HW 3 Advanced Linear Algebra

Jonathan Fogel

March 2021

1 Idempotent Operators

Setup: Suppose that T is an idempotent operator on a vector space V− that is, $T = T^2$.

1.1 Eigenvalues

Question: Find all possible eigenvalues of T.

Answer: Suppose the idempotent operator T has eigenvector v and eigenvalue λ. We can then show that:

$$Tv = \lambda v$$

and we can show:

$$Tv = T^2 v = TTv = T\lambda v = \lambda Tv = \lambda^2 v$$

So we know that:

$$\lambda v = \lambda^2 v$$
$$\lambda = \lambda^2$$

Since λ is an eigenvalue, and therefore a scalar, we than know that λ can only be 1 or 0. So the only possible eigenvalues are 1 and 0.

1.2 Examples

Question: Give two examples of idempotent operators on $\mathbb{R}^2$.

Answer: One clearly idempotent operator on $\mathbb{R}^2$ is I. This is shown below.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Another idempotent matrix is $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$. This is shown below.

$$\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 4-2 & 4-2 \\ -2+1 & -2+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

1.3 Relationship between Ker(T), Range(T), and V

Question: Prove that V=ker(T)⊕Range(T).

Proof: For all $v \in V$:

$$v = v - T(v) + T(v)$$

$$v = (v - T(v)) + T(v)$$

First we look at the first term of the equation, (v-T(v)). If we apply T to this term, we see that:

$$T(v - T(v)) = T(v) - T(T(v)) = T(v) - T^2(v) = T(v) - T(v) = 0$$

So (v-T(v)) ∈ Ker(T). Now we look at the second term, T(v). T(v) is clearly in the image of T, by definition, so we now know that any vector v in space V can be decomposed into one part that is in the image and one part in the kernel, so:

$$V = ker(T) \oplus Range(T)$$

2 Fibonacci Sequence

Setup: (Compare Axler's problem 5.C.16) Define the Fibonacci sequence recursively: $F_1 = F_2 = 1$; $F_n = F_{n-1} + F_{n-2}$ when n≥3. Define T on $\mathbb{R}^2$ by T(x, y) = (y, x+y).

2.1 $T^n(0, 1)$

Question: Show by induction:$T^n(0, 1) = (F_n, F_{n+1})$ for each $n \in \mathbb{N}$.

Proof: Let transformation T be defined as T(x, y) = (y, x+y). Start with the base case of n=1. We then have:

$$T(0, 1) = (1, 1) = (F_1, F_1) = (F_1, F_2)$$

So for the base case, n=1, $T^n(0, 1) = (F_n, F_{n+1})$ is true. Now suppose the equality is true for n=k. Then for n=k+1, we find:

$$T^{k+1}(0, 1) = T(T^k(0, 1)) = T(F_k, F_{k+1}) = (F_{k+1}, F_k + F_{k+1})$$

From the recursive definitions, we know that $F_k + F_{k+1} = F_{k+2}$, so we rewrite:

$$T^{k+1}(0, 1) = (F_{k+1}, F_{k+2})$$

So the equality is true for n=k+1 if it is true for n=k. Therefore, by induction, $T^n(0,1) = (F_n, F_{n+1})$.

2.2 Eigenvalues

Question: Find the eigenvalues of T, and a corresponding basis of $\mathbb{R}^2$ of eigenvectors.

Answer: Let λ be an eigenvalue corresponding to eigenvector (a, b). Then we can show:

$$T(a,b) = \lambda(a,b) = (\lambda a, \lambda b)$$

But we also know, by definition of T, that:

$$T(a,b) = (b, a+b)$$

So we combine these formulas to find:

$$\lambda a = b$$
$$\lambda = \frac{b}{a}$$

And also that:

$$\lambda b = a + b$$
$$\lambda = \frac{a}{b} + 1$$

Substituting $\frac{a}{b}$ for $\frac{1}{\lambda}$ then gives us:

$$\lambda = \frac{1}{\lambda} + 1$$

We can then represent this as a quadratic and solve to find that for:

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

So the eigenvalues for T are $\frac{1+\sqrt{5}}{2}$, and $\frac{1-\sqrt{5}}{2}$.

To find their corresponding eigenvectors, I will start by creating a matrix representation of T. Because we know that T(0,1)=(1,1), and that T(1,1)=(1,2), it is straightforward to write T in the form:

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Using $\lambda = \frac{1+\sqrt{5}}{2}$ we find the first eigenvector by using:

$$(T - \frac{1+\sqrt{5}}{2}I)\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$T = \begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1+\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\text{And so we find that: } \begin{cases} -\frac{1+\sqrt{5}}{2}x + y = 0 \\ x + \frac{1-\sqrt{5}}{2}y = 0 \end{cases}$$

