Polynomials

degree, $n = 2$

$$3x^3 + 5x^1 + 3 \cdot x^0 = 0 \Rightarrow \boxed{\underbrace{3}_{a_0}x^0 + \underbrace{5}_{a_1}x^1 + \underbrace{3}_{a_2}x^{\boxed{2}} = 0}$$

General form:

$$\underset{\text{degree}}{P_n}(x) = \underbrace{a_0}x^0 + \underbrace{a_1}x^1 + \underbrace{a_2}x^2 + \ldots\ldots + \underbrace{a_n}x^n$$

Coefficient

$$P_{\underline{3}}(x) = \underbrace{a_0}x^0 + \underbrace{a_1}x^1 + \underbrace{a_2}x^2 + \underbrace{a_3}x^3$$

coeff = $[a_0, a_1, a_2, a_3] = 4$

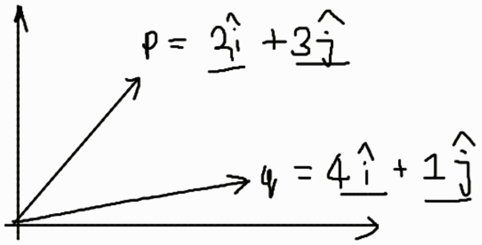
$$\boxed{\text{number of coeff} = \text{degree} + 1}$$

$$P_{\underline{\underline{25}}}(x) = 2x^0 + 3x + \cdot \, - - - \; 25x^{25}$$

number of coeff = $25 + 1 = 26$

Vector space:

A region where we can add $(\pm)$ or multiply with scalers



$$p+q = 6\hat{i} + 4\hat{j}$$
$$p+(-q) = -2\hat{i} + 2\hat{j}$$
$$2*p = 4\hat{i} + 6\hat{j}$$

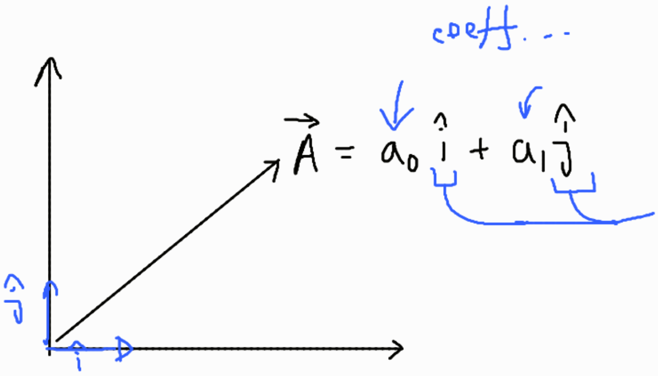
Polynomial:

$$P_2(x) = 1 + x + 2x^2$$
$$P_3(x) = x^3$$

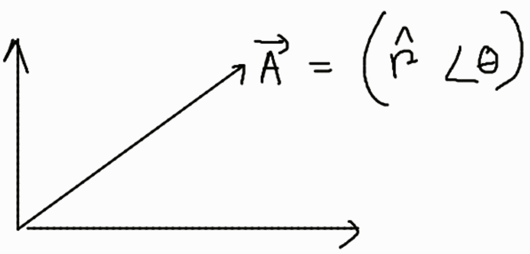
$$P_2(x) + P_3(x) = 1 + x + 2x^2 + x^3$$
$$P_3(x) * 5 = 5x^3$$
$$P_2(x) * 3 = 3 + 3x + 6x^2$$

Basis:

is a set of vectors that spans the space.



Basis = $\{\hat{i}, \hat{j}\}$



Basis = $\{\hat{r}, \hat{\theta}\}$

Polynomial:

Coeff

$$P_3(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$$

$$\text{Basis} = \{x^0, x^1, x^2, x^3\} = \underbrace{\{1, x, x^2, x^3\}}_{\text{Natural Basis}}$$

Dimensional Space:

how many dimensions?

3 degree Polynomial, Basis = $\{1, x, x^2, x^3\}$

basis's dimension = 4

Basis's dimension = degree + 1

Ex:

$P_{37}(x) = \text{------ --}$

degree, number of coeff, basis, dimension?

$n=37$ $\quad n+1 = 38$ $\quad \{1, x, x^2, \ldots\ldots x^{37}\}$ $\quad n+1 = 38$

Functional Space:

→ Natural functions ⟶ has ∞ degree, ∞ dimension

$f(x) = 2 + 3x + 10x^2 + 14x^3 + \ldots\ldots\ldots$

basis = $\{1, x, x^2, x^3, \ldots\ldots\}$

We can reproduce it using polynomials with some error.

$P_2(x) = 2 + 3x + 10x^2$

$$\begin{cases} f(x) \in V^{\infty} & \leftarrow \text{function belongs to } \infty \text{ dimension vector space} \\ P_n(x) \in V^{n+1} & \leftarrow \text{Polynomial with degree } n\text{, belongs to } (n+1) \text{ dimension vector space.} \end{cases}$$

$$f(x) = \boxed{2 + 3x + 10x^2} \underbrace{+ 14x^3 + \cdots\cdots}_{\text{Error}}$$

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

Error ↳ Truncation Error

$$P_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

if we increase degree, we will able to reduce the error.

