1 Solutions

Proof 1.4 For $y = (a, b)$, set $y_1 = (a, -b)$, $y_2 = (-a, b)$, $y_3 = (-a, -b)$, then

$$G(x, y) = \Gamma(x, y) - \Gamma(x, y_1) - \Gamma(x, y_2) + \Gamma(x, y_3) \qquad \square$$

Proof 1.5 Set $x = (x_1, \cdots, x_n)$, $y = (a, 0, \cdots, 0)$, then we use Poisson integral and maximum principle for harmonic functions to get

$$\begin{aligned}
&\int_{\partial B_1} \left[(x_1 - a)^2 + x_2^2 + \cdots + x_n^2)\right]^{-n/2} \, \mathrm{d}S_x \\
= &\int_{\partial B_1} \frac{\mathrm{d}S_x}{|x - y|^n} = \frac{1 - |y|^2}{n\omega_n} \int_{\partial B_1} \frac{n\omega_n}{1 - |y|^2} \frac{1}{|x - y|^n} \, \mathrm{d}S_x \\
= &\frac{n\omega_n}{1 - |y|^2}
\end{aligned} \qquad \square$$

Proof 1.7 We use the mean value inequalities for $u(x)$, for any ball $B(x, r) \subset\subset \mathbb{R}^n$, we have

$$\begin{aligned}
|u| &= |\frac{1}{\omega_n r^n} \int_{B(x,r)} u \, \mathrm{d}x| \\
&\leq \frac{1}{\omega_n r^n} \int_{B(x,r)} |u| \, \mathrm{d}x \\
&\leq (\omega_n r^n)^{-1/p} \|u\|_{L_p(B(x,r))} \\
&\leq (\omega_n r^n)^{-1/p} \|u\|_{L_p(\mathbb{R}^n)} \to 0 \quad \text{as} \quad r \to \infty
\end{aligned}$$

Hence $u(x) = 0$ $\square$

Proof 1.9 Set

$$f(x) = x \log x,$$

we have

$$f'(x) = \log x + 1 \leq 0 \quad \text{in} \quad (0, 1/e],$$

hence $f(x)$ is nonincreasing in $[0, 1/e]$. Since $f(x) \in C^1((1/e, 1])$, we consider $f(x)$ in $[0, 1/e]$.
Set

$$g(h) = f(x + h) - f(x) - f(h),$$

we have

$$g'(h) = \log \frac{x + h}{h} \geq 0 \quad \text{in} \quad (0, 1 - x],$$

since $g(0) = 0$, we have

$$f(x) - f(x + h) \leq -f(h).$$

Since $f(x)$ is nonincreasing in $[0, 1/e]$, we have

$$|\frac{f(x) - f(x + h)}{h^\alpha}| = \frac{f(x) - f(x + h)}{h^\alpha} \leq -\frac{f(h)}{h^\alpha} = -h^{1-\alpha} \log h.$$

hence $f(x) \in C^{0,\alpha}[0, 1]$, where $\alpha \in (0, 1)$, and $f(x) \notin C^{0,1}[0, 1]$. $\square$

Proof 1.10 Since A, B are both positive-definite matrix, we have

$$A = P\sqrt{\bar{A}}\sqrt{\bar{A}}P^T, \qquad B = Q\bar{B}Q^T.$$

Then

$$\begin{aligned}
\det A \det B &= \det\left(\sqrt{\bar{A}}P^T Q\bar{B}Q^T P\sqrt{\bar{A}}^T\right) \\
&\le \left(\frac{\operatorname{tr}\left(\sqrt{\bar{A}}P^T Q\bar{B}Q^T P\sqrt{\bar{A}}^T\right)}{n}\right)^n \\
&= \left(\frac{\operatorname{tr}(AB)}{n}\right)^n
\end{aligned}$$

since $\sqrt{\bar{A}}P^T Q\bar{B}Q^T P\sqrt{\bar{A}}^T$ is positive-definite, and use AM-GM inequality. $\square$

Proof 1.12 WLOG $\forall x > 0$, $\eta \in (0,1)$, we consider $(x\eta, x)$, set

$$u = \log x, \quad \bar{u} = \fint_{x\eta}^{x} \log x \,\mathrm{d}x, \quad \tilde{u} = \fint_{x\eta}^{x} |u - \bar{u}| \,\mathrm{d}x, \quad \bar{u}_0 = \fint_{\eta}^{1} \log t \,\mathrm{d}t.$$