These are actually equivalent equations. Using the first equation, we set the free variable x to 1, which gives us y=$\frac{1+\sqrt{5}}{2}$. We now have an eigenvector, e_1:

$$e_1 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

We can do the same process to $\lambda = \frac{1-\sqrt{5}}{2}$ to find the other eigenvector, e_2, to be:

$$e_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

This then gives us the basis:

$$\left[\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} \right]$$

2.3 $T^n(0,1)$ Continued

Question: Use the solution of 2.2 to compute $T^n(0,1)$. Conclude (that is, prove) that:

$$F_n = \frac{1}{\sqrt{5}}\left[\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n \right]$$

Proof: First, we will break down (0,1) with the above basis.

$$(0,1) = C_1 e_1 + C_2 e_2$$

$$(0,1) = C_1 \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$(0,1) = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$(0,1) = \frac{1}{\sqrt{5}}(e_1 - e_2)$$

We now look at $T^n(0,1)$:

$$T^n(0,1) = T^n(\frac{1}{\sqrt{5}}(e_1 - e_2))$$

$$T^n(0,1) = \frac{1}{\sqrt{5}}(T^n(e_1) - T^n(e_2))$$

$$T^n(0,1) = \frac{1}{\sqrt{5}}(\lambda_1^n(e_1) - \lambda_2^n(e_2))$$

$$T^n(0,1) = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n(e_1) - (\frac{1-\sqrt{5}}{2})^n(e_2))$$

We then focus on the just first component of $T^n(0,1)$, F_n (proven in subsection 2.1). Since 1 is the first component of both e_1 and e_2, we then find:

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

3 Cauchy-Schwarz Application

Question: (Compare Axler 6.A.11) Use the Cauchy-Schwarz inequality to prove that:

$$(a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \geq 9$$

for all positive numbers a, b, c.

Proof: One example of the Cauchy Schwarz inequality is that if $x_1, x_2 ... x_n$ and $y_1, y_2 ... y_n \in \mathbb{R}$, then:

$$|x_1 y_1 + x_2 y_2 ... x_n y_n|^2 \leq (x_1^2 + x_2^2 ... x_n^2)(y_1^2 + y_2^2 ... y_n^2)$$

Now take a,b,c$\in \mathbb{R}$, and suppose that $x_1 = \sqrt{a}, x_2 = \sqrt{b}, x_3 = \sqrt{c}, y_1 = \frac{1}{\sqrt{a}}, y_2 = \frac{1}{\sqrt{b}}, y_3 = \frac{1}{\sqrt{c}}$. We can then show that:

$$\left|\frac{\sqrt{a}}{\sqrt{a}} + \frac{\sqrt{b}}{\sqrt{b}} + \frac{\sqrt{c}}{\sqrt{c}}\right|^2 \leq (\sqrt{a}^2 + \sqrt{b}^2 + \sqrt{c}^2)(\frac{1}{\sqrt{a}}^2 + \frac{1}{\sqrt{b}}^2 + \frac{1}{\sqrt{c}}^2)$$

$$3^2 \leq (a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})$$

$$9 \leq (a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})$$

4 $\mathcal{P}_2(\mathbb{R})$ and its Inner Product

Setup: Let V=$\mathcal{P}_2(\mathbb{R})$.

4.1 Inner Products

Question: Show that the pairing $\langle f, g\rangle = \int_0^2 f(x)g(x)dx$ defines an inner product on V.

Answer: We will now check each property of an inner product:

$$\begin{cases} \langle v,v\rangle \geq 0 \text{ For all v in V} \\ \langle v,v\rangle = 0 \text{ iff v=0} \\ \langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle \text{ For all u,v,w in V} \\ \langle \lambda u,v\rangle = \lambda\langle u,v\rangle \text{ For all u,v in V} \\ \langle u,v\rangle = \overline{\langle v,u\rangle} \text{ For all u,v in V} \end{cases}$$

We now check positivity. Because the coefficients in V are all real, than for any v in V, $v(x)^2$ has to have non-negative coefficients. So the function $v(x)^2$ must be ≥ 0. So:

$$v(x)^2 \geq 0$$

$$\int_0^2 v(x)^2 dx \geq \int_0^2 0 dx$$

$$\int_0^2 v(x)v(x)dx \geq 0$$

$$\langle v,v\rangle \geq 0$$

We now check definiteness.

$$\text{If} \langle v,v\rangle = 0, \text{then:}$$

$$\int_0^2 v(x)v(x)dx = 0$$

$$\int_0^2 v(x)^2 dx = 0$$

But $v(x)^2$ is a positive function unless v(x)=0. So if $\langle v,v\rangle = 0$, then v(x)=0. Now we show the converse. If v(x)=0, then:

$$\langle v,v\rangle = \langle 0,0\rangle = \int_0^2 0dx = 0$$

So the converse is true as well. the inner product then has definiteness.

