# HW 2 Advanced Linear Algebra

Jonathan Fogel

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## 1 Intersections of Subspaces

**Question:** Suppose that U,W are subspaces of a vector space V. Prove that U∩W is also a subspace.

**Proof:** Since U is a subspace, it must contain the zero vector. Since W is a subspace, it must also contain the zero vector. The zero vector then is contained in U∩W.

Now choose x,y such that:

$$x, y \in U \cap W$$

Then :

$$x, y \in U$$
$$x, y \in W$$

Since U is a subspace, and therefore closed under addition:

$$x + y \in U$$

Similarly:

$$x + y \in W$$

Since (x+y) is contained in both U and W:

$$x + y \in U \cap W$$

x and y were arbitrary within U∩W, and so the set U∩W is closed under addition.

Choose scalar s and vector z such that $s \in \mathbb{R}$, and $z \in U \cap W$. Because U is a subspace, and therefore closed under scalar multiplication:

$$sz \in U$$

Similarly:

$$sz \in W$$

And so:

$$sz \in U \cap W$$

s and z were arbitrary within their bounds, so U∩W is closed under scalar multiplication. U∩W is then a subspace.

## 2 Even and odd functions

**Setup:** In general, the notation $Y^X$ denotes the collection of all functions from a set X to a set Y. So in particular, $\mathbb{R}^{\mathbb{R}}$ denotes the set of all functions f:$\mathbb{R} \to \mathbb{R}$, and note that $\mathbb{R}^{\mathbb{R}}$ is a vector space in a natural way. Recall that a function f is called even if f(x) =f(-x) for all x$\in \mathbb{R}$, and odd if f(x) =-f(-x) for all x$\in \mathbb{R}$.

### 2.1 Even Functions as a Subspace

**Question:** Let $U_e$ denote the set of all even functions in $\mathbb{R}^{\mathbb{R}}$, and let $U_o$ denote the set of odd functions. Prove that $U_e$ is a subspace of $\mathbb{R}^{\mathbb{R}}$. (Make sure you see that the argument for $U_o$ would be very similar.)

**Proof:** The zero function is both an odd and even function, so $0 \in U_e$(and $0 \in U_o$).

Now take $f_1(x), f_2(x) \in U_e$. Then:

$$f_1(x) = f_1(-x)$$

$$f_2(x) = f_2(-x)$$

Define $g(x) = f_1(x) + f_2(x)$. Because $f_1(x), f_2(x) \in U_e$, we can then show:

$$g(-x) = f_1(-x) + f_2(-x) = f_1(x) + f_2(x) = g(x)$$

$$g(-x) = g(x)$$

The sum $g(x) = f_1(x) + f_2(x)$ is then even. Since $f_1(x), f_2(x)$ were arbitrary even functions, $U_e$ is closed under addition.

Now choose scalar $s \in \mathbb{R}$ and $f_3(x) \in U_e$.

$$sf_3(-x) = sf_3(x)$$

So even functions are closed under scalar multiplication. Thus, The set of all even functions, $U_e$ is a subspace.

### 2.2 The Direct Sum of all Even and Odd Functions

**Question:** Prove that $\mathbb{R}^{\mathbb{R}}$=$U_e \oplus U_o$ (Hint: The tricky part is figuring out how to express an arbitrary function as a sum of an even function and an odd function. An inspiring example might be the function f(x) =$e^x$, which can be expressed as the sum $f_{even}(x) = (e^x + e^{-x})/2$ and $f_{odd}(x) = (e^x - e^{-x})/2$.)

**Proof:** Choose the function $f(x) \in \mathbb{R}^{\mathbb{R}}$. We can show:

$$f(x) = \frac{2}{2} f(x)$$

$$f(x) = \frac{2f(x)}{2}$$

$$f(x) = \frac{2f(x) + (f(-x) - f(-x))}{2}$$
$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

We now look at each term of the above equation, defining them as:

$$f_e(x) = \frac{f(x) + f(-x)}{2} \text{ and } f_o(x) = \frac{f(x) - f(-x)}{2}$$

.

