# Contents



# Counting & Random Events

Bags of red & black balls

- With replacement

- Without replacement

Bag with 10 balls - how many ways can we take out 2 balls?

1. With replacement - $10 \times 10 = 100$
2. Without replacement - $10 \times 9 = 90$

3 red & 7 black balls - how many way can we take out:

- 1 red then 1 black
    1. With replacement - $\frac{3\times 7}{10\times 10}$
    2. Without replacement - $\frac{3\times 7}{10\times 9}$
- 2 reds
    1. With replacement - $\frac{3\times 3}{10\times 10}$
    2. Without replacement - $\frac{3\times 2}{10\times 9}$
- 2 red & 3 black
    1. With replacement - $\frac{(3\times 3)(7\times 7\times 7)}{10^5} = (\frac{3}{10})^2(\frac{7}{10})^3$
    2. Without replacement - $\frac{(3\times 2)(7\times 6\times 5)}{10\times 9\times 8\times 7\times 6} \times \frac{5!}{2!3!}$

# Definitions

- Sample space
- Random event
- Random variable - Mapping from $S \to \mathbb{R}$

## Conditional Probability

$P(E \mid F) =$ the probability of $E$ given $F$ has already happened

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

## Chain Rule

$$P(E \cap F) = P(E \mid F)P(F)$$

## Marginalisation

$$P(E) = P(E \cap F_1) + P(E \cap F_2) + \cdots + P(E \cap F_n)$$
$$P(E) = P(E \mid F_1)P(F_1) + P(E \mid F_2)P(F_2) + \cdots + P(E \mid F_n)P(F_n)$$

given

- $F_1, F_2, \ldots, F_n$ are mutually exclusive
- $F_1 \cup F_2 \cup \cdots \cup F_n = S$

## Bayes Rule

$$P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F)}$$

## Independence

$$P(E \cap F) = P(E)P(F)$$
$$P(E \mid F) = P(E)$$

### Example Question

$X \in \{1,2,3\}, Y \in \{1,2,3\}, V = X + Y \in \{2,3,4,5,6\}$

Are $X$ & $V$ Independent?

$P(V = 2 \land X = 2) = 0$

$P(V = 2) = (\frac{1}{3})^2$

$P(X = 2) = \frac{1}{3}$

$P(V = 2) \times P(X = 2) \neq P(V = 2 \land X = 2)$

## Expected Value

$$E[X] = \sum_{i=1}^{n} x_i P(X = x_i)$$

### Linearity

Random variable $X$ takes values $x_1, x_2, \dots, x_n$ so,

$E[aX+b] = \sum_{i=1}^{n}(ax_i + b)P(X = x_i)$

$= \sum_{i=1}^{n} ax_i P(X = x_i) + \sum_{i=1}^{n} bP(X = x_i)$

$= a\sum_{i=1}^{n} x_i P(X = x_i) + b\sum_{i=1}^{n} P(X = x_i)$

$= aE[X] + b$

### Two Random Variables

$E[aX+bY] = \sum_x \sum_y (ax+by)P(X = x \cap Y = y)$

$= a\sum_x \sum_y xP(X = x \cap Y = y) + b\sum_x \sum_y yP(X = x \cap Y = y)$

$= a\sum_x xP(X = x) + b\sum_y yP(Y = y)$

$= aE[X] + bE[Y]$

since $\sum_y P(X = x \cap Y = y) = P(X)$

### Independent Random Variables

$E[XY] = \sum_x \sum_y xyP(X = x \text{ and } Y = y)$

$= \sum_x \sum_y xyP(X = x)P(Y = y)$

$= \sum_x xP(X = x) \sum_y yP(Y = y)$

$= E[X]E[Y]$

## Variance

$$Var(X) = \sum_{i=1}^{n}(x_i - \mu)^2 P(x_i)$$

with $\mu = E[X]$

### Non Linearity

$Var(aX+b) = E[(aX+b)^2] - E[aX+b]^2$

$= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2$

$= a^2E[X^2] + 2abE[X] + b^2 - a^2E[X]^2 - 2abE[X] - b^2$

$= a^2E[X^2] - a^2E[X]^2$

$= a^2(E[X^2] - E[X]^2)$

$= a^2 Var(X)$

**Indepedent Random Variables**

$Var(X+Y) = E[(X+Y)^2] - E[X+Y]^2$

$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$

$= E[X^2] + 2E[XY] + E[Y^2] - E[X]^2 - 2E[X]E[Y] - E[Y]^2$

$= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 + 2E[XY] - 2E[X]E[Y]$

$= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2$

$= Var(X) + Var(Y)$

# Inequalities

- Markov
- Cheyshev
  - Law of large numbers
- Chernoff
  - Frequency interpretation of probability

**Will give us Chebyshev and Chernoff if needed**

## Markov

$$P(X \geq a) \leq \frac{E(X)}{a} \text{ for all } a > 0$$

## Chebyshev

$$P(\mid X - \mu \mid \geq k) \leq \frac{\sigma^2}{k^2} \text{ for all } k > 0$$

## Chernoff

$$P(X \geq a) \leq \min_{t>0} e^{ta} e^{\log E(e^{tX})}$$

# Confidence Intervals

$P(a \leq X \leq b)$

- Inequalities (esp. chernoff & cheybshev)
- Cental limit theorem
- Bootstrapping

## CLT

$$X = \frac{1}{N}\sum_{k=1}^{N} X_k \sim N(E(X_i), \frac{Var(X_i)}{n})$$

# Linear Regression Model

$$Y = \sum_{i=1}^{m} \Theta^{(i)} x^{(i)} + M, M \sim N(0,1), \Theta^{(i)} \sim N(0, \lambda)$$

$$f_{D|\Theta}(d \mid \vec{\theta}) \propto L(\theta) = \exp(-\sum_{j=1}^{n}(y_j - \sum_{i=1}^{m} \theta^{(i)} x_j^{(i)} * (i))^2/2), f_{\Theta^{(i)}}(\theta) \propto \exp(-\theta^2/2\lambda)$$