

back to needham pg. 32,  $\delta s = |\delta z|$   
 here  $s$  is the arc length and  $z$  is a point on the complex plane, how do we know this for sure?

Great question — this is exactly the kind of subtlety Needham likes to sneak in. Let's pin it down carefully.

### What's going on on p.32

On that page,  $z$  lives in the **complex plane** and we are measuring *arc length in the plane itself*, not on the curved surface anymore.

So here's the key idea up front:

**In the complex plane, arc length is defined using the Euclidean metric.**

Under that metric, an infinitesimal arc length is exactly the magnitude of the complex displacement.

That's why

$$\delta s = |\delta z|$$

No magic — it's a definition + geometry.

### Step 1: What is $\delta z$ ?

Write

$$z = x + iy$$

so

$$\delta z = \delta x + i \delta y$$

This is just a tiny displacement vector in the plane.

## Step 2: What does arc length mean in the plane?

Arc length  $s$  is measured using the **Euclidean distance**:

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

Now look at the complex magnitude:

$$|\delta z| = \sqrt{(\delta x)^2 + (\delta y)^2}$$

So immediately:

$$\boxed{\delta s = |\delta z|}$$

This is not something we *prove* — it's how length works in the complex plane.

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## Step 3: Why Needham is allowed to do this

Earlier in the book, Needham carefully distinguishes:

- **Length on the surface** → curved geometry, hats, scale factors, etc.
- **Length in the complex plane** → flat, Euclidean geometry

At this point on p.32:

- We are **already in the plane**
- The "map" has happened
- We are measuring separation **after mapping**

So:

- $s$  = arc length *in the plane*
- $z$  = coordinate in the plane
- Euclidean metric applies
- Therefore  $\delta s = |\delta z|$

Later, when he writes

$$\hat{\delta s} \approx \Lambda(z, \gamma) \delta s$$

he is **explicitly correcting** for the fact that the surface length is *not* Euclidean.

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## Intuition check (very important)

Think physically:

- You take a tiny step in the complex plane
- That step is a vector
- The length of that step is **the vector's magnitude**

If this weren't true, the complex plane would not be a flat geometry — and all of complex analysis would collapse.

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## One-sentence summary

We know  $\delta s = |\delta z|$  because in the complex plane arc length is defined by the Euclidean metric, and the Euclidean length of a displacement is exactly the magnitude of the corresponding complex number.

If you want, next we can:

- connect this directly to the metric tensor language, or
- contrast it explicitly with how  $\hat{\delta}s$  is computed on the surface, or
- tie it to conformality and why angles survive but lengths don't.



I am failing to see the difference between  $s$  and  $\delta s$ .

Totally fair — this is one of those “once it clicks, it clicks forever” distinctions. Let's slow it way down and strip it of symbols.

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## Big picture first (no math)

- $s = a \text{ length}$

Think: “How far along the curve am I?”

- $\delta s = a \text{ tiny change in that length}$

Think: “If I move just a little bit, how much farther did I go?”

So:

- $s$  is an **accumulated total**
- $\delta s$  is an **increment**

Same relationship as:

- position vs displacement
- time vs elapsed time
- money vs the change in money

## Concrete physical analogy

Imagine a hiking trail.

- $s$ : the number on the trail marker (e.g. "3.2 miles from start")
- $\delta s$ : the tiny step you just took

You **never** confuse:

- "I am at mile 3.2"
- with "I just walked 2 feet"

That's exactly the distinction here.

## Now put it in math terms

Let a curve be parameterized by something (time  $t$ , angle, whatever).

- $s(t)$  = arc length from the start up to parameter value  $t$
- $\delta s$  = arc length between two *nearby* points on the curve

Formally:

$$\delta s = s(t + \delta t) - s(t)$$

In the limit, this becomes:

$$ds = \left| \frac{dz}{dt} \right| dt$$

But Needham is deliberately staying **pre-calculus-notation** here.

## Why Needham uses $\delta$ instead of $d$

This is important.

