# Contents



# Conditional Probability

**Conditional probability** is the probability that event $E$ occurs *given* that event $F$ has already occured. Call this "conditioning" on $F$. Writen as $P(E \mid F)$.

- Same meaning as $P(E$ given $F$ already observed)
- Its a probability (satisfies all the axioms, will see this shortly), with:
  - Sample space $S$ resticted to those outcomes consistent with $F$, i.e. $S \cap F$
  - Event space $E$ restricted to those outcomes consistent with $F$, i.e. $E \cap F$

With equally likely outcomes

$P(E \mid F) = \frac{\text{number of outcomes in } E \text{ consistent with } F}{\text{number of outcomes in } S \text{consistent with } F} = \frac{|E \cap F|}{|S \cap F|}$

Note that $\mid S \cap F \mid = \mid F \mid$ so

$P(E \mid F) = \frac{|E \cap F|}{|S \cap F|} = \frac{|E \cap F|}{|F|} = \frac{|E \cap F|}{|S|} \frac{|S|}{|F|} = \frac{\frac{|E \cap F|}{|S|}}{\frac{|F|}{|S|}} = \frac{P(E \cap F)}{P(F)}$

General definition: $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$ where $P(F) > 0$.

Implies $P(E \cap F) = P(E \mid F)P(F)$ known as the *chain rule* - it's important!

If $P(F) = 0$?

- $P(E \mid F)$ is undefined
- Can't condition on something that can't happened

P(E|F) is a probability - it satisfies all the properties of ordinary probabilities

- $0 \leq P(E \mid F) \leq 1$
- $P(S \mid F) = 1$
- If $E_1$, $E_2$ are mutually exclusive events then $P(E_1 \cup E_2 \mid F) = P(E_1 \mid F) + P(E_2 \mid F)$

## Marginalisation

Suppose we have mutually exclusive events $F_1$, $F_2$, ..., $F_n$ such that $F_1 \cup F_2 \cup \cdots \cup F_n = S$ then

$$P(E) = P(E \cap F_1) + P(E \cap F_2) + \cdots + P(E \cap F_n)$$

Marginalisation is very handy, example:

- Roll two coins. What is the probability that the first coin is heads?
- Event E is the first coin heads, $F_1$ is second coin heads, $F_2$ is second coin tails
- $P(E) = P(E \cap F_1) + P(E \cap F_2) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$

## Bayes Rule

Recall $P(E \cap F) = P(E \mid F)P(F)$

Clearly, and also $P(F \cap E) = P(F \mid E)P(E)$

But $P(E \cap F) = P(F \cap E)$, so $P(E \mid F)P(F) = P(F \mid E)P(E)$

i.e. $P(E \mid F) = \frac{P(F|E)P(E)}{P(F)}$