Discrete Fourier Transformation (DFT)

Discrete Fourier Transform (DFT)

The **Discrete Fourier Transform (DFT)** is a mathematical tool used in image processing to analyze the frequency components of discrete signals or images.

It transforms an image from the **spatial domain** (pixel intensity) to the **frequency domain**, representing the image in terms of its frequency components.

Applications of DFT in Image Processing

Filtering: Removing noise or enhancing certain features.

Compression: Reducing image size by discarding less important frequency components.

Edge Detection: Identifying edges by analyzing high-frequency components.

Pattern Recognition: Extracting features for image classification.

FREQUENCY IN 2D IMAGES

A 1D signal (e.g., a sine wave) varies in one direction (time or space). A 2D signal (e.g., an image) varies in two directions (rows and columns of pixels in the spatial domain). In 1D, we deal with variations over time/space, and in 2D, these variations occur along horizontal (x) and vertical (y) axes (both spatial in the case of images).

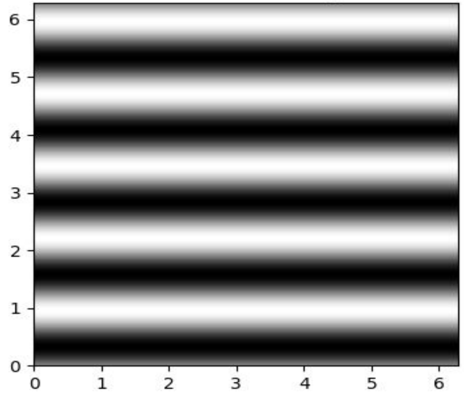
In 1D, frequency measures how rapidly the signal oscillates over time/space. In 2D, frequency now measures variations in both horizontal and vertical directions:

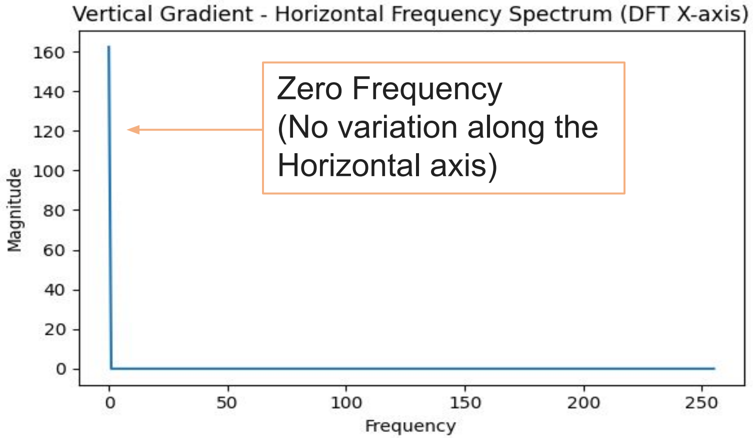
- **Horizontal frequency:** How rapidly pixel intensity changes horizontally (e.g., vertical stripes in an image).
- **Vertical frequency:** How rapidly pixel intensity changes vertically (e.g., horizontal stripes in an image).

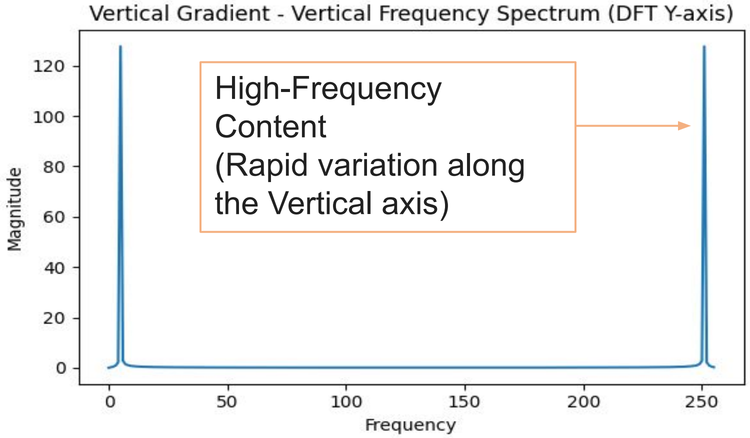
The horizontal and vertical components together can capture variations along any direction in a 2D plane.

FREQUENCY IN 2D IMAGES

Let us take a look at the image here and the DFTs along the horizontal (X) and Vertical (Y) axis below. As we can see, there is no frequency component (apart from the DC component) in the horizontal frequency spectrum (since there's no change in intensity along the X-axis). But there's a high-frequency component in the vertical frequency spectrum as intensity changes periodically along the Y-axis.

