HW 1 Advanced Linear Algebra

Jonathan Fogel

February 2021

1 Warmup Computations

1.1 Solve the System Hand

The System of equations:

$$5x_1 - 3x_2 + x_3 + x_4 - 4x_5 = 1$$
$$5x_1 - 10x_2 - x_3 + 2x_4 + 3x_5 = 2$$
$$3x_1 - 6x_2 - x_3 + x_4 + 2x_5 = 3$$

Can be represented as the augmented matrix:

$$\begin{bmatrix} 5 & -3 & 1 & 1 & -4 & 1 \\ 5 & -10 & -1 & 2 & 3 & 2 \\ 3 & -6 & -1 & 1 & 2 & 3 \end{bmatrix}$$

Dividing Row 1 by 5:

$$\begin{bmatrix} 1 & -3/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 5 & -10 & -1 & 2 & 3 & 2 \\ 3 & -6 & -1 & 1 & 2 & 3 \end{bmatrix}$$

Subtracting 5 times row 1 from row 2, and subtracting 3 times row 1 from row 3:

$$\begin{bmatrix} 1 & -3/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 0 & -7 & -2 & 1 & 7 & 1 \\ 0 & -21/5 & -8/5 & 2/5 & 22/5 & 12/5 \end{bmatrix}$$

Divide row 2 by - 7:

$$\begin{bmatrix} 1 & -3/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 0 & 1 & 2/7 & 1/7 & -1 & -1/7 \\ 0 & -21/5 & -8/5 & 2/5 & 22/5 & 12/5 \end{bmatrix}$$

Add 21/5 times row 2 to row 3:

$$\begin{bmatrix} 1 & -3/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 0 & 1 & 2/7 & 1/7 & -1 & -1/7 \\ 0 & 0 & -2/5 & -1/5 & 1/5 & 9/5 \end{bmatrix}$$

Multiply row 3 by -5/2:

$$\begin{bmatrix} 1 & -3/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 0 & 1 & 2/7 & 1/7 & -1 & -1/7 \\ 0 & 0 & 1 & 1/2 & -1/2 & -9/2 \end{bmatrix}$$

Add -2/7 times row 3 to row 2:

$$\begin{bmatrix} 1 & -3/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 0 & 1 & 0 & -2/7 & -6/7 & 8/7 \\ 0 & 0 & 1 & 1/2 & -1/2 & -9/2 \end{bmatrix}$$

Add 3/5 times row 2 to row 1 and subtract 1/5 times row 3 to row 1 (A lot of arithmetic in this step!):

$$\begin{bmatrix} 1 & 0 & 0 & -1/14 & -17/14 & 25/14 \\ 0 & 1 & 0 & -2/7 & -6/7 & 8/7 \\ 0 & 0 & 1 & 1/2 & -1/2 & -9/2 \end{bmatrix}$$

Reverting back to equations gives the following system of equations:

$$x_1 - \frac{1}{14}x_4 - \frac{17}{14}x_5 = \frac{25}{14}$$
$$x_2 - \frac{2}{7}x_4 - \frac{6}{7}x_5 = \frac{8}{7}$$
$$x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_5 = -\frac{9}{2}$$

