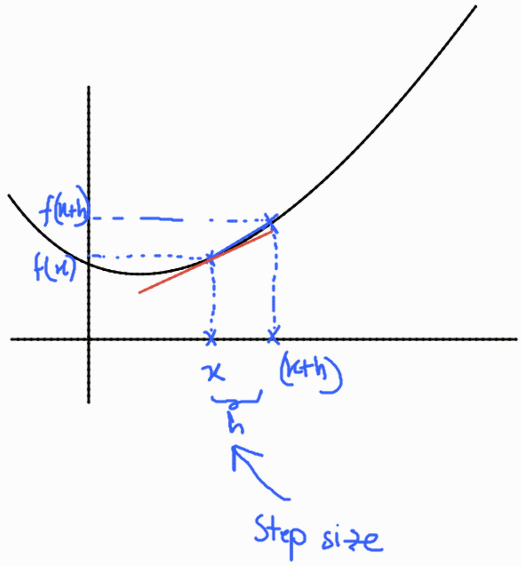
Numerical Differentiation:

Forward Difference:



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{x+h - x}$$

$$\text{FD, } f'(x) = \frac{f(x+h) - f(x)}{h}$$

Example: given $f(x) = x^2 + 10x$. Find $f'(x)$ using FD at $x = 2$, $h = 0.1$.

Actual D., $f'(x) = 2x + 10$

$$f'(2) = 2*2 + 10 = 14$$

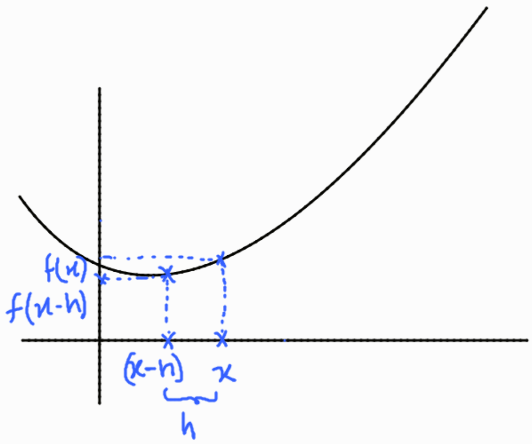
Forward Difference, $f'(x) = \dfrac{f(x+h) - f(x)}{h}$

$$= \frac{f(2+0.1) - f(2)}{0.1}$$

$$= \frac{(2.1)^2 + 10*2.1 - (2^2 + 10*2)}{0.1}$$

FD, $f'(2) = 14.1$

Backward Difference:



$$f'(x) = \frac{f(x) - f(x-h)}{x - (x-h)}$$

$$= \frac{f(x) - f(x-h)}{x - x + h}$$

$$BD, \ f'(x) = \frac{f(x) - f(x-h)}{h}$$

Example: given $f(x) = x^2 + 10x$. Find $f'(x)$ using BD at $x = 2$, $h = 0 \cdot 1$.

Actual $f'(2) = 14$

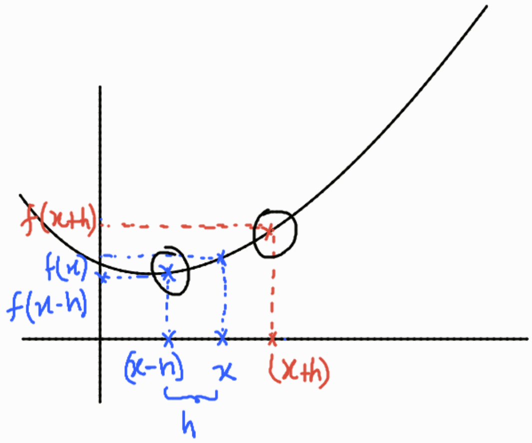
$$BD, \ f'(2) = \frac{f(2) - f(2 - 0.1)}{0.1}$$

$$= \frac{2^2 + 10 * 2 - (1.9^2 + 10 * 1.9)}{0.1}$$

$$= 13.9$$

Central Difference:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



$$CD,\ f'(x) = \frac{f(x+h) - f(x-h)}{x+h-(x-h)}$$

$$CD,\ f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Example: given $f(x) = x^2 + 10x$. Find $f'(x)$ using CD at $x = 2$, $h = 0.1$.