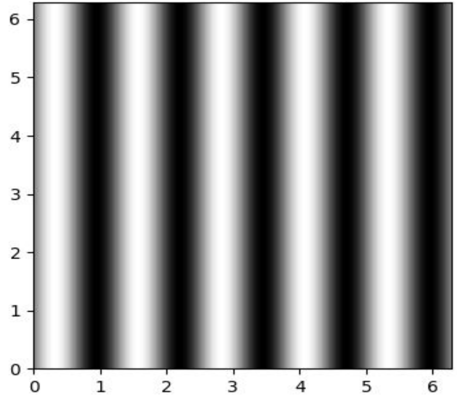


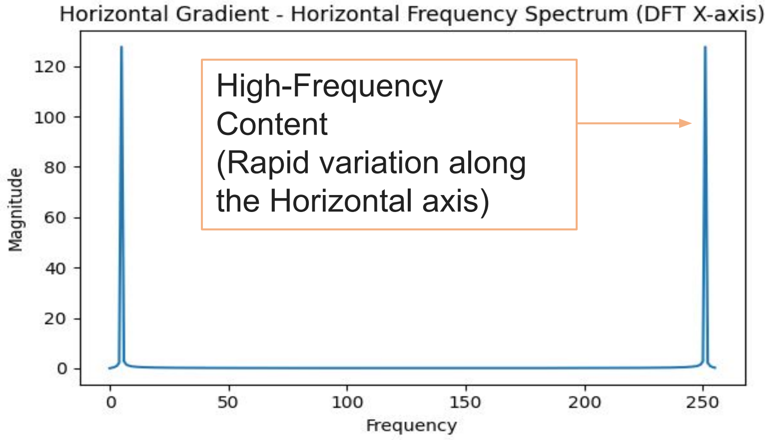


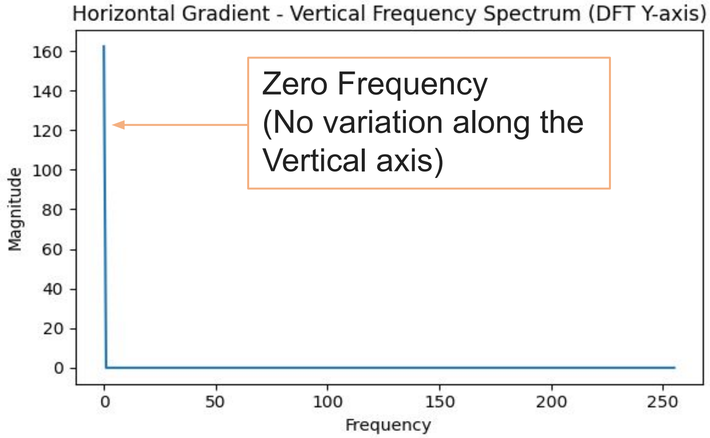


FREQUENCY IN 2D IMAGES

Now take a look at the image here and the DFTs along the horizontal (X) and Vertical (Y) axis below. As we can see, there is no frequency component (apart from the DC component) in the vertical frequency spectrum (since there's no change in intensity along the Y-axis). But there's a high-frequency component in the horizontal frequency spectrum as intensity changes periodically along the X-axis.

