f(x)$ using a polynomial $P_n(x)$

$$I(f) = \int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$

Now we know,

Lagrange polynomial $P_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + \dots + l_n(x) f(x_n)$

$$= \sum_{k=0}^{n} l_k(x) f(x_k)$$

So, $\underbrace{\int_a^b P_n(x)\,dx} = \int_a^b \sum_{k=0}^{n} l_k(x) f(x_k)\,dx$

$$\Rightarrow I_n = \int_a^b \sum_{k=0}^{n} l_k(x) \underbrace{f(x_k)}_{\text{value}}\,dx$$

$$\Rightarrow I_n = \sum_{k=0}^{n} f(x_k) \underbrace{\int_a^b l_k(x)\,dx}$$

This part is called <u>weight function</u>.

It is defined as $\sigma_k$.

$$\boxed{\therefore \sigma_k = \int_a^b l_k(x)\,dx}$$

$$\Rightarrow \boxed{I_n = \sum_{k=0}^{n} f(x_k)\,\sigma_k} \longrightarrow$$ This is the general quadrature rule.

If the nodes $x_k$ are equally spaced in $[a,b]$, then the formula becomes the Newton Cotes family (which includes the trapezoidal rule, Simpson's rule etc.)

## Closed Newton's Cotes formula

→ Interval: $[a, b]$

→ The integration nodes includes the endpoints,

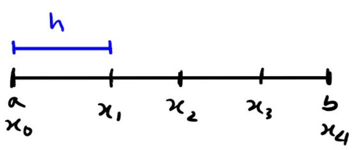
$$[a, b] \quad a \to x_0, \quad b \to x_n$$

$\therefore$ nodes are,

$$\boxed{a = x_0} < x_1 < x_2 < \cdots\cdots < x_{n-1} < \boxed{x_n = b}$$

The nodes are equally spaced. The spacing,

$$\boxed{h = \frac{b-a}{n}}$$



Now, for $n = 1$ we use Trapezium Rule

for $n = 2$ we use Simpson Rule

## Open Newton's Cotes formula

→ Interval: $[a, b]$

→ The integration nodes doesn't include the endpoints,

∴ nodes are,

$$\boxed{a < x_0} < x_1 < x_2 < \ldots\ldots < x_{n-1} < \boxed{x_n < b}$$

The nodes are equally spaced. The spacing,

$$\boxed{h = \frac{b-a}{n+2}}$$

## Trapezium Rule Closed Newton's Cotes formula with $n = 1$:

Since degree $n = 1$, there will be 2 nodes.

As $x \in [\underset{\substack{\downarrow \\ x_0}}{a}, \underset{\substack{\downarrow \\ x_1}}{b}]$ $\qquad h = \dfrac{b-a}{n} = \dfrac{b-a}{1} = b-a$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-b}{a-b}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x-b}{b-a}$$

So, $\sigma_0 = \int_a^b l_0(x)\,dx$

$$= \int_a^b \frac{x-b}{a-b}\,dx$$

$$= \frac{1}{a-b} \int_a^b x - b\,dx$$

$$= \frac{1}{a-b} \left[\frac{x^2}{2} - bx\right]_a^b$$

$$= \frac{1}{a-b} \left[\frac{b^2}{2} - b^2 - \frac{a^2}{2} - ab\right]$$

$$= \frac{1}{a-b} \left[-\frac{b^2}{2} - \frac{a^2 - 2ab}{2}\right]$$

$$= \frac{1}{a-b} \left[\frac{-b^2 - a^2 + 2ab}{2}\right]$$

$$= \frac{1}{a-b} \; \frac{-(a-b)^2}{2}$$

$$= -\frac{1}{\cancel{a-b}} \; \frac{\cancel{(a-b)^2}^{\,(a-b)}}{2}$$

$$= \frac{-(a-b)}{2} \Rightarrow \sigma_0 = \frac{b-a}{2}$$

$$\sigma_1 = \int_a^b l_1(x)\, dx$$

$$= \int_a^b \frac{x-a}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b (x-a)\, dx$$

$$= \frac{1}{b-a} \left[ \frac{x^2}{2} - ax \right]_a^b$$

$$= \frac{1}{b-a} \left[ \frac{b^2}{2} - ab - \frac{a^2}{2} + a^2 \right]$$

$$= \frac{1}{b-a} \left[ \frac{b^2 - 2ab - a^2 + 2a^2}{2} \right]$$

$$= \frac{1}{b-a} \left[ \frac{a^2 - 2ab + b^2}{2} \right]$$

$$= \frac{1}{b-a} \frac{(a-b)^2}{2}$$

$$= -\frac{1}{\cancel{a-b}} \cdot \frac{\cancel{(a-b)^2}^{\,(a-b)}}{2}$$

$$= \frac{-(a-b)}{2}$$

$$\sigma_1 = \frac{b-a}{2}$$

Now, we know,

$$I_n = \int_a^b P_1(x)\, dx$$

$$= \int_a^b l_0(x) f(x_0) + l_1(x) f(x_1)\, dx$$

$$= \int_a^b l_0(x) f(x_0)\, dx + \int_a^b l_1(x) f(x_1)\, dx$$

$$= f(x_0) \int_a^b l_0(x)\, dx + f(x_1) \int_a^b l_1(x)\, dx$$

$$= f(x_0)\, \sigma_0 + f(x_1)\, \sigma_1$$

$$= f(x_0)\,\frac{b-a}{2} + f(x_1)\,\frac{b-a}{2}$$

$$= \frac{b-a}{2}\left(f(x_0) + f(x_1)\right)$$

$$\therefore \boxed{I_n = \frac{b-a}{2}\left(f(a) + f(b)\right)} \longrightarrow$$

Trapezium Rule aka Closed Newton Cotes formula with $n = 1$

Example:

Given that, $f(x) = e^x$ and $[0, 2]$

a) Find Numerical Integration using Trapezium Rule.

b) Find Actual Integration value.

c) Find Relative Percentage error.

