Bistable Configurations of Compliant Mechanisms Modeled Using Four Links and Translational Joints

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Bistable mechanical devices remain stable in two distinct positions without power input. They find application in valves, switches, closures, and clasps. Mechanically bistable behavior results from the storage and release of energy, typically in springs, with stable positions occurring at local minima of stored energy. Compliant mechanisms offer an elegant way to achieve this behavior by incorporating both motion and energy storage into the same flexible element. Interest in compliant bistable mechanisms has also recently increased because of the advantages of bistable behavior in many micro-electro-mechanical systems (MEMS). Design of compliant or rigid-body bistable mechanisms typically requires simultaneous consideration of both energy storage and motion requirements. This paper simplifies this process by developing theory that provides prior knowledge of mechanism configurations that guarantee bistable behavior. Configurations which include one or more translational, or slider, joints are studied in this work. Several different mechanism types are analyzed to determine compliant segment placement that will ensure bistable mechanism operation. Examples demonstrate the power of the theory in design. [DOI: 10.1115/1.1760776]

1 Introduction

In many devices, such as switches, closures, and clasps, mechanisms are desired which experience stable equilibrium in two distinct positions. Several authors have explored bistable mechanism characteristics, including the design of particular bistable mechanisms [1–3]. There has also been considerable effort devoted to the design and fabrication of bistable micro-mechanisms for micro-valves [4–7], micro-switches or relays [8–17], and fiber-optic switches [18–20]. Work on a mechanically bistable display system [21] and multi-stable mechanisms [22] has also been presented. Recent work has even focused on using mechanically bistable devices in a binary reconfigurable device, which uses the stable states of multiple bistable mechanisms to create many stable positions for the system [23]. Much of this research relies on residual stress to induce beam buckling, a well-known bistable phenomenon. However, the difficulty of accurately controlling residual stress in micromachined materials complicates reproducibility of such devices [24–25]. Devices that do not require beam buckling often suffer from a complicated design process, in which computer models are manipulated until desired behavior is achieved. Hence, a need exists to develop simplified design methods for bistable mechanisms.

Compliant bistable mechanisms are a particular class of bistable mechanisms which use deflections of their members to gain motion, rather than relying solely on traditional rigid-body joints. Compliant mechanisms represent an elegant way to achieve bistable behavior because the flexible members allow both motion and energy storage to be incorporated into one element. In addition, compliance offers several other advantages, such as reduction in part-count, reduced friction, and less backlash and wear [26]. However, the design of compliant bistable mechanisms is often not straightforward or easy, requiring the simultaneous analysis of both the motion and energy storage of the mechanism.

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This paper addresses this problem by developing theory specifying the placement of compliant segments within several different mechanism types to result in bistable behavior. With a known bistable mechanism configuration, dimensional synthesis may then be performed to meet motion requirements. Previous work discusses this topic for the four-link mechanism class [27]; this paper expands the theory to include four-link mechanisms with translational (slider) joints. Examples are presented to demonstrate the ease of design made possible by this theory.

2 Approach to Mechanism Modeling

The pseudo-rigid-body model provides a convenient tool to use in the analysis and synthesis of compliant bistable mechanisms [28]. This model approximates the force-deflection characteristics of a compliant segment using two or more rigid segments joined by rigid-body joints, with springs at the joints to model the segment's stiffness. Flexures which approximate the motion of a pin joint, including small-length flexural pivots and fixed-pinned segments, are modeled with one pin joint and one torsional spring [28–30]. The location of the pin joint is determined by loading conditions, and the value of the torsional spring stiffness depends on geometry and material properties of the flexible segment. Functionally binary pinned-pinned (FBPP) segments, consisting of a compliant segment loaded only at the pin joints on its ends, approximate the motion of a slider joint because the segment can only oppose a force directed along the line between its pin joints. Thus, these segments are modeled using a slider attached to a translational spring [31]. Although the force-deflection behavior of the segment is generally non-linear, a linear spring approximates this behavior reasonably well and will be used here for simplicity.

Several simple equations involving a few model constants have been developed to express link lengths and spring stiffnesses within the pseudo-rigid-body model [28–31]; however, it is sufficient here to state that many types of compliant mechanisms can be treated as rigid-body mechanisms which incorporate springs at the joints. Thus, to create a compliant mechanism from a rigid-body mechanism model, one or more pin joints would be replaced by a small-length flexural pivot or a fixed-pinned segment, or a slider joint would be replaced by a functionally-binary pinned-pinned segment. Extensive testing using physical mechanisms has demonstrated the validity of this model [32–36]. Therefore, the theory developed in this paper will treat compliant mechanisms and rigid-body mechanisms at the same time by allowing any joint with a spring to represent the appropriate compliant segment.

2.1 The Stability of Compliant Mechanisms. As compliant mechanisms move, they store or release strain energy in their flexible members. This storage and release of energy gives them one or more distinct stable equilibrium positions [3]. A mechanism is at an equilibrium position when no external forces are required to maintain the mechanism's position. For this discussion, an equilibrium position is stable if the mechanism returns to that position after small disturbances, but it is unstable if small disturbances cause the mechanism to assume a different position. In the absence of other energy input, the stable equilibrium positions of a mechanism will correspond to local minima in the strain energy storage of the mechanism [37]. Hence, a knowledge of the strain energy equation for a compliant mechanism allows calculation of stable positions.

Using the pseudo-rigid-body model, the strain energy equation of a compliant mechanism is easily generated. For a segment which approximates a pin joint, the potential energy V stored in the segment is

$$V=\frac{1}{2}K\Theta^2 \tag{1}$$

where K is the torsional spring constant, calculated using model equations, and Θ is the pseudo-rigid-body angle, or the angle of deflection of the compliant segment. The strain energy stored in a FBPP segment is

$$V=\frac{1}{2}K_s(\Delta x)^2 \tag{2}$$

where Δx is the change in distance between the segment's two pin joints, and K_s is the linear spring constant. Because each compliant segment stores energy independently of the others, the total strain energy in the mechanism is simply the sum of the energy stored in each compliant segment [38].

2.2 Method of Identification of Bistable Mechanism Configurations. To find mechanism configurations resulting in bistable behavior, the strain energy equations for several types of mechanisms will be studied. Each joint in the mechanism can be examined independently by choosing a non-zero spring constant for a spring operating at the joint, while spring constants for all other joints are zero. For a mechanism to be bistable in a given configuration, it must meet three criteria. First, the first derivative of the potential energy equation must have at least three solutions, or mechanism positions that make the first derivative of energy equal to zero. Second, the second derivative of energy must be positive at two of these solutions, indicating two stable states, while it must be negative at all of the other solutions, indicating unstable positions. Third, the two stable positions as well as at least one of the unstable positions must be viable mechanism positions—that is, the mechanism must be able to assume these positions during normal motion. The results of this analysis for several mechanism classes allow determination of mechanism configurations which are bistable.