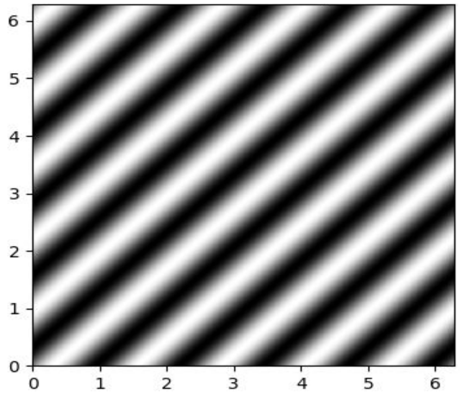


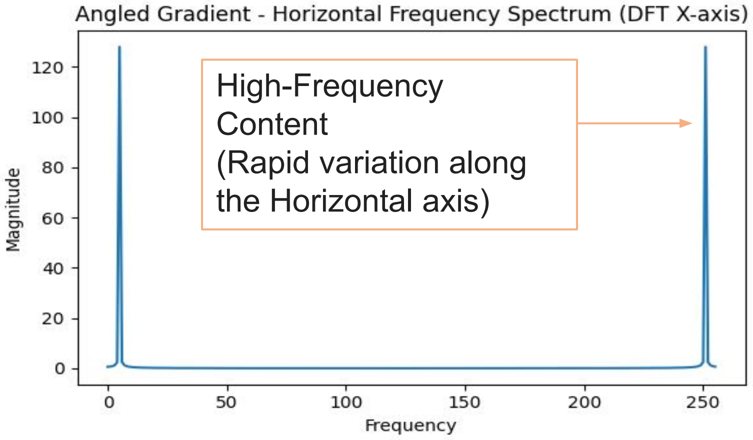


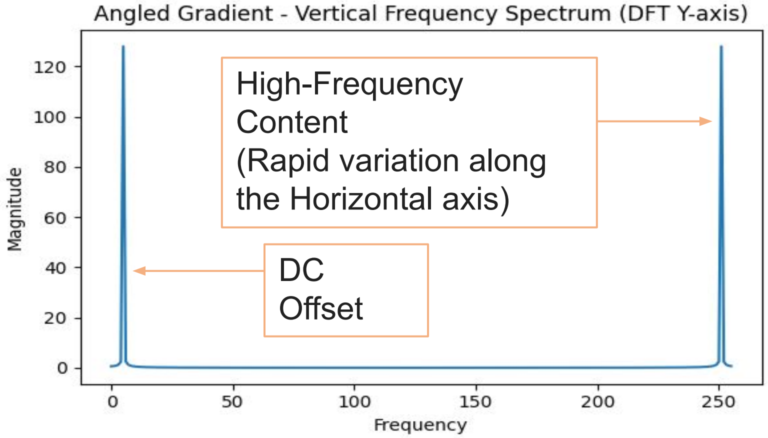


FREQUENCY IN 2D IMAGES

Now take a look at the image here and the DFTs along the horizontal (X) and Vertical (Y) axis below. As we can see, there's a high-frequency component in both cases, the horizontal frequency spectrum, and the vertical frequency spectrum, as intensity changes periodically along both the X-axis and the Y-axis.



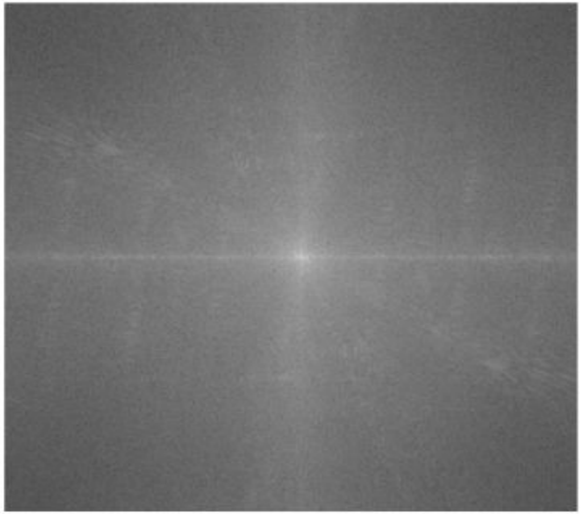




2D DFT: REAL LIFE IMAGES

The DFT of this image below once again shows that the image contains both low & high frequency components. This is because the nighttime cityscapes with sharp lights from windows and neon signs shows strong high frequencies. There are many bright regions throughout the frequency spectrum, even far from the center, which implies the existence of high-frequency components in the image.

Original Image



DFT

For a 2D image $f(x, y)$ of size $M \times N$, the DFT is defined as:

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

Where:

- $F(u, v)$ is the frequency representation of the image.
- $f(x, y)$ is the spatial intensity at position (x, y).
- u, v: Frequency indices.
- j: Imaginary unit.

The complex exponential that represents sinusoidal basis functions.

The **inverse DFT (IDFT)** converts the frequency domain back to the spatial domain:

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

Implementation of DFT in Image Processing

a. Transforming an Image

Convert the image to grayscale if it is in color.

Apply the DFT to obtain the frequency spectrum.

Visualize the magnitude spectrum

Filtering in the Frequency Domain

1. **Low-Pass Filtering**:
    - Removes high-frequency components (details/noise).
    - Retains smooth regions.

High-Pass Filtering:

- Retains edges and fine details.
- Removes smooth regions.

Low-pass filtering (preserving low frequencies) and **high-pass filtering** (preserving high frequencies) in the frequency domain using DFT.

Steps:

- Apply a filter (mask) in the frequency domain.
- Perform inverse DFT to return to the spatial domain.

DC Component

The DFT transforms an image into a frequency domain where each point represents a specific frequency of variation in the image.