This is called,

Weierstrass Appronimation Theorem:

$$\text{Error} = \left| f(x) - P_n(x) \right|$$

$n \uparrow$

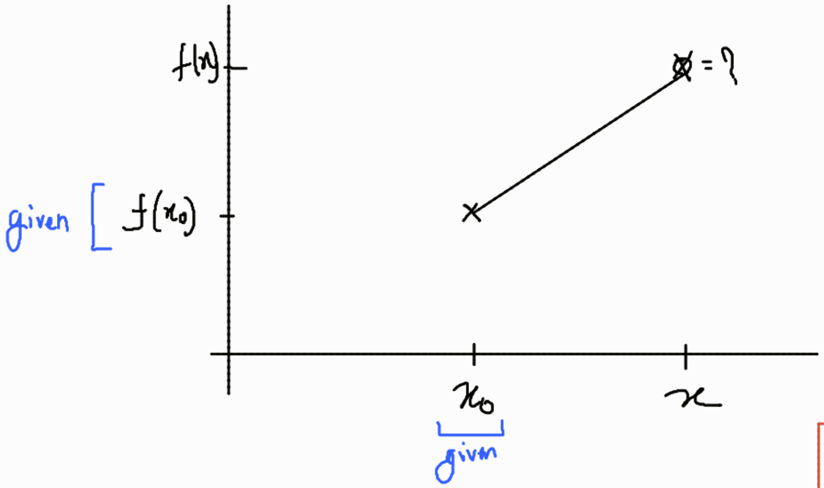
error $\downarrow$

For any $f \in C([0,1])$ and any $\underline{\epsilon > 0}$, there exists a polynomial $p(x)$ such that

$$\max_{0 \le x \le 1} \left| f(x) - P_n(x) \right| \le \epsilon \leftarrow \text{epsilon}$$

$$f(x) \approx P_n(x)$$

Taylor Series:



gradient, $f'(x_0) = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\Rightarrow f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f'(x_0)(x - x_0) = f(x) - f(x_0)$$

$$\boxed{\Rightarrow f(x) = f'(x_0)(x - x_0) + f(x_0)}$$

→ Taylor Series for straight line.

Actual taylor series ⟶ more complex & infinitly large

Taylor Series, (Not finite)

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)\,(x - x_0)^2}{2!} + \frac{f'''(x_0)\,(x - x_0)^3}{3!} + \cdots$$

Proof of taylor series:

Step 1 :

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots\cdots$$

$$f'(x) = 0 + \boxed{a_1} + \underbrace{2a_2(x-x_0)}_{0} + \underbrace{3a_3(x-x_0)^2}_{0} + \underbrace{\cdots\cdots}_{0}$$

$$f''(x) = 0 + 0 + 2a_2 + \underbrace{3*2\,a_3(x-x_0)}_{0} + \cdots\cdots$$

$$f'''(x) = 0 + 0 + 0 + 3*2\,a_3 + \underbrace{\cdots\cdots\cdots}_{0}$$

Step 3 Let, $x = x_0$,

$$f(x_0) = a_0 + a_1\overbrace{(x_0-x_0)}^{0} + a_2\overbrace{(x_0-x_0)^2}^{0} + a_3\overbrace{(x_0-x_0)^3}^{0} + \cdots\cdots$$

$f(x_0) = a_0$

$f'(x_0) = a_1$

$f''(x_0) = 2a_2$

$f'''(x_0) = 3*2*a_3$

Now,

$a_0 = f(x_0)$

$a_1 = f'(x_0)$

$a_2 = \dfrac{f''(x_0)}{2 \to 2!} = \dfrac{f''(x_0)}{2!}$

$a_3 = \dfrac{f'''(x_0)}{3*2 \to 3!} = \dfrac{f'''(x_0)}{3!}$

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots\cdots$$

$$\Rightarrow f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

[showed]

Example: $\sin(x) = x - \frac{x^3}{3!} + \frac{5x^5}{5!} - \cdots\cdots$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \frac{f^4(x_0)(x-x_0)^4}{4!} + \frac{f^5(x_0)(x-x_0)^5}{5!} + \cdots$$

Let, $x_0 = 0$,

$$f(x) = \underbrace{\sin(0)}_{0} + \underbrace{\cos(0)}_{1}(x) + \frac{\underbrace{-\sin(0)}_{0}(x)^2}{2!} + \frac{\underbrace{-\cos(0)}_{1}x^3}{3!} + \frac{\underbrace{\sin(0)}_{0}x^4}{4!} + \frac{\underbrace{\cos(0)}_{1}x^5}{5!} + \cdots\cdots$$

$$= 0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\cdots$$

Example:

Find $\underbrace{\sin(0.1)}_{x}$ using Taylor Series,

$$\sin(x) = \underbrace{x}_{1st} - \underbrace{\frac{x^3}{3!}}_{2nd} + \underbrace{\frac{x^5}{5!}}_{3rd} \cdots\cdots$$

Upto 1 term,

$\sin(0.1) \approx 0.1$

Upto 2nd term, $\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} = 0.0998333\cdots\cdots$

Upto 3rd term, $\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} = 0.0\underline{998334166}7\cdots$

Actual value:

$$\sin(0.1) = 0.0\underbrace{998334166}_{9\ s.f}5$$