3 Double-Slider Mechanisms With a Link Joining the Sliders

This mechanism type consists of a link joined by pin joints to two slider joints, as shown in Fig. 1. The figure shows springs placed at each joint—torsional springs at pin joints and translational springs at sliders. x_2 and x_4 are measured from the undeflected state. The displacement equations in terms of θ_3 are

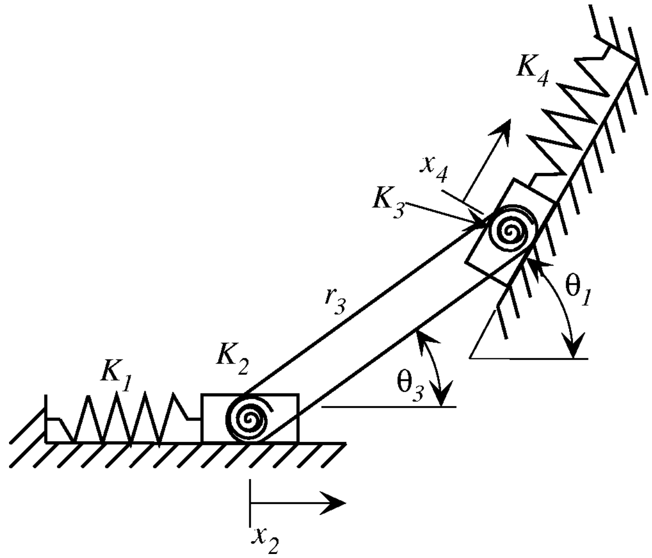

Fig. 1 A double-slider mechanism model with the two sliders joined by a link. Springs at each joint represent compliant segments modeled with the pseudo-rigid-body model.

$$x_2=\frac{r_3[\sin(\theta_1-\theta_{30})+\sin(\theta_3-\theta_1)]}{\sin\theta_1} \tag{3}$$

$$x_4=\frac{r_3(\sin\theta_3-\sin\theta_{30})}{\sin\theta_1} \tag{4}$$

where θ_{30} is the initial angle of the link. All springs are assumed to be undeflected when $\theta_3=\theta_{30}$.

3.1 Analysis. The energy equation for this mechanism is

$$V=\frac{1}{2}(K_1\psi_1^2+K_2\psi_2^2+K_3\psi_3^2+K_4\psi_4^2) \tag{5}$$

with

$$\begin{aligned}\psi_1&=x_2\\ \psi_2&=\theta_3-\theta_{30}\\ \psi_3&=\theta_3-\theta_{30}\\ \psi_4&=x_4\end{aligned} \tag{6}$$

3.1.1 Analysis for the Spring Labeled K_1. If K_1 is chosen to be the only non-zero spring constant,

$$\frac{dV}{d\theta_3}=0=K_1x_2\frac{dx_2}{d\theta_3}=K_1\left(\frac{r_3}{\sin\theta_1}\right)^2[\sin(\theta_1-\theta_{30})+\sin(\theta_3-\theta_1)]\cos(\theta_3-\theta_1) \tag{7}$$

The solutions to this first derivative equation are

$$\begin{aligned}\theta_3&=\theta_{30}+\pi n,\quad n=0,\pm 2,\pm 4,\ldots\\ \theta_3&=2\theta_1-\theta_{30}+\pi m,\quad m=\pm 1,\pm 3,\pm 5,\ldots\\ \theta_3&=\theta_1+\frac{\pi}{2}m,\quad m=\pm 1,\pm 3,\pm 5,\ldots\end{aligned} \tag{8}$$

where, for the three solutions to represent distinct mechanism positions,

$$\theta_{30}\neq\theta_1+\frac{\pi}{2}m \tag{9}$$

The second derivative is

$$\frac{d^2V}{d\theta_3^2}=K_1\left(\frac{r_3}{\sin\theta_1}\right)^2\{\cos^2(\theta_3-\theta_1)-\sin(\theta_3-\theta_1)[\sin(\theta_1-\theta_{30})+\sin(\theta_3-\theta_1)]\}\tag{10}$$

For the first two solutions in Eq. (8), the sin term in Eq. (10) is zero, leaving only a $\cos^2$ term, which is always positive. Therefore, the first two solutions are stable positions for the mechanism. For the last solution, the $\cos^2$ term is zero, and the remainder of the expression will be negative for all values of θ_1, given the restriction in Eq. (9). Therefore, the last solution, which is really two physical positions for the mechanism, corresponds to two unstable positions. The restriction in Eq. (9) says that the initial mechanism position cannot be at either of the unstable positions; these correspond to positions of extreme motion for the slider attached to K_1. Because this type of mechanism can rotate through a complete revolution in θ_3, all of the solutions in Eq. (8) are viable positions for the mechanism, so that a translational spring K_1 leads to bistable behavior.

3.1.2 Analysis for the Springs Labeled K_2 or K_3. If either K_2 or K_3 is exclusively non-zero,

$$\frac{dV}{dx_4}=0=K_n(\theta_3-\theta_{30})\frac{d\theta_3}{dx_4}=K_n(\theta_3-\theta_{30})\frac{\sin\theta_1}{r_3\sqrt{1-\left(\frac{x_4}{r_3}\sin\theta_1+\sin\theta_{30}\right)^2}}\quad n=2,3\tag{11}$$

The only solution to this equation, solved simultaneously with Eq. (4), is $x_4=0$, which is the initial position. Therefore, there are no other equilibrium positions for the mechanism, proving that a torsional spring K_2 or K_3 does not result in a bistable mechanism.

3.1.3 Analysis for the Spring Labeled K_4. If K_4 is exclusively non-zero, then the first derivative equation is

$$\frac{dV}{d\theta_3}=0=K_4x_4\frac{dx_4}{d\theta_3}=K_4\left(\frac{r_3}{\sin\theta_1}\right)^2(\sin\theta_3-\sin\theta_{30})\cos\theta_3\tag{12}$$

with solutions

$$\begin{aligned}\theta_3&=\theta_{30}+\pi n,\quad n=0,\pm2,\pm4,\dots\\ \theta_3&=\pi m-\theta_{30},\quad m=\pm1,\pm3,\pm5,\dots\\ \theta_3&=\frac{\pi}{2}m,\quad m=\pm1,\pm3,\pm5,\dots\end{aligned}\tag{13}$$

where, for the three solutions to represent three distinct mechanism positions,

$$\theta_{30}\neq\frac{\pi}{2}m.\tag{14}$$

In the interest of space, the second derivative of energy will not be explicitly stated in this or subsequent proofs. However, its derivation, followed by substitution of the solutions from Eq. (13) reveals that the first two solutions give positive values for the second derivative, while the last solution gives negative values. Moreover, because the link has full rotation, each of the solutions represents a viable mechanism position. Therefore, a translational spring K_4 will produce a bistable mechanism, unless the restriction in Eq. (14) is not met—that is, if the initial position is an extreme position for the slider attached to K_4.

3.1.4 Analysis Summary. Hence, for a double-slider mechanism with a link joining the sliders, the mechanism will be bistable if a spring is placed at either of the sliders and the initial position is not an extreme position for the spring. This result applies to either a rigid-body mechanism with springs or to a compliant mechanism modeled as such, like the mechanisms shown in Fig. 2. However, a torsional spring placed at either pin joint does not cause bistable behavior. The figure also shows one of the unstable positions and the second stable position. The compliant mechanism is shown with a functionally-binary pinned-pinned segment replacing the spring.

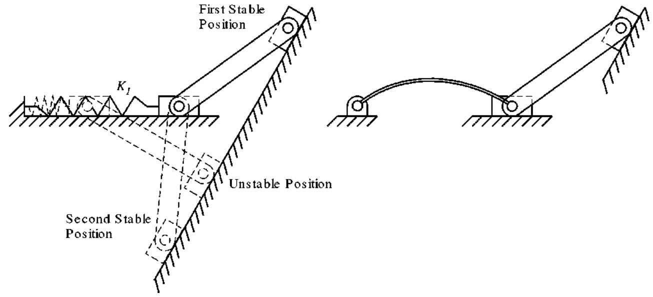

Fig. 2 A bistable double-slider mechanism with a link joining the sliders and a compliant equivalent. The second stable position and one of the unstable positions are shown.

