GLOBAL ATTRACTORS OF WEIGHTED p-LAPLACE EQUATION WITH DAMPING TERM

Abstract. We study the long-time behavior of the solutions to a nonlinear damped and weighted p-laplace equation. We prove the existence and uniqueness of weak solutions and the existence of the global attractors.

1. Introduction

This article is devoted to study the asymptotic dynamics for a dynamical system generated by a nonlinear weighted p-laplace equation that reads

$$
\begin{aligned}
&u_t = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) - b(x)|\nabla u|^2 && \text{in } \Omega \times \mathbb{R}^+,\\
&u(x,0) = u_0 && \text{in } \Omega,\\
&u = 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, $p > 1$. $a(x)$, $b(x) \in C^1(\bar{\Omega})$, $b(x) \geq 0$, and $a(x) > 0$ in Ω, $a(x) = 0$ on $\partial\Omega$.

p-Laplacian parabolic equations have been considered for the past several decades (see [29, 38, 2, 34, 1, 21, 7, 17, 28, 20, 6, 12, 13], and reference there in). In particular, the class of weighted p-laplace type equations have been of great interest for recent years because of the development of PDEs theory and applications of such equations in many branches of applied sciences such as image denoising, model growing/collapsing sandpiles, model Newtonian and non Newtonian Fluids, type-II superconductivity theory etc (see [8, 9, 31, 35] and reference therein). Therefore, many researchers began to do many works about such equations ([14, 33, 18, 26, 30, 11, 10, 25, 32, 15, 16, 36, 37]). We usually allow the weight $a(x)$ vanish at the boundary to simulate the insulation of air. Recently, there are many papers (see [4, 3, 5, 22, 24, 23]) have considered global attractors for some classes of weighted p-laplace type equations.

In this paper, our model as same as model in [37], but we release the condition of initial data as $u_0 \in L^2(\Omega)$ which in [37] is positive initial data in $L^\infty(\Omega) \cap W_0^{1,p}(a,\Omega)$ and get the existence and uniqueness of the weak solution. Futhermore, we use these solutions to define a semigroup $\{S(t)\}_{t\geq 0}$ in $L^2(\Omega)$ and study the asymptotic behavior of this equation. However, the damping term brings a lot of trouble on some estimates, firstly in the proof of existence of solution, we lack of Grönwall's inequality to get convergence,

2010 *Mathematics Subject Classification.* 74H40, 35B41, 35K65.

Key words and phrases. Global attractor, global solution, weighted p-laplacian.

and then the lackness of embedding theorem when we estimate $\|u_t(s)\|_2^2$ in theorem 4.3 which is important to proof the existence of global attractor in $W_0^{1,p}(\Omega)$.

The content of the paper is as follows. In section 2 we state some basic results that will be used later. In section 3 we proof the existence and uniqueness of the weak solution. In section 4 we give the proof of the existence of global attractors in $L^2(\Omega)$ and $W_0^{1,p}(\Omega)$.

2. Preliminaries and hypotheses

In this section, we will introduce the functional spaces and some useful lemmas, useful results in other papers will also put in here.

Now we give the definition of weighted sobolev spaces. Let Ω be a bounded domain in $\mathbb{R}^n$ and $a\colon \mathbb{R}^n \to [0,\infty)$ be a locally summable nonnegative function, i.e. a weight.

Definition 2.1. A weighted Lebesgue space $L^p(a,\Omega)$, $1 \le p < \infty$, as a Banach space of locally summable functions $u\colon \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|u\|_{L^p(a,\Omega)} = \left(\int_\Omega a|u|^p\right)^{\frac{1}{p}}. \tag{2.1}$$

Definition 2.2. A weighted Sobolev space $W^{k,p}(a,\Omega)$, $1 \le k < \infty$, $1 \le p < \infty$, as a normed space of locally summable, k times weakly differentiable functions $u\colon \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|u\|_{W^{k,p}(a,\Omega)} = \left(\int_\Omega a|u|^p\right)^{\frac{1}{p}} + \sum_{|\alpha|=k}\left(\int_\Omega a|D^\alpha u|^p\right)^{\frac{1}{p}}, \tag{2.2}$$

where α is a multi-index.