Calculate

$$\begin{aligned}
\bar{u} &= \fint_{\eta}^{1} \log xt \,\mathrm{d}t = \bar{u}_0 + \log x \\
\tilde{u} &= \fint_{\eta}^{1} |\log t + \log x - \bar{u}| \,\mathrm{d}t = \fint_{\eta}^{1} |\log t - \bar{u}_0| \,\mathrm{d}t \\
\bar{u}_0 &= \frac{\eta \log \eta - \eta + 1}{\eta - 1}.
\end{aligned}$$

Since

$$\fint_{\eta}^{1} (\log t - \bar{u}_0) \,\mathrm{d}t = 0$$

there exist $t_0 \in (\eta, 1)$ s.t. $\log t_0 = \bar{u}_0$, hence

$$\frac{1}{1-\eta}\int_{t_0}^{1} (\log t - \bar{u}_0) \,\mathrm{d}t = -\frac{1}{1-\eta}\int_{\eta}^{t_0} (\log t - \bar{u}_0) \,\mathrm{d}t \ge 0.$$

Then we have

$$\tilde{u} = \frac{1}{1-\eta}\left(\int_{t_0}^{1} - \int_{\eta}^{t_0} (\log t - \bar{u}_0)\right) \mathrm{d}t = \frac{2}{1-\eta}\int_{t_0}^{1} (\log t - \bar{u}_0) \,\mathrm{d}t$$

since

$$\log t - \bar{u}_0 \le \log 1 - \bar{u}_0 \le 1$$

we have

$$\tilde{u} \le \frac{2(1-t_0)}{1-\eta} \le 2.$$

Hence $\log x \in BMO(0, \infty)$. $\square$

2 G-T Chapter 2 Solutions

Proof 2.2 Fix $x_0 \in T$, where T is the open, smooth portion of $\partial\Omega$, then we can find $r > 0$, s.t. $B_r(x_0)$ is divided into two parts by T, then we extend u with zero in $B_r(x_0) - \Omega$.

Now we prove u is harmonic in $B_r(x_0)$. Since u is harmonic in $B_r(x_0) \cap \Omega$ and $u \equiv 0$ in $B_r(x_0) - \Omega$, all we need to show is u is harmonic in $B_r(x_0) \cap T$, so we need prove for every ball $B_R(y) \subset\subset \Omega \cup B_r(x_0)$ it satisfies the mean value property, where $y \in B_r(x_0) \cap T$. By the proof of the mean value inequalities, we only need to show

$$\int_{\partial B_R(y)} \frac{\partial u}{\partial \nu} \,\mathrm{d}s = 0. \tag{1}$$

$u = \partial u/\partial\nu = 0$ on the smooth portion T guarantees that $u \in C^1$ in T, then we have the integral (1) is well-defined, since $u \equiv 0$ in $B_r(x_0) - \Omega$, we have

$$\int_{\partial B_R(y)} \frac{\partial u}{\partial \nu} \,\mathrm{d}s = \int_{B_R(x_0)\cap\Omega} \triangle u \,\mathrm{d}x = 0$$

hence u is harmonic in $\Omega \cup B_r(x_0)$, and $u \equiv 0$ in $B_r(x_0) - \Omega$, by analytic, $u \equiv 0$ in Ω. $\square$

Proof 2.3 1. Fix $x, y \in \Omega$, $x \neq y$, write

$$v(z) := G(x,z), \quad w(z) := G(y,z), \quad z \in \Omega,$$

then

$$\begin{aligned} \triangle v(z) &= 0, & z &\neq x, \\ \triangle w(z) &= 0, & z &\neq y \end{aligned}$$

and $w = v = 0$ on $\partial\Omega$. Consider Green's identity on $\tilde{\Omega} := \Omega \setminus [B_\epsilon(x) \cup B_\epsilon(y)]$

$$\begin{aligned} \int_{\tilde{\Omega}} (v \triangle w - w \triangle v) \,\mathrm{d}z &= \int_{\partial\Omega} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) \mathrm{d}s \\ &+ \int_{\partial B_\epsilon(x)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) \mathrm{d}s + \int_{\partial B_\epsilon(y)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) \mathrm{d}s, \end{aligned}$$

since v, w is harmonic in $\tilde{\Omega}$, and vanishes on $\partial\Omega$, we have

$$\int_{\partial B_\epsilon(x)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) \mathrm{d}s + \int_{\partial B_\epsilon(y)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) \mathrm{d}s = 0,$$

since

$$\left| \int_{\partial B_\epsilon(x)} v\frac{\partial w}{\partial \nu} \,\mathrm{d}s \right| \le C \int_{\partial B_\epsilon(x)} |v| \,\mathrm{d}s \le C\epsilon$$

and

$$\lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} \frac{\partial v}{\partial \nu} w \,\mathrm{d}s = \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} \frac{\partial \Gamma}{\partial \nu}(x-z) w(z) \,\mathrm{d}s = w(x).$$

Similarly we deduce the other integral to get $w(x) = v(y)$, hence $G(x,y) = G(y,x)$.

