LU Decomposition

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}}_{b}$$

LU Decomposition idea:

$$A = LU$$

L = Lower Triangular Matrix (with 1's on diagonal)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{22} & l_{23} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

U = Upper Triangular Matrix

To solve,

$$L \cdot \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix}}_{\text{temporary value}}^{\boxed{a}} = \overbrace{b}^{\text{known constants}} \quad \Rightarrow La = b$$

$$U \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{Solution}} = \overset{a}{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}} \quad \Rightarrow Ux = a$$

<u>Example:</u>

number of unknown,
$n = 3$

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

<u>Step 1:</u> Find coefficient Matrix

$$A^{(1)} = \begin{bmatrix} 1 & 2 & 1 \\ \underline{1} & -2 & 2 \\ \underline{2} & 12 & -2 \end{bmatrix}$$

multiplier

$$R_2 = R_2 - \boxed{\frac{1}{1}} R_1$$
$$R_3 = R_3 - \boxed{\frac{2}{1}} R_1$$

<u>Step 2:</u> Find Frobneus Matrix $F^{(1)}$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

diagonally the values will be 1.

Negative value of multiplier

In this step, we did not get any multipliar for this position.

<u>Step 3:</u> Find $A^{(2)} = \underbrace{F^{(1)} \times A^{(1)}}$

This sequence is fixed.

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & \underline{8} & -4 \end{bmatrix}$$

$$R_3 = R_3 - \left(\frac{8}{-4}\right) R_2$$

Multiplier $= -2$

<u>Step 4:</u> Find Frobenius Matrix, $F^{(2)}$

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{2} & 1 \end{bmatrix}$$

diagonally 1.

In this step, we did not do any row operation. So, No multiplier

From step 3, we got (-2) as multiplier, So, the negative value of multiplier (-2) is written.

step 5: Find $A^{(3)} = \underbrace{F^{(2)} \times A^{(2)}}$ This sequence will not change.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix}$$

$$U = A^{(3)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

When we reach $A^{(n)}$ where n is number of unknown, We can say, $U = A^{(n)}$

Step 6: Find L,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

Actual values of multipliers

$$\boxed{A = LU}$$

Step 7: Compute

$L\underbrace{a}_{\text{Temporary values}} = b$ → y values / Known constants / f(x)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$$\boxed{a_1 = 0}$$

$$a_1 + a_2 = 4 \quad \therefore \quad \boxed{a_2 = 4}$$

$$2a_1 - 2a_2 + a_3 = 4 \quad \therefore \quad a_3 = 4 + 2(4)$$

$$\boxed{a_3 = 12}$$

temporary values, $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Step 8: Compute $Ux = a$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$$-2x_3 = 12, \quad x_3 = \frac{12}{-2} = -6$$

$$-4x_2 + x_3 = 4, \quad x_2 = -2.5$$

$$x_1 + 2x_2 + x_3 = 0, \quad x_1 = 11$$

Comparision

$$A x = \underbrace{b}_{\text{for changes in } b}$$

Gaussian Elimination we need to recalculate everything from the beginning.

Gaussian Elimination

- Converting to upper triangular form $\rightarrow O(n^3)$
- Back substitution $\rightarrow O(n^2)$

Overall, $O(n^3)$ each time

$$A x = \underbrace{b}_{\text{for changes in } b}$$

$A = LU$ needs to calculate only once. The forward and backward substitution is only needed for changes in b.

LU Decomposition

- → $A = LU$ takes $O(n^3)$
- → Forward & Backward Substitution $\} \; O(2n^2) = O(n^2)$

$O(n^3)$ for only first time.
For changes in b, it takes $O(n^2)$ each time.

Advantage: Less calculation required for changes in b or for calculating with multiple b.

Example:

$$2x + 3y + z = 9$$
$$4x + 7y + 5z = 23$$
$$-2x + 4y + 5z = 9$$

Find x, y, z using LU decomposition.

Step 1:

$$A^{(1)} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ -2 & 4 & 5 \end{bmatrix}$$

$R_2 = R_2 - \left(\frac{4}{2}\right) R_1$ multiplier $= 2$

$R_3 = R_3 - \left(\frac{-2}{2}\right) R_1$ " $= -1$

Step 2:

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 3: $A^{(2)} = F^{(1)} \times A^{(1)}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ -2 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 7 & 6 \end{bmatrix}$$

$R_3 = R_3 - \frac{7}{1} R_2$ multiplier $= 7$

Step 4: $F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix}$

Step 5: $A^{(3)} = F^{(2)} \times A^{(2)}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 7 & 6 \end{bmatrix}$$

$$U = A^{(3)} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -15 \end{bmatrix}$$

Step 6:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 7 & 1 \end{bmatrix}$$

Step 7

$$L\,a = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 7 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \\ 9 \end{bmatrix}$$

$a_1 = 9$

$2 \cdot a_1 + a_2 = 23$, $\quad a_2 = 23 - 2(9) = 5$

$-a_1 + 7a_2 + a_3 = 9$, $\quad a_3 = 9 + 9 - 7(5) = -17$

Step 8:

$$Ux = a$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -17 \end{bmatrix}$$

$$-15z = -17 \quad , \quad z = \frac{17}{15}$$

$$y + 3z = 5 \quad , \quad y = 5 - 3 \cdot \left(\frac{17}{15}\right) = \frac{8}{5}$$

$$2x + 3y + z = 9 \quad , \quad x = \frac{9 - 3\left(\frac{8}{5}\right) - \left(\frac{17}{15}\right)}{2} = \frac{23}{15}$$

(Ans)