Furthermore, $W_0^{k,p}(a,\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{L^p(a,\Omega)} = \left(\int_\Omega a|D^\alpha u|^p\right)^{\frac{1}{p}}. \tag{2.3}$$

Remark 2.3. *Generally $W_0^{k,p}(a,\Omega)$ is not dense in $W^{k,p}(a,\Omega)$ unless the Muckenhoupt condition $a \in A_{p^-}$ is satisfied. See [19] for detailed discussing.*

Next we give a lemma which is helpful to deal with p-laplacian.

Lemma 2.4. *Let α and β denoting vectors in $\mathbb{R}^n$, if $p \ge 4$, then we have*

$$|\alpha^2 - \beta^2|^{\frac{p}{2}} \le C\langle|\alpha|^{p-2}\alpha - |\beta|^{p-2}\beta, \alpha - \beta\rangle \tag{2.4}$$

Proof. when $p \ge 1$

$$|\alpha - \beta|^p \le C||\alpha|^{p-1}\alpha - |\beta|^{p-1}\beta|, \tag{2.5}$$

to prove this, firstly, for $n = 1$ and α, $\beta \geq 0$, we have

$$|\alpha - \beta|^p \leq |\alpha^p - \beta^p|. \tag{2.6}$$

Without loss of generality, we assume $\alpha \leq \beta$, let

$$f(x) = (\beta - \alpha + x)^p - x^p, \tag{2.7}$$

then

$$f'(x) = p\left((\beta - \alpha + x)^{p-1} - x^{p-1}\right) \geq 0, \tag{2.8}$$

hence

$$(\beta - \alpha)^p = f(0) \leq f(\alpha) = \beta^p - \alpha^p. \tag{2.9}$$

Applying jensen's inequality, we also have

$$(\alpha + \beta)^p \leq 2^{p-1}\left(\alpha^p + \beta^p\right), \tag{2.10}$$

hence we proved eq. (2.5) while $n = 1$, for $n > 1$, we need use the law of cosines, assume γ is the angle contained between sides of lengths α and β, we have

$$\begin{aligned}
\left(|\alpha - \beta|^p\right)^2 &= \left(\alpha^2 + \beta^2 - 2|\alpha||\beta|\cos\gamma\right)^p \\
&\leq \left(\alpha^2 + \beta^2\right)^p - 2^p|\alpha|^p|\beta|^p\cos^p\gamma \\
&\leq 2^{p-1}\left(\alpha^{2p} + \beta^{2p} - 2|\alpha|^p|\beta|^p\cos^p\gamma\right) \\
&\leq \left(2^{p-1} + C\right)\left(\alpha^{2p} + \beta^{2p}\right) - \left(2^{p-1}\cos^p\gamma + C\right)2|\alpha|^p|\beta|^p \\
&\leq \left(2^{p-1} + C\right)\left(\alpha^{2p} + \beta^{2p} - 2|\alpha|^p|\beta|^p\cos\gamma\right) \\
&= \left(2^{p-1} + C\right)||\alpha|^{p-1}\alpha - |\beta|^{p-1}\beta|^2,
\end{aligned} \tag{2.11}$$

when C large enough, indeed

$$\begin{aligned}
&2^{p-1}\cos^p\gamma + C - \left(2^{p-1} + C\right)\cos\gamma \\
= {}&2^{p-1}\left(\cos^p\gamma - \cos\gamma\right) + C\left(1 - \cos\gamma\right) \geq 0
\end{aligned} \tag{2.12}$$

when C large enough. Hence we proved eq. (2.5).

On the other hand, we have

$$||\beta|^{p-1}\beta - |\alpha|^{p-1}\alpha| \leq p|\beta - \alpha|\int_0^1 |\alpha + t(\beta - \alpha)|^{p-1}dt. \tag{2.13}$$

Note that, if $p \geq 2$, we have

$$\langle|\alpha|^{p-2}\alpha - |\beta|^{p-2}\beta, \alpha - \beta\rangle \geq |\alpha - \beta|^2\int_0^1 |\beta + t(\alpha - \beta)|^{p-2}dt, \tag{2.14}$$