We can then find $f_e(-x)$ and $f_o(-x)$.

$$f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x)$$
$$f_o(-x) = \frac{f(-x) - f(x)}{2} = \frac{-f(x) + f(-x)}{2} = -f_o(x)$$

So then, by definition, $f_e(x)$ is even, and $f_o(x)$ is odd. But $f_e(x) + f_o(x) = f(x)$ and $f(x)$ was an arbitrary function, so any function in $\mathbb{R}^{\mathbb{R}}$ can be written as the direct sum of an even and an odd function.

Suppose now, for contradiction,that $\mathbb{R}^{\mathbb{R}} \neq U_e \oplus U_o$. Since $U_e$ and $U_o$ are subspaces of $\mathbb{R}^{\mathbb{R}}$, then this could only be true if there exists a member of $\mathbb{R}^{\mathbb{R}}$ that cannot be expressed as a direct sum of members of $U_e$ and $U_o$. But, we just showed that any member of $\mathbb{R}^{\mathbb{R}}$ can be expressed as a direct sum of an even and odd function. So $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

# 3 Axler 4.3

**Question:** Suppose m and n are positive integers with m$\leq$n, and suppose $\lambda_1$...,$\lambda_m$ $\in \mathbb{C}$. Prove that there exists a polynomial p $\in \mathcal{P}(\mathbb{C})$ with deg p=n such that each $\lambda_i$ is a root of p, and p has no other roots.

**Proof:** Take polynomial p $\in \mathcal{P}(\mathbb{C})$ defined by:

$$p(a+bi) = p(z) = \Big((z-\lambda_1)(z-\lambda_2)...(z-\lambda_{m-1})\Big)(z-\lambda_m)^{n-m+1}$$

We can then show:

$$deg(p) = 1 + 1 + ... + 1 + (n-m+1) = (m-1) + (n-m+1) = n$$

The polynomial p then has degree n. We can also see that if p(z)=0, then z=$\lambda_i$.

So there exists a polynomial p $\in \mathcal{P}(\mathbb{C})$ with deg p=n such that each $\lambda_i$ is a root of p, and p has no other roots.

# 4 Axler 4.9

**Question:** Suppose p $\in \mathcal{P}(\mathbb{C})$. Define q:$\mathbb{C} \rightarrow \mathbb{C}$ by q(z) =p(z)$\overline{p(\overline{z})}$. Prove that q is a polynomial with real coefficients.

**Proof:** For p $\in \mathcal{P}(\mathbb{C})$, let deg(p)=n and choose $a_0, a_1...a_n \in \mathbb{C}$. Then p can be written in the form:

$$p(z) = a_0 + a_1 z + a_2 z^2 ... a_n z^n$$

We can then write $p(\overline{z})$ as the polynomial:

$$p(\overline{z}) = a_0 + a_1 \overline{z} + a_2 \overline{z}^2 ... a_n \overline{z}^n$$

We then can also show $\overline{p(\overline{z})}$ as the polynomial:

$$\overline{p(\overline{z})} = \overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 ... \overline{a_n} z^n$$

Define q:$\mathbb{C} \rightarrow \mathbb{C}$ by q(z) =p(z)$\overline{p(\overline{z})}$. Since we have shown that both p(z) and $\overline{p(\overline{z})}$ are polynomials, then their product, q, is also a polynomial.

We now will determine $\overline{q(\overline{z})}$:

$$\overline{q(\overline{z})} = \overline{p(\overline{z})\overline{p(\overline{\overline{z}})}} = \overline{p(\overline{z})\overline{p(z)}} = \overline{p(\overline{z})}p(z) = p(z)\overline{p(\overline{z})} = q(z)$$

$$\overline{q(\overline{z})} = q(z)$$

Now take $b_0, b_1 ... b_2 n \in \mathbb{C}$. Using the above result, and that deg(q)=2n, we can write:

$$\overline{q(\overline{z})} = \overline{b_0} + \overline{b_1} z ... \overline{b_{2n}} z^{2n} = b_0 + b_1 z + b_{2n} z^{2n} = q(z)$$

So for each coefficient, $b_i$, $\overline{b_i} = b_i$. Since this equivalence is only true for strictly real input, we know that each coefficient in q(z) is real. Thus, q(z) is a polynomial with strictly real coefficients.