Universidade Federal do Ceará

Master's in Teleinformatics Engineering

Applied Electromagnetics: Assignment 2

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For the following plot (Figure 1):

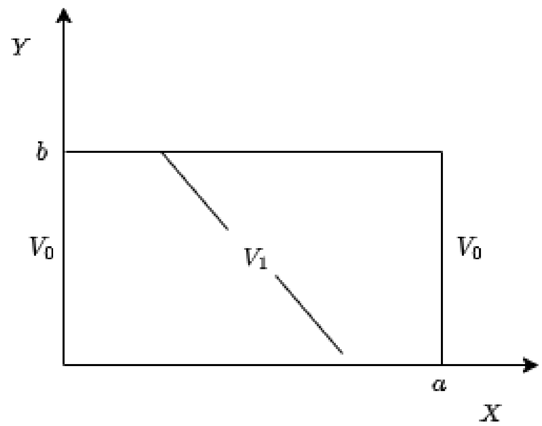

Figure 1: Potential Plot

- (a) Find the solution plot $V(x, y)$
- (b) Find the solution plot of $|E|$

For $n = 10$ and $n = 50$, where $a = 1$ m, $b = 2$ m, $V_0 = 10$ V and $V_1 = V_0 sin(\pi x/a) = 10 sin(\pi x)$ V.

(a) Solution:

We start with the following Boundary Conditions:

- $V(x = 0, y) = 10$ V
- $V(x = 1, y) = 10$ V
- $V(x, y = 0) = 10 sin(\pi x)$ V

- $V(x, y = 2) = 10sin(\pi x)$ V

To solve this problem, we apply the Superposition Principle to compute the four potentials separately and sum them, as shown in Figure 2.

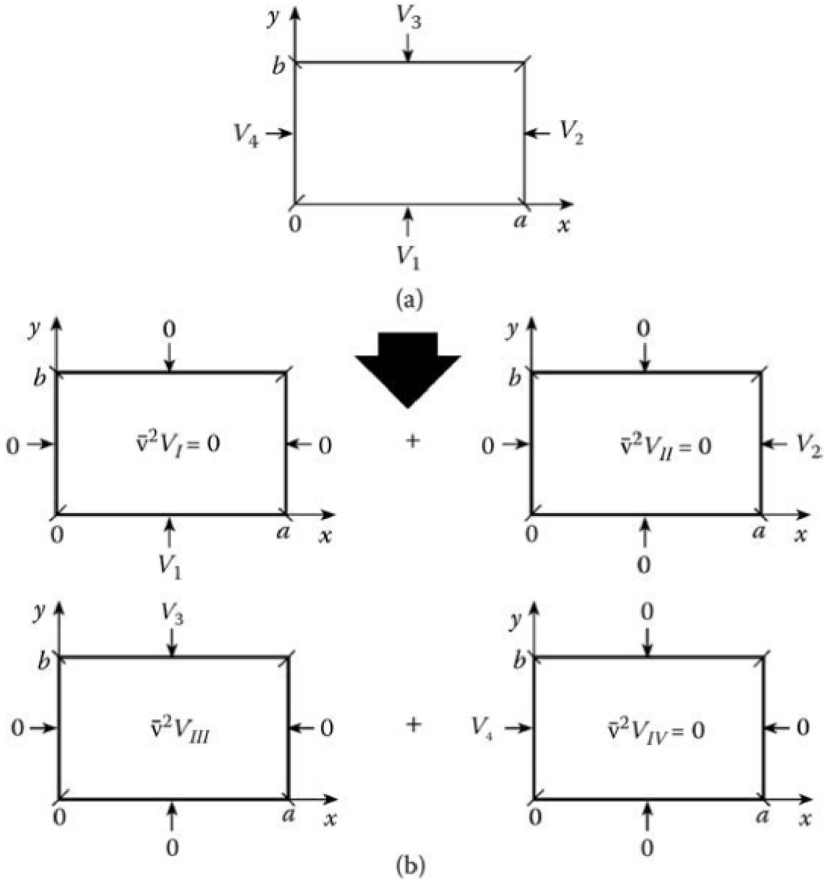

Figure 2: Superposition of Potentials ([2])

That is, we want $V(x, y) = V_1 + V_2 + V_3 + V_4$.