the proof of eq. (2.14) could find in [27]. Hence when $p \geq 4$ we have

$$
\begin{aligned}
|\alpha^2-\beta^2|^{\frac{p}{2}} &\leq 2^{\frac{p}{2}}|\alpha-\beta|^{\frac{p}{2}}\left(\int_0^1|\beta+t(\alpha-\beta)|dt\right)^{\frac{p}{2}} \\
&\leq C\frac{p-2}{2}2^{\frac{p}{2}}|\alpha-\beta|^2\left(\int_0^1|\beta+t(\alpha-\beta)|^{\frac{p-4}{2}}dt\right) \\
&\times\left(\int_0^1|\beta+t(\alpha-\beta)|dt\right)^{\frac{p}{2}} \\
&\leq C\frac{p-2}{2}2^{\frac{p}{2}}|\alpha-\beta|^2\int_0^1|\beta+t(\alpha-\beta)|^{p-2}dt \\
&\leq C\langle|\alpha|^{p-2}\alpha-|\beta|^{p-2}\beta,\alpha-\beta\rangle
\end{aligned}
\tag{2.15}
$$

□

Finally, we need the following theorem to help us proof existence and uniqueness of the weak solution,

Theorem 2.5. *[37, thm 1.3 and 1.6] If $p>4$, $a(x)$, $b(x)$ satisfies*

$$\int_\Omega b^{\frac{2p}{p-4}}a^{-\frac{4}{p-4}} \leq c, \tag{2.16}$$

and u_0 satisfies

$$0 \leq u_0 \in L^\infty(\Omega), a(x)u_0 \in W_0^{1,p}(\Omega), \tag{2.17}$$

then eq. (1.1) *has unique nonnegative weak solution satisfies*

$$u \in L^\infty(Q_T), a(x)|\nabla u|^p \in L^1(Q_T). \tag{2.18}$$

The initial value is satisfied in the sense of that

$$\lim_{t\to 0}\int_\Omega|u(x,t)-u_0(x)|dx = 0. \tag{2.19}$$

3. Existence and uniqueness of the weak solution

Now we prove the existence of the weak solution. Since we already have the well-posedness when initial data satisfied strictly conditions (see theorem 2.5), we can use these initial data to approach $u_0 \in L^2$.

Theorem 3.1. *If $p>4$ in eq.* (1.1)*, $a(x)$, $b(x)$ satisfies*

$$\int_\Omega b^{\frac{2p}{p-4}}a^{-\frac{4}{p-4}} \leq c, \tag{3.1}$$

and $u_0 \in L^2(\Omega)$, then it has a weak solution satisfies

$$u \in L^p(0,T;W_0^{1,p}(a,\Omega)), \quad u \in C([0,T];L^2(\Omega)). \tag{3.2}$$

Proof. First of all, we prove a stronger existence conclusion based on [37], which limits u_0 as

$$u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(a, \Omega). \tag{3.3}$$

We now consider the following regularized problem

$$u_{\epsilon t} - \operatorname{div}\left((a(x)+\epsilon)\left(|\nabla u_\epsilon|^2 + \epsilon\right)^{\frac{p-2}{2}} \nabla u_\epsilon\right) + b(x)\,|\nabla u_\epsilon|^2 = 0, (x,t) \in Q_T \tag{3.4}$$

$$u_\epsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \tag{3.5}$$

$$u_\epsilon(x,0) = u_{\epsilon,0}(x), \quad x \in \Omega \tag{3.6}$$

it is well-know that above problem has a unique classical solution.

Using the method of convolution a sequence of mollifiers, we could find a sequence $\|u_{\epsilon,0}\|_\infty$ bounded uniformly and $u_{\epsilon,0} \in C_0^\infty(\Omega)$, $au_{\epsilon,0} \to au_0$, in $W_0^{1,p}(\Omega)$, by maximum principle we have $|u_\epsilon| \le C$, where C only depends on $\|u_0\|_\infty$.

Multiplying eq. (3.4) by u_ϵ and integrating it over Q_T, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_\epsilon^2 + \iint_{Q_T} (a(x)+\epsilon)\left(|\nabla u_\epsilon|^2 + \epsilon\right)^{\frac{p-2}{2}} |\nabla u_\epsilon|^2 \\ &\qquad + \iint_{Q_T} b(x)\,|\nabla u_\epsilon|^2\, u_\epsilon = \frac{1}{2}\int_\Omega u_{\epsilon,0}^2. \end{aligned} \tag{3.7}$$