2. Consider $w(x) = G(x,y) = \Gamma(x-y) + h$ is harmonic in $\Omega \setminus \{y\}$, since h is harmonic in Ω, and $|\Omega| < \infty$, we have $|h| < \infty$ in Ω. But when $x \to y$, $\Gamma(x-y) \to -\infty$, hence there exist $r > 0$ s.t. $w < 0$ in $\partial B_r(y)$

   By the strong maximum principle, since $w(x) \equiv 0$ in $\partial\Omega$, we have $w < 0$ in $\Omega \setminus B_r(y)$.

3. Fix $x_0 \in \partial\Omega$, since f is bounded, w.l.o.g, we assume $f(y) \equiv 1$, then

$$\int_\Omega |G(x,y)|\,\mathrm{d}y = \int_{\Omega\cap B_{2\epsilon}(x_0)} |G(x,y)|\,\mathrm{d}y + \int_{\Omega\setminus B_{2\epsilon}(x_0)} |G(x,y)|\,\mathrm{d}y =: I + J.$$

   To estimate I, since $G(x_0,y) = 0$, for ϵ sufficiently small, we have $h(x,y) > 0$ where x in $B_\epsilon(x_0)$, hence $|G(x,y)| < |\Gamma(x,y)|$, and $B_{2\epsilon}(x_0) \subset B_{3\epsilon}(x)$, so we have

$$I \le \int_{\Omega\cap B_{3\epsilon}(x_0)} |G(x,y)|\,\mathrm{d}y \le \int_{\Omega\cap B_{3\epsilon}(x_0)} |\Gamma(x,y)|\,\mathrm{d}y \le C\epsilon^2,$$

   then we have

$$\lim_{x\to x_0} I = 0.$$

   To estimate J, we have

$$|G(x,y)| \le |\Gamma(\epsilon)| \quad \forall y \in \Omega \setminus B_{2\epsilon}(x_0), x \in B_\epsilon(x_0),$$

   and for any fixed y

$$G(x,y) \to 0 \quad \text{as} \quad x \to x_0,$$

   use Lebesgue's dominated convergence theorem, we have

$$\lim_{x\to x_0} J = 0,$$

   hence we completes the proof. □

Proof 2.4 It is suffices to show that U is harmonic on T, $\forall x \in T$.

There is a ball $B = B(x)$, by the reflection, we have U is continuous on ∂B, thus we can find a harmonic functions v in B, and $v \equiv U$ on ∂B by Poisson integral formula.

Since U is defined as odd reflection, use Poisson integral we have $v \equiv u \equiv 0$ on T, use the maximum principle on $\Omega^+ \cap B$ and $\Omega^- \cap B$, we get $U \equiv v$ in B. Hence U is harmonic in $\Omega^+ \cup T \cup \Omega^-$. □

Proof 2.5 WLOG we set the annular region as $B_R(0) \setminus B_r(0)$, where $R > r > 0$. All we need to do is let the Green's function vanishes on $\partial B_R \cup \partial B_r$ by combine the fundamental solution Γ.

1. Let it vanishes on ∂B_R, we have

$$h_1 = \Gamma(x,y) - \frac{|y|}{R}\Gamma(x, \frac{R^2}{|y|^2}y),$$

2. let it vanishes on ∂B_r, we have

$$h_2 = h_1 - \frac{|y|}{r}\Gamma(x, \frac{r^2}{|y|^2}y) + \frac{R}{r}\Gamma(x, \frac{r^2}{R^2}y),$$

3. let it vanishes on ∂B_R, we have

$$h_3 = h_2 + \frac{r}{R}\Gamma(x, \frac{R^2}{r^2}y) - \frac{r|y|}{R^2}\Gamma(x, \frac{R^4}{r^2|y|^2}y),$$