4 Double-Slider Mechanisms With a Pin Joining the Sliders

This class consists of mechanisms with four joints, including two slider joints. The two sliders are joined by a pin joint, as shown in Fig. 3. Using θ as the independent variable,

$$r_2=\frac{e_1+e_2\cos\theta}{\sin\theta}\tag{15}$$

and

$$r_4=\frac{e_1\cos\theta+e_2}{\sin\theta}\tag{16}$$

For ease of analysis, we require both e_1 and e_2, the slider eccentricities, to be non-negative, and

$$e_1\geqslant e_2\tag{17}$$

These requirements may be made without loss of generality because mechanisms which violate these conditions are merely ki-

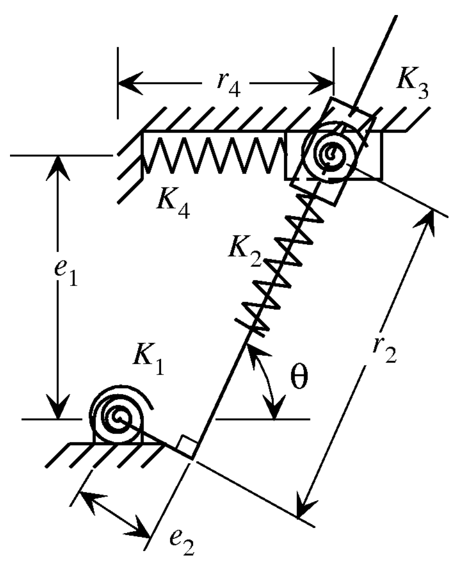

Fig. 3 A model of a fully compliant double-slider mechanism. Each compliant segment is modeled by a joint with a spring attached to it.

nematic inversions of a mechanism which does satisfy the criteria. Also, note that mechanism motion using these conditions requires that θ remain between 0 and π.

4.1 Case 1: Eccentricities Not Equal. The energy equation is the same as Eq. (5), where the K's are the spring constants as noted in Fig. 3, and the ψ's are the deflections of each spring, given by

$$\begin{aligned}\psi_1&=\theta-\theta_0\\ \psi_2&=r_2-r_{20}\\ \psi_3&=\theta-\theta_0\\ \psi_4&=r_4-r_{40}\end{aligned} \tag{18}$$

where a "0" subscript indicates the initial position at which all springs are undeflected.

4.1.1 Analysis for the Springs Labeled K_1 or K_3. If either K_1 or K_3 is chosen to be the only non-zero spring constant, and r_4 is the independent variable, the first derivative equation is

$$\frac{dV}{dr_4}=0=K_n(\theta-\theta_0)\frac{d\theta}{dr_4}\quad n=1,3 \tag{19}$$

Solutions to this equation require either θ to equal θ_0 or the derivative term to be zero. For θ to equal θ_0,

$$\frac{r_4\sqrt{r_4^2+e_1^2-e_2^2}-e_1e_2}{r_4^2+e_1^2}=\frac{r_{40}\sqrt{r_{40}^2+e_1^2-e_2^2}-e_1e_2}{r_{40}^2+e_1^2} \tag{20}$$

The sole solution to this equation is $r_4=r_{40}$, or the initial position. The derivative term is given by

$$\frac{d\theta}{dr_4}=-\frac{e_1}{r_4^2+e_1^2}-\frac{e_2r_4}{(r_4^2+e_1^2)\sqrt{r_4^2+e_1^2-e_2^2}}=0 \tag{21}$$

There are no real solutions in r_4 to this equation. Thus, only one position satisfies the first derivative equation, proving that a torsional spring placed either at spring location 1 or 3 will not result in a bistable mechanism.

4.1.2 Analysis for the Spring Labeled K_2. If K_2 is exclusively non-zero, the first derivative equation is

$$\frac{dV}{d\theta}=0=K_2(r_2-r_{20})\frac{dr_2}{d\theta}=K_2\left(\frac{e_1+e_2\cos\theta}{\sin\theta}-\frac{e_1+e_2\cos\theta_0}{\sin\theta_0}\right)\left(-\frac{e_2+e_1\cos\theta}{\sin^2\theta}\right) \tag{22}$$

The solutions in terms of θ for this equation are

$$\begin{aligned}\theta&=\theta_0\\ \theta&=a\cos\left(-\frac{(e_1^2+e_2^2)\cos\theta_0+2e_1e_2}{e_1^2+e_2^2+2e_1e_2\cos\theta_0}\right)\\ \theta&=\pi-a\cos\left(\frac{e_2}{e_1}\right)\end{aligned} \tag{23}$$

where, for three distinct mechanism positions,

$$\theta_0\neq\pi-a\cos\left(\frac{e_2}{e_1}\right) \tag{24}$$

The second solution in Eq. (23) also has the condition

$$\sin\theta=\frac{\sin\theta_0(e_1^2-e_2^2)}{e_1^2+e_2^2+2e_1e_2\cos\theta_0} \tag{25}$$

However, for the case where $e_2<e_1$, the right hand side of Eq. (25) will always be positive, so that θ lies between 0 and π. These are also the limits for physical values that θ can take, indicating that this solution is within the physical range of the mechanism. In addition, because $e_2<e_1$, the third solution is also physically realizable. The stability of each solution position can be determined from the second derivative of potential energy. Substitution of the solutions in Eq. (23) into the second derivative reveals that the first two solutions give positive values, while the third gives negative values. Thus, the first two solutions are stable positions, while the third is an unstable mechanism position which lies between the other two. Therefore, a spring attached to the slider with the smaller eccentricity will create a bistable mechanism, unless the initial position corresponds to the third solution in Eq. (23), which is the extreme position for the slider.

4.1.3 Analysis for the Spring Labeled K_4. If K_4 is exclusively non-zero, the first derivative equation is

$$\frac{dV}{d\theta}=0=K_4(r_4-r_{40})\frac{dr_4}{d\theta}=K_4\left(\frac{e_1\cos\theta+e_2}{\sin\theta}-\frac{e_1\cos\theta_0+e_2}{\sin\theta_0}\right)\left(-\frac{e_2\cos\theta+e_1}{\sin^2\theta}\right) \tag{26}$$

Note that the equation is identical to the equation with non-zero K_2, with e_1 and e_2 swapped. The solutions in terms of θ for this equation are

$$\begin{aligned}\theta&=\theta_0\\ \theta&=a\cos\left[-\frac{(e_1^2+e_2^2)\cos\theta_0+2e_1e_2}{e_1^2+e_2^2+2e_1e_2\cos\theta_0}\right]\\ \theta&=\pi-a\cos\left(\frac{e_1}{e_2}\right)\end{aligned} \tag{27}$$

The second solution in Eq. (23) also has the condition

$$\sin\theta=\frac{\sin\theta_0(e_2^2-e_1^2)}{e_1^2+e_2^2+2e_1e_2\cos\theta_0} \tag{28}$$

For the case where $e_2<e_1$, the right hand side of Eq. (28) will always be negative, so that the second solution given in Eq. (23) lies between $-\pi$ and 0. This solution to the equation is thus out of the range of physically realizable mechanism positions. In addition, because $e_2<e_1$, the third solution is not defined mathematically. Hence, the mechanism could be taken apart and reassembled in a different assembly configuration that would include the second solution in Eq. (27), but it cannot reach this position through ordinary mechanism motion. Hence, a spring placed at spring location 4, which is attached to the slider with the larger eccentricity, will not create a bistable mechanism.

4.1.4 Analysis Summary. Therefore, for this class of double-slider mechanisms, only a spring placed at the slider with the smaller eccentricity will result in bistable behavior, assuming the initial position is not an extreme position for the slider. Figure 4 illustrates an example. A bistable compliant mechanism may also be constructed as illustrated, where the spring and slider have been replaced by a FBPP segment. The figure represents only one possible compliant configuration.

