Chaotic Dynamics in Circuits

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I. ABSTRACT

In this report we show that when an alternating current of varying input voltages runs through an inductor and diode (LD) circuit, the system bifurcates until it becomes chaotic. Our system bifurcates at V_0 =1.17 ± 0.01V, 1.46 ± 0.01V, 1.53 ± 0.01V, 1.83 ± 0.01V with added systematic error due to the inaccuracy of the experimental setup. Using these bifurcations, we approximated Feigenbaum's constant to be $\delta = 4.14 \pm 0.28$ which is within 11% of the accepted value $\delta = 4.67$. Our experimental results are consistent with the conclusion that period doubling systems are governed by Feigenbaum's Constant, and that period doubling bifurcation is a route to chaos. Repeating the experiment using a more precise oscilloscope and function generator would reduce error in the experimental results and lead to a more accurate approximation of Feigenbaum's Constant.

II. INTRODUCTION

Period doubling bifurcation has incredible ties to patterns found in many complex systems in nature, including plant growth, and population dynamics [1]. As a system bifurcates over time, each bifurcation is the real portion of the Mandelbrot set, which is used to describe fractal geometry [2]. Mitchell Feigenbaum discovered that the ratio of bifurcations in a period doubling system is linear, and can be described by the Feigenbaum constant [3]. Furthermore, after several iterations of period doubling bifurcation, our system becomes chaotic [4]. In an attempt to explore chaos and period doubling bifurcation, we use an inductor diode circuit to observe these behaviors.

In this lab we step an alternating current voltage source through an LD circuit between 1.00V and 4.00V in order to observe both period doubling bifurcation and chaos. We observed four bifurcations at 1.17 ± 0.01V, 1.46 ± 0.01V, 1.53 ± 0.01V, and 1.83 ± 0.01V. Near 1.83V, the circuit becomes difficult to read with this experimental setup, but does eventually become chaotic.

III. THEORY

An RLD circuit consists of an inductor, a diode, and a resistor wired in series and connected to a sinusoidal drive function. RLD circuits display period doubling bifurcation due to the nonlinear dynamics associated with a diode [4]. Period doubling bifurcation is observed when a change in initial conditions leads to a new pattern with twice the period of the original system [1]. This is illustrated in Figure 1.

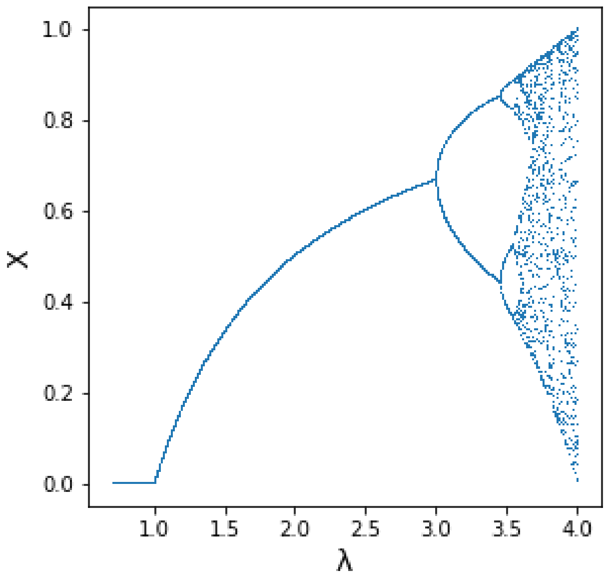

FIG. 1: A theoretical bifurcation map of equation III.1

This pattern is mathematically described by Equation III.1.

$$x_{n+1} = rx_n(1 - x_n) \tag{III.1}$$

Sections of the graph where one trend-line splits into two diverging lines represents a bifurcation. The horizontal distance between these bifurcations follows an intriguing pattern described by the Feigenbaum constant. The Feigenbaum constant is given by

$$\delta = \lim_{b \to +\infty} \frac{\lambda_{b+1} - \lambda_b}{\lambda_{b+2} - \lambda_{b+1}} \tag{III.2}$$

Where λ_b are the values of λ at the b^{th} period doubling. As the system bifurcates, the ratio used to attain Feigenbaum's constant approaches the accepted value of $\delta = 4.67$ [3]. After multiple iterations of bifurcation, the system becomes chaotic, which is clear in the right half of Figure 1. In this map, bifurcations occur so frequently they are difficult to distinguish, signaling chaos [5].