By the fact

$$\begin{aligned} |\int_0^T \int_\Omega b|\nabla u_\epsilon|^2 u_\epsilon| &\le C\int_0^T \left(\int_\Omega b^{\frac{p}{p-2}} a^{-\frac{2}{p-2}} |u_\epsilon|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \left(\int_\Omega a|\nabla u_\epsilon|^p\right)^{\frac{2}{p}} \\ &\le C\int_0^T \left(\int_\Omega |u_\epsilon|^2\right)^{\frac{p}{2(p-2)}} + \frac{1}{2}\int_0^T\int_\Omega a|\nabla u_\epsilon|^p \\ &\le \eta \int_0^T \int_\Omega |u_\epsilon|^2 + \frac{1}{2}\int_0^T \int_\Omega a|\nabla u_\epsilon|^p + C(\eta) \end{aligned} \tag{3.8}$$

we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_\epsilon^2 + \iint_{Q_T} (a(x)+\epsilon)\left(|\nabla u_\epsilon|^2 + \epsilon\right)^{\frac{p-2}{2}} |\nabla u_\epsilon|^2 \\ &\qquad \le \frac{1}{2}\iint_{Q_T} a|\nabla u_\epsilon|^p + C. \end{aligned} \tag{3.9}$$

As well as

$$\iint_{Q_T} a|\nabla u_\epsilon|^p \le \iint_{Q_T} (a(x)+\epsilon)\left(|\nabla u_\epsilon|^2+\epsilon\right)^{\frac{p-2}{2}} |\nabla u_\epsilon|^2 \le \frac{1}{2}\iint_{Q_T} a|\nabla u_\epsilon|^p + C \tag{3.10}$$

hence we have

$$\iint_{Q_T} a|\nabla u_\epsilon|^p \le C. \tag{3.11}$$

Indeed, eq. (3.11) correspond to [37, thm 1.3 eq. (2.5)], the remaining prove is similar as the prove in [37, thm 1.3], we omit it here and give the temporary conclusion: eq. (1.1) has a unique weak solution when the initial data satisfies conditions (3.3).

Second, we prove the existence conclusion with the initial data in $L^2(\Omega)$. Choose $C_c^\infty(\Omega) \supset \{u_{n,0}\}_{n=1}^\infty$ convergence to u_0 in $L^2(\Omega)$ as $n \to \infty$. $\forall u_{n,0}$, by the first part of this proof, there exist a unique u_n as a weak solution satisfy eq. (1.1). Hence we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega (u_n - u_m)^2(t) \\ &+\int_0^t\int_\Omega a(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n - \nabla u_m) \\ &= \int_0^t\int_\Omega b(x)\left(|\nabla u_n|^q - |\nabla u_m|^q\right)(u_n - u_m) + \frac{1}{2}\int_\Omega (u_n - u_m)^2(0). \end{aligned} \tag{3.12}$$

Use lemma 2.4, we have

$$\begin{aligned} &\int_0^t\int_\Omega b\left(|\nabla u_n|^2 - |\nabla u_m|^2\right)(u_n - u_m) \\ &\le \left(\int_0^t\int_\Omega a\left(|\nabla u_n|^2 - |\nabla u_m|^2\right)^{\frac{p}{2}}\right)^{\frac{2}{p}}\left(\int_0^t\int_\Omega \left(ba^{-\frac{2}{p}}(u_n - u_m)\right)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \\ &\le C\left(\int_0^t\int_\Omega a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n - \nabla u_m)\right)^{\frac{2}{p}} \\ &\times\left(\int_0^t\int_\Omega b^{\frac{2p}{p-4}}a^{-\frac{4}{p-4}}\right)^{\frac{p-4}{2p}}\left(\int_0^t\int_\Omega (u_n - u_m)^2\right)^{\frac{1}{2}} \\ &\le \frac{1}{2}\int_0^t\int_\Omega a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n - \nabla u_m) \\ &+C\left(\int_0^t\int_\Omega (u_n - u_m)^2\right)^{\frac{p}{2(p-2)}}. \end{aligned} \tag{3.13}$$

Combine above inequalities we get

$$\begin{aligned} &\int_\Omega (u_n - u_m)^2(t) \\ &+\int_0^t\int_\Omega a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n - \nabla u_m) \\ &\le C\left(\int_0^t\int_\Omega (u_n - u_m)^2\right)^{\frac{p}{2(p-2)}} + \int_\Omega (u_n - u_m)^2(0). \end{aligned} \tag{3.14}$$