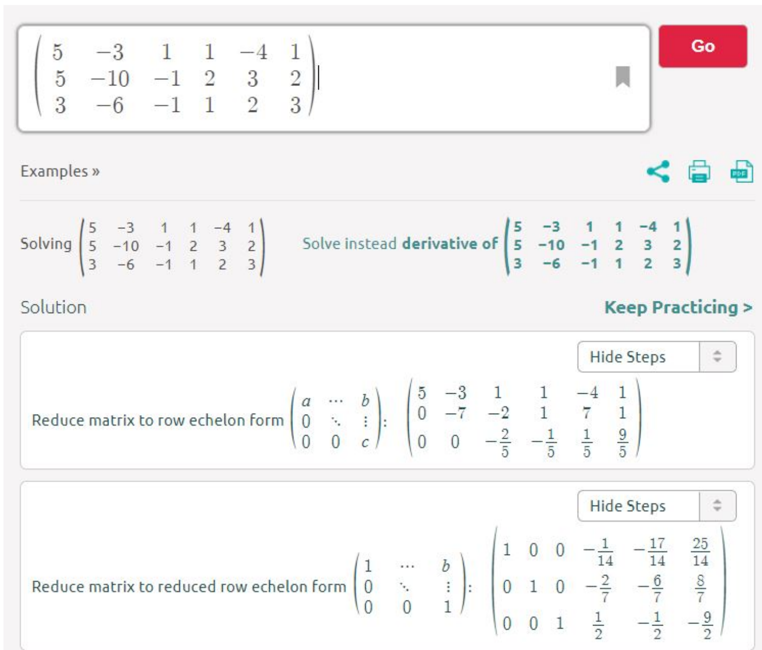
Solving for x_1, x_2, and x_3 gives:

$$x_1 = \frac{1}{14}x_4 + \frac{17}{14}x_5 + \frac{25}{14}$$
$$x_2 = \frac{2}{7}x_4 + \frac{6}{7}x_5 + \frac{8}{7}$$
$$x_3 = -\frac{1}{2}x_4 + \frac{1}{2}x_5 - \frac{9}{2}$$

Where both x_4 and x_5 are free variables.

1.2 Solve the Same System with Software

Now the same system will be solved using software. Using the free resource symbolab.com, you can insert a matrix of any size, and have it spit out information, such as the size, order, row echelon form, reduced row echelon form, etc. Below is the above system inserted into the program. The bottom line then gives the RREF.



The final step would be to then define variables x_1, x_2, and x_3, and set both x_4 and x_5 as free variables.

1.3 Invert the Matrix by hand

Given the matrix:

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$$

We then augment the appropriately sized identity matrix onto it:

$$\begin{bmatrix} 1 & -3 & 4 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Swap row 2 and 3:

$$\begin{bmatrix} 1 & -3 & 4 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Add 2 times row 1 to row 2:

$$\begin{bmatrix} 1 & -3 & 4 & 1 & 0 & 0 \\ 0 & -9 & 8 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Add 2/9 times row 2 to row 3:

$$\begin{bmatrix} 1 & -3 & 4 & 1 & 0 & 0 \\ 0 & -9 & 8 & 2 & 0 & 1 \\ 0 & 0 & 25/9 & 4/9 & 1 & 2/9 \end{bmatrix}$$

Multiply row 3 by 9/25, and divide row 2 by -9:

$$\begin{bmatrix} 1 & -3 & 4 & 1 & 0 & 0 \\ 0 & 1 & -8/9 & -2/9 & 0 & -1/9 \\ 0 & 0 & 1 & 4/25 & 9/25 & 2/25 \end{bmatrix}$$

Add 8/9 times row 3 to row 2:

$$\begin{bmatrix} 1 & -3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/25 & 8/25 & -1/25 \\ 0 & 0 & 1 & 4/25 & 9/25 & 2/25 \end{bmatrix}$$

Add 3 times row 2 and 4 times row 3 both to row 1:

$$\begin{bmatrix} 1 & 0 & 0 & 3/25 & -12/25 & -11/25 \\ 0 & 1 & 0 & -2/25 & 8/25 & -1/25 \\ 0 & 0 & 1 & 4/25 & 9/25 & 2/25 \end{bmatrix}$$

We now have the identity augmented with the inverse of the original matrix. The inverse is then:

$$\begin{bmatrix} 3/25 & -12/25 & -11/25 \\ -2/25 & 8/25 & -1/25 \\ 4/25 & 9/25 & 2/25 \end{bmatrix}$$