4. let it vanishes on ∂B_r, we have

$$h_4 = h_3 - \frac{R|y|}{r^2}\Gamma(x, \frac{r^4}{R^2|y|^2}y) + \frac{R^2}{r^2}\Gamma(x, \frac{r^4}{R^4}y),$$

5. let it vanishes on ∂B_R, we have

$$h_5 = h_4 + \frac{r^2}{R^2}\Gamma(x, \frac{R^4}{r^4}y) - \frac{r^2|y|}{R^3}\Gamma(x, \frac{R^6}{r^4|y|^2}y),$$

6. $\cdots$

Set

$$\begin{aligned} g_n = & \left(\frac{r}{R}\right)^{n-1}\Gamma(x, \left(\frac{r}{R}\right)^{2(1-n)}y) - \left(\frac{r}{R}\right)^{n-1}\frac{|y|}{R}\Gamma(x, \left(\left(\frac{r}{R}\right)^{n-1}\frac{|y|}{R}\right)^{-2}y) \\ & - \left(\frac{R}{r}\right)^{n-1}\frac{|y|}{r}\Gamma(x, \left(\left(\frac{R}{r}\right)^{n-1}\frac{|y|}{r}\right)^{-2}y) + \left(\frac{R}{r}\right)^{n}\Gamma(x, \left(\frac{R}{r}\right)^{-2n}y), \end{aligned}$$

we get the Green's function

$$G(x, y) = \sum_{n=1}^{\infty} g_n$$

where the RHS is convergence by M-test. $\square$

Proof 2.6 Fix $y \in B_R$, $\forall x \in B_R$

$$\left|\frac{x-y}{R-|y|}\right| \geq 1 = \left|\frac{x}{R}\right|,$$

hence

$$(R-|y|)^n \int_{\partial B_R} \frac{u\,\mathrm{d}s}{|x-y|^n} \leq R^n \int_{\partial B_R} \frac{u\,\mathrm{d}s}{|x|^n},$$

it implies that

$$u(y) \leq \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}}u(0).$$

Similarly we can get the lower bound. $\square$

Proof 2.7 $\forall z \in \partial\Omega$, w.l.o.g we assume $z = 0$ and $x_n = 0$ is the tangent hyperplane of $\partial\Omega$ at z. By the definition of C^2 boundary, we can choose $x_n = 0$ as the supporting hyperplane and define a support function f in B_ϵ is $C^2(\mathbb{R}^{n-1})$ and satisfying

$$f(x') = x_n > 0 \quad \text{in} \quad B_\epsilon \setminus \{0\}, \text{ and } \quad f(0) = 0,$$

where $x' = (x_1, \cdots, x_{n-1})$, and $(x', x_n) \in \partial\Omega$. Then we consider a ball $B_r(x_0)$ where $r \leq \epsilon$ tangent with the supporting hyperplane $x_n = 0$ at z, we can define the support function

$$g(x') = r - \sqrt{r^2 - |x'|^2},$$

set $h = g - f$, and $h(0) = 0$, all we need to do is to prove there exist r s.t. $D^2h(0)$ is positive-definite. Since

$$h_{ij} = (r^2 - |x'|^2)^{-3/2} x_i x_j + \delta_{ij}(r^2 - |x'|^2)^{-1/2} - \partial_{ij} f(x'),$$

we have

$$(h_{ij}(0)) = r^{-1} I - (\partial_{ij} f(0)),$$

hence we choose r s.t.

$$\frac{1}{r} \geq \max |\lambda_i|,$$

where λ_i are eigenvalues of $D^2 f(0)$. □

4 G-T Chapter 4 Solutions

Proof 4.7

$$\begin{aligned}
\triangle_x u(x/|x|^2) &= \sum_i \frac{\partial}{\partial x_i}\left(\sum_k \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i}\right) = \sum_i \sum_l \frac{\partial}{\partial y_l}\left(\sum_k \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i}\right)\frac{\partial y_l}{\partial x_i} \\
&= \sum_i \sum_l \sum_k \left(\frac{\partial^2 u}{\partial y_l \partial y_k}\frac{\partial y_k}{\partial x_i} + \frac{\partial^2 y_k}{\partial x_i \partial y_l}\frac{\partial u}{\partial y_k}\right)\frac{\partial y_l}{\partial x_i} \qquad (2) \\
&= \sum_i \sum_l \sum_k \frac{\partial^2 u}{\partial y_l \partial y_k}\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_i} + \sum_i \sum_k \frac{\partial^2 y_k}{\partial x_i^2}\frac{\partial u}{\partial y_k}
\end{aligned}$$