(a)

We know,

$$I_n = \frac{b-a}{2}\left(f(a) + f(b)\right)$$

$$= \frac{2-0}{2}\left(e^0 + e^2\right)$$

$$= 8.3891$$

(b)

$$\int_0^2 e^x\,dx$$

$$= \left[e^x\right]_0^2$$

$$= e^2 - e^0$$

$$= 6.3891$$

(c)

$$\text{Relative Percentage Error} = \left| \frac{\text{Approximate} - \text{Actual}}{\text{Actual}} \right| \times 100\%$$

$$= \left| \frac{8.3891 - 6.3891}{6.3891} \right| \times 100\%$$

$$= 31.65\%$$

## Upper Bound of Interpolation Error

Cauchy's Theorem,

$$\text{Upper Bound of Interpolation Error} = \left| \frac{f^{n+1}(\xi)}{(n+1)!} \right| \left| \int_a^b \overbrace{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}^{\omega(x)}\, dx \right|$$

where interval $\in [a, b]$

for $n = 1$, $f(x) = e^x$ interval $[0, 2]$,

$$\left| \frac{f^{n+1}(\xi)}{(n+1)!} \right| = \left| \frac{f^2(\xi)}{2!} \right|$$

$$= \left| \frac{e^x}{2!} \right|$$

for maximum, $= \left| \dfrac{e^2}{2!} \right| = \dfrac{1}{2} e^2$

$$\omega(x) = (x - x_0)(x - x_1)$$
$$= (x - 0)(x - 2)$$
$$= x(x-2)$$
$$= x^2 - 2x$$

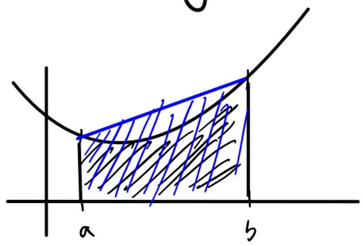
$$\left| \int_0^2 \omega(x)\, dx \right| = \left| \int_0^2 (x^2 - 2x)\, dx \right|$$
$$= \left| \left[ \frac{x^3}{3} - \frac{2x^2}{2} \right]_0^2 \right|$$
$$= \left| \left[ \frac{2^3}{3} - \frac{2(2)^2}{2} - 0 + 0 \right] \right|$$
$$= \left| -\frac{4}{3} \right| = \frac{4}{3}$$

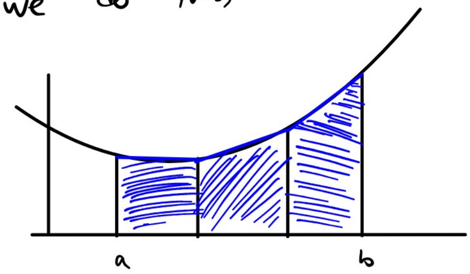
$\therefore$ Upper bound Error $= \frac{1}{\cancel{2}} e^2 \times \frac{\cancel{4}^{2}}{3}$

$$= \frac{2}{3} e^2$$

(Ans)

# Composite Newton's Cotes formula

Instead of doing this  , we do this,




This method improves result without increasing the actual node numbers.

Here, we divide $[a,b]$ into $m$ subintervals with equal length,

$$h = \frac{b-a}{m}$$

So,

$C_{1,m}$ ⟵ Composite Newton's Cotes formula notation

$$C_{1,m}(f) = \sum_{i=0}^{m} I_{1i} = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right]$$

Example: find $\int_0^2 e^x\,dx$ using composite Newton cotes formula where number of sub intervals, $m = 2$ and $m = 4$

$m = 2$ $\qquad f(x) = e^x$

Here, $[a,b] = [0,2]$

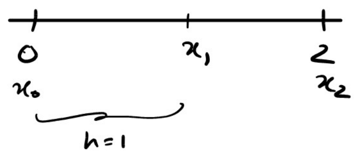
$\therefore$ length of subinterval, $h = \dfrac{b-a}{m}$

$$= \frac{2-0}{2} = 1$$

$\therefore x_0 = 0$

$\therefore x_1 = 0 + h = 0 + 1 = 1$

$\therefore x_2 = 2$



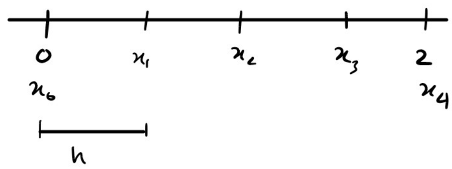
$$C_{1,2}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

$$= \frac{h}{2}\left[e^0 + 2e^1 + e^2\right]$$

$$= 6.9128$$

Now for $m=4$,

$$h = \frac{b-a}{m} = \frac{2-0}{4} = 0.5$$




$x_0 = 0$

$x_1 = 0 + h = 0.5$

$x_2 = 0.5 + 0.5 = 1$

$x_3 = 1 + 0.5 = 1.5$

$x_4 = 2$

$$C_{1,4} = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{0.5}{2}\left(e^0 + 2e^{0.5} + 2e^1 + 2e^{1.5} + e^2\right)$$

$$= 6.5216$$

(Ans)

From early examples, we saw, the more subinterval we take, the smaller the error becomes.

$\therefore$ The error decreases as $m$ increases.

- The error decreases by a factor of 4 indicating that it is quadratic convergence.