Chaos Theory is used to describe complex, multi-component systems that are extremely sensitive to initial conditions. In chaotic systems, one event depends on the preceding event, meaning a small initial change leads to profoundly different final results. This chronological dependency, combined with the idea of error makes a chaotic system completely random after a certain number of iterations [6]. In a frequency domain, chaos looks like a collection of random spikes, and on this bifurcation map, chaos looks like the more solid colored portion of Figure 1. There are periods of stability on this graph, shown in the large white regions, which eventually deteriorate into the more blue chaotic sections of the graph.

IV. EXPERIMENTAL SETUP

An Alternating sine-wave current, starting at V_0=1.00V with a frequency of 627.00 kHz, was put through a diode and inductor in series as seen in Figure 2. While a traditional RLD circuit has a resistor, this experimental setup does not, because the combined internal resistance of the diode and the inductor provides resistance in the system.

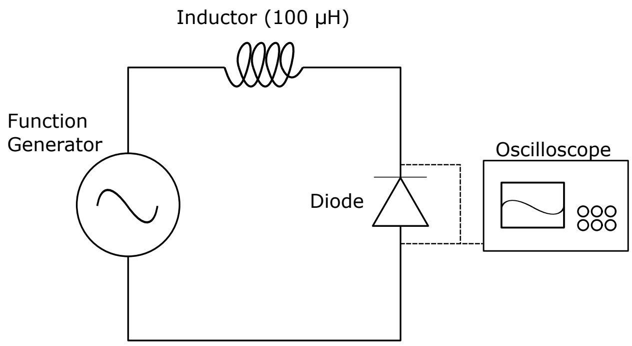

FIG. 2: Experimental setup for a period doubling circuit consisting of an AC drive function, an inductor and a diode all being recorded using an oscilloscope

The time-dependent voltage drop measured by the oscilloscope is characterized by having periodic spikes, with heights determined by period doubling bifurcation. An example of this signal is shown in Figure 3.

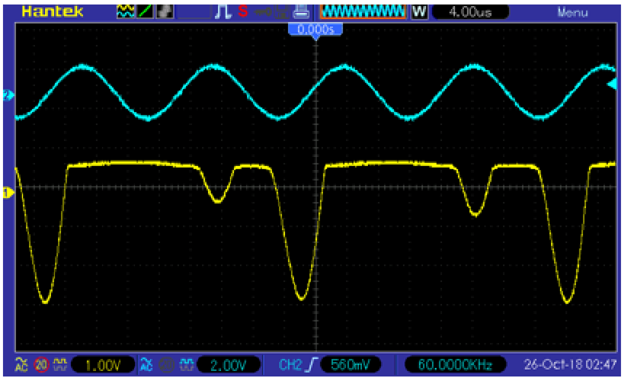

FIG. 3: Voltage spikes of a time domain RLD circuit with one bifurcation (yellow) next to the original drive function (blue)[7]

The heights of these spikes were recorded as V_0 swept from 1.00V to 4.00V to create an experimental bifurcation map as seen in Figure 5. In order to accurately determine the voltages at which bifurcation occurs, the input voltage was graphed as a function of the output voltage. An example of this plot before and after a bifurcation is shown in Figure 4.

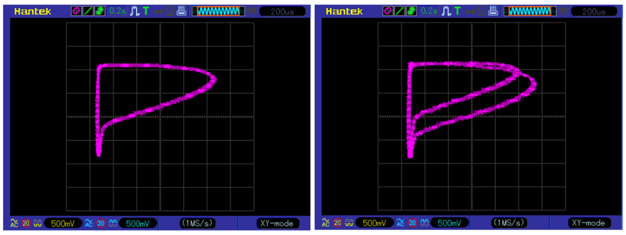

FIG. 4: Left: a graph of input voltage as a function of output voltage of an RLD circuit with a single frequency. Right: a graph of input voltage as a function of output voltage of an RLD circuit after a single bifurcation has occurred.[7]

Due to the inaccuracy associated with estimating the height of the voltage peaks in Figure 3, we chose to use the more precise method for finding a bifurcation described in Figure 4. At values of V_0 between bifurcations, the voltage drop's waveform was recorded for use in spectral analysis. The smallest voltage step that the function generator can take is 0.01 V, which limits the number of bifurcations that can be accurately observed. In addition, the resolution of the oscilloscope is low for what is necessary to precisely measure bifurcations, adding to the systematic error associated with this experiment.