For $p = 4$, we use Grönwall's inequality have

$$\int_\Omega (u_n - u_m)^2(t) \le \int_\Omega (u_n - u_m)^2(0)e^{Ct}, \tag{3.15}$$

for a.e. $0 \le t \le T$. For $p > 4$, we can't use Grönwall's inequality directly, since $\frac{p}{2(p-2)} < 1$. But we can use Young's inequality up exponential to 1, indeed we can deduce eq. (3.14) as

$$\begin{aligned} &\int_\Omega (u_n - u_m)^2(t) \\ &+ \int_0^t \int_\Omega a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n - \nabla u_m) \\ &\le \epsilon \int_0^t \int_\Omega (u_n - u_m)^2 + \epsilon^{-\frac{p}{p-4}} C + \int_\Omega (u_n - u_m)^2(0), \end{aligned} \tag{3.16}$$

hence we can use Grönwall's inequality to get

$$\int_\Omega (u_n - u_m)^2(t) \le \left(\epsilon^{-\frac{p}{p-4}} C + \int_\Omega (u_n - u_m)^2(0)\right) e^{\epsilon t} \tag{3.17}$$

for a.e. $0 \le t \le T$. Choose $\epsilon = \delta t^{-1}$, where $\delta > 0$ is arbitrary, for a.e. fixed $0 \le t \le T$, let $n, m \to \infty$ and $\delta \to 0$, we have

$$\|u_n(t,\cdot) - u_m(t,\cdot)\|_{L^2(\Omega)} \to 0, \tag{3.18}$$

hence for a.e. fixed $0 \le t \le T$, there exists $u(t,\cdot) \in L^2(\Omega)$ such that

$$\|u_n(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)} \to 0 \tag{3.19}$$

as $n \to \infty$, applying continuous of $u_n(t,\cdot)$ in $L^2(\Omega)$, we have

$$\lim_{t' \to t} \|u(t,\cdot) - u(t',\cdot)\|_{L^2(\Omega)} = 0. \tag{3.20}$$

Applying eq. (3.18) in eq. (3.14), we have

$$\int_0^t \int_\Omega a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n - \nabla u_m) \to 0, \tag{3.21}$$

note that

$$\begin{aligned} &\int_0^T \int_\Omega a|\nabla u_n - \nabla u_m|^p \\ &\le 2^{p-2} \int_0^T \int_\Omega a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n - \nabla u_m), \end{aligned} \tag{3.22}$$

hence there exists $u \in L^p(0,T; W_0^{1,p}(a,\Omega)) \cap C([0,T]; L^2(\Omega))$ such that

$$u_n \to u \tag{3.23}$$

in $L^p(0,T; W_0^{1,p}(a,\Omega)) \cap C([0,T]; L^2(\Omega))$ as $n \to \infty$.

Finally, we consider the damping term, applying Hölder's inequality, for any $v \in C_c^\infty$,

$$
\begin{aligned}
&\int_0^T \int_\Omega b\left(|\nabla u_n|^2 - |\nabla u|^2\right) v \\
&\le C_p \left(\int_0^T \int_\Omega a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\right)^{\frac{2}{p}} \qquad (3.24)\\
&\times \left(\int_0^T \int_\Omega b^{\frac{2p}{p-4}} a^{-\frac{4}{p-4}}\right)^{\frac{p-4}{2p}} \left(\int_0^T \int_\Omega v^2\right)^{\frac{1}{2}} \to 0
\end{aligned}
$$

as $n \to \infty$, hence we have completed the proof. □

Remark 3.2. *For $p > 4$, we also could estimate eq. (3.14) as*

$$
\begin{aligned}
&\int_\Omega (u_n - u_m)(t) \\
&+ \int_0^t \int_\Omega a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n - \nabla u_m) \qquad (3.25)\\
&\le C \int_0^t \int_\Omega (u_n - u_m)^{\frac{p}{p-2}} + \int_\Omega (u_n - u_m)^2 (0),
\end{aligned}
$$

since $\frac{p}{p-2} > 1$, we could use Grönwall's inequality, then applying similar steps we could get eq. (3.21). By weak compactness theorem we could also get the existence of solutions.

Uniqueness can use the same way in [37] to get. Now we can use these solutions to define a semigroup $\{S(t)\}_{t>0}$, by setting

$$S(t)u_0 = u(t), \tag{3.26}$$

which is continuous on u_0 in $L^2(\Omega)$.

