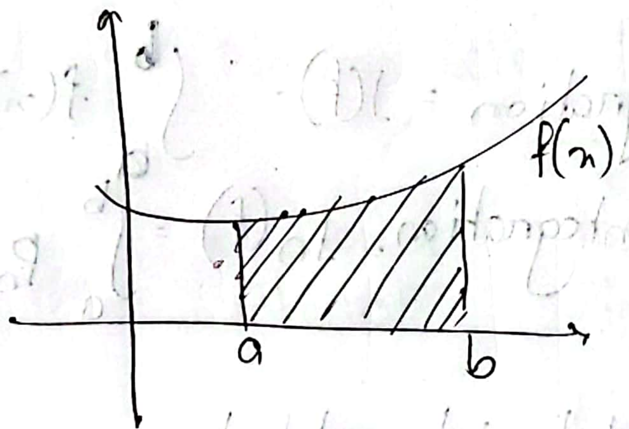
Chapter 7

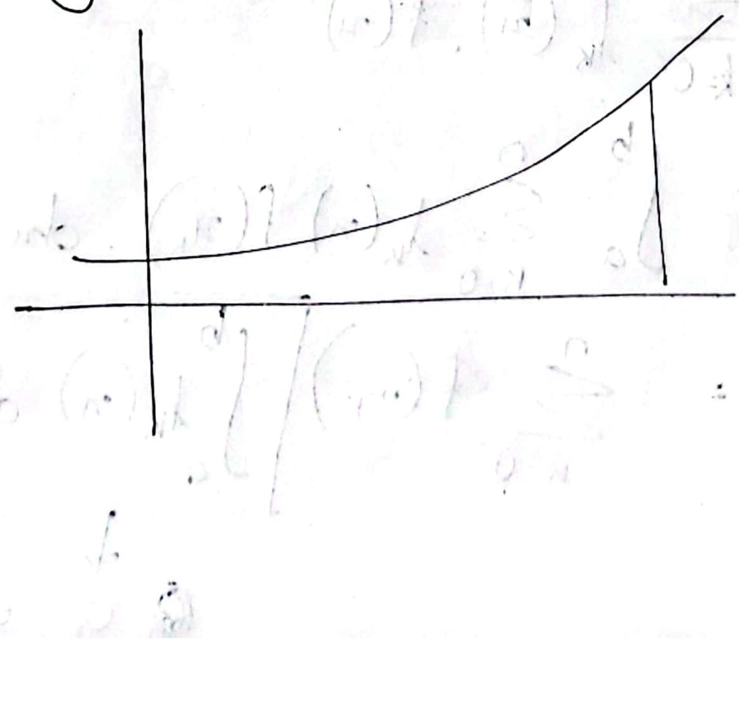
Integration:

$$I(f) = \int_b^a f(x)\, dx.$$



→ Integration gives the area under $f(x)$ within the bound a & b.

→ by defination integration is an infinite sum



→ Numerical integration replace function $f(x)$ with interpolating polynomial of degree n that passes through $(n+1)$ nodes

Actual Integration $= I(f) = \int_a^b f(x)\,dx$

Numerical Integration $= I_n(f) = \int_a^b P_n(x)\,dx$

Here $P_n(x)$ must be interpolated with equidistant nodes $x_0, x_1, x_2 \dots x_n$

$$P_n(x) = \sum_{k=0}^{n} l_k(x) . f(x)$$

$$\therefore I_n(f) = \int_a^b \sum_{k=0}^{n} l_k(x) . f(x_k) . dx$$

$$= \sum_{k=0}^{n} f(x_k) \boxed{\int_a^b l_k(x) . dx}$$

σ_k : weighted factors

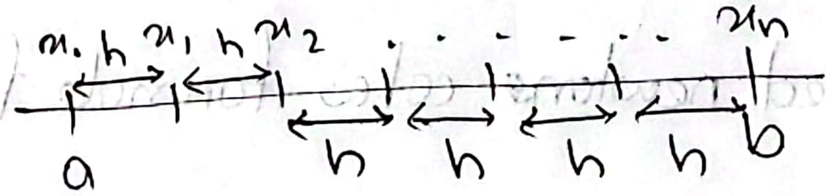
$$\therefore I_n(f) = \sum_{k=0}^{n} \sigma_k \cdot f(x_k)$$