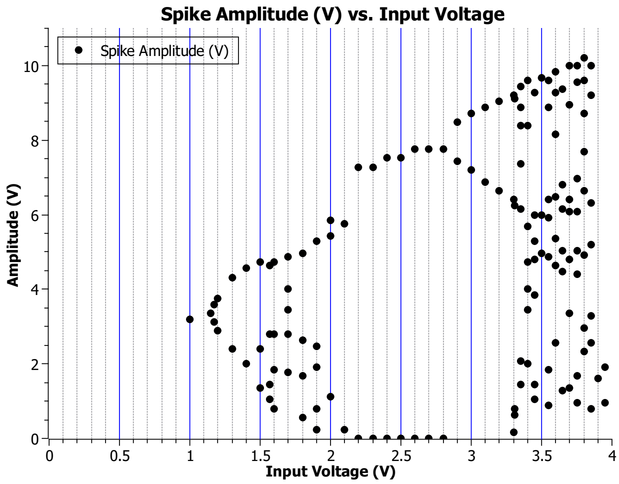

FIG. 5: Experimental bifurcation map for a inductor diode circuit. Each point represents a voltage peak associated with a specified input voltage

V. DATA AND ANALYSIS

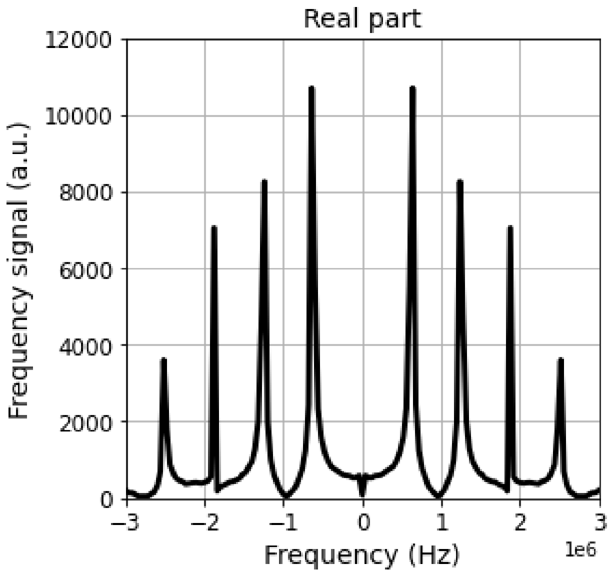

FIG. 6: The frequency domain graph of a single period signal RLD circuit at V_0 = 1.1V

Figures 6, 7, and 8 are frequency spectra of data taken over voltages ranging from V_0 =1.00 to 4.00V in the circuit described above. Before the first bifurcation, the spectrum at 1.00V has one clear frequency spike at approximately 0.06×10^{-3} Hz, as seen in Figure 6. The first bifurcation can be observed in Figure 7 at 1.17 ± 0.01V. You can tell that a bifurcation occurs because there is a new frequency peak at half of the original system's main frequency

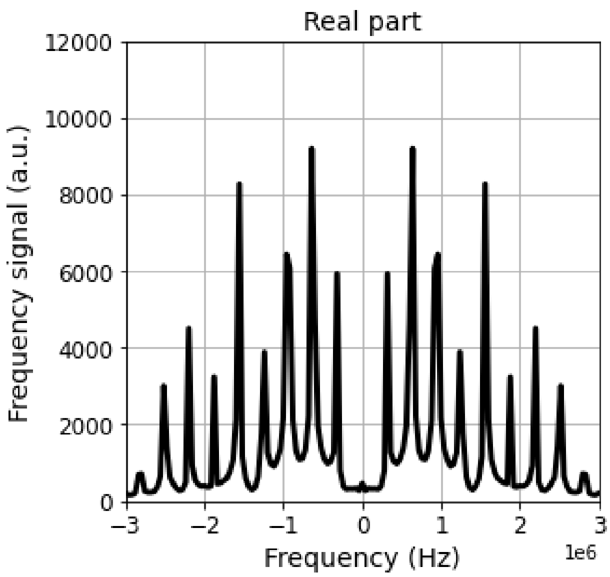

FIG. 7: Frequency domain graph of the same RLD circuit that has bifurcated and now contains two frequency spikes that represent a period doubling at $V_0 = 1.3$V