4. Existence of the global attractors

Theorem 4.1. *The semigroup $\{S(t)\}_{t\ge 0}$ possesses a bounded absorbing set in L^2 and $W_0^{1,p}(a,\Omega)$ respectively, i.e. for any bounded subset $B \subset L^2(\Omega)$, there exist constants $T(\|u_0\|_2)$ and $\rho > 0$, such that*

$$\|u(t)\|_2^2 + \int_\Omega a|\nabla u|^p \le \rho, \tag{4.1}$$

for all $t \ge T$ and $u_0 \in B$, where $u(t) = S(t)u_0$.

Proof. If we multiply eq. (1.1) by u and integrate over Ω, then we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \int_\Omega a|\nabla u|^p + \int_\Omega b|\nabla u|^2 u = 0, \tag{4.2}$$

since $p \ge 4$, use Hölder's inequality,

$$\int_\Omega |\nabla u|^2 = \int_\Omega a^{-\frac{2}{p}} a^{\frac{2}{p}} |\nabla u|^2 \le \left(\int_\Omega a^{-\frac{2}{p-2}}\right)^{\frac{p-2}{p}} \left(\int_\Omega a|\nabla u|^p\right)^{\frac{2}{p}} \tag{4.3}$$

and use Poincarè's inequality, we have

$$\left(\int_\Omega |u|^2\right)^{\frac{p}{2}} \le C\int_\Omega a|\nabla u|^p, \tag{4.4}$$

where C only related with Ω and p. For the 3rd term, we could use Hölder's inequality and Young's inequality to get the bound respectively,

(4.5)

$$\begin{aligned}
|\int_\Omega b|\nabla u|^2 u| &\le \left(\int_\Omega a|\nabla u|^p\right)^{\frac{2}{p}} \left(\int_\Omega |ba^{-\frac{2}{p}}u|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \\
&\le \frac{1}{2}\int_\Omega a|\nabla u|^p + 2^{\frac{p-2}{2}}\int_\Omega |ba^{-\frac{2}{p}}u|^{\frac{p}{p-2}} \\
&\le \frac{1}{2}\int_\Omega a|\nabla u|^p + 2^{\frac{p-2}{2}}\left(\int_\Omega b^{\frac{2p}{p-4}}a^{-\frac{4}{p-4}}\right)^{\frac{p-4}{2(p-2)}}\left(\int_\Omega u^2\right)^{\frac{p}{2(p-2)}} \\
&\le \frac{1}{2}\int_\Omega a|\nabla u|^p + \epsilon\left(\int_\Omega u^2\right)^{\frac{p}{2}} + C_\epsilon\left(\int_\Omega b^{\frac{2p}{p-4}}a^{-\frac{4}{p-4}}\right)^{\frac{p-4}{2(p-3)}},
\end{aligned}$$

choose ϵ small enough and combine above estimates, we get

$$\frac{d}{dt}\|u\|_2^2 + \|u\|_2^p \le C \tag{4.6}$$

where C independents on u. By the Grönwall's inequality, we get the bounded absorbing set in $L^2(\Omega)$, i.e. $\exists\rho_0$ and $T_0 = T_0(\|u_0\|_2)$ such that

$$\|u(t)\|_2^2 \le \rho_0 \text{ for } t \ge T_0. \tag{4.7}$$

Applying eqs. (4.2), (4.5) and (4.7), we have

$$\frac{d}{dt}\|u\|_2^2 + \int_\Omega a|\nabla u|^p \le C. \tag{4.8}$$

If we multiply eq. (1.1) by u_t and integrate over Ω, then we have

$$\|u_t\|_2^2 + \frac{1}{p}\frac{d}{dt}\int_\Omega a|\nabla u|^p + \int_\Omega b|\nabla u|^2 u_t = 0, \tag{4.9}$$

estimate the 3rd term like eq. (4.5), then we could get

$$C\|u_t\|_2^2 + \frac{d}{dt}\int_\Omega a|\nabla u|^p \le \int_\Omega a|\nabla u|^p + C, \tag{4.10}$$

eq. (4.8) + eq. (4.10) implies that

$$C\|u_t\|_2^2 + \frac{d}{dt}\left(\int_\Omega a|\nabla u|^p + \|u\|_2^2\right) \le C. \tag{4.11}$$

