Contents

Cumulative Distribution Functions

Suppose X and Y are two random variables

- $F_{XY}(x,y) = P(X \leq x \text{ and } Y \leq y)$ is the cumulative distribution function for X and Y
- When X and Y are independent then

  $$P(X \leq x \text{ and } Y \leq y) = P(X \leq x)P(Y \leq y)$$

  i.e.

  $$F_{XY}(x,y) = F_X(x)F_Y(y)$$

When X and Y are discrete random variable taking values $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$

- $F_{XY}(x,y) = \sum_{i:x_i \leq x} \sum_{j:y_j \leq y} P(X = x_i \text{ and } Y = y_j)$

When X and Y are jointly continuous-valued random variables there exists a probability density function (PDF) $f_{XY}(x,y) \geq 0$ such that

$$F_{XY}(x,y) \int_{\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v) du\, dv$$

Can think of $P(u \leq X \leq u + du \text{ and } v \leq Y \leq v + dv) \approx f_{XY}(u,v) du\, dv$ when du, dv are infinitesimally small.

Conditional Probability Density Function

Suppose X and Y are two continuous random variables with joint PDF $f_{XY}(x,y)$. Define conditional PDF:

$$f_{X\sim Y}(x\sim y)=\frac{f_{XY}(x,y)}{f_Y(y)}$$

Compare with conditional probability for discrete RVs:

$$P(X=x\mid Y=y)=\frac{P(X=x\text{ and }Y=y}{P(Y=y)}$$

Chain Rule for PDFs

Since

$$f_{X|Y}(x\mid y)=\frac{f_{XY}(x,y)}{f_Y(y)}$$

the chain rule also holds for PDFs:

$$f_{XY}(x,y)=f_{X|Y}(x\mid y)f_Y(y)=f_{Y|X}(y\mid x)f_X(x)$$

Also

- $\int_{-\infty}^{\infty} f_{X|Y}(x\mid y)dx=\frac{\int_{-\infty}^{\infty} f_{XY}(x,y)dx}{f_Y(y)}=\frac{f_Y(y)}{f_Y(y)}=1$
- We can marginalise PDFs:

$$\int_{-\infty}^{\infty} f_{XY}(x,y)dy=\int_{-\infty}^{\infty} f_{Y|X}(y\mid x)f_X(x)dy=d_X(x)\int_{-\infty}^{\infty} f_{Y|X}(y\mid x)dy=f_X(x)$$

Bayes Rule for PDFs

Since

$$f_{X|Y}(x\mid y)=\frac{f_{XY}(x,y)}{f_Y(y)}$$

then

$$f_{X|Y}(x\mid y)f_Y(y)=f_{XY}(x,y)=f_{Y|X}(y\mid x)f_X(x)$$

and so we have Bayes Rule for PDFs:

$$f_{Y|X}(Y\mid X)=\frac{f_{X|Y}(x\mid y)f_Y(y)}{f_X(x)}$$

Independence

Suppose X and Y are two continuous random variables with joint PDF $f_{XY}(x,y)$. Then X and Y are independent when

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

Why?

$$P(X \leq x \text{ and } Y \leq y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)dudv = \int_{-\infty}^{x} f_X(u)du \int_{-\infty}^{y} f_Y(v)dv = P(X \leq x)P(Y \leq y)$$