4.2 Case 2: Eccentricities Equal. The preceding section has shown that a torsional spring placed at either pin joint will not cause bistable behavior regardless of the eccentricities. However, for either non-zero K_2 or K_4, if $e_2=e_1$, the solutions in Eq. (23) and Eq. (27) to the first derivative equation are both

$$\begin{aligned}\theta&=\theta_0\\ \theta&=\pi\end{aligned} \tag{29}$$

Unfortunately, substitution into the second derivative of potential energy shows that while the first solution is definitely stable, the value $\theta=\pi$ results in a singular mechanism position. In this position, the lines of action of the two sliders lie on top of each other

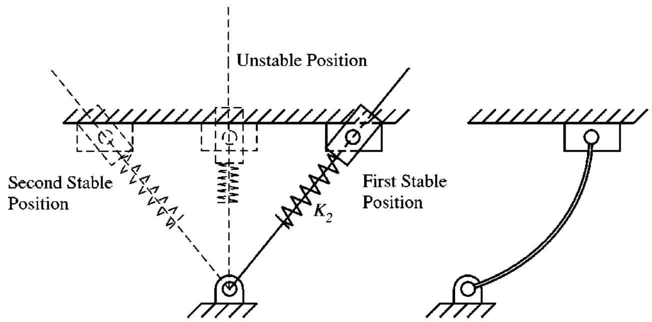

Fig. 4 A bistable double-slider mechanism with a pin joint joining the sliders, and a compliant equivalent. e_2 is zero in this illustration. The unstable and second stable positions are shown in dashed lines.

due to their equal eccentricities, and the sliders can move anywhere along their line of action in the absence of springs. Thus, a translational spring attached to either slider will be free to expand to its undeflected length when the singular position is reached, making the position stable. As this mechanism position lies within the feasible range for the mechanism class (0 to π), the mechanism will always be able to take on this position during motion. Therefore, the mechanism will have bistable behavior if a translational spring is placed at either sliding joint.

5 Slider-Crank or Slider-Rocker Mechanisms

The slider-crank or slider-rocker mechanism type is shown in Fig. 5 with springs placed at each joint. For this analysis, r_2 is arbitrarily chosen as the shortest link. Also, e is constrained to the range $e \geqslant 0$. This may be done without loss of generality because the case where $r_2 > r_3$ is merely a kinematic inversion of the case where $r_2 < r_3$, and a negative value for e represents a rotation of the entire mechanism by 180 deg. If

$$r_3 - r_2 > e \tag{30}$$

then the mechanism is a slider-crank. If the two sides in Eq. (30) are equal, then the mechanism is a change-point slider-crank, and if the left side is less than the right side, then the mechanism is a slider-rocker. In addition, the displacement equations are

$$e = r_2 \sin \theta_2 + r_3 \sin \theta_3 \tag{31}$$

$$r_1 = r_2 \cos \theta_2 + r_3 \cos \theta_3 \tag{32}$$

5.1 Slider-Rocker or Change-Point Slider-Crank Mechanisms. The energy equation is the same as Eq. (5), with

$$\psi_1 = \theta_2 - \theta_{20}$$

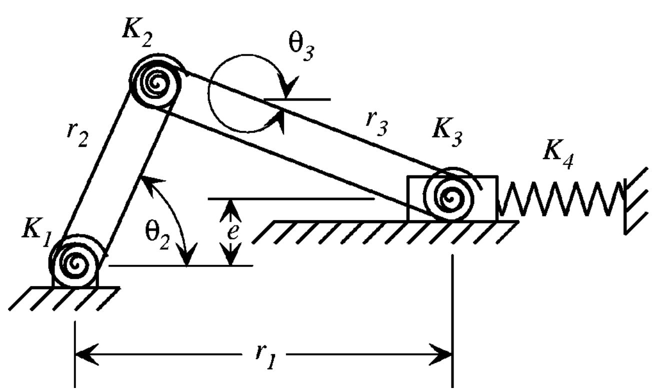

Fig. 5 A model of a general compliant slider-crank or slider-rocker mechanism

$$\begin{aligned} \psi_2 &= \theta_2 - \theta_{20} - (\theta_3 - \theta_{30}) \\ \psi_3 &= \theta_3 - \theta_{30} \\ \psi_4 &= r_1 - r_{10} \end{aligned} \tag{33}$$

5.1.1 Analysis for the Spring Labeled K_1. If K_1 is exclusively non-zero, the first derivative of energy with respect to θ_3 is

$$\frac{dV}{d\theta_3} = 0 = K_1(\theta_2 - \theta_{20}) \frac{d\theta_2}{d\theta_3} = K_1(\theta_2 - \theta_{20}) \left[\frac{-r_3 \cos \theta_3}{r_1 - r_3 \cos \theta_3} \right] \tag{34}$$

The first part, $\theta_2 - \theta_{20}$, gives two distinct solutions when solved simultaneously with Eq. (31). The second part, the derivative, gives a third solution. These are

$$\begin{aligned} &\theta_3 = \theta_{30} \text{ and } \theta_2 = \theta_{20} \\ &\theta_3 = \pi - \theta_{30} \text{ and } \theta_2 = \theta_{20} \\ &\theta_3 = \pm \frac{\pi}{2} \end{aligned} \tag{35}$$

where, for each solution to represent a distinct mechanism position,

$$\theta_{30} \neq \pm \frac{\pi}{2} \tag{36}$$

For the first two solutions in Eq. (35), the second derivative is positive, indicating that they are each stable positions. For the third solution, the second derivative is negative for a slider-rocker mechanism, resulting in an unstable equilibrium position. For a change-point slider-crank mechanism, the limit of the second derivative as the third solution is approached is also negative. Therefore, for either type, the mechanism will be bistable with a spring K_1 as long as each of the solutions in Eq. (35) represents a viable mechanism position.

The first two solutions in Eq. (35) are viable for any assembly configuration because they each satisfy Eq. (31), assuming the mechanism may be assembled—that is, there exist values θ_{20} and θ_{30} that satisfy Eq. (31). The third solution corresponds to two different mechanism positions, where $\theta_3 = \pi/2$ and $\theta_3 = -\pi/2$. For $\theta_3 = \pi/2$, Eq. (31) may be written

$$\sin \theta_2 = \frac{e - r_3}{r_2} \tag{37}$$

For a slider-rocker, with $r_3 - r_2 < e$, the right-hand side of this equation lies between 1 and -1, assuming the mechanism can be assembled $(r_2 + r_3 > e)$. Similarly, for a change-point slider-crank, with $r_3 - r_2 = e$, Eq. (37) reduces to $\sin \theta_2 = -1$. Thus, in either case, θ_2 is a real number, indicating that the mechanism can assume this position. On the other hand, for $\theta_3 = -\pi/2$, Eq. (31) is

$$\sin \theta_2 = \frac{e + r_3}{r_2} \tag{38}$$

The right-hand side is greater than one for $r_3 \geqslant r_2$ and $r_3 - r_2 < e$, indicating that no real mechanism position corresponds to this solution for a slider-rocker. For a change-point slider-crank, no real solution exists unless $e = 0$, in which case $r_3 = r_2$, and $\sin \theta_2 = 1$. Nevertheless, in either mechanism type, at least one of the two unstable positions may be reached; therefore, we may conclude that a slider-rocker or a change-point slider-crank mechanism with a torsional spring at location 1 will be bistable unless the initial mechanism position corresponds to the extreme position for the spring, as given by Eq. (36). Figure 6 shows a sample mechanism with a spring at location 1.