Applying eq. (4.6), we have

$$\frac{d}{dt}\|u\|_2^2 + \|u\|_2^2 \le C \tag{4.12}$$

hence

$$\|u(t+1)\|_2^2+\int_t^{t+1}\|u\|_2^2\le C+\|u(t)\|_2^2, \tag{4.13}$$

similarity, applying eq. (4.8), we have

$$\|u(t+1)\|_2^2+\int_t^{t+1}\int_\Omega a|\nabla u|^p\le C+\|u(t)\|_2^2, \tag{4.14}$$

combine eqs. (4.13) and (4.14), we have

$$\int_t^{t+1}\int_\Omega a|\nabla u|^p+\|u\|_2^2\le C \text{ for } t\ge T_0. \tag{4.15}$$

Applying eqs. (4.11) and (4.15), use Uniform Grönwall's Lemma we get the bounded absorbing set in $W_0^{1,p}(a,\Omega)$, i.e $\exists\rho$ and $T\ge T_0$ such that

$$\|u(t)\|_2^2+\int_\Omega a|\nabla u|^p\le\rho \text{ for } t\ge T, \tag{4.16}$$

hence we completed this proof. □

Since $p\ge 4$, by the compact embedding $W_0^{1,p}(\Omega)\subset\subset L^2(\Omega)$ and theorem 3.1 yield the existence of a global attractor in $L^2(\Omega)$ immediately.

Theorem 4.2. *The semigroup $\{S(t)\}_{t\ge0}$ generated by the weak solution of eq.* (1.1) *possesses a global attractor $\mathcal{A}_2$ in $L^2(\Omega)$.*

Next we prove the existence of attractor in $W_0^{1,p}(\Omega)$. Firstly, we have to obtain u_t bounded in $L^2(\Omega)$. Unfortunately, we cannot get uniform boundedness of u_t when $|\nabla u|$ small enough, hence we deal $|\nabla u|$ as $|\nabla u|+1$.

Theorem 4.3. *For any bounded subset B in $L^2(\Omega)$, if $|\nabla u|>1$, there exists a constant $T'=T'(B)>0$, such that*

$$\|u_t(s)\|_2^2\le M,\quad \forall u_0\in B \text{ and } s\ge T'. \tag{4.17}$$

Proof. By differentiating eq. (1.1) in time and denoting $v=u_t$, we have

$$v_t=\operatorname{div}\left(a|\nabla u|^{p-2}\nabla v\right)+\operatorname{div}\left((p-2)\,a|\nabla u|^{p-4}\left(\nabla u\cdot\nabla v\right)\nabla u\right)-2b\nabla u\cdot\nabla v. \tag{4.18}$$

We multiply above equation by v and integrate over Ω, then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_2^2+\int_\Omega a|\nabla u|^{p-2}|\nabla v|^2+\int_\Omega (p-2)\,a|\nabla u|^{p-4}\left(\nabla u\cdot\nabla v\right)^2&\\ +2\int_\Omega b\nabla u\cdot\nabla v v=0.&\end{aligned} \tag{4.19}$$

Now we estimate the damping term,

$$
\begin{aligned}
\int_\Omega b\nabla u\cdot\nabla v v &\le \left(\int_\Omega b^2\left(\nabla u\cdot\nabla v\right)^2\right)^{\frac12}\|v\|_2\\
&\le \|b^2a^{-1}\|_\infty^{\frac12}\left(\int_\Omega a|\nabla u|^2|\nabla v|^2\right)^{\frac12}\|v\|_2\\
&\le \|b^2a^{-1}\|_\infty^{\frac12}\left(\int_\Omega a\left(|\nabla u|+1\right)^2|\nabla v|^2\right)^{\frac12}\|v\|_2\\
&\le \|b^2a^{-1}\|_\infty^{\frac12}\left(\int_\Omega a\left(|\nabla u|+1\right)^{p-2}|\nabla v|^2\right)^{\frac12}\|v\|_2\\
&\le \epsilon\int_\Omega a\left(|\nabla u|+1\right)^{p-2}|\nabla v|^2+C(\epsilon)\|v\|_2^2,
\end{aligned}
\tag{4.20}
$$

for the 2nd term

$$
\int_\Omega a|\nabla u|^{p-2}|\nabla v|^2\ge 2^{3-p}\int_\Omega a\left(|\nabla u|+1\right)^{p-2}|\nabla v|^2-\int_\Omega a|\nabla v|^2. \tag{4.21}
$$

hence we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|v\|_2^2+C\int_\Omega a\left(|\nabla u|+1\right)^{p-2}|\nabla v|^2\\
&+\int_\Omega (p-2)\,a|\nabla u|^{p-4}\left(\nabla u\cdot\nabla v\right)^2\le C\|v\|_2^2+C.
\end{aligned}
\tag{4.22}
$$