- $d$ : exact differential (calculus limit)
- $\delta$ : *small but finite* separation

Needham is saying:

"Take two nearby points, not an infinitesimal yet."

So:

- $\delta s$  = actual small measured distance
- not yet a derivative

That's why he can talk about **directions**  $\gamma$  and **finite distortions**.

## Why your confusion is reasonable

Because both are called "arc length."

But they play **different roles**:

Symbol	Meaning	Type
$s$	total arc length along curve	function / accumulated quantity
$\delta s$	small piece of arc length	increment

And crucially:

$$s \neq \delta s$$

just like

$$x \neq \delta x$$

## One killer sentence (remember this)

$s$  is a coordinate along the curve;  $\delta s$  is a small ruler measurement along that coordinate.

If you want, next we can:

- draw the exact picture Needham has in mind,
- translate this into strict differential-geometry language,
- or connect it directly to why conformal maps preserve angles but not lengths.



is it true that as soon as you have the points  $q$  and  $z$  then you have  $s$ ?

Short answer: **yes — but with an important qualifier.**

Long answer: this is exactly where the geometry lives.

## Precise statement

Given two points  $q$  and  $z$ , you *do not* automatically have  $s$ .

You *do* automatically have  $\delta s$ .

That distinction matters a lot.

## Why you *do* get $\delta s$

If  $q$  and  $z$  are two nearby points **in the complex plane**, then the arc length *between them* is just the straight-line distance:

$$\delta s = |z - q|$$

No extra structure needed. The plane is flat.

So:

- points  $\rightarrow$  displacement  $\rightarrow$  length

That's immediate.

## Why you *don't* automatically get $s$

$s$  is **not** just "distance between two points."

$s$  means:

"distance *along the curve* measured from a chosen origin."

To define  $s$ , you need **three things**:

1. A curve
2. A starting point (where  $s = 0$ )
3. A path along the curve

Two points alone do not tell you:

- which path to take (if the curve loops),
- or where the counting starts.

## Example that breaks the idea

Take a circle.

- Two points  $q$  and  $z$  on the circle
- There are **two** arc lengths between them:
  - clockwise
  - counter-clockwise

Same points, different  $s$ .

So:

- $\delta s$ : well-defined (local separation)
- $s$ : ambiguous without extra choices

## How this fits Needham exactly

On p.32 Needham is doing **local geometry**.

That's why he works with:

- $\delta z$
- $\delta s$
- directions  $\gamma$

He is *deliberately avoiding* global quantities like  $s$ .

## Ultra-compact takeaway

Points determine local separation  $\delta s$ .

A curve + origin determine the accumulated length  $s$ .

If you want, next we can:

- redraw this using Needham's hemisphere  $\rightarrow$  plane picture,
- explain how  $s$  becomes a coordinate,
- or show how this leads directly to the scale factor  $\Lambda(z, \gamma)$ .



do 1 and 2

Perfect — let's do (1) the picture and (2) how  $s$  becomes a coordinate, *strictly in Needham's language*.

## 1 The picture Needham has in mind (hemisphere $\rightarrow$ plane)

Think of **three layers**.

### Layer A: the surface (hemisphere)

- You have a curved surface  $S$  (Needham's hemisphere).
- Pick a curve drawn on that surface.
- Choose a starting point  $P$  on the curve.
- Define

$\hat{s}$  = arc length along the surface from  $P$



So on the **surface**:

- $\hat{s}$  = accumulated length
- $\hat{\delta}s$  = small surface separation between nearby points

This length depends on the surface's metric.

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## Layer B: the map to the complex plane

Now you **map** the surface to the complex plane:

$$\text{surface point} \longrightarrow z \in \mathbb{C}$$

- Curves on the surface become curves in the plane.
  - Distances generally change.
  - Angles survive (conformal map).
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## Layer C: the complex plane

In the plane:

- The image curve is just an ordinary plane curve.
- Arc length is Euclidean.