5.1.2 Analysis for the Spring Labeled K_2. If K_2 is exclusively non-zero, then the first derivative equation using θ_2 as the independent variable is

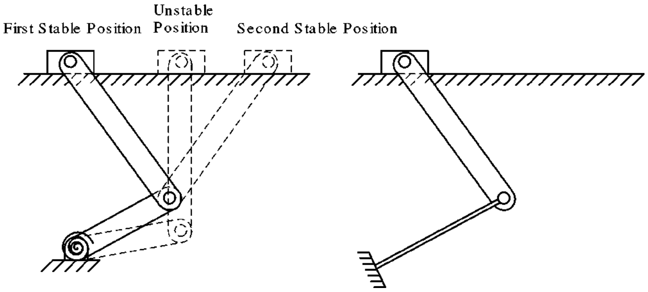

Fig. 6 A bistable slider-rocker with a spring at location 1. The unstable position and second stable position are also shown, as well as a sample compliant mechanism.

$$\frac{dV}{d\theta_2}=0=K_2[\theta_2-\theta_3-(\theta_{20}-\theta_{30})]\left(\frac{r_1}{r_1-r_2\cos\theta_2}\right) \quad (39)$$

The solutions to Eq. (39) which satisfy Eq. (31) are

$$\theta_2=\theta_{20} \text{ and } \theta_3=\theta_{30}$$
$$\theta_2=\theta_{20}-2a\tan\left(\frac{e}{r_{10}}\right)+\pi \text{ and } \theta_3=\theta_{30}-2a\tan\left(\frac{e}{r_{10}}\right)+\pi$$
$$\sin\theta_2=\frac{e^2+r_2^2-r_3^2}{2er_2} \text{ and } r_1=0 \quad (40)$$

where, for the three solutions to represent distinct mechanism positions,

$$\sin\theta_{20}\neq\frac{e^2+r_2^2-r_3^2}{2er_2} \text{ and } r_{10}\neq 0 \quad (41)$$

As expected, the first two solutions in Eq. (40) give positive results in the second derivative for any configuration, whereas the third solution gives a negative result for a slider-rocker mechanism. For a change-point slider-crank, the second derivative gives zero divided by zero for the third solution, but the limit goes to negative infinity as the third solution is approached. Thus, for either a slider-rocker or a change-point slider-crank mechanism, a spring placed at location 2 will result in a bistable mechanism as long as each of the solutions in Eq. (40) corresponds to a viable mechanism position.

The first two solutions represent feasible mechanism positions for any mechanism position because, in either case, the solutions satisfy Eq. (31). For the third solution to be feasible, we require, from Eq. (31),

$$-1\leqslant\frac{e^2+r_2^2-r_3^2}{2er_2}\leqslant 1 \quad (42)$$

For a change-point slider-crank, the term in the center of this inequality reduces to 1, indicating that this is a viable position. The only exception is when $e=0$, indicating that $r_2=r_3$, so that the center term is undefined. This is because this position is the change-point position for the mechanism, which is a singular position. Because a change-point mechanism will always be able to assume the change-point position, any change-point slider crank will be able to reach the position corresponding to the third solution. For a slider-rocker, the inequalities used in the previous section ($r_3-r_2<e$ and $r_3+r_2>e$) may be manipulated to result in the inequality in Eq. (42). Therefore, a slider-rocker or a change-point slider-crank with a spring K_2 will also result in a bistable mechanism, unless the initial position of the mechanism is an extreme position for the spring, as given by Eq. (41). An example mechanism with the spring in this location is shown in Fig. 7.

5.1.3 Analysis for the Spring Labeled K_3. If K_3 is exclusively non-zero, then the first derivative equation is

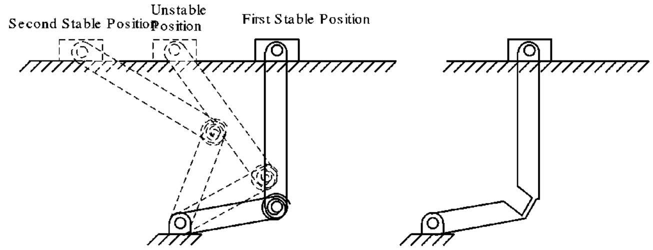

Fig. 7 A bistable slider-rocker with a spring placed at location 2, and a compliant equivalent

$$\frac{dV}{d\theta_2}=0=-K_3(\theta_3-\theta_{30})\frac{r_2\cos\theta_2}{r_3\cos\theta_3} \quad (43)$$

This equation has three solutions which also satisfy Eq. (31). They are

$$\theta_2=\theta_{20} \text{ and } \theta_3=\theta_{30}$$
$$\theta_2=\pi-\theta_{20} \text{ and } \theta_3=\theta_{30} \quad (44)$$
$$\theta_2=\pm\frac{\pi}{2}$$

where, for the three solutions to be distinct mechanism positions,

$$\theta_{20}\neq\pm\frac{\pi}{2} \quad (45)$$

The second derivative of energy is positive for the first two solutions in Eq. (44) and negative for the third solution for a slider-rocker. Its limit is negative for a change-point slider-crank. Therefore, the first two solutions are stable positions, while the third is an unstable position. Furthermore, it is easy to show that the first two solutions in Eq. (44) represent feasible mechanism positions for any configuration of slider-rocker or change-point slider crank because each solution satisfies Eq. (31). The third solution represents two possible mechanism positions: $\theta_2=\pm\pi/2$. For $\theta_2=\pi/2$, there is a real value of θ_3 to satisfy Eq. (31) for any set of r_2, r_3, and e that satisfy the conditions outlined earlier ($r_3\geqslant r_2$ and $r_3+r_2\geqslant e$). However, for $\theta_2=-\pi/2$, a real value for θ_3 only exists for a change-point slider-crank. Because there are two feasible stable positions and at least one feasible unstable position, the mechanism is bistable with a spring K_3, provided that the initial position is not an extreme position for the spring, as given by Eq. (45).

5.1.4 Analysis for the Spring Labeled K_4. If K_4 is exclusively non-zero, the first derivative of energy is

$$\frac{dV}{d\theta_2}=0=K_4(r_1-r_{10})(r_2\cos\theta_2\tan\theta_3-r_2\sin\theta_2) \quad (46)$$

There are four solutions that satisfy this equation and Eq. (31). They are

$$\theta_2=\theta_{20} \text{ and } \theta_3=\theta_{30}$$
$$\theta_2=2a\tan\left(\frac{e}{r_{10}}\right)-\theta_{20} \text{ and } \theta_3=2a\tan\left(\frac{e}{r_{10}}\right)-\theta_{30}$$
$$\theta_2=\theta_3=a\sin\left(\frac{e}{r_2+r_3}\right) \quad (47)$$
$$\theta_2=\pi+\theta_3=a\sin\left(\frac{e}{r_2-r_3}\right)$$

where, for the solutions to represent distinct mechanism positions,

$$\theta_{20}\neq\theta_{30} \text{ and } \theta_{20}\neq\pi+\theta_{30} \quad (48)$$

The second derivative is positive for the first two solutions in Eq. (47) and negative for the second two solutions, as long as the condition in Eq. (48) is met. Therefore, the mechanism has two stable positions and two unstable positions. Feasibility of the solutions is all that remains to be proved.

The first solution in Eq. (47) is feasible because it is the initial position. The second solution is also feasible because it satisfies Eq. (31). For the third solution to be feasible, we require

$$-1 \leqslant \frac{e}{r_3+r_2} \leqslant 1 \tag{49}$$

which will be satisfied for any assembled mechanism, for which $r_2+r_3>e$. However, the fourth solution will not be feasible for a slider-rocker mechanism because $r_3-r_2<e$, making the argument of the asin function out of its domain. On the other hand, for *a* change-point slider-crank, the fourth solution will reduce to $\theta_2 = a\sin(-1)$, which is feasible. Therefore, because in either mechanism type at least one unstable position is feasible, any slider-rocker or change-point slider-crank mechanism with a translational spring attached to the slider will be bistable, as long as the initial position is not an extreme position, or one which violates Eq. (48).

