Constraint Satisfaction Problem [Var, Dom, Con]

- a list $\mathrm{Var} = [X_1, \ldots, X_n]$ of *variables* X_i
- a list $\mathrm{Dom} = [D_1, \ldots, D_n]$ of finite sets D_i of size s_i
- a finite set Con of *constraints* that may or may not be satisfied by (a node) instantiating X_i with a value in D_i (search space size $\prod_{i=1}^{n} s_i$)

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X = a, Y = b
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Order-independent unification (Martelli-Montanari)

Input: set $\mathcal{E}$ of pairs $[t, t']$
Output: substitution $[[X1, t1], \ldots, [Xk, tk]]$ unifying pairs in $\mathcal{E}$

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Simplify $\mathcal{E}$ non-deterministically until no longer possible

1. $[f(s1, \ldots, sk), f(t1, \ldots, tk)]$ (allowing $k = 0$)
   $\Longrightarrow$ replace by pairs $[s1, t1], \ldots, [sk, tk]$
2. $[f(s1, \ldots, sk), g(t1, \ldots, tm)]$ where $f \neq g$ or $k \neq m$
   $\Longrightarrow$ halt with failure
3. $[X, X] \Longrightarrow$ delete
4. $[t, X]$ where t is not a var $\Longrightarrow$ replace by $[X, t]$
5. $[X, t]$ where $X \notin Var(t)$ and X occurs elsewhere
   $\Longrightarrow$ apply $[X, t]$ to all other pairs
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N.B. Prolog omits *occurs check* $X \in Var(t)$ in 5, 6 for speed-up

Instantiate before negating (as failure)

```
% \+p :- (p,!,fail); true.
```

```
p(X) :- \+q(X), r(X).
q(a).  q(b).
r(a).  r(c).
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| ?- p(X).                          % contra ?- p(c).
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Generate-and-test

brute force: instantiate all variables before testing constraints

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genTest(D1...Dn) :- node(X1...Xn,D1...Dn),
                    constraint(X1...Xn).
node(X1...Xn,D1...Dn) :- member(X1,D1),...,
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For each of the $\prod_{i=1}^{n} s_i$-choices of `X1...Xn` such that

`node(X1...Xn,D1...Dn)`

(with `Di` of size `si`), assume

`constraint(X1...Xn)`

can be checked within a polynomial of `X1...Xn`.

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Nodes are generated in lexicographic order with*out* regard to constraints.

Inferring changes

Horn-SAT by minimal changes to $00\cdots0$ (all variables 0/false)

CSAT	definite clause	list encoding
$\overline{u} \vee x \vee \overline{z}$	x :- u, z.	$[x, u, z]$
$\overline{u} \vee \overline{v}$	*false* :- u, v.	$[\mathit{false}, u, v]$

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For each stage i, collect the variables set at stage i to 1/true in A_i

$$
\begin{aligned}
A_0 &:= \emptyset \qquad \text{(all variables false)} \\
A_{i+1} &:= \{x \mid \underbrace{\text{member}([x|T], KB)}_{x \text{ :- } t_1 \ldots t_k \text{ in } KB} \text{ and } \underbrace{\text{all}(T, A_i)}_{\{t_1 \ldots t_k\} \subseteq A_i}\}
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No minimal set for non-Horn $x \vee y$ (or xor).

Instantiate one variable at a time

allow node to map X_i to ?, raising search space size from

$$\prod_{i=1}^{n} s_i \quad \text{to} \quad \prod_{i=1}^{n}(s_i + 1) \quad \text{from adding ? to } D_i$$

PAY-OFF: search tree of depth n and branching factor $\max_i s_i$
with start node instantiating no variable, and
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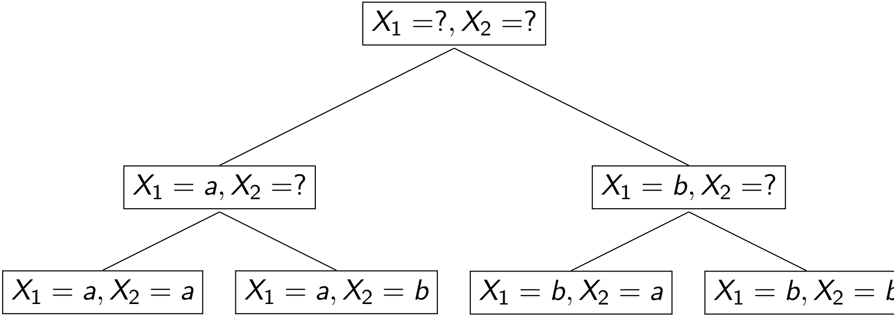
E.g. $n = 2, \quad D_1 = D_2 = \{a, b\}$



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whenever $arc(N0, N1)$,

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Reduce domains of un-instantiated variables via constraints
Constraint Graph: node = variable (e.g. 3-Color)

$$arc(X_i, X_j) \iff \mathrm{Con}[X_i, X_j] \neq \emptyset$$

Arc Consistency: for $arc(X_i, X_j)$ and $i < j$,

$$(\forall d \in D(X_i))(\exists d' \in D(X_j))\ d, d' \text{ satisfy } \mathrm{Con}[X_i, X_j]$$

removing d from $D(X_i)$ when no such d' exists

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Optimizing the backtracking search

- MRV: instantiate variable with minimum remaining values
  (to minimize branching/cases)
- LCV: assign value that is least constraining (for greatest
  chance of success)