The **DC component** corresponds to the frequency indices (u=0,v=0) in the DFT.

In practical terms:

- F(0,0) contains the sum of all pixel values in the image.
- When normalized (e.g., divided by the total number of pixel, it represents the average intensity of the image.

Example Problems

Problem 1: Given a 2x2 image matrix:

$$f(x,y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Compute the 2D DFT *F(u,v)*

Problem 2: An image has the following 2x2 DFT in the frequency domain:

$$F(u,v) = \begin{bmatrix} 50 & 10 \\ 5 & 2 \end{bmatrix}$$

Apply a low-pass filter that retains only *F(0,0)* and compute the reconstructed image.

Problem 3: Given the following 2x2 image:

$$F(u,v) = \begin{bmatrix} 10 & -2 \\ -2 & 0 \end{bmatrix}.$$

1. Apply a **low-pass filter** that retains only the DC component F(0,0)
2. Reconstruct the image using the inverse DFT.

The low-pass filter smooths the image, removing all high-frequency variations, leaving a uniform intensity of 10 across the image.

Problem 4: Given the following 2x2 image:

$$F(u,v) = \begin{bmatrix} 10 & -2 \\ -2 & 0 \end{bmatrix}.$$

1. Apply a **high-pass filter** that retains only the DC component F(0,0)
2. Reconstruct the image using the inverse DFT.

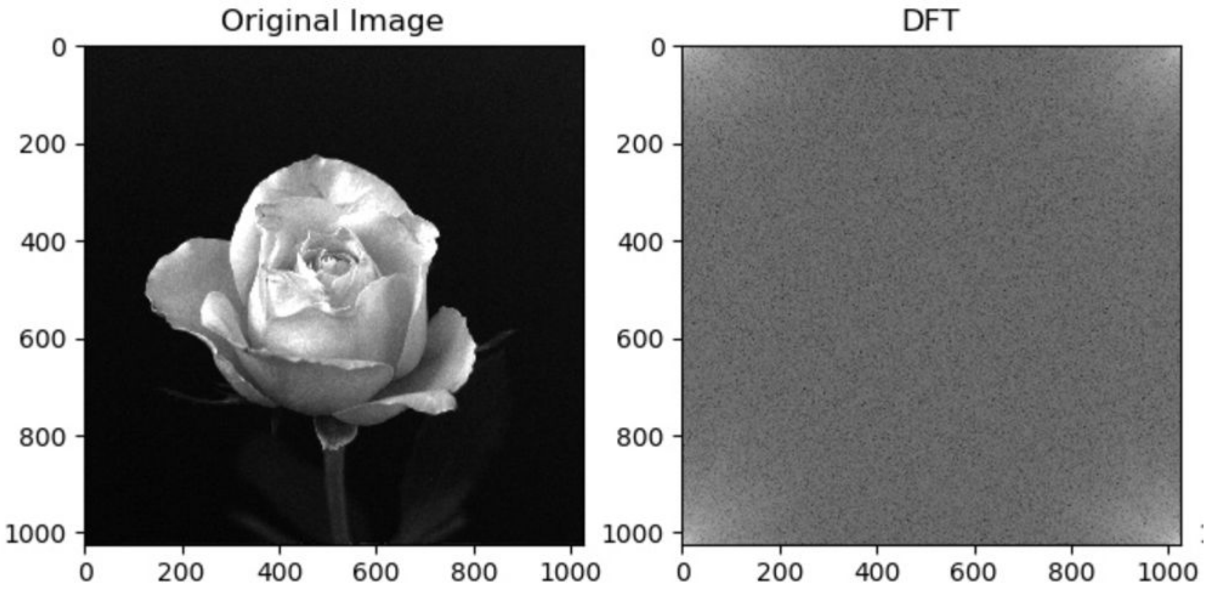
The high-pass filter removes the average intensity (DC component) from the image, enhancing high-frequency details such as edges or sharp changes.

Original Image:

- This is the input grayscale image of a rose. The intensities in this image represent pixel values in the **spatial domain**.

DFT (Magnitude Spectrum):

- Result of applying the **Discrete Fourier Transform (DFT)** to the original image.
- This transforms the image from the spatial domain into the **frequency domain**, representing the image in terms of sinusoidal frequencies.
- Features of the DFT image:
    - Bright regions indicate strong frequency components.
    - The center represents the **DC component** (low-frequency content or average brightness).
    - Diminishing brightness toward the edges represents higher frequency content.



[Sample Code]

Thank You