1.4 Solve the System of Equations using its Inverse

We begin with the system of equations:

$$3x - 2y + z = -3$$

$$-x + y = 2$$

$$7x - 6z = 9$$

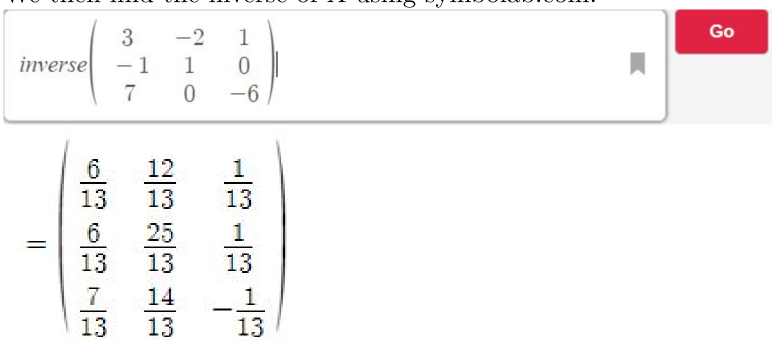
Which can be rewritten in the following form Ax=B, where:

$$A : \begin{bmatrix} 3 & -2 & 1 \\ -1 & 1 & 0 \\ 7 & 0 & -6 \end{bmatrix}$$

$$x : \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B : \begin{bmatrix} -3 \\ 2 \\ 9 \end{bmatrix}$$

We then find the inverse of A using symbolab.com:



We then multiply A inverse to the front of both sides of our equation (order matters!):

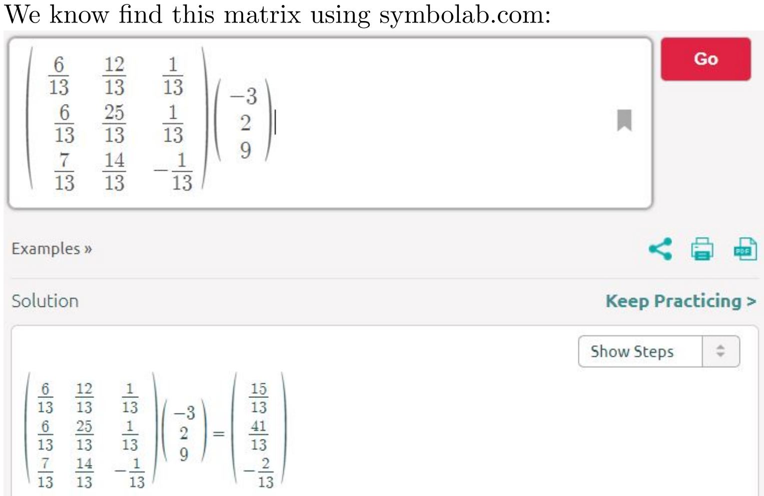
$$Ax = B$$

$$A^{-1}Ax = A^{-1}B$$

And since we know that $A^{-1}A = I$, the identity matrix, we can simplify to:

$$x = A^{-1}B$$

We know find this matrix using symbolab.com:



This tells us that x=15/13, y=41/13,and z=-2/13.

1.5 Multiplying Matrices, and When you Can't

Given the matrices A,B such that:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 4 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1+0 & -1+8 & -1+6 \\ -3+0 & -3+0 & -3+0 \\ 4+0 & 4+4 & 4+3 \end{bmatrix} = \begin{bmatrix} -1 & 7 & 5 \\ -3 & -3 & -3 \\ 4 & 8 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1-3+4 & -2+0-1 \\ 0+12-12 & 0+0+3 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 0 & 3 \end{bmatrix}$$

A is a 3x2 matrix, and B is a 2x3 matrix. For matrix multiplication to work, the inner two dimensions need to be equal, and the outer two are the dimensions of the product. For instance, in AB, it is (3x2)(2x3). 2=2, so it is allowed, and the product will have size 3x3.

If we let there be matrix C of size 2x2, AB (3x2)(2x2) would work, but BA (2x3)(2x2) would not work.