For two nearby image points:

$$\delta s = |\delta z|$$

⚠ This  $s$  is **not** the same as  $\hat{s}$ .

Needham's entire game is to relate:

$$\hat{\delta}s \text{ to } \delta s$$

That's where  $\Lambda(z, \gamma)$  enters.

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## 2 How $s$ becomes a *coordinate*

This is subtle and important.

## Step 1: Pick an origin

Choose a point  $z_0$  on the plane curve and declare:

$$s(z_0) = 0$$

This choice is arbitrary but necessary.

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## Step 2: Accumulate local lengths

Move along the curve in the plane.

Each tiny step contributes:

$$\delta s = |\delta z|$$

Adding them up:

$$s = \sum |\delta z|$$

In the limit:

$$s = \int |dz|$$

So:

- $s$  labels *where you are along the curve*
  - it is **intrinsic** to the curve
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## Step 3: Why this is a coordinate, not just a number

Once defined,  $s$  behaves like:

- a 1D coordinate along the curve
- independent of how the curve is parameterized

If you reparameterize the curve:

- $s$  does not change
- only how fast you move along it does

That's why arc length is special.

## Key contrast (this is the heart of your confusion)

Object	Meaning
$z$	position in the plane
$\delta z$	tiny displacement
$\delta s$	length of that displacement
$s$	total accumulated length

So:

- $q, z \Rightarrow \delta s$
- curve + origin + accumulation  $\Rightarrow s$

## One sentence that locks it in

$s$  is not given by points; it is built by adding up infinitely many  $\delta s$ 's along a chosen path.

If you want, next we can:

- explicitly connect this to  $\gamma$  (direction dependence), or
- rewrite this in full differential-geometry metric language so it matches what you'll later see in graphics and PDEs.



must arc length contain an orientation (CW, CCW)? I  
thought arc length is just a single number

This is an *excellent* question — you're poking exactly at the boundary between **length** and **coordinate**. The short answer is:

Arc length itself does *not* contain an orientation.

Arc-length *parameterization* does.

Let's separate those cleanly.

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## 1 Arc length as a number (no orientation)

If I ask:

"What is the length of this curve segment?"

the answer is just a **non-negative number**:

$$L \geq 0$$

No CW, no CCW, no sign.

Length never points anywhere.

So your intuition is **100% correct** here.

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## 2 Where orientation sneaks in (this is the trick)

Orientation appears **only** when arc length is used as a coordinate.

The moment you say:

"Let  $s$  measure distance *along the curve* from a chosen start point"

you have secretly added:

1. a **starting point**, and
2. a **direction in which  $s$  increases**

That's orientation.

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## 3 Same curve, two valid arc-length coordinates

Take a circle.

- Choose a start point  $P$

- Walk **clockwise**  $\rightarrow s$  increases
- Walk **counter-clockwise**  $\rightarrow s$  also increases

Both are legitimate arc-length parameters.

They differ by:

$$s_{CCW} = L - s_{CW}$$

Same geometry. Different coordinates.

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#### 4 How Needham uses this (important)

Needham is careful:

- $\delta s \rightarrow$  **unsigned length**
- $\gamma \rightarrow$  **direction information**

That's why he writes:

$$\delta z = e^{i\gamma} \delta s$$

- $\delta s \geq 0$  (pure magnitude)
- $e^{i\gamma}$  carries direction

So orientation is **factored out**, not baked into length.

This is very clean geometry.

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#### 5 Why your confusion was unavoidable

In physics and geometry books:

- sometimes  $s$  means "distance"
- sometimes it means "distance-along-curve coordinate"

Same letter. Different roles.

Needham is switching roles *mid-chapter*, but correctly.

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### Final lock-in sentence

**Length is a number; orientation appears only when you decide how that number is accumulated along a curve.**

If you want, next we can:

- rewrite Needham's formulas with explicit "unsigned length + direction" everywhere, or
- show how this exact idea becomes tangent vectors and line integrals later in the book and in graphics (hair, curves, ribbons).