Newton Cote's formula

for closed interval

for open interval

Finding $x_0, x_1, \dots x_n$ using closed Newton Cote's formula:



$[a, b]$

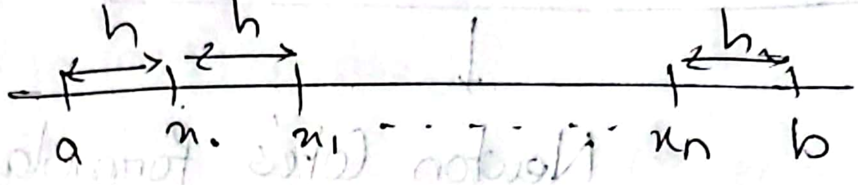
$$h = \frac{b-a}{n}$$

$x_0 = a$

$x_1 = x_0 + h = a + h$

$x_2 = x_1 + h = a + h + h = a + 2h$

Finding $x_0, x_1, \dots x_n$ using open Newton cote's Formula:



$$h = \frac{b-a}{n+2}$$

$x_0 = a+h$

$x_1 = a+2h$

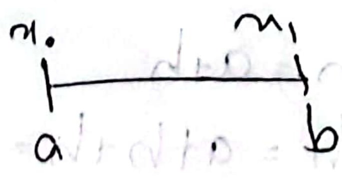
$x_2 = a+3h$

Trapezium/ Trapezoid rule:

→ Closed newtons cotes formula for $n=1$

$n =$ degree of polynomial $= 1$

∴ no. of nodes = 2



$$I_n(f) = \int_a^b P_n(x)\,.dx$$

$$I_1(f) = \int_a^b P_1(x)\,dx$$

$$\downarrow$$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$I_1(f) = \int_a^b \left[l_0(x) f(x_0) + l_1(x) f(x_1) \right] dx$$

$$= \int_a^b l_0(x)\,dx \times f(x_0) + \int_a^b l_1(x)\,dx \times f(x_1)$$

$$I_1(f) = \sigma_0 f(x_0) + \sigma_1 f(x_1)$$

$$\sigma_0 = \int_a^b f_0(x)\, dx$$

$$= \int_a^b \frac{x - x_1}{x_0 - x_1}\, dx$$

$$= \int_a^b \frac{x - b}{a - b}\, dx.$$

$$= \frac{1}{a-b} \int_a^b (x-b)\, dx.$$

$$= \frac{1}{a-b} \left[\frac{x^2}{2} - bx\right]_a^b$$

$$= \frac{1}{a-b} \left(\frac{b^2}{2} - b^2 - \frac{a^2}{2} + ab\right)$$

$$= \frac{b-a}{2}$$

$$\sigma_1 = \int_a^b l_1(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b (x-a)\,dx$$

$$\vdots$$

$$= \frac{b-a}{2}$$

$$I_1(f) = \int_a^b P_1(x)\,dx$$

$$= \sigma_0 f(x_0) + \sigma_1 f(x_1)$$

$$= \sigma_0 f(a) + \sigma_1 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

$$= \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$\therefore I_1 = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Example:

Find $I(f)$ and $I_1(f)$ for $f(x) = e^x$ on interval $[0,2]$
actual numerical

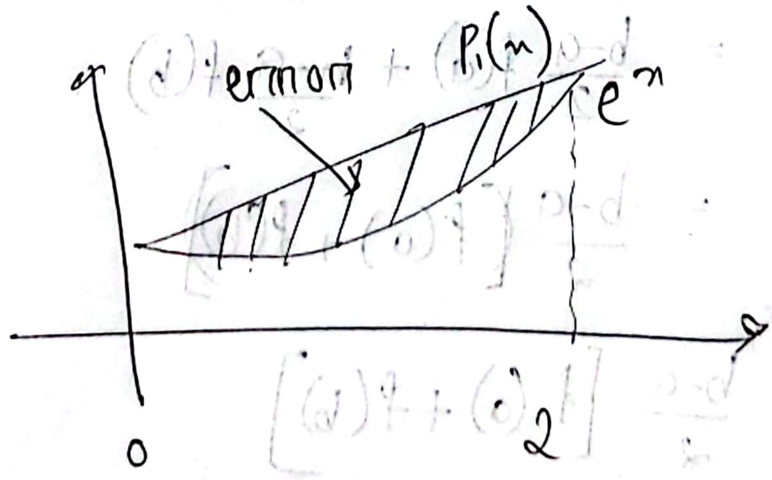
Solution:

$$I(f) = \int_0^2 e^x \, dx = \left[e^x\right]_0^2 = e^2 - e^0 = 6.389$$

$$I_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{2-0}{2}\left[e^0 + e^2\right] = 8.389$$

$$\% \text{ error} = \frac{I - I_1}{I} \times 100 = 31.3\%$$



We can also find the upper bound of the error

If a function $f(x)$ is interpolated by a n degree polynomial, error_{max} can be found using cauchy's theorem

upper bound error

$$|f(x) - P_n(x)| \leq \left| \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\ldots(x-x_n) \right|$$

$$|I - I_n| \leq \left| \frac{f^{n+1}(\xi)}{(n+1)!} \right| \left| \int_a^b (x-x_0)(x-x_1)\ldots(x-x_n)\, dx \right|$$

$$\int_a^b \left| (x-x_0)(x-x_1) \right| dx.$$

$$= \int_a^b \left| (x-a)(x-b) \right| dx = \int_0^2 \left| x^2 - 2x \right| dx.$$

$$= \left[\frac{x^3}{3} - \frac{2x^2}{2} \right]_0^2$$

$$= 4/3$$

upper bound of error $\leq \frac{e^2}{2!} \times 4/3.$

Simpson's Rule:

Trapezium rule $= \int_a^b P_1(x)\,dx$

Simpson's rule $= \int_a^b P_2(x)\,dx$

$$I_2(f) = \int_a^b P_2(x)\,dx.$$

$$\downarrow$$

$$l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

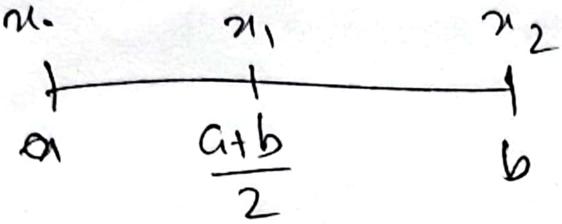
$$I_2(f) = \int_a^b \left[l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) \right] dx$$