2 Ker(T) and Range(T) as subspaces

Prompt: Let T:V→W be a linear transformation. Prove that Ker(T) is a subspace of V and that Range(T)is a subspace of W.

Important definitions:

- A subset of a vector field is a subspace iff it is closed under addition and scalar multiplication, and if its nonempty.
- Ker(T)=$\{v \epsilon V | T(v) = 0_W\}$
- Range(T)=$\{T(v) | v \epsilon V\}$

Proof:

Let T:V→W be a linear transformation. Ker(T) is a subset of the vector field V by definition. Since T is a linear transformation, T(0_V) must get mapped to 0_W (property of linear transformations). So Ker(T) contains 0_V, and is thus nonempty. Now choose x,y ϵ Ker(T).

$$x + y = 0_W + 0_W = 0_W$$

So (x+y) ϵ Ker(T). Ker(T) is then closed under addition. Now take scalar C.

$$Cx = C0_W = 0_W$$

So (Cx) ϵ Ker(T). Ker(T) is then closed under scalar multiplication. Ker(T) is then a subspace of V.

Since T(0_V)=0_W, $0_W\epsilon$Range(T). Thus, Range(T) is nonempty, as it contains at least 0_W. We also know that Range(T) is a subset of vectors in W, by definition.

Now take vectors $\vec{v_1}, \vec{v_2}\epsilon V$ and $\vec{w_1}, \vec{w_2}\epsilon W$ such that $T(\vec{v_1}) = \vec{w_1}$, and $T(\vec{v_2}) = \vec{w_2}$. by definition of linear transformation:

$$T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2}) = \vec{w_1} + \vec{w_2}$$

So $(\vec{w_1} + \vec{w_2})\epsilon$ Range(T). Range(T) is then closed under addition, as the vectors were arbitrary within their bounds.Wow take scalar C. By definition of linear transformation:

$$T(C\vec{v_1}) = CT(\vec{v_1}) = C\vec{w_1}$$

$C\vec{w_1}\epsilon Range(T)$, so Range(T) is then closed under scalar multiplication. Range(T) is then a subspace of W.

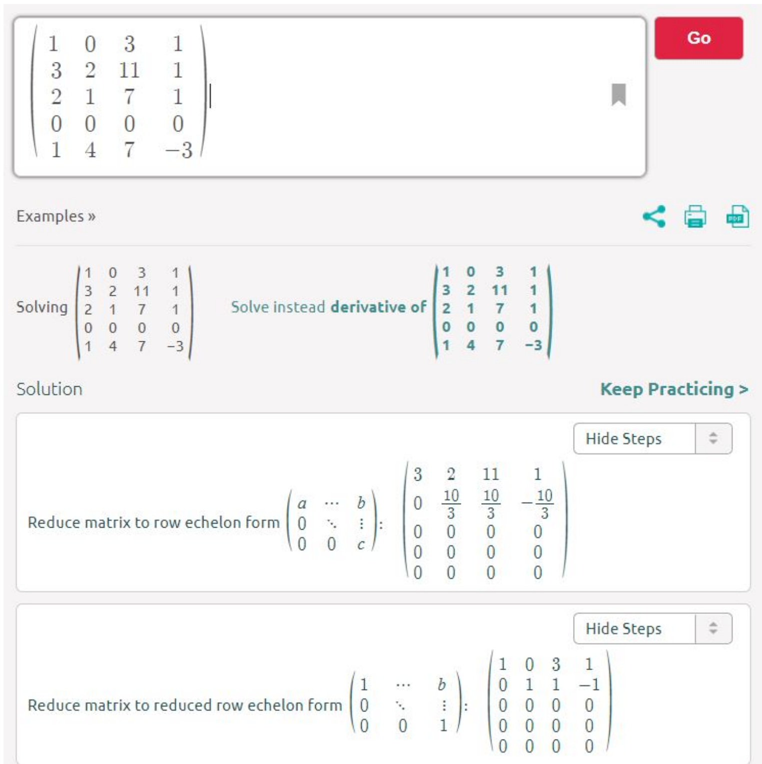
3 $T : \mathbb{R}^4 \rightarrow \mathbb{R}^5$ Example

