Introduction

1. What it's all about

The practical problem, solving systems of linear equations, that motivates the subject of Linear Algebra, is introduced in chapter 4. Although the problem is concrete and easily understood, the methods and theoretical framework required for a deeper understanding of it are abstract. Linear algebra can be approached in a completely abstract fashion, untethered from the problems that give rise to the subject.

Such an approach is daunting for the average student so we will strike a balance between the practical, specific, abstract, and general.

Linear algebra plays a central role in almost all parts of modern technology. Systems of linear equations involving hundreds, thousands, even *billions* of unknowns are solved every second of every day in all corners of the globe. One of the more fantastic uses is the way in which Google prioritizes pages on the web. All web pages are assigned a *page rank* that measures its importance. The page ranks are the unknowns in an enormous system of linear equations. To find the page rank one must solve the system of linear equations. To handle such large systems of linear equations one uses sophisticated techniques that are developed first as abstract results about linear algebra.

Systems of linear equations are rephrased in terms of matrix equations, i.e., equations involving matrices. The translation is straightforward but after mastering the basics of "matrix arithmetic" one must interpret those basics in geometric terms. That leads to linear geometry and the language of vectors and vector spaces.

Chapter 1 provides a brief account of linear geometry. As the name suggests, linear geometry concerns lines. The material about lines in the plane is covered in high school. Unless you know that material backwards and forwards linear algebra will be impossible for you. Linear geometry also involves higher dimensional analogues of lines, for examples, lines and planes in 3-space, or $\mathbb{R}^3$ as we will denote it. I am assuming you met that material in a multivariable calculus course. Unless you know *that* material backwards and forwards linear algebra will be impossible for you.[1]

After reviewing linear geometry we will review basic facts about matrices in chapter 2. That will be familiar to some students. Very little from chapter 2 is required to understand the initial material on systems of linear equations in chapter 4.

2. Some practical applications

(1) solving systems of linear equations
(2) birth-death processes, Leslie matrices (see Ch. 6 in Difference equations. From rabbits to chaos), Lotka-Volterra
(3) Google page rank
(4) X-ray tomography, MRI, density of tissue, a slice through the body is pixellated and each pixel has a different density to be inferred/computed from the input/output of sending various rays through the body. Generally many more measurements than unknowns so almost always obtain an inconsistent system but want the least-squares solution.

3. The importance of asking questions

You will quickly find that this is a difficult course. Everyone finds it difficult. The best advice I can give you is to ask questions. When you read something you don't understand don't just skim over it thinking you will return to it later and then fail to return to it. Ask, ask, ask. The main reason people fail this course is because they are afraid to ask questions. They think asking a question will make them look stupid. Smart people aren't afraid to ask questions. No one has all the answers.

The second most common reason people fail this course is that they wait till the week before the exam to start thinking seriously about the material. If you are smart you will try to understand every topic in this course at the time it first appears.

Linear Geometry

1. Linearity

The word linear comes from the word "line".

The geometric aspect of linear algebra involves lines, planes, and their higher dimensional analogues: e.g., lines in the plane, lines in 3-space, lines in 4-space, planes in 3-space, planes in 4-space, 3-planes in 4-space, 5-planes in 8-space, and so on, ad infinitum. Such things form the subject matter of linear geometry.

Curvy things play no role in linear algebra or linear geometry. We ignore circles, spheres, ellipses, parabolas, etc. All is linear.

1.1. What is a line? You already "know" what a line is. More accurately, you know something about lines in the plane, $\mathbb{R}^2$, or in 3-space, $\mathbb{R}^3$. In this course, you need to know something about lines in n-space, $\mathbb{R}^n$.

1.2. What is $\mathbb{R}^n$? $\mathbb{R}^n$ is our notation for the set of all n-tuples of real numbers. We run out of words after *pair, triple, quadruple, quintuple, sextuple, ...* so invent the word *n-tuple* to refer to an ordered sequence of n numbers where n can be any positive integer.

