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A generalisation

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$$Q \;=\; \lim_{n\to\infty} Q_n$$

$$Q_{n+1}(s,s') \;:=\; R(s,s') + \frac{1}{2}\max\{Q_n(s',s'') \mid arc_{=}(s',s'')\} \tag{1}$$

$\leadsto$

$$\begin{aligned} Q_{n+1}(s,a) \;:=\; & \alpha\big[R(s,a) + \gamma\max\{Q_n(s',a') \mid a' \in A\}\big] \\ & +\; (1-\alpha)Q_n(s,a) \end{aligned} \tag{2}$$

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(1)	(2)
s'	a
1	α
$\frac{1}{2}$	γ

(1) is (2) with action a resulting in s' deterministically for $\alpha = 1$, with $\gamma = \frac{1}{2}$

Markov decision process (MDP)

a 5-tuple $\langle S, A, p, r, \gamma \rangle$ consisting of

- a finite set S of states $s, s', \ldots$
- a finite set A of actions $a, \ldots$
- a function $p : S \times A \times S \to [0, 1]$

$$p(s, a, s') \;=\; \text{prob}(s'|s, a) \;=\; \text{how likely is } s' \text{ after doing } a \text{ at } s$$

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Missing: policy $\pi : S \to A$ (what to do at s)

Exercise (Poole & Mackworth, chap 9)

Sam is either fit or unfit

$$S = \{\text{fit, unfit}\}$$

and has to decide whether to exercise or relax

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$p(s, a, s')$ and $r(s, a, s')$ are a-table entries for row s, col s'

exercise	fit	unfit
fit	.99, 8	
unfit	.2, 0	

relax	fit	unfit
fit	.7, 10	
unfit	0, 5	

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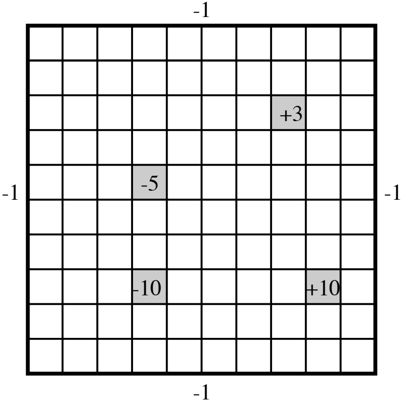
unfit	.2, 0	.8, 0

relax	fit	unfit
fit	.7, 10	.3, 10
unfit	0, 5	1, 5

Entries in red follow from assuming immediate rewards do not depend on the resulting state, and

$$\sum_{s'} p(s, a, s') = 1$$

Grid World



Poole & Mackworth, 9.5

states: 100 positions
actions: up, down, left, right
punish -1 when banging into wall
& 4 reward/punish states
prob: 0.7 as directed (if possible)
...

Policy from an MDP

Given state s, pick action a that maximizes return

$$Q(s,a) \ := \ \overbrace{\sum_{s'} p(s,a,s')}^{\text{different outcomes } s'} \left(\underbrace{r(s,a,s')}_{\text{immediate}} + \overbrace{\gamma V(s')}^{\text{discounted future}} \right)$$

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e.g., the greedy Q-policy above

$$\pi(s) \;:=\; \arg\max_a Q(s,a)$$

for

$$Q(s,a) \;=\; \sum_{s'} p(s,a,s')\big(r(s,a,s') + \gamma \max_{a'} Q(s',a')\big)$$

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Mutual recursion between Q/V and π

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Focus on Q, approached in the limit

$$\lim_{n \to \infty} q_n$$

from iterates

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In case $p(s,a,s') = 1$ for some s' (necessarily unique), the iterates simplify to

$$\begin{aligned} q_0(s,a) &:= r(s,a,s') \\ q_{n+1}(s,a) &:= r(s,a,s') + \gamma \max_{a'} q_n(s',a') \end{aligned}$$

Determinstic actions and absorbing states (game over)

Fix an MDP with min reward m.

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$$
\begin{aligned}
Q(s, a) &= r(s, a, s) + \gamma V(s) \\
V(s) &= \frac{r_s}{1 - \gamma} \quad \text{where } r_s = \max_a r(s, a, s)
\end{aligned}
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A state s is a *sink* if it is absorbing and $r(s, a, s) = m$ for all a.

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Let

$$A(s) \;:=\; \{a \in A \mid a \text{ is not an } s\text{-drain}\}$$

so if $A(s) \neq \emptyset$,

$$V(s) \;=\; \max\{Q(s,a) \mid a \in A\} \;=\; \max\{Q(s,a) \mid a \in A(s)\}$$

Arcs & goals as a deterministic MDP ($p \in \{0, 1\}$)

Given *arc* and goal set G, let

$$A \;=\; \{s \mid (\exists s')\; arc_{=}(s', s)\}$$

where for each $a \in A$,

$$p(s, a, s') \;=\; \begin{cases} 1 & \text{if } a = s' \text{ and } arc_{=}(s, s') \\ 0 & \text{otherwise} \end{cases}$$

$$r(s, a, s') \;=\; \begin{cases} R(s, s') & \text{if } a = s' \text{ and } arc_{=}(s, s') \\ \text{anything} & \text{otherwise} \end{cases}$$

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Satisfy prob constraint $\sum_{s'} p(s, a, s') \;=\; 1$ via sink state $\perp \notin A \cup dom(arc)$, requiring of every $a \in A$ and $s \in S$,

$$p(s, a, \perp) \;=\; \begin{cases} 1 & \text{if not } arc_{=}(s, a) \\ 0 & \text{otherwise} \end{cases}$$

$$p(\perp, a, s) \;=\; \begin{cases} 1 & \text{if } s = \perp \\ 0 & \text{otherwise} \end{cases}$$

$$r(s, a, \perp) \;=\; \text{min reward}$$