Take the following transformation:

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_3 + x_4 \\ 3x_1 + 2x_2 + 11x_3 + x_4 \\ 2x_1 + x_2 + 7x_3 + x_4 \\ 0 \\ x_1 + 4x_2 + 7x_3 - 3x_4 \end{bmatrix}$$

3.1 Finding a basis for Ker(T)

To find a basis for Ker(T), we first want to represent the system of equations as an augmented matrix in reduced row echelon form. This is done with symbolab.com. The results of the software are shown below. Note that the calculation shown is without the extra augmented zeroes, so that the computing time is lowered. We will augment this row in after calculation.



The software shows that in row echelon form the matrix is reduced to:

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To find the Ker(T) is to find the set of solutions where T(v)=0. So we set up:

$$x_1 + 3x_3 + x_4 = 0$$

$$x_2 + x_3 - x_4 = 0$$

Which can be arranged to show:

$$x_1 = -3x_3 - x_4$$

$$x_2 = -x_3 + x_4$$

$$x_3 = free$$

$$x_4 = free$$

So Ker(T) can be expressed as:

$$\begin{bmatrix} -3x_3 - x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

And a basis for Ker(T) is:$\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \}$

3.2 Extending the Basis to $\mathbb{R}^4$

To extend this basis to create a basis for $\mathbb{R}^4$, we must find enough vectors to span the space, while being linearly independent from one another. Since we are talking about $\mathbb{R}^4$, we know there must be 4 total basis vectors. Two easy vectors to add are:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ And } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

This then makes the basis:

$$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \}$$

You check whether this is a valid basis by creating a matrix with the basis vectors as columns and reducing it to reduced row echelon form. If there are pivots in each column/row. This basis, when reduced, has this trait, and so we know it each vector is linearly independent, and that the basis is valid.

3.3 The Basis of Range(T)

From part (a), we found the columns corresponding to x_1 andx_2 to be linearly independent. The image of these columns give us the basis for Range(T). This basis is:

$$\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} \}$$

3.4 Extending the Basis to $\mathbb{R}^5$

To create a basis for $\mathbb{R}^5$, we do a similar procedure to part (b). Because we are working in $\mathbb{R}^5$, we need a total of 5 basis vectors, meaning we need to find 3 additional vectors which are linearly independent. The following is this created basis:

$$\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \}$$

You check whether this is a valid basis by creating a matrix with the basis vectors as columns and reducing it to reduced row echelon form. If there are pivots in each column/row. This basis, when reduced, has this trait, and so we know it each vector is linearly independent, and that the basis is valid.

3.5 What is $[T]_\alpha^\beta$?

$[T]_\alpha^\beta$ is the notation for the linear transformation, T, in a form that takes input from the α basis, and return output in the β basis. In this case, α refers to the basis found in part b, and β refers to part d. The equation for $[T]_\alpha^\beta$ is:

$[T]_\alpha^\beta = [[T(v1)]_\beta|[T(v2)]_\beta|\cdots|[T(vn)]_\beta]$

To apply this to our basis and transformation, first we replace the vectors with our basis vectors for α:

$$[T]_\alpha^\beta = [[T(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix})]_\beta, [T(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix})]_\beta, [T(\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix})]_\beta, [T(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix})]_\beta]$$

We then apply T to these vectors as shown at the start of question 3.

$$[T]_\alpha^\beta = [[\begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}]_\beta, [\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}]_\beta, [\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}]_\beta, [\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}]_\beta]$$

We now express these vectors as linear combinations of our β basis vectors, where each row corresponds to the ordered basis vectors of$/beta$.

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is then $[T]_\alpha^\beta$.