1. Starting with $V_3 = V(x, y = 2) = 10sin(\pi x)$ V and others zero, we apply Laplace's equation:

$$\nabla^2 V = 0 \implies \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

   Using separation of variables $V(x, y) = X(x)Y(y)$:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \implies \frac{-X''}{X} = \frac{Y''}{Y} = \lambda$$

   Which leads to:

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases} \tag{1}$$

Applying boundary conditions:

- $X(0) = 0$, $X(1) = 0$
- $Y(0) = 0$
- $Y(2) =$ given by $10sin(\pi x)$

Testing $\lambda = 0$ and $\lambda < 0$ lead to trivial solutions. Therefore, use $\lambda = \beta^2$:

$$X(x) = h_n sin(n\pi x)$$

$$Y(y) = k_n sinh(n\pi y)$$

Thus:

$$V_3(x, y) = \sum_{n=0}^{\infty} c_n sin(n\pi x) sinh(n\pi y)$$

Applying the top boundary:

$$10sin(\pi x) = c_1 sinh(2\pi) \implies c_1 = \frac{10}{sinh(2\pi)}$$

Final form:

$$V_3(x, y) = \frac{10sin(\pi x)^2 sinh(\pi y)}{sinh(2\pi)} \tag{2}$$

2. By symmetry, for bottom boundary:

$$V_1(x, y) = \frac{10sin(\pi x)^2 sinh[\pi(2 - y)]}{sinh(2\pi)} \tag{3}$$

3. For $V_2 = V(x = 1, y) = 10$ V, use:

$$V_2(x, y) = \sum_{n=1}^{\infty} \begin{cases} \frac{40sinh(n\pi x/2)sin(n\pi y/2)}{n\pi sinh(n\pi/2)}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \tag{4}$$

4. Similarly for left boundary:

$$V_4(x, y) = \sum_{n=1}^{\infty} \begin{cases} \frac{40sinh(n\pi(1-x)/2)sin(n\pi y/2)}{n\pi sinh(n\pi/2)}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \tag{5}$$

The total potential:

$$
\begin{aligned}
V(x,y) &= \frac{10sin(\pi x)^2[sinh(\pi y) + sinh(\pi(2-y))]}{sinh(2\pi)} \\
&+ \sum_{n=1}^{\infty} \begin{cases} \frac{40sinh(n\pi x/2)sin(n\pi y/2)}{n\pi sinh(n\pi/2)} + \frac{40sinh(n\pi(1-x)/2)sin(n\pi y/2)}{n\pi sinh(n\pi/2)}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
\end{aligned} \tag{6}
$$

Plotting the potential:

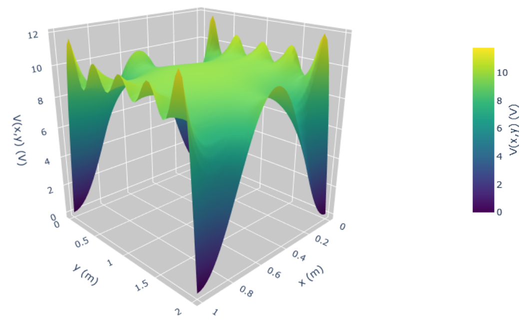

Figure 3: Potential for $n \leq 10$

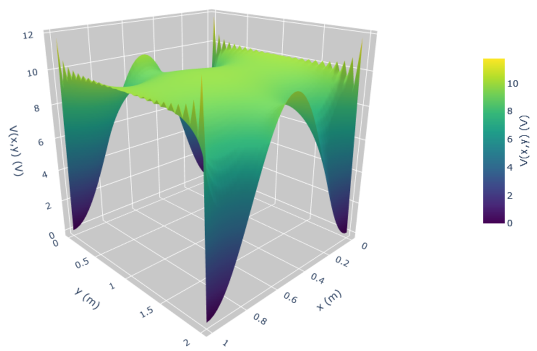

Figure 4: Potential for $n \leq 50$

These plots were generated using Python [5], Sympy [3], and Plotly [4]. Code is available in the GitHub repository: Notebook Plotting.

