CSE330- Numerical Methods
Quiz 04; Fall'24

Name: ______________________ **ID:** ____________ **Section:** _____

Marks: 15 points **Time: 20 minutes**

Instructions: **Answer all questions on the space provided below for each.**

Question 1: CO3 (4+4+2+5 points): A linear system is described by the following equations:

$$x_1 + 2x_3 = 10$$
$$3x_1 = 6$$
$$2x_1 + 5x_2 + 2x_3 = 9$$

Based on these equations, answer the questions below.

(a) From the given linear equations, identify the matrices A, x and b such that the linear system can be expressed as a matrix equation. Find the value of det(A).
(b) Construct the Frobenius matrices $F^{(1)}$ and $F^{(2)}$ from this system.
(c) Compute the unit lower triangular matrix L.
(d) Now find the solution of the linear system using the LU decomposition method. Use the unit lower triangular matrix found in the previous question.

a) $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 0 \\ 2 & 5 & 2 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $b = \begin{bmatrix} 10 \\ 6 \\ 9 \end{bmatrix}$

b) $F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ $A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 0 \\ 2 & 5 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 2+0+0 \\ -3+3+0 & 0+0+0 & -6+0+0 \\ -2+0+2 & 0+0+5 & -4+0+2 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 5 & -2 \end{bmatrix}$$

Here we have pivoting problem. We swap R_2 and R_3

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & -6 \end{bmatrix}$$

$\therefore$ ~~$P^{(1)}$ became~~ $\begin{bmatrix} 1 & 0 & 0 \\ & & \\ & & \end{bmatrix}$

$$P^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $$A^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 0+0+0 & 2+0+0 \\ 0+0+0 & 0+5+0 & 0-2+0 \\ 0+0+0 & 0+0+0 & 0+0-6 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

[Here 2 and 3 switched placed as we changed/swapped rows for pivoting] b $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 6 \end{bmatrix}$$

$a_0 = 10$

$2 \times 10 + a_1 = 9$

$\therefore a_1 = -11$

$3 \times 10 + a_2 = 6$

$a_2 = 6 - 30$

$= -24$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -11 \\ -24 \end{bmatrix}$$

$-6x_3 = -24$

$x_3 = 4$

$5x_2 - 2x_3 = -11$

or, $x_2 = \dfrac{-11 + 2 \times 4}{5}$

$= -3/5$

$x_1 + 2x_3 = 10.$

$x_1 = 10 - 2 \times 4$

$= 2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3/5 \\ 4 \end{bmatrix}$$

Instructions: Answer all questions on the space provided ...

Question 1: CO3 (4+4+2+5 points): A linear system is described by the following equations:

$$
\begin{aligned}
2x_1 + 6x_2 + 2x_3 &= 6 \\
4x_1 + 2x_2 + 3x_3 &= 10 \\
2x_1 + 5x_2 &= 15
\end{aligned}
$$

Based on these equations, answer the questions below.

(a) From the given linear equations, identify the matrices A, x and b such that the linear system can be expressed as a matrix equation. Find the value of det(A).

(b) Construct the Frobenius matrices $F^{(1)}$ and $F^{(2)}$ from this system.

(c) Compute the unit lower triangular matrix L.

(d) Now find the solution of the linear system using the LU decomposition method. Use the unit lower triangular matrix found in the previous question.

a) $\underbrace{\begin{bmatrix} 2 & 6 & 2 \\ 4 & 2 & 3 \\ 2 & 5 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 10 \\ 15 \end{bmatrix}}_{b}$

$\det(A) =$

b) $F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$m_{21} = \dfrac{a_{21}}{a_{11}} = \dfrac{4}{2} = 2$

$m_{31} = \dfrac{2}{2} = 1$

$$A^{(2)} = F^{(1)} \times A^{(1)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 6 & 2 \\ 4 & 2 & 3 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 2+0+0 & 6+0+0 & 2+0+0 \\ -4+4+0 & -12+2+0 & -4+3+0 \\ -2+0+2 & -6+0+5 & -2+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 2 \\ 0 & -10 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$A^{(3)} = F^{(2)} \times A^{(2)}$$

$$= \begin{bmatrix} 2 & 6 & 2 \\ 0 & -10 & -1 \\ 0 & 0 & -\frac{19}{10} \end{bmatrix} = U$$

$0+1+\frac{1}{1}\;0+\frac{1}{10}-2$

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{-1}{-10} = 1/10$$

$$\begin{matrix} 2 & 6 & 2 \\ 0 & -10 & -1 \\ 0 & 0 & -19 \end{matrix}$$

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/10 & 1 \end{bmatrix}$$

c) $$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/10 & 1 \end{bmatrix}$$

d) ~~A = LU~~ $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/10 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 15 \end{bmatrix}$$

$y_1 = 6$

$2y_1 + y_2 = 10$

$\Rightarrow y_2 = -2$

$y_1 + 1/10\, y_2 + y_3 = 15$

$\Rightarrow 6 + 1/10(-2) + y_3 = 15$

$\therefore y_3 = 9.2 = \frac{46}{5}$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 9.2 \end{bmatrix}$$

$Ux = y$

$$\cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/10 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ \frac{46}{5} \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 6 & 2 \\ 0 & -10 & -1 \\ 0 & 0 & -\frac{19}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ \frac{46}{5} \end{bmatrix}$$

$-19/10\, x_3 = 46/5 \Rightarrow x_3 = -4.84$

$-10x_2 - x_3 = -2 \Rightarrow x_2 = 0.684$

$2x_1 + 6x_2 + 2x_3 = 6 \Rightarrow x_1 = 5.78$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5.78 \\ 0.68 \\ -4.84 \end{bmatrix}$$