since

$$\begin{aligned}
\frac{\partial y_k}{\partial x_i} &= \partial_i \frac{x_k}{|x|^2} = \frac{\delta_{ik}}{|x|^2} - 2\frac{x_i x_k}{|x|^4}, \\
\frac{\partial^2 y_k}{\partial x_i^2} &= \frac{-2\delta_{ik} x_i}{|x|^4} + 8\frac{x_i^2 x_k}{|x|^6} - 2\frac{x_k}{|x|^4} - 2\frac{x_i \delta_{ik}}{|x|^4} \\
&= -4\frac{\delta_{ik} x_i}{|x|^4} + 8\frac{x_i^2 x_k}{|x|^6} - 2\frac{x_k}{|x|^4},
\end{aligned}$$

we have

$$(2) = \sum_i \sum_l \sum_k \frac{\partial^2 u}{\partial y_l \partial y_k} \left(\frac{\delta_{ik}}{|x|^2} - 2\frac{x_i x_k}{|x|^4} \right) \left(\frac{\delta_{il}}{|x|^2} - 2\frac{x_i x_l}{|x|^4} \right) \tag{3}$$

$$+ \sum_i \sum_k \frac{\partial u}{\partial y_k} \left(-4\frac{\delta_{ik} x_i}{|x|^4} + 8\frac{x_i^2 x_k}{|x|^6} - 2\frac{x_k}{|x|^4} \right). \tag{4}$$

For (3) we have

$$\begin{aligned}
(3) &= \sum_i \sum_l \sum_k \frac{\partial^2 u}{\partial y_l \partial y_k} \frac{\delta_{ik}\delta_{il}}{|x|^4} - 4\sum_l \sum_k \frac{\partial^2 u}{\partial y_l \partial y_k} \frac{x_k x_l}{|x|^6} + 4\sum_i \sum_l \sum_k \frac{x_i^2 x_k x_l}{|x|^8} \\
&= \sum_i \frac{\partial^2 u}{\partial y_i^2} \frac{1}{|x|^4} - 4\sum_l \sum_k \frac{\partial^2 u}{\partial y_l \partial y_k} \frac{x_k x_l}{|x|^6} + 4\sum_i \frac{x_i^2}{|x|^2} \sum_l \sum_k \frac{x_k x_l}{|x|^6} \\
&= \sum_i \frac{\partial^2 u}{\partial y_i^2} \frac{1}{|x|^4},
\end{aligned}$$

and for (4) we have

$$\begin{aligned}
(4) &= -4\sum_k \frac{\partial u}{\partial y_k} \frac{x_k}{|x|^4} + 8\sum_k \frac{\partial u}{\partial y_k} \frac{x_k}{|x|^4} - 2n\sum_k \frac{\partial u}{\partial y_k} \frac{x_k}{|x|^4} \\
&= 2(2-n)\sum_k \frac{\partial u}{\partial y_k} \frac{x_k}{|x|^4}.
\end{aligned}$$

Consider

$$\triangle v(x) = \triangle u(x)|x|^{2-n} + u(x)\,\triangle |x|^{2-n} + 2\nabla u \nabla |x|^{2-n},$$

since

$$\begin{aligned}
\nabla u \nabla |x|^{2-n} &= \sum_i \frac{\partial u}{\partial x_i} \frac{|x|^{2-n}}{x_i} = \sum_i \left(\sum_j \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right) (2-n)|x|^{-n} x_i \\
&= (2-n)|x|^{-n} \sum_i \sum_j \frac{\partial u}{\partial y_j} \left(\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4} \right) x_i \\
&= (2-n) \left(|x|^{-2-n} \sum_j \frac{\partial u}{\partial y_j} x_j - 2|x|^{-2-n} \sum_j \frac{\partial u}{\partial y_j} x_j \right) \\
&= -(2-n)|x|^{-2-n} \sum_j \frac{\partial u}{\partial y_j} x_j,
\end{aligned}$$

hence

$$\begin{aligned}
\triangle v(x) &= \triangle_y u |x|^{-2-n} + 2(2-n)\sum_k \frac{\partial u}{\partial y_k} \frac{x_k}{|x|^{2+n}} - 2(2-n)\sum_k \frac{\partial u}{\partial y_k} \frac{x_k}{|x|^{2+n}} \\
&= \triangle_y u |x|^{-2-n}.
\end{aligned}$$

□