Actual $f'(2) = 14$

$$CD,\ F'(x) = \frac{f(2+0.1) - f(2-0.1)}{2 * 0.1}$$

$$= \frac{(2.1)^2 + 10 * 2.1 - (1.9^2 + 10 * 1.9)}{0.2}$$

$$= 14$$

$$\text{FD}, \quad f'(x) = \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{Approximate value}} - \underbrace{\frac{f^2(\xi)\,h}{2!}}_{\text{Truncation Error}}$$

$$\text{BD}, \quad f'(x) = \frac{f(x)-f(x-h)}{h} - \frac{f^2(\xi)\,h}{2!}$$

$$\text{CD}, \quad f'(x) = \frac{f(x+h)-f(x-h)}{2h} - \frac{f^3(\xi)\,h^2}{3!}$$

So, we get,

forward & backward

Error $\propto h$

Central, Error $\propto h^2$

Example: Given $f(x) = \ln(x)$, $x = 2$, $h = [1, 0.1, 0.01]$

find $f'(2)$ and truncation error using forward difference.

$$f(x) = \ln(x) \qquad f'(x) = \frac{1}{x} \Rightarrow f'(2) = \frac{1}{2} = 0.5$$

$$\text{FD, } f'(2) = \frac{f(2+h) - f(2)}{h} = \frac{\ln(2+h) - \ln(2)}{h}$$

h	$f'(2)$ [FD]	Truncation Error
1	$\frac{\ln(2+1) - \ln(2)}{1} = 0.405465$	$0.5 - 0.405465 = 0.09453$
0.1	$\frac{\ln(2+0.1) - \ln(2)}{0.1} = 0.487902$	$0.5 - 0.487902 = 0.012098$
0.01	$\frac{\ln(2+0.01) - \ln(2)}{0.01} = 0.498754$	$0.5 - 0.498754 = 0.001246$
0.001	0.499875	$= 0.00012\ldots$

Here, h is decreasing and also error is decreasing.

Ex: $f(x) = 2x^2 - e^x$. find $f'(2)$ using CD and find truncation error. $[h = 0.1, 0.01, 0.001]$

$\boxed{\text{DIY}}$

Example:

x_0	4	4.1	4.2	4.3	4.4
$f(x_0)$	16	18	20	21	22

0.1 0.1

* Using forward & backward, calculate $f'(4.2)$.

$\therefore h = 0.1$

Forward, $$f'(4.2) = \frac{f(4.2+0.1) - f(4.2)}{0.1} = \frac{21-20}{0.1} = 10$$

Backward, $$f'(4.2) = \frac{f(4.2) - f(4.2-0.1)}{0.1} = \frac{20-18}{0.1} = 20$$

Example: $$v(t) = 2000 \ln\left(\frac{14 \times 10^4}{14\times 10^4 - 2100t}\right) - 9.8t$$

Calculate the value of $v'(16)$ using FD, BD and CD, given $\boxed{\Delta t = 2s}$ → step size, h

FD, $$v'(16) = \frac{v(16+2) - v(16)}{2} = \frac{453.0214897 - 392.073}{2}$$

$$\boxed{= 30.474}$$

BD, $$v'(16) = \frac{v(16) - v(16-2)}{2} = \frac{392.073 - 334.244}{2}$$

$$\boxed{= 28.9145}$$

CD, $$v'(16) = \frac{v(16+2) - v(16-2)}{2 * 2} = \frac{453.0214897 - 334.244}{4}$$

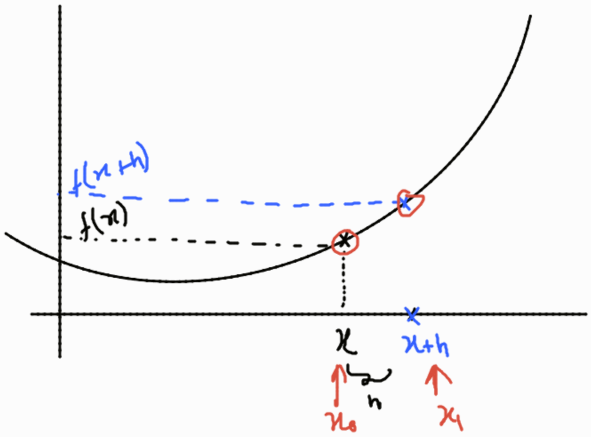
$$\boxed{= 29.69437}$$

Proof of formulas:

$$\text{FD}, \quad f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{f^2(\xi)\, h}{2!}$$

