FD, $f'(x) = \dfrac{f(x+h) - f(x)}{h} - \dfrac{f^2(\xi)}{2} h$

BD, $f'(x) = \dfrac{f(x) - f(x-h)}{h} - \dfrac{f^2(\xi)}{2} h$

$\Big\}$ error $\downarrow$ $h \downarrow$, error $\propto h$

CD, $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} - \dfrac{f^3(\xi)}{3!} h^2$ $\Big\}$ error $\propto h^2$

Considering, CD,

Error = Truncation Error + Rounding Error

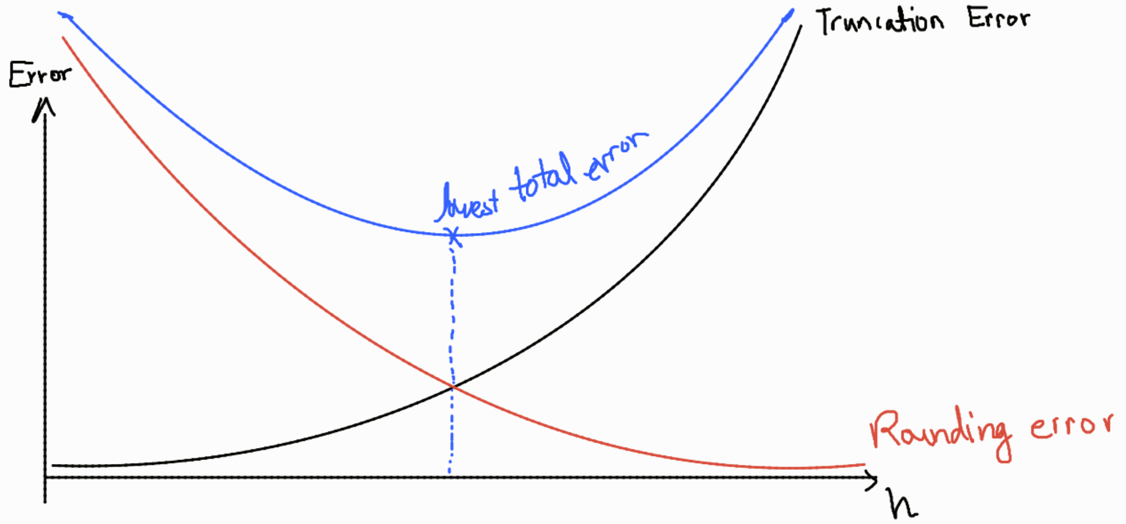
$$= \frac{f^3(\xi)}{3!} \cdot h^2 + \epsilon_m \cdot \frac{f(x+h) - f(x-h)}{2h}$$

$h \downarrow$ error $\downarrow$

$h \uparrow$ error $\uparrow$

$h \downarrow$ error $\uparrow$

$h \uparrow$ error $\downarrow$



Truncation Error:

Central: $\dfrac{f^3(\xi)}{3!} \cdot h^2$ $\Rightarrow$ Error is order of $h^2 = O(h^2)$

Forward, Backward: $\dfrac{f^2(\xi)}{2} h$ $\Rightarrow$ Error is order of $h = O(h)$

Richardson Extrapolation:

- Defines error in a better way
- * Reduces the order of error.

Expand using taylor series.
[5th order approx.] (h^5)

$$CD_1 \longrightarrow D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$f(x+h) = f(x) + f'(x).h + \frac{f''(x)}{2!}h^2 + \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)}{4!}h^4 + \frac{f^5(x)}{5!}h^5 + O(h^6)$$

$$f(x-h) = \cancel{f(x)} - f'(x).h + \cancel{\frac{f''(x)}{2!}h^2} - \frac{f^3(x)}{3!}h^3 + \cancel{\frac{f^4(x)}{4!}h^4} - \frac{f^5(x)}{5!}h^5 + O(h^6)$$

$$f(x+h) - f(x-h) = 2f'(x).h + 2.\frac{f^3(x)}{3!}h^3 + 2\frac{f^5(x)}{5!}h^5 + O(h^7)$$

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{1}{2h}\left[2f'(x).h + 2.\frac{f^3(x)}{3!}h^3 + 2\frac{f^5(x)}{5!}h^5 + O(h^7)\right]$$

$$CD \rightarrow D_h = \underbrace{f'(x)}_{\text{Actual derivation}} + \underbrace{\frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}h^4 + O(h^6)}_{\text{Truncation Error}}$$

term order $O(h^2)$

term order $O(h^4)$

$h < 1$

$\hookrightarrow 0.0001 \ldots$

We will try to remove $(h^2, h^4, \ldots)$ terms one by one.

Let's start with (h^2).

$$D_h = f'(x) + \frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}h^4 + O(h^6) \quad \text{—— (1)}$$

$$D_{\frac{h}{2}} = f'(x) + \frac{f^3(x)}{3!}\cdot\frac{h^2}{4} + \frac{f^5(x)}{5!}\,\frac{h^4}{16} + O(h^6) \quad \text{—— (II)}$$

$4*(II) - (I) \Rightarrow$

$$4D_{\frac{h}{2}} - D_h = 3f'(x) + 0 + \left(\frac{1}{4}-1\right)\frac{f^5(x)}{5!}h^4 + O(h^6)$$

$$\frac{4D_{\frac{h}{2}} - D_h}{3} = f'(x) + \left(-\frac{3}{4}\right)\cdot\frac{1}{3}\,\frac{f^5(x)}{5!}h^4 + O(h^6)$$

1 term is removed.

$$\frac{4D_{\frac{h}{2}} - D_h}{3} = f'(x) - \frac{1}{4}\,\frac{f^5(x)}{5!}h^4 + O(h^6) = D_h^{(1)}$$

$$\therefore\quad D_h^{(1)} = \frac{4D_{\frac{h}{2}} - D_h}{3}$$

What about removing h^4 term / $D_h^{(2)}$.

$$D_h^{(1)} = f'(x) - \frac{1}{4}\,\frac{f^5(x)}{5!}h^4 + O(h^6) \quad \text{—— (I)}$$

$$D_{\frac{h}{2}}^{(1)} = f'(x) - \frac{1}{4}\,\frac{f^5(x)}{5!}\,\frac{h^4}{16} + O(h^6) \quad \text{—— (II)}$$

$16*(II) - (I) \Rightarrow$

$$16D_{\frac{h}{2}}^{(1)} - D_h^{(1)} = 15f'(x) + 0 + O(h^6)$$

$$\frac{16D_{\frac{h}{2}}^{(1)} - D_h^{(1)}}{15} = f'(x) + O(h^6) = D_h^{(2)}$$

$$\therefore\quad D_h^{(2)} = \frac{16D_{\frac{h}{2}}^{(1)} - D_h^{(1)}}{15}$$