(b) Solution:
Compute the electric field magnitude from $E = -\nabla V$:

$$\vec{E}(x,y) = -\frac{\partial V(x,y)}{\partial x}\hat{a_x} - \frac{\partial V(x,y)}{\partial y}\hat{a_y}$$

Partial derivative with respect to x:

$$\begin{aligned}\frac{\partial V(x,y)}{\partial x} &= \frac{20\pi sin(\pi x)cos(\pi x)[sinh(\pi y) + sinh(\pi(2-y))]}{sinh(2\pi)} \\ &+ \sum_{n=1}^{\infty}\begin{cases}\frac{40cosh(n\pi x/2)sin(n\pi y/2) - 40cosh(n\pi(1-x)/2)sin(n\pi y/2)}{2sinh(n\pi/2)}, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}\end{aligned} \tag{7}$$

Partial derivative with respect to y:

$$\begin{aligned}\frac{\partial V(x,y)}{\partial y} &= \frac{10\pi sin(\pi x)^2[cosh(\pi y) - cosh(\pi(2-y))]}{sinh(2\pi)} \\ &+ \sum_{n=1}^{\infty}\begin{cases}\frac{40sinh(n\pi x/2)cos(n\pi y/2) + 40sinh(n\pi(1-x)/2)cos(n\pi y/2)}{2sinh(n\pi/2)}, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}\end{aligned} \tag{8}$$

Plotting the electric field magnitude:

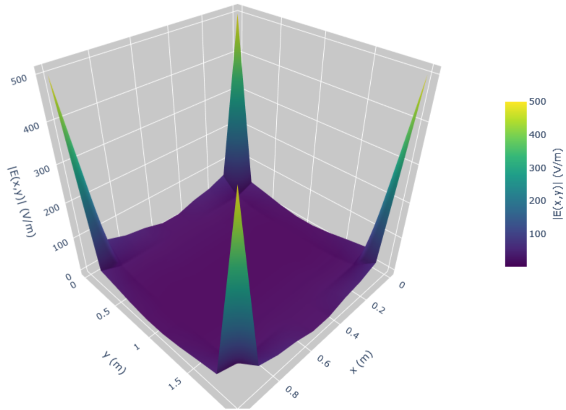

Figure 5: Electric Field Magnitude for $n \leq 10$

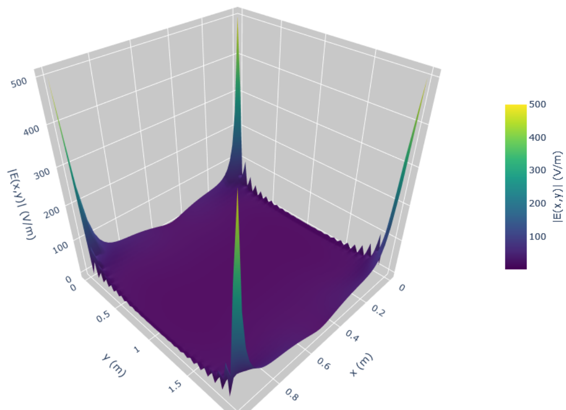

Figure 6: Electric Field Magnitude for $n \leq 50$

These plots were generated using Python [5], Sympy [3], and Plotly [4]. Code is available in the GitHub repository: Notebook Plotting.

References

[1] Article title - Section 8.3 : Periodic Functions & Orthogonal Functions. (n.d.). https://tutorial.math.lamar.edu/classes/de/PeriodicOrthogonal.aspx

[2] Sadiku, M.N.O. (2018). *Computational Electromagnetics with MATLAB*, Fourth Edition. CRC Press.

[3] Meurer, A., et al. (2017). SymPy: Symbolic computing in Python. *PeerJ Computer Science*, 3, e103.

[4] Collaborative data science. (2015). https://plot.ly

[5] Python 3 Reference Manual. (2009).

[6] Sadiku, M. N. O. (2018). *Elements of Electromagnetics*, 7th Edition. Oxford University Press.