5.1.5 Analysis Summary. The preceding sections prove that a slider-rocker or change-point slider-crank mechanism with one spring placed at exactly one joint of the mechanism will be bistable as long as the initial position does not represent an extreme position for the spring, regardless of which of the four joints is used. The slider-crank mechanism remains to be examined.

5.2 Slider-Crank Mechanisms. The previous analysis showed that stable and unstable positions exist for springs at any of the four spring locations. Moreover, the stable positions are feasible mechanism positions for any slider-crank mechanism because no knowledge of the precise mechanism type was required to demonstrate their feasibility. All that remains is to show whether or not the unstable positions corresponding to a spring placed at each joint are feasible for a slider-crank mechanism.

For a spring K_1, which is attached to the shortest link, the unstable positions are, from Eq. (35),

$$\theta_3 = \pm\frac{\pi}{2} \tag{50}$$

Substitution into Eq. (31) gives

$$\sin\,\theta_2 = \frac{e\pm r_3}{r_2} \tag{51}$$

Knowing that $r_3 \geqslant r_2$ and, for a slider-crank, $r_3-r_2>e$, this equation reduces to

$$\sin\,\theta_2 > 1 \quad \text{or} \quad \sin\,\theta_2 < -1 \tag{52}$$

for which neither condition can be met. Hence, the mechanism cannot assume the unstable position, preventing it from switching between the two feasible stable positions. In other words, the mechanism can only switch from one stable position to the other if it is disassembled and reassembled in the other position—it cannot do so through normal mechanism motion.

The other spring attached to the shortest link is K_2. The unstable position for this spring is characterized by, from Eq. (40),

$$\sin\,\theta_2 = \frac{e^2+r_2^2-r_3^2}{2er_2} \tag{53}$$

For a slider-crank, the right-hand side of this equation is less than -1, indicating that this mechanism position is not feasible. Therefore, springs located at joints adjacent to the shortest link will not result in bistable behavior.

K_3 is the torsional spring not adjacent to the shortest link. The unstable positions for this spring location, from Eq. (44), are

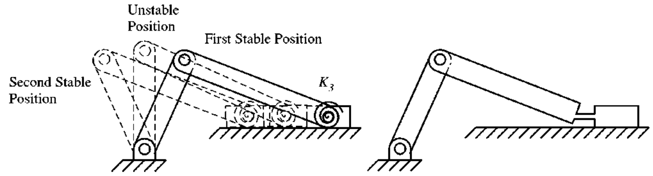

Fig. 8 A bistable slider-crank with a spring at location 3. The second stable position and one of the unstable positions are shown in dashed lines. An equivalent compliant mechanism is also shown.

$$\theta_2 = \pm\frac{\pi}{2} \tag{54}$$

which may be reached for any slider-crank for which link 2 is the shortest link. This is because the crank, which is the shortest link, may fully rotate for a slider-crank. To specifically show that this angle may be reached, substitute Eq. (54) into Eq. (31) to get

$$\sin\,\theta_3 = \frac{e\pm r_2}{r_3} \tag{55}$$

That either position is defined may be proved using the known inequalities $r_3-r_2>e$, $r_3 \geqslant r_2$, and $r_2+r_3>e$. Hence, the mechanism will be bistable with a spring placed opposite the shortest link. For example, Fig. 8 shows a bistable slider-crank with a spring at position 3.

Finally, for a translational spring K_4, the unstable positions, from Eq. (47), are

$$\theta_2 = a\sin\left(\frac{e}{r_2\pm r_3}\right) \tag{56}$$

Again, it is easy to show that either position is feasible from the basic inequalities. Figure 9 shows a bistable slider-crank with a spring placed at the slider.

Note that if $r_2>r_3$, then the shortest link will be link 3, so that a spring at joint 1 will cause bistable behavior, while a spring at joint 3 will not. This may easily be seen by repeating the preceding analysis using $r_2>r_3$. Therefore, a slider-crank mechanism with a spring placed at one joint will be bistable only if the spring is located at the slider or opposite the shortest link, assuming the initial mechanism position is not an extreme position for the spring.

6 Summary of Spring Locations Necessary for Bistable Mechanisms

Table 1 summarizes the spring locations for each mechanism type which will result in a bistable mechanism if no other springs

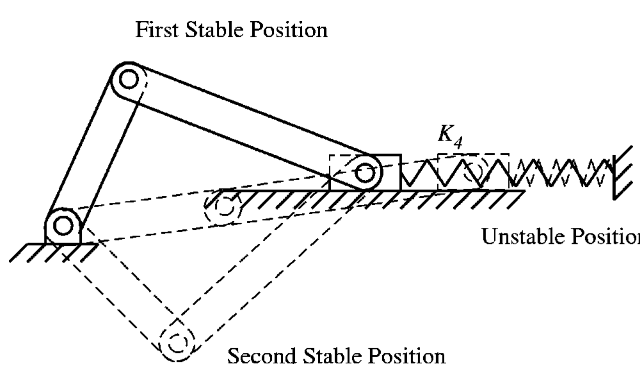

Fig. 9 A bistable slider-crank with the two stable positions and one unstable position shown. In this case, the spring is placed in position 4.

Table 1 The spring locations necessary for each mechanism type to cause bistable behavior

Mechanism Class	*Location of Springs for Bistable Mechanism*
Double-Slider (pin joint joining), $e_1 \neq e_2$	translational joint with shorter eccentricity
Double-Slider (pin joint joining), $e_1 = e_2$	either translational joint
Double-Slider (link joining)	either translational joint
Slider-Crank	translational joint or pin joint not attached to shortest link
Change-Point Slider-Crank	any joint
Slider-Rocker	any joint

act on the mechanism. The table applies to either rigid-body mechanisms or compliant mechanisms modeled using the appropriate rigid-body mechanism. If multiple springs are present, the potential energy equation must be solved for a full understanding of the location of the unstable and stable positions. However, the analyses presented here give a designer knowledge of which spring positions will work for or against bistable behavior. Moreover, in many cases it is possible to make one spring significantly stiffer than the other springs in the mechanism, allowing the stiff spring to dominate the others. Then, the mechanism behaves much as if the dominating spring were the only spring in the mechanism.

7 Examples in Design

The information regarding placement of springs to produce bistable behavior is very useful for design. By knowing beforehand the spring locations which will produce bistable behavior, a designer can quickly generate many viable mechanism configurations. The examples here demonstrate the process.

7.1 Example: Bistable CD Ejection Actuator. A bistable mechanism is desired to eject compact discs or similar media from a case. The mechanism must move in a straight line to push the CD out of the case; hence, one or more translational joints is desirable. Thus, any of the mechanism classes discussed in this paper may be used. To develop designs, each mechanism type may be considered in a configuration using one of the spring locations specified in Table 1. Figure 10 shows four example mechanisms. In each case, one of the joints has a spring attached to it with the spring locations given by Table 1. One of these designs may then be chosen for further development. For example, if design (a) is chosen, a compliant mechanism like that shown in Fig. 11 could result. In this mechanism, the pin joints are approximated with very small, thin flexural hinges, known as living hinges. The spring and slider joint are approximated using a functionally binary pinned-pinned segment. Dimensions and materials can be chosen to meet any other design constraints.

7.2 Example: Bistable Electrical Switch. A bistable electrical switch with a rotating link used to toggle the mechanism between states is desired. Figure 12 shows five different mechanism configurations which could be used. These configurations are chosen by investigating various inversions of the slider-rocker mechanism type with different spring locations. This figure illustrates how mechanism inversions can be used to create many different possible configurations. Figure 12(*c*) is developed further here because of its simplicity, allowing it to be constructed with only one link and one slider. In addition, by replacing the spring and slider with a FBPP segment, and by using living hinges in place of pin joints, the mechanism can be made fully compliant. The design is shown in Fig. 13. Again, dimensions and materials can be chosen from any other design constraints.