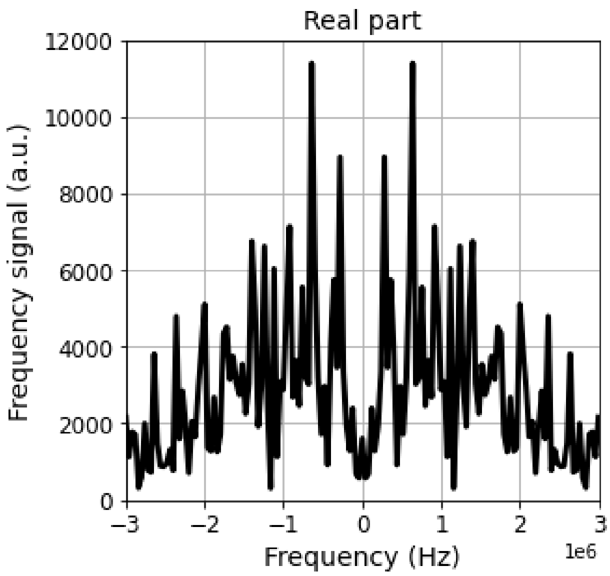

FIG. 8: Frequency domain graph of the same RLD circuit that has bifurcated until it has become chaotic at $V_0 = 3.6$V.

spike. This illustrates a period doubling of the system, as discussed in the theory section. As the voltage is increased, the system continues to bifurcate at $V_0 = 1.46 \pm 0.01$V, 1.53 ± 0.01V, 1.83 ± 0.01V and then becomes difficult to read with this experimental setup near V_0=1.83V and beyond.

After several bifurcations this system becomes chaotic, which is illustrated in Figure 8. It is clear in Figure 5, as well as Figure 8, that the voltage peaks become seemingly random, and the frequency spikes become difficult to distinguish, which agrees with the time evolution of a chaotic system discussed in the theory [6].

With these bifurcation points we can calculate Feigenbaum's constant using Equation III.2, where λ_b, λ_{b+1} and λ_{b+2} are voltages where a bifurcation occurred. Our data shows $\delta_{b=1} = 4.14 \pm 0.28$ and $\delta_{b=2} = 0.22 \pm 0.06$.

Since the points of bifurcation were not clear after the third recorded value, it is likely that the fourth value, $\lambda_4 = 1.94 \pm 0.01$V, is not accurate. This would only affect the calculated value for $\delta_{b=2}$. This explains the inaccurate value that we calculated for $\delta_{b=2} = 0.22 \pm 0.06$, which is off by a factor of 20.64. Disregarding our measurement for λ_4, the only value we can calculate for δ is $\delta_{b=1}$. Using this as an approximation for δ gives $\delta = 4.14 \pm 0.28$. The accepted value is $\delta = 4.67$, which is not within our range [3].

A clear point of error within this experiment was the lack of higher order measurements for δ. Feigenbaum's constant, shown in Equation III.2 is the limit as $b \to \infty$, but the experiment only allowed for the accurate measurement of 3 points of bifurcation, which translates to 1 value for δ. Without a trend to follow, the uncertainty in our value for δ is difficult to accurately state but is certainly larger than what is presented.

VI. CONCLUSION

In conclusion, we have demonstrated the use of a chaotic circuit to approximate Feigenbaum's constant, and proven that period doubling is a route to chaos. We used an alternating current set at 627 kHz and several input voltages between 1.00V and 4.00V to determine at which input voltages bifurcation and chaos occur. Using these points of bifurcation, it is possible to approximate Feigenbaum's constant. With the data collected, the experiment approximated $\delta = 4.14 \pm 0.28$, which the accepted value is not within.

A possible reason for this discrepancy is the inability for the equipment available to accurately detect bifurcations past the 3^{rd} period doubling. This experiment should be repeated with a higher resolution oscilloscope and a more accurate function generator to ensure that each bifurcation is detected. Not only is this circuit useful as an introduction to nonlinear dynamics, but chaotic electronic systems have also become prominent as a means of increasing security in information transmission [8].

VII. REFERENCES

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