2node, degree = 1

$x_0 \rightarrow$	x	$f(x)$	$\leftarrow$	$f(x_0)$
$x_1 \rightarrow$	$x+h$	$f(x+h)$	$\leftarrow$	$f(x_1)$



$P_1(x) =$

$$P_1(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x)$$

from Cauchy theorems,

$$\left| f(x) - P_n(x) \right| = \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n)$$

$$f(x) - P_1(x) = \frac{f^2(\xi)}{2!}(x-x_0)(x-x_1)$$

$$\Rightarrow f(x) = P_1(x) + \frac{f^2(\xi)}{2!}(x-x_0)(x-x_1)$$

$$= f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + \frac{f^2(\xi)}{2!}(x-x_0)(x-x_1)$$

$$(x-x_0)(x-x_1) = x^2 - x x_0 - x x_1 + x_0 x_1$$

$$\Rightarrow 2x - x_0 - x_1$$

$$f(x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0} + \underbrace{\frac{f^2(\xi)}{2!}}_{u}\underbrace{(x-x_0)(x-x_1)}_{v}$$

$$\frac{d}{dx} uv = u'v + uv'$$

Now,

$$f'(x) = \frac{f(x_0)}{x_0-x_1}\cdot 1 + \frac{f(x_1)}{x_1-x_0}\cdot 1 + \frac{f^3(\xi)}{2!}\frac{d}{dx}(\xi)(x-x_0)(x-x_1) + \frac{f^2(\xi)}{2!}(2x-x_0-x_1)$$

Now, $x_0 = x$, $\quad x_1 = x + h$

$$f'(x) = \frac{f(x)}{x-x-h} + \frac{f(x+h)}{x+h-x} + \frac{f^3(\xi)}{2!}\frac{d}{dx}(\xi)\underbrace{(x-x)}_{0}(x-x-h) + \frac{f^2(\xi)}{2!}(2x-x-x-h)$$

$$= -\frac{f(x)}{h} + \frac{f(x+h)}{h} + \frac{f^2(\xi)}{2!}(-h)$$

$$\boxed{\underset{\text{Actual derivative}}{f'(x)} = \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{FD Approximation}} - \underbrace{\frac{f^2(\xi)}{2!}.h}_{\text{Truncation Error}}}$$

Proved.

For

BD, $$f'(x) = \frac{f(x)-f(x-h)}{h} - \frac{f^2(\xi)\,h}{2!}$$

$x_0 \rightarrow$	$x-h$	$f(x-h)$
$x_1 \rightarrow$	x	$f(x)$

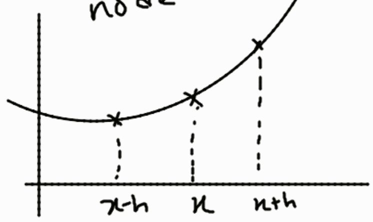
DIY.

For CD,

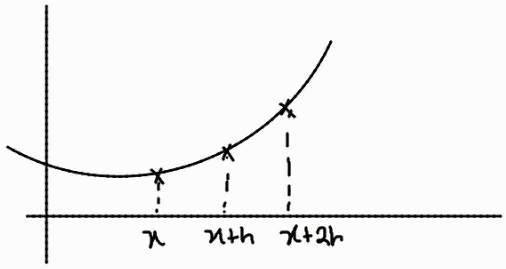
CD, $$f'(x) = \frac{f(x+h)-f(x-h)}{2h} - \frac{f^3(\xi)\,h^2}{3!}$$

$\cancel{x_0 = x-h}$
$\cancel{x_1 = x}$
$\cancel{x_2 = x+h}$

node = 3



for proof,



node 3,
degree 2

	x	$f(x)$	
$x_0 \rightarrow$	x	$f(x)$	$f(x_0)$
$x_1 \rightarrow$	$x+h$	$f(x+h)$	$f(x_1)$
$x_2 \rightarrow$	$x+2h$	$f(x+2h)$	$f(x_2)$

$$f(x) - P_2(x) = \frac{f^3(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$f(x) = P_2(x) + \frac{f^3(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$= l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) + \frac{f^3(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$= \vdots$$