$$= \int_a^b l_0(x)\,dx \times f(x_0) + \int_a^b l_1(x)\,dx \times f(x_1)$$

$$+ \int_a^b l_2(x)\,dx \times f(x_2).$$

$$= \sigma_0 f(x_0) + \sigma_1 f(x_1) + \sigma_2 f(x_2)$$

here $n+1 = 3 \Rightarrow \{x_0, x_1, x_2\}$



$$I_0 = \int_a^b l_0(x)\,dx$$

$$= \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx.$$

$$= \int_a^b \frac{(x-m)(x-b)}{(a-m)(a-b)}\,dx.$$

$$= \frac{1}{(a-m)(a-b)} \int_a^b (x-m)(x-b)\,dx.$$

$$= \frac{1}{6}(b-a)$$

$$\sigma_1 = \int_a^b l_1(x)\, dx$$

$$= \frac{2}{3}(b-a)$$

$$\sigma_2 = \int_a^b l_2(x)\, dx.$$

$$= \frac{1}{6}(b-a)$$

$$I_2(f) = \sigma_0 f(x_0) + \sigma_1 f(x_1) + \sigma_2 f(x_2)$$

$$= \sigma_0 f(a) + \sigma_1 f(m) + \sigma_2 f(b)$$

$$= \frac{1}{6}(b-a) f(a) + \frac{2}{3}(b-a) f(m) + \frac{1}{6}(b-a) f(b)$$

$$= \frac{b-a}{6}\left[f(a) + 4f(m) + f(b)\right]$$

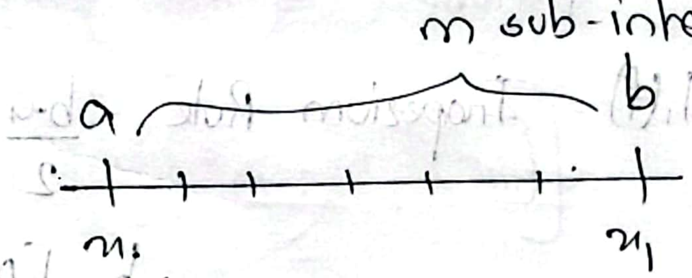
$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Composit- Newton cotes formula:

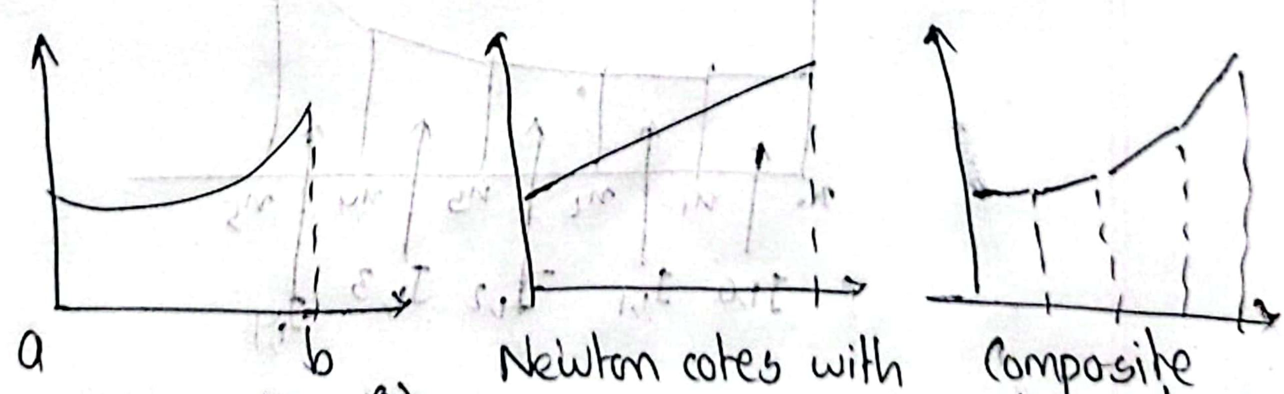
This method improves result without increasing number of nodes

→ Basic idea is to divide the interval ~~into~~ [a,b] m subintervals.

when n = 1 :



For each sub-interval, we apply trapezium rule, then add up.



Actual integration $I(f)$ — Newton cotes with $N = 1$ — Composite newton cotes $n = 1$ $C_{1,m}(f)$