8 Conclusion

Using knowledge of mechanism motion, analyses have been presented regarding the placement of compliant segments in a pseudo-rigid-body model to guarantee a mechanism's bistable behavior. This paper has focused on identifying such mechanism configurations that contain one or more slider joints. The work simplifies design by giving the designer prior knowledge of the compliant mechanism configurations that will lead to a bistable mechanism. Once the mechanism configuration is selected, its dimensions can be chosen to meet force or motion requirements of a design problem without concern for compromising the bistable behavior of the mechanism. Two examples were presented to demonstrate the use of the theory in design.

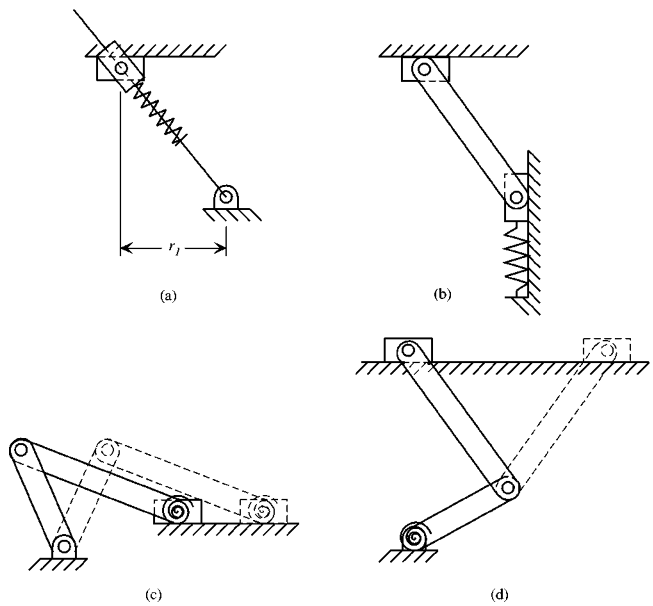

Fig. 10 Possible mechanisms that could be used to make a bistable CD ejection actuator. (*a*) and (*b*) are the two types of double-slider mechanisms; (*c*) and (*d*) are a slider-crank and slider-rocker mechanism, respectively.

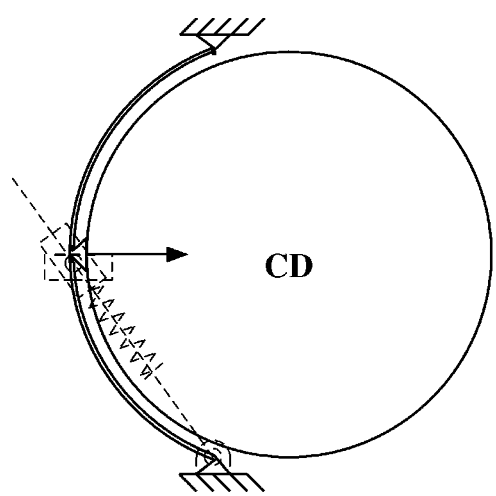

Fig. 11 The resulting compliant bistable mechanism, based on the double-slider with a pin joint joining the sliders. A pseudo-rigid-body model mechanism is shown in dashed lines.

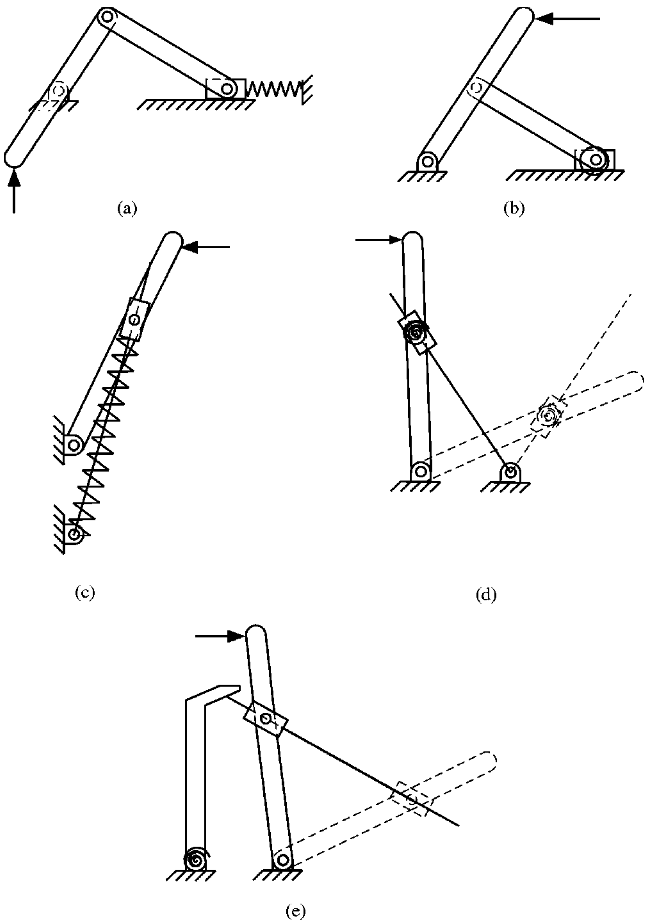

Fig. 12 Five different possible configurations of the slider-crank or slider-rocker class which could meet the design specifications. The second positions of (*d*) and (*e*) are included to aid in visualization.

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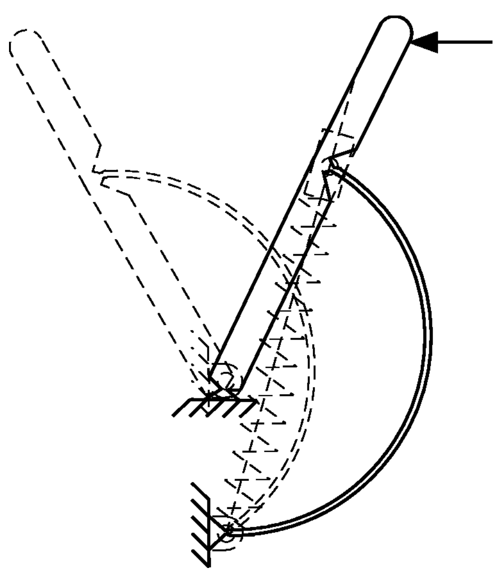

Fig. 13 The conceptual design for the bistable electrical switch

References

[1] Schulze, E. F., 1955, ''Designing Snap-Action Toggles,'' Prod. Eng. (N.Y.), pp. 168–170.

[2] Howell, L. L., Rao, S. S., and Midha, A., 1994, ''The Reliability-Based Optimal Design of a Bistable Compliant Mechanism,'' ASME J. Mech. Des., **117**(1), pp. 156–165.

[3] Opdahl, P. G., Jensen, B. D., and Howell, L. L., 1998, ''An Investigation into Compliant Bistable Mechanisms,'' *Proc. 1998 ASME Design Engineering Technical Conferences*, DETC98/MECH-5914.

[4] Wagner, B., Quenzer, H. J., Hoershelmann, S., Lisec, T., and Juerss, M., 1996, ''Bistable Microvalve with Pneumatically Coupled Membranes,'' *Proc. IEEE Micro Electro Mechanical Systems* pp. 384–388.

[5] Goll, C., Bacher, W., Buestgens, B., Maas, D., Menz, W., and Schomburg, W. K., 1996, ''Microvalves with Bistable Buckled Polymer Diaphragms,'' J. Micromech. Microeng., **6**(1), pp. 77–79.

