Newtons Divided Difference Form

x	$f(x)$
x_0	$f(x_0)$
x_1	$f(x_1)$
$\vdots$	
x_n	$f(x_n)$

$$
\begin{aligned}
P_n(x) = \underline{a_0} + \underline{a_1}(x-x_0) &+ \underline{a_2}(x-x_0)(x-x_1) \\
&+ \underline{a_3}(x-x_0)(x-x_1)(x-x_2) \\
&+ \cdots \\
&\cdots \\
&+ \underline{a_n}(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)
\end{aligned}
$$

Here,

$$
\left.
\begin{aligned}
a_0 &= f[x_0] \\
a_1 &= f[x_0, x_1] \\
a_2 &= f[x_1, x_1, x_2] \\
&\vdots \\
a_n &= f[x_0, x_1 \cdots x_n]
\end{aligned}
\right]
$$

$$
\begin{aligned}
P_n(x) = \underline{f[x_0]} + \underline{f[x_0, x_1]}(x-x_0) &+ \underline{f[x_0, x_1, x_2]}(x-x_0)(x-x_1) \\
&+ \underline{f[x_0, \cdots x_3]}(x-x_0)(x-x_1)(x-x_2) \\
&+ \cdots \\
&+ \underline{f[x_0 \cdots x_n]}(x-x_0)(x-x_1)\cdots(x-x_{n-1})
\end{aligned}
$$

Example:

node $= 4$

degree $= 4-1$, $n = 3$

node $= 5$

degree $= 5-1 = 4$

	x	$f(x)$	
x_0	-1	5	$f(x_0)$
x_1	0	1	$f(x_1)$
x_2	1	3	$f(x_2)$
x_3	2	11	$f(x_3)$
x_4	4	20	$f(x_4)$

$$\underset{P_4(x)}{P_3(x)} = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$
$$+ f[x_0, x_1 \cdots x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

$$\frac{y_2 - y_1}{x_2 - x_1}$$

$x_0 = -1$ $\quad f[x_0] = 5$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{1-5}{0-(-1)} = -4$$

$x_1 = 0$ $\quad f[x_1] = 1$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{3-1}{1-0} = 2$$

$x_2 = 1$ $\quad f[x_2] = 3$

$$f[x_2, x_3] = \frac{11-3}{2-1} = 8$$

$x_3 = 2$ $\quad f[x_3] = 11$

$$f[x_3, x_4] = \frac{20-11}{4-2} = \frac{9}{2}$$

$x_4 = 4$ $\quad f[x_4] = 20$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = 3$$
$$= \frac{2-(-4)}{1-(-1)} =$$

$$f[x_1, x_2, x_3] = \frac{8-2}{2-0} = 3$$

$$f[x_2, x_3, x_4] = \frac{9/2 - 8}{4-1} = \frac{7}{6}$$

$$f[x_0, x_1 \cdots x_3] = \frac{3-3}{2-(-1)} = 0$$

$$f[x_1, x_2, x_3, x_4] = \frac{7/6 - 3}{4-0} = -\frac{25}{24}$$

$$f[x_0, x_1, x_2 \cdots x_4] = \frac{-25/24 - 0}{4-(-1)} = -\frac{5}{24}$$

$$P_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$P_3(x) = 5 - 4(x+1) + 3(x+1)(x) + 0 \cdot (x+1)(x)(x-1)$$

Updated Polynomial =

$$P_4(x) = 5 - 4(x+1) + 3(x+1)(x) + 0 \cdot (x+1)(x)(x-1)$$
$$- \frac{5}{24}(x+1)(x)(x-1)(x-2)$$