degree

sub-divisions

→ Total sum is denoted by $C_{j,m}(f)$ and called composite Newton cote's

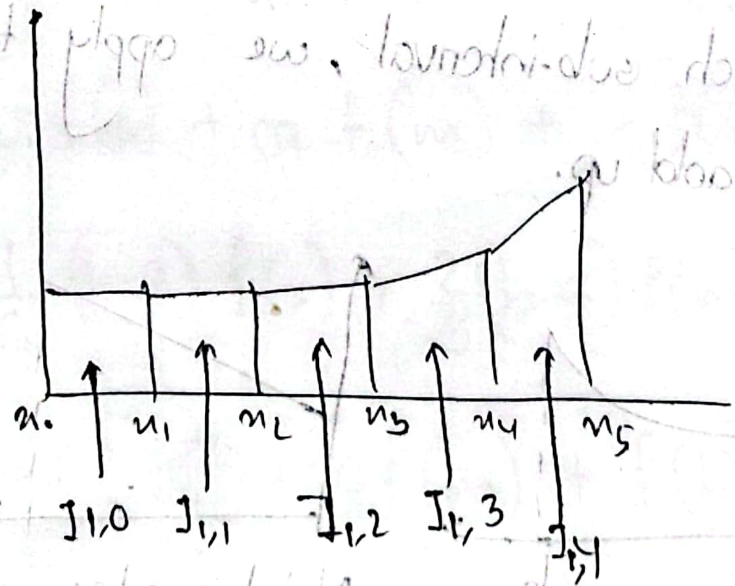
For m intervals we define,

$$h = \frac{b-a}{m}$$

Apply trapezium for each subinterval

$$I_1(f) = \text{Trapezium Rule} = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{h}{2}\left[f(a) + f(b)\right]$$



$$I_{1,0} = h/2 \left[f(x_0) + f(x_1) \right]$$

$$I_{1,1} = h/2 \left[f(x_1) + f(x_2) \right]$$

$$I_{1,2} = h/2 \left[f(x_2) + f(x_3) \right]$$

$$\vdots$$

$$I_{1,m-1} = h/2 \left[f(x_{m-2}) + f(x_{m-1}) \right]$$

$$I_{1,m} = h/2 \left[f(x_{m-1}) + f(x_m) \right]$$

$$C_{1,m} = h/2 \left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots 2f(x_{m-1}) + f(x_m) \right]$$

Example

$f(x) = e^x$

$a = 0$

$b = 2$

$$I(f) = \int_0^2 e^x \, dx = 6.389056$$

Composite Newton Cotes with num of subinterval = 2

$$h = \frac{b-a}{m} = \frac{2-0}{2} = 1$$

as $m = 2$, we have to find x_0, x_1, x_2

[If $m = 3$, then we had to find x_0, x_1, x_2, x_3]

$x_0 = a = 0$

$x_1 = x_0 + h = 0+1 = 1$

$x_2 = x_1 + h = 1+1 = 2$

$C_{1,m}(f)$

$$C_{1,2}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

$$= \frac{1}{2}\left[e^0 + 2e^1 + e^2\right]$$

$$= 6.91281$$

Composite Newton cotes with num of sub interval = 3

$$h = \frac{b-a}{3} = \frac{2-0}{3} = 2/3$$

Find x_0 to x_3:

$x_0 = a = 0$

$x_1 = x_0 + h = 0 + 2/3 = 2/3$

$x_2 = x_1 + h = 2/3 + 2/3 = 4/3$

$x_3 = x_2 + h = 4/3 + 2/3 = 6/3 = 2$

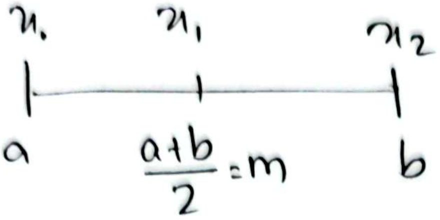
$$C_{1,3}(f) = h/2 \left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3) \right]$$

$$= \frac{2/3}{2} \left[e^0 + 2e^{2/3} + 2e^{4/3} + e^2 \right]$$

$$= 6.62395$$

So, as m increase error decrease.

Weighted factor derivation for Simpson's.



$$\sigma_1 = \int_a^b \frac{x-x_0}{x_1-x_0} \times \frac{x-x_2}{x_1-x_2}\, dx$$

$$= \int_a^b \frac{x-a}{m-a} \times \frac{x-b}{m-b}\, dx$$

$$= \frac{1}{(m-a)(m-b)} \int_a^b (x-a)(x-b)\, dx$$

$$= \frac{1}{(m-a)(m-b)} \int_a^b \left(x^2 - ax - bx + ab\right) dx$$

$$= \frac{1}{(m-a)(m-b)} \left[\frac{x^3}{3} - \frac{ax^2}{2} - \frac{bx^2}{2} + abx\right]_a^b$$