[6] Shinozawa, Y., Abe, T., and Kondo, T., 1997, ''Proportional Microvalve Using a Bi-Stable Magnetic Actuator,'' *Proc. 1997 IEEE Micro Electro Mechanical Systems* (MEMS), pp. 233–237.

[7] Schomburg, W. K., and Goll, C., 1998, ''Design Optimization of Bistable Microdiaphragm Valves,'' Sens. Actuators, A, **64**(3), pp. 259–264.

[8] Hälg, B., 1990, ''On A Nonvolatile Memory Cell Based on Micro-?electro-mechanics,'' IEEE Micro Electro Mechanical Systems, pp. 172–176.

[9] Matoba, H., Ishikawa, T., Kim, C., and Muller, R. S., 1994, ''A Bistable Snapping Mechanism,'' IEEE Micro Electro Mechanical Systems, pp. 45–50.

[10] Kruglick, E. J. J., and Pister, K. S. J., 1998, ''Bistable MEMS Relays and Contact Characterization,'' *1998 Solid-State Sensor and Actuator Workshop*, pp. 333–337.

[11] Sun, X., Farmer, K. R., and Carr, W., 1998, ''Bistable Microrelay Based on Two-Segment Multimorph Cantilever Actuators,'' *Proc. 1998 IEEE Micro Electro Mechanical Systems (MEMS)*, pp. 154–159.

[12] Vangbo, M., and Bäklund, Y., 1998, ''A Lateral Symmetrically Bistable Buckled Beam,'' J. Micromech. Microeng., **8**, pp. 29–32.

[13] Jensen, B. D., Howell, L. L., and Salmon, L. G., 1999, ''Design of Two-Link, In-Plane, Bistable Compliant Micro-Mechanisms,'' ASME J. Mech. Des., **121**(3), pp. 416–423.

[14] Saif, M. T. A., 2000, ''On a Tunable Bistable MEMS—Theory and Experiment,'' J. MEMS, **9(2)**, pp. 157–169.

[15] Qiu, J., Lang, J. H., and Slocum, A. H., 2001, ''A Centrally-Clamped Parallel-Beam Bistable MEMS Mechanism,'' *Proc. IEEE Micro Electro Mechanical Systems (MEMS) 2001*, CH37090, pp. 353–356.

[16] Baker, M. S., and Howell, L. L., 2002, ''On-Chip Actuation of an In-Plane Compliant Bistable Micro-Mechanism,'' J. Microelectromech. Syst., **11**, pp. 566–573.

[17] Masters, N. D., and Howell, L. L., 2003, ''A Self-Retracting Fully-Compliant Bistable Micromechanism,'' J. MEMS, **12**, pp. 273–280.

[18] Hoffman, M., Kopka, P., Gross, T., and Voges, E., 1999, ''Optical Fibre Switches Based on Full Wafer Silicon Micromachining,'' J. Micromech. Microeng., **9**(2), pp. 151–155.

[19] Pieri, F., and Piotto, M., 2000, ''A Micromachined Bistable 1×2 Switch for Optical Fibers,'' Microelectron. Eng., **53**(1), pp. 561–564.

[20] Maekoba, H., Helin, P., Reyne, G., Bourouina, T., and Fujita, H., 2001, ''Self-Aligned Vertical Mirror and V-grooves Applied to an Optical-Switch: Modeling and Optimization of Bi-Stable Operation by Electromagnetic Actuation,'' Sens. Actuators, A, **87**(3), pp. 172–178.

[21] Fleming, J. G., 1998, ''Bistable Membrane Approach to Micromachined Displays,'' *Flat Panel Display Materials-1998 MRS Symposium*, Vol. 508, pp. 219–224.

[22] Limaye, A. A., King, C. W., and Campbell, M. I., 2003, ''Analysis of Multiple Equilibrium Positions in Magnetostatic Field,'' *Proc. ASME 2003 Design Engineering Technical Conf.*, DETC2003/DAC-48841.

[23] Hafez, M., Lichter, M. D., and Dubowsky, S., 2003, ''Optimized Binary Modular Reconfigurable Robotic Devices,'' IEEE/ASME Trans. Mechatron., **8**(1), pp. 18–25.

[24] Fang, W., and Wickert, J. A., 1995, ''Comments on Measuring Thin-Film Stresses Using Bi-layer Micromachined Beams,'' J. Micromech. Microeng., **5**, pp. 276–281.

[25] Ziebart, V., Paul, O., Münch, U., Jürg, S., and Baltes, H., 1998, ''Mechanical Properties of Thin Films from the Load Deflection of Long Clamped Plates,'' J. MEMS, **7**(3), pp. 320–328.

[26] Shoup, T. E., and McLarnan, C. W., 1971, ''A Survey of Flexible Link Mechanisms Having Lower Pairs,'' J. Mec., **6**(3), pp. 97–105.

[27] Jensen, B. D., and Howell, L. L., , ''Identification of Compliant Pseudo-Rigid-Body Four-Link Mechanism Configurations Resulting in Bistable Behavior,'' ASME J. Mech. Des., in press.

[28] Howell, L. L., and Midha, A., 1994, ''A Method for the Design of Compliant Mechanisms with Small-Length Flexural Pivots,'' ASME J. Mech. Des., **116**(1), pp. 280–290.

[29] Howell, L. L., and Midha, A., 1995, ''Parametric Deflection Approximations for End-Loaded, Large-Deflection Beams in Compliant Mechanisms,'' ASME J. Mech. Des., **117**(1), pp. 156–165.

[30] Howell, L. L., Midha, A., and Norton, T. W., 1996, ''Evaluation of Equivalent Spring Stiffness for Use in a Pseudo-Rigid-Body Model of Large-Deflection Compliant Mechanisms,'' ASME J. Mech. Des., **118**(1), pp. 126–131.

[31] Edwards, B. T., Jensen, B. D., and Howell, L. L., 2001, ''A Pseudo-Rigid-Body Model for Initially-Curved Pinned-Pinned Segments Used in Compliant Mechanisms,'' ASME J. Mech. Des., **123**(3), pp. 464–468.

[32] Derderian, J. M., Howell, L. L., Murphy, M. D., Lyon, S. M., and Pack, S. D., 1996, ''Compliant Parallel-Guiding Mechanisms,'' *Proc. 1996 ASME Design Engineering Technical Conferences* 96-DETC/MECH-1208.
[33] Howell, L. L., and Midha, A., 1996, ''A Loop-Closure Theory for the Analysis and Synthesis of Compliant Mechanisms,'' ASME J. Mech. Des., **118**, pp. 121–125.
[34] Jensen, B. D., Howell, L. L., Gunyan, D. B., and Salmon, L. G., 1997, ''The Design and Analysis of Compliant MEMS Using the Pseudo-Rigid-Body Model,'' *Microelectro-mechanical Systems (MEMS) 1997, 1997 ASME Int. Mech. Eng. Congress and Exposition*, DSC-Vol. 62, pp. 119–126.
[35] Lyon, S. M., Evans, M. S., Erickson, P. A., and Howell, L. L., 1999, ''Prediction of the First Modal Frequency of Compliant Mechanisms Using the Pseudo-Rigid-Body Model,'' ASME J. Mech. Des., **121**(2), pp. 309–313.
[36] Mettlach, G. A., and Midha, A., 1996, ''Using Burmester Theory in the Design of Compliant Mechanisms,'' *Proc. 1996 ASME Design Engineering Technical Conferences*, 96-DETC/MECH-1181.
[37] Leipholz, H., 1970, *Stability Theory*, Academic Press, New York and London.
[38] Howell, L. L., 2001, *Compliant Mechanisms*, John Wiley and Sons, New York, NY.