Gaussian Elimination

1. Transform A into Triangular Matrix (Upper / Lower)
2. Using Backward / Forward substitution to solve

Example:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

Row operations

$R_{\square} = R_{\square} - \left(\frac{\square}{\square}\right) R_{\square}$ (Multiplier)

Augmented Matrix $=$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ \boxed{1} & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right] \quad \begin{aligned} R_2 &= R_2 - \left(\tfrac{1}{1}\right) R_1 \\ R_2 &= R_2 - R_1 \end{aligned}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ \boxed{2} & 12 & -2 & 4 \end{array}\right] \quad \begin{aligned} R_3 &= R_3 - \left(\tfrac{2}{1}\right) R_1 \\ R_3 &= R_3 - 2R_1 \end{aligned}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & \boxed{8} & -4 & 4 \end{array}\right] \quad \begin{aligned} R_3 &= R_3 - \frac{8}{-4} R_2 \\ R_3 &= R_3 + 2R_2 \end{aligned}$$

$$= \left[\begin{array}{ccc|c} \overset{x_1}{\downarrow} & \overset{x_2}{\downarrow} & \overset{x_3}{\downarrow} & \\ 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right]$$

$$-2x_3 = 12 \Rightarrow x_3 = \frac{12}{-2} \Rightarrow x_3 = -6$$

$$-4x_2 + x_3 = 4 \Rightarrow x_2 = \frac{4 - (-6)}{-4} = -2.5$$

$$x_1 + 2x_2 + x_3 = 0$$

$$\Rightarrow x_1 = -2x_2 - x_3$$

$$= -2(-2.5) - (-6)$$

$$= 5 + 6 = 11$$

$\therefore$ $x_1 = 11$

$x_2 = -2.5$

$x_3 = -6$

Example: The upward velocity of a rocket given at three different time in the following table:

Time (s)	5	8	12
Velocity (ms^{-1})	106.8	177.2	279.2

* The velocity data is approximated by a polynomial as

$$v(t) = b_1 t^2 + b_2 t + b_3 \qquad \text{for} \quad 5 \le t \le 12$$

a) Find the values of b_1, b_2 & b_3 using the gaussian elimination method.

b) Find the velocity at $t = 7$ seconds.

Time (s)	5	8	12
Velocity (ms^{-1})	106.8	177.2	279.2

(a)

$$v(t) = b_1 t^2 + b_2 t + b_3$$

$$v(5) = b_1 (5^2) + b_2 (5) + b_3 = 106.8$$

$$\Rightarrow 25 b_1 + 5 b_2 + b_3 = 106.8 \quad \ldots\ldots (I)$$

$$v(8) = b_1 (8^2) + b_2 (8) + b_3 = 177.2$$

$$\Rightarrow 64 b_1 + 8 b_2 + b_3 = 177.2 \quad \ldots\ldots (II)$$

$$v(12) = b_1 (12)^2 + b_2 (12) + b_3 = 279.2$$

$$\Rightarrow 144 b_1 + 12 b_2 + b_3 = 279.2 \quad \ldots\ldots (III)$$

$\therefore$ Augmented Matrix,

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 1 & 177.2 \\ 144 & 12 & 1 & 279.2 \end{array}\right]$$

$$R_2 = R_2 - \frac{64}{25} R_1$$

$$R_3 = R_3 - \frac{144}{25} R_1$$

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 0 & -4.8 & -1.56 & -96.208 \\ 0 & -16.8 & -4.76 & -335.968 \end{array}\right] \quad R_3 = R_3 - \left(\frac{-16.8}{-4.8}\right) R_2$$

$b_1 \quad b_2 \quad b_3$

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 0 & -4.8 & -1.56 & -96.208 \\ 0 & 0 & 0.7 & 0.76 \end{array}\right]$$

$$0.7 b_3 = 0.76 \quad \Rightarrow b_3 = \frac{0.76}{0.7} = 1.085$$

$$-4.8 b_2 - 1.56 b_3 = -96.208 \Rightarrow b_2 = \frac{-96.208 + 1.56(1.085)}{-4.8}$$

$$= 19.69$$

$$25 b_1 + 5 b_2 + b_3 = 106.8 \Rightarrow b_1 = \frac{106.8 - 5(19.69) - (1.085)}{25}$$

$$= 0.2597$$

$$v(t) = b_1 t^2 + b_2 t + b_3$$

$$v(t) = 0.2597\, t^2 + 19.69\, t + 1.085$$

(b)

$$v(7) = 0.2597(7^2) + 19.69(7) + 1.085$$

$$= 151.643 \ ms^{-1}$$

(Ans)

Pivoting

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 64 & 0 & 1 & 177.2 \\ 144 & 12 & 0 & 279.2 \end{array}\right]$$

Row wise pivoting

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 144 & 12 & 0 & 279.2 \\ 64 & 0 & 1 & 177 \end{array}\right]$$

$b_1 \quad b_2 \quad b_3$

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 64 & 0 & 1 & 177.2 \\ 144 & 12 & 0 & 279.2 \end{array}\right]$$

Column wise pivoting

$b_1 \quad b_3 \quad b_2$

$$\left[\begin{array}{ccc|c} 25 & 1 & 5 & 106.8 \\ 64 & 1 & 0 & 177.2 \\ 144 & 0 & 12 & 279.2 \end{array}\right]$$