We now check additivity in first slot.

$$\langle u+v,w\rangle = \int_0^2 (u(x)+v(x))w(x)dx = \int_0^2 (u(x)w(x)+v(x)w(x))dx$$

$$\langle u+v, w\rangle = \int_0^2 u(x)w(x)dx + \int_0^2 v(x)w(x)dx$$

$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$

We now check homogeneity in first slot.

$$\langle \lambda u, v\rangle = \int_0^2 \lambda u(x)v(x)dx$$

$$\langle \lambda u, v\rangle = \lambda \int_0^2 u(x)v(x)dx$$

$$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$$

We now check conjugate symmetry. Since we only have real coefficients, we can write:

$$\overline{\langle u, v\rangle} = \langle u, v\rangle$$

We continue to show:

$$\langle u, v\rangle = \int_0^2 u(x)v(x)dx = \int_0^2 v(x)u(x)dx$$

$$\langle u, v\rangle = \langle v, u\rangle = \overline{\langle v, u\rangle}$$

This definition holds for all of the properties, so it is an acceptable inner product on V.

4.2 Norms

Question: Find$\|2x^2 - x - 7\|$, using the norm induced by this inner product.

Answer: First we are reminded of the definition of norm, with respect to an inner product.

$$\|a\| = \sqrt{\langle a, a\rangle}$$

So we find:

$$\|2x^2 - x - 7\| = \sqrt{\langle (2x^2 - x - 7), (2x^2 - x - 7)\rangle}$$

$$\|2x^2 - x - 7\| = \sqrt{\int_0^2 (2x^2 - x - 7)(2x^2 - x - 7)dx}$$

$$\|2x^2 - x - 7\| = \sqrt{\int_0^2 (4x^4 - 4x^3 - 27x^2 + 14x + 49)dx}$$

$$\|2x^2 - x - 7\| = \sqrt{[\frac{4x^5}{5} - x^4 - 9x^3 + 7x^2 + 49x\Big|_0^2}$$

$$\|2x^2 - x - 7\| = \sqrt{\frac{318}{5}}$$

4.3 Gram-Schmidt Procedure

Question: Use the Gram-Schmidt procedure to transform the list 1, x, x^2 to an orthonormal basis. Explain your steps.

Answer: We start with the list 1, x, x^2. First, we notice that every element of the list exists in vector space V. Now suppose x_n form a basis for V. Let v_1=1, v_2=x and v_3=x^2. We now find x_1 by normalizing it.

$$x_1 = \frac{v_1}{\|v_1\|}$$

$$x_1 = \frac{1}{\sqrt{\int_0^2 1dx}} = \frac{1}{\sqrt{2}}$$

We now have x_1. We then use the following projection formula to find x_2. Note that integration for the inner product will no longer be explicitly shown.

$$x_2 = \frac{v_2 - \langle v_2, x_1\rangle x_1}{\|v_2 - \langle v_2, x_1\rangle x_1\|}$$

$$x_2 = \frac{x - \langle x, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}}}{\|x - \langle x, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}}\|}$$

$$x_2 = \frac{x - \sqrt{2}\frac{1}{\sqrt{2}}}{\|x - \sqrt{2}\frac{1}{\sqrt{2}}\|}$$

$$x_2 = \frac{x - 1}{\|x - 1\|}$$

$$x_2 = \frac{x - 1}{\sqrt{\langle x - 1, x - 1\rangle}}$$

$$x_2 = \frac{x - 1}{\sqrt{\frac{2}{3}}}$$

$$x_2 = \frac{\sqrt{6}(x - 1)}{2}$$

We now have found x_2. We do a similar, but longer projection procedure to find our third and final basis vector below.

$$x_3 = \frac{v_3 - \langle v_3, x_1\rangle x_1 - \langle v_3, x_2\rangle x_2}{\|v_3 - \langle v_3, x_1\rangle x_1 - \langle v_3, x_2\rangle x_2\|}$$

$$x_3 = \frac{x^2 - \langle x^2, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} - \langle x^2, \frac{\sqrt{6}(x-1)}{2}\rangle \frac{\sqrt{6}(x-1)}{2}}{\|x^2 - \langle x^2, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} - \langle x^2, \frac{\sqrt{6}(x-1)}{2}\rangle \frac{\sqrt{6}(x-1)}{2}\|}$$

$$x_3 = \frac{x^2 - \frac{4\sqrt{2}}{3}\frac{1}{\sqrt{2}} - \frac{2\sqrt{6}}{3}\frac{\sqrt{6}(x-1)}{2}}{\|x^2 - \frac{4\sqrt{2}}{3}\frac{1}{\sqrt{2}} - \frac{2\sqrt{6}}{3}\frac{\sqrt{6}(x-1)}{2}\|}$$

$$x_3 = \frac{x^2 - 2x + \frac{2}{3}}{\|x^2 - 2x + \frac{2}{3}\|}$$

$$x_3 = \frac{x^2 - 2x + \frac{2}{3}}{\sqrt{\langle x^2 - 2x + \frac{2}{3}\rangle}}$$

$$x_3 = \frac{x^2 - 2x + \frac{2}{3}}{\sqrt{\frac{8}{45}}}$$

$$x_3 = \frac{\sqrt{10}\left(3x^2 - 6x + 2\right)}{4}$$

And so the Gram-Schmidt Procedure gives us the orthonormal basis:

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}} \\ x_2 = \frac{\sqrt{6}(x-1)}{2} \\ x_3 = \frac{\sqrt{10}\left(3x^2-6x+2\right)}{4} \end{cases}$$