$$= \frac{1}{(m-a)(m-b)} \left(\frac{b^3}{3} - \frac{ab^2}{2} - \frac{b^3}{2} + ab^2 - \frac{a^3}{3} + \frac{a^3}{2} + \frac{a^2b}{2} - a^2b\right)$$

$$= \frac{1}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} \left(\frac{2b^3 - 3ab^2 - 3b^3 + 6ab^2 - 2a^3 + 3a^3 + 3a^2b - 6a^2b}{6}\right)$$

$$= \frac{1}{\left(\frac{a+b-2a}{2}\right)\left(\frac{a+b-2b}{2}\right)} \left(\frac{-b^3 + 3ab^2 + a^3 - 3a^2b}{6}\right)$$

$$= \frac{1}{\left(\frac{b-a}{2}\right)\left(\frac{a-b}{2}\right)}\left\{\frac{-(b^3-3ab^2-a^3+3a^2b)}{6}\right\}$$

$$= \frac{4}{(b-a)(b-a)}\left\{\frac{b^3-a^3-3ab(b-a)}{6}\right\}$$

$$= \frac{4(b-a)^3}{6(b-a)^2}$$

$$= \frac{2(b-a)}{3}$$

$$\sigma_2 = \int_a^b l_2(x)\,dx$$

$$= \int_a^b \frac{(x-a)(x-m)}{(b-a)(b-m)}\,dx$$

$$= \frac{1}{(b-a)(b-m)}\int_a^b (x-a)(x-m)\,dx$$

$$= \frac{1}{(b-a)(b-m)}\int_a^b \cancel{x^2-ax+am-a}\,(x^2-xm-ax+am)\,dx$$

$$= \frac{1}{(b-a)(b-m)}\left[\frac{x^3}{3}-\frac{x^2m}{2}-\frac{ax^2}{2}+amx\right]_a^b$$

$$= \frac{1}{(b-a)(b-m)}\left\{\left(\frac{b^3}{3}-\frac{b^2m}{2}-\frac{ab^2}{2}+abm\right)-\left(\frac{a^3}{3}-\frac{a^2m}{2}-\frac{a^3}{2}+a^2m\right)\right]$$

$$=\frac{1}{(b-a)(b-m)}\left(\frac{b^3}{3}-\frac{b^2m}{2}-\frac{ab^2}{2}+amb-\frac{a^3}{3}+\frac{a^2m}{2}+\frac{a^3}{2}-a^2m\right)$$

$$=\frac{1}{(b-a)\left(b-\frac{a+b}{2}\right)}\left\{\frac{b^3}{3}-\frac{b^2}{2}\times\frac{a+b}{2}-\frac{ab^2}{2}+ab\times\frac{(a+b)}{2}-\frac{a^3}{3}+\frac{a^2}{2}\left(\frac{a+b}{2}\right)+\frac{a^3}{2}-a^2\left(\frac{a+b}{2}\right)\right\}$$

$$=\frac{1}{(b-a)\left(\frac{b-a}{2}\right)}\left\{\frac{b^3}{3}-\frac{ab^2}{4}-\frac{b^3}{4}-\frac{ab^2}{2}+\frac{a^2b}{2}+\frac{ab^2}{2}-\frac{a^3}{3}+\frac{a^3}{4}+\frac{a^2b}{4}+\frac{a^3}{2}-\frac{a^3}{2}-\frac{a^2b}{2}\right\}$$

$$=\frac{2}{(b-a)^2}\left(\frac{b^3}{12}-\frac{3ab^2}{4}+\frac{3a^2b}{4}+\frac{ab^2}{2}-\frac{a^3}{12}-\frac{a^2b}{2}\right)$$

$$=\frac{2}{(b-a)^2}\times\left(\frac{b^3}{12}-\frac{ab^2}{4}+\frac{a^2b}{4}-\frac{a^3}{12}\right)$$

$$=\frac{2}{(b-a)^2}\times\frac{b^3-3ab^2+3a^2b-a^3}{12}$$

$$=\frac{(b-a)^3}{6(b-a)^2}$$

$$=\frac{1}{6}(b-a)$$

find σ, yourself.