The Quadratic Discriminant

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1 Introduction

The discriminant is a crucial tool in solving problems related to quadratics, or polynomials of degree two.

Recall that the general form of a quadratic is $ax^2 + bx + c = 0$ with $a \neq 0$. To solve for the discriminant, we first divide both sides by a.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Adding $(\frac{b}{2a})^2$, or $\frac{b^2}{4a^2}$, to both sides of the equation allows us to complete the square.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2}{4a^2}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

We arrive at the key equation:

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Definition 1.1. The discriminant of a quadratic in the form $ax^2+bx+c = 0$ is the number $b^2 - 4ac$.

We know that a number $c \in \mathbb{R}$ (c is a real number) has $c^2 > 0$. However, if $c^2 < 0$, $c \in \mathbb{C}$ and $c \notin \mathbb{R}$ (c is a non-real complex number). The left-hand side of the key equation, $(x + \frac{b}{2a})^2$, follows the same rules. Note that since $4a^2 > 0$, the discriminant is the deciding factor for whether $(x + \frac{b}{2a})^2 \in \mathbb{R}$ or not. We get the following theorem:

Theorem 1.2.

- If the discriminant is less than 0, then $ax^2 + bx + c$ does not have real roots.
- If the discriminant is equal to 0, then $ax^2 + bx + x$ has one real root.
- If the discriminant is greater than 0, then $ax^2 + bx + c$ has two real roots.

2 Exercises