An Introduction to Quadratics

Alexander Chen

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1 Fundamentals

Quadratic equations can be found in almost every type of math, and in many math competitions.

Definition 1.1. A quadratic equation is a special kind of polynomial that has a degree of 2, which means that it highest power exponent of x is 2.

Definition 1.2. The standard form of a quadratic is as follows:

$$ax^2 + bx + c = 0$$

Definition 1.3. The roots (or zeros) of a quadratic are the values of x such that $ax^2 + bx + c = 0$.

Definition 1.4. The "leading coefficient" of a quadratic is the coefficient of x^2, or a.

You will often be required to derive the roots of a quadratic. There are few ways to accomplish this.

1.1 Factoring

Factoring is the most common method for finding the roots of a quadratic. The key is to find two numbers whose product equals $a \cdot c$ and whose sum equals b, rewrite the quadratic, and then factor. This is best explained with an example. Consider the quadratic:

$$2x^2 + 3x + 1$$

We see that $a \cdot c = 2$ and $b = 3$. Two numbers that satisfy this are 1 and 2. Now we rewrite the quadratic as:

$$2x^2 + x + 2x + 1$$

The first two terms can be rewritten as $x(2x+1)$, and the last two terms as $1(2x+1)$. Factor to get:

$$(x+1)(2x+1)$$

Remember, we want to find the values of x such that the above expression equals 0. Since 0 times anything is zero, we can set each term equal to 0. $x+1=0$ gives us $x=-1$ as a root, and $2x+1=0$ gives us $x=-\frac{1}{2}$ as a root.

Note: In cases where the leading coefficient (a) is just 1, we can directly write the final form. Take x^2+2x+3. The two numbers whose sum is 2 and product is 3 are clearly 1 and 2. We slap these two numbers on the end of two $(x+\ldots)$'s to get $(x+1)(x+2)$. The roots are therefore $x=-1,-2$.

Exercise 1.5. Find the roots of $x^2+9x+20$.

1.2 Completing the Square

Completing the square is a handy method used not only for factoring but also for many other things. The key is to transform the original quadratic as such:

$$ax^2+bx+c=a(x+d)^2+e$$

2 The Discriminant

The discriminant is a crucial tool in solving problems related to quadratics, or polynomials of degree two.

Recall that the general form of a quadratic is $ax^2+bx+c=0$ with $a\neq 0$. To solve for the discriminant, we first divide both sides by a.

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Adding $(\frac{b}{2a})^2$, or $\frac{b^2}{4a^2}$, to both sides of the equation allows us to complete the square.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{c}{a}=\frac{b^2}{4a^2}$$

$$(x+\frac{b}{2a})^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

We arrive at the key equation:

$$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$$

Definition 2.1. The discriminant of a quadratic in the form $ax^2+bx+c = 0$ is the number $b^2 - 4ac$.

We know that a number $c \in \mathbb{R}$ (c is a real number) has $c^2 > 0$. However, if $c^2 < 0$, $c \notin \mathbb{R}$ (c is a non-real complex number). The left-hand side of the key equation, $(x + \frac{b}{2a})^2$, follows the same rules. Note that since $4a^2 > 0$, the discriminant is the deciding factor for whether $(x + \frac{b}{2a})^2 \in \mathbb{R}$ or not. We get the following theorem:

Theorem 2.2.

- If the discriminant is less than 0, then $ax^2 + bx + c$ does not have real roots.
- If the discriminant is equal to 0, then $ax^2 + bx + x$ has one real root.
- If the discriminant is greater than 0, then $ax^2 + bx + c$ has two real roots.

Exercise 2.3. Check if $5x^2 + 6x + 8$ has real roots and if so, the number of real roots.

Solution. The discriminant of the quadratic equals $6^2 - 4 \cdot 8 \cdot 5 = -124$. Since $-124 < 0$, there are no real roots.