Now applying eqs. (4.10), (4.16) and (4.22), we could get

$$
\frac{d}{dt}\left(\|v\|_2^2+C\int_\Omega a|\nabla u|^p\right)\le C, \tag{4.23}
$$

next we integrating eq. (4.11) from t to $t+1$, we have

$$
\int_\Omega a|\nabla u(t+1)|^p+\|u(t+1)\|_2^2+C\int_t^{t+1}\|v\|_2^2\le C+\int_\Omega a|\nabla u(t)|^p+\|u(t)\|_2^2, \tag{4.24}
$$

hence for t large enough, we have

$$
\int_t^{t+1}\left(\|v\|_2^2+C\int_\Omega a|\nabla u|^p\right)\le C, \tag{4.25}
$$

then we use Uniform Grönwall's Lemma could get, there exists $M>0$ such that

$$
\|v(t)\|_2^2+\int_\Omega a|\nabla u|^p\le M \text{ for } t\ge T'\ge T, \tag{4.26}
$$

hence we completed this proof. □

Now we can prove the existence of global attractor in $W_0^{1,p}(\Omega)$.

Theorem 4.4. *The semigroup* $\{S(t)\}_{t\geq 0}$ *associated with eq.* (1.1) *possesses a global attractor* $\mathcal{A}_V$ *in* $W_0^{1,p}(\Omega)$*, i.e.* $\mathcal{A}_V$ *is compact in* $W_0^{1,p}(\Omega)$*, invariant and attracts the bounded sets of* $L^2(\Omega)$ *in the topology of* $W_0^{1,p}(\Omega)$*.*

Proof. In this prove, all we need to show is that $\{S(t)\}_{t\geq 0}$ is asymptotically compact in $W_0^{1,p}(\Omega)$. Let B_0 be a bounded absorbing set in $W_0^{1,p}(\Omega)$ obtained in theorem 3.1, then we need to verify for any sequence $\{u_{0n}\}_{n=1}^{\infty} \subset B_0$, $\{u_n(t_n)\}_{n=1}^{\infty}$ possesses a convergence subsequence in $W_0^{1,p}(\Omega)$.

Indeed thanks to theorem 4.2, we have $\{u_n(t_n)\}_{n=1}^{\infty}$ is precompact in $L^2(\Omega)$. So we could take a subsequence in $\{u_n(t_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$ and still denote as $\{u_n(t_n)\}_{n=1}^{\infty}$. Since $p \geq 4$ and applying Hölder's inequality, we have

$$\begin{aligned}
&\int_\Omega a|\nabla u_m(t_m) - \nabla u_n(t_n)|^p \\
&\leq C \int_\Omega a\left(|\nabla u_m(t_m)|^{p-2}\nabla u_m(t_m) - |\nabla u_n(t_n)|^{p-2}\nabla u_n(t_n)\right)\nabla\left(u_m(t_m) - u_n(t_n)\right) \\
&= C\left(\int_\Omega (u_n - u_m)_t\,(u_m - u_n) - \int_\Omega b\left(|\nabla u_m|^2 - |\nabla u_n|^2\right)(u_m - u_n)\right) \\
&\leq C\|(u_n - u_m)_t\|_2\|u_m - u_n\|_2 \\
&+C\left(\int_\Omega a\left(|\nabla u_m|^p + |\nabla u_n|^p\right)\right)^{\frac{2}{p}}\left(b^{\frac{2p}{p-4}}a^{-\frac{4}{p-4}}\right)^{\frac{p-4}{2p}}\|u_m - u_n\|_2 \to 0
\end{aligned}$$

as m, $n \to \infty$. Hence applying theorem 3.1, the semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor $\mathcal{A}_V$ in $W_0^{1,p}(\Omega)$. □

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