5. Linear System of Equations

↳ How do we solve a linear system numerically?

Linear system of equations → The highest power of all variables must be 1

<u>Example:</u>

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + 2x_3 &= 4 \\ 2x_1 + 12x_2 - 2x_3 &= 4 \end{aligned}\right\}$$

A Linear System of equations

Number of unknown/variables, $n = 3$

General Form:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right]$$

a_{ij} = Coefficient

b_i = known constant

x_j = Unknown/Variables

Matrix Form:

$$\underbrace{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{A} \underbrace{\begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix}}_{x} = \underbrace{\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}}_{b}$$

$$\Rightarrow Ax = b$$

Basic Linear Algebra facts:

1. A^T is the transpose matrix of A.

$\therefore (A^T)_{ij} = A_{ji}$ Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$A_{\underbrace{12}} = 2$ $A^T_{\underbrace{21}} = 2$

2. A is a symmetric matrix if $A = A^T$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $\therefore$ A is symmetric

3. A is non-singular (invertible) if for every vector b, the equation $Ax = b$ has a solution.

4. A is non-singular (invertible) if & only if $\det(A) \neq 0$

— if $\det(A) = 0$, the system has no unique solution. (None or infinite solution)

5. A is non singular (invertible) if & only if,

there exists a unique inverse A^{-1} such that, $AA^{-1} = A^{-1}A = \underbrace{I}_{\text{Identity Matrix}}$.

Solving Linear System

$Ax = b$

Two rules for solving:

1. Matrix A must be a square Matrix. $(n \times n)$
2. Matrix A must be non singular (invertible)

$$Ax = b$$

$$x = A^{-1} b$$

But, finding inverse of A is computationally expensive when number of unknown (n) is large.

So, instead of directly solving $Ax = b$, we try to transform the system in the following ways,

→ Diagonal Matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$x_1 = \frac{2}{2} = 1$$

$$x_2 = \frac{3}{3} = 1$$

→ Orthogonal Matrix (Q such that, $Q^{-1} = Q^T$)

→ Triangular Matrix (Upper or Lower)

5.1: Triangular System:

→ Upper Triangular Matrix

→ Lower " "

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\det(u) = u_{11} * u_{22} * u_{33}$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$\det(L) = l_{11} * l_{22} * l_{33}$

$$\det(T) = \prod_{i=1}^{n} t_{ii}$$

Solving Triangular System,

Using a lower triangular Matrix, $(n = 4)$

$$Ax = b$$

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$\Rightarrow l_{11} x_1 = b_1 \qquad \Rightarrow x_1 = \dfrac{b_1}{l_{11}}$

$\Rightarrow l_{21} x_1 + l_{22} x_2 = b_2 \qquad \Rightarrow x_2 = \dfrac{b_2 - l_{21} x_1}{l_{22}}$

$\Rightarrow l_{31} x_1 + l_{32} x_2 + l_{33} x_3 = b_3 \qquad \Rightarrow x_3 = \dfrac{b_3 - l_{31} x_1 - l_{32} x_2}{l_{33}}$

$\Rightarrow l_{41} x_1 + l_{42} x_2 + l_{43} x_3 + l_{44} x_4 = b_4 \qquad \Rightarrow x_4 = \dfrac{b_4 - l_{41} x_1 - l_{42} x_2 - l_{43} x_3}{l_{44}}$

This is called forward substitution.

Similarly for upper triangular matrix, we use backward substitution.

The total number of operation required for forward or backward substitution is n^2. [Operation means $+, -, *, /$]

Prove that the total number of operation required for backward substitution is n^2.

let, $n = 4$,

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$\Rightarrow l_{11} x_1 = b_1$

$\Rightarrow l_{21} x_1 + l_{22} x_2 = b_2$

$\Rightarrow l_{31} x_1 + l_{32} x_2 + l_{33} x_3 = b_3$

$\Rightarrow l_{41} x_1 + l_{42} x_2 + l_{43} x_3 + l_{44} x_4 = b_4$

$\Rightarrow x_1 = \dfrac{b_1}{l_{11}}$ $\qquad x_1 \Rightarrow 1$ division

$\Rightarrow x_2 = \dfrac{b_2 - l_{21} x_1}{l_{22}}$ $\qquad x_2 \Rightarrow 1$ division, 1 multiplication, 1 subtraction

$\Rightarrow x_3 = \dfrac{b_3 - l_{31} x_1 - l_{32} x_2}{l_{33}}$ $\qquad x_3 \Rightarrow 1$ div, 2 mul, 2 sub

$\Rightarrow x_4 = \dfrac{b_4 - l_{41} x_1 - l_{42} x_2 - l_{43} x_3}{l_{44}}$ $\qquad x_4 \Rightarrow 1$ div, 3 mul, 3 sub

$\vdots$

$x_n \Rightarrow 1$ div, $(n-1)$ mul, $(n-1)$ sub

$\therefore$ Total number of operation $= \sum_{j=1}^{n} \left[1 + (j-1) + (j-1)\right]$

$$= \sum_{j=1}^{n} \left[1 + 2j - 2\right]$$

$$= \sum_{j=1}^{n} \left[2j - 1\right]$$

$$= \sum_{j=1}^{n} 2j - \sum_{j=1}^{n} 1 \leftarrow \left[1+1+\dots 1 = n\right]$$

$$= 2\sum_{j=1}^{n} j - n \qquad \left[1+2+3+\dots+n = \frac{n(n+1)}{2}\right]$$

$$= \cancel{2} \frac{n*(n+1)}{\cancel{2}} - n$$

$$= n^2 + n - n$$

$\therefore$ Total number of operation $= n^2$ [Proved]

Same for upper triangular matrix's backward substitution.