For example, (8,7,6,5,4,3,2,1) is an 8-tuple. It is not the same as the 8-tuple (1,2,3,4,5,6,7,8). We think of these 8-tuples as labels for two different points in $\mathbb{R}^8$. We call the individual entries in (8,7,6,5,4,3,2,1) the coordinates of the point (8,7,6,5,4,3,2,1); thus 8 is the first coordinate, 7 the second coordinate, etc.

Often we need to speak about a point in $\mathbb{R}^n$ when we don't know its coordinates. In that case we will say something like this: *let* $(a_1, a_2, \dots, a_n)$ *be a point in* $\mathbb{R}^n$. Here $a_1, a_2, \dots, a_n$ are some arbitrary real numbers.

1.3. The origin. The origin is a special point in $\mathbb{R}^n$: it is the point having all its coordinates equal to 0. For example, (0,0) is the origin in $\mathbb{R}^2$; (0,0,0) is the origin in $\mathbb{R}^3$; (0,0,0,0) is the origin in $\mathbb{R}^4$; and so on. We often write $\underline{0}$ for the origin.

There is a special notation for the set of all points in $\mathbb{R}^n$ except the origin, namely $\mathbb{R}^n - \{\underline{0}\}$. We use the minus symbol because $\mathbb{R}^n - \{\underline{0}\}$ is obtained by taking $\underline{0}$ away from $\mathbb{R}^n$. Here "taking away" is synonymous with "removing". This notation permits a useful brevity of expression: *"suppose $p \in \mathbb{R}^n - \{\underline{0}\}$"* means the same thing as *"suppose p is a point in $\mathbb{R}^n$ that is not $\underline{0}$"*.

1.4. Adding points in $\mathbb{R}^n$. We can add points in $\mathbb{R}^n$. For example, in $\mathbb{R}^4$, $(1,2,3,4)+(6,4,2,0)=(7,6,5,4)$. The origin is the unique point with the property that $\underline{0}+p=p$ for all points p in $\mathbb{R}^n$. For that reason we also call $\underline{0}$ zero.

We can also subtract: for example, $(6,4,2,0)-(1,2,3,4)=(5,2,-1,-4)$.

We can not add a point in $\mathbb{R}^4$ to a point in $\mathbb{R}^5$. More generally, if $m \neq n$, we can't add a point in $\mathbb{R}^m$ to a point in $\mathbb{R}^n$.

1.5. Multiplying a point in $\mathbb{R}^n$ by a number. First, some examples. Consider the point $p=(1,1,2,3)$ in $\mathbb{R}^4$. Then $2p=(2,2,4,6)$, $5p=(5,5,10,15)$, $-3p=(-3,-3,-6,-9)$; more generally, if t is any real number $tp=(t,t,2t,3t)$. In full generality, if λ is a real number and $p=(a_1,\dots,a_n)$ is a point in $\mathbb{R}^n$, λp denotes the point $(\lambda a_1,\dots,\lambda a_n)$; thus, the coordinates of λp are obtained by multiplying each coordinate of p by λ. We call λp a multiple of p.

1.6. Lines. We will now define what we mean by the word *line*. If p is a point in $\mathbb{R}^n$ that is not the origin, the line through the origin in the direction p is the set of all multiples of p. Formally, if we write $\mathbb{R}p$ to denote that line,[1]

$$\mathbb{R}p=\{\lambda p \mid \lambda \in \mathbb{R}\}.$$

If q is another point in $\mathbb{R}^n$, the line through q in the direction p is

$$\{q+\lambda p \mid \lambda \in \mathbb{R}\};$$

we often denote this line by $q+\mathbb{R}p$.

When we speak of a line in $\mathbb{R}^n$ we mean a subset of $\mathbb{R}^n$ of the form $q+\mathbb{R}p$. Thus, if I say L is a line in $\mathbb{R}^n$ I mean there is a point $p \in \mathbb{R}^n - \{\underline{0}\}$ and a point $q \in \mathbb{R}^n$ such that $L=q+\mathbb{R}p$. The p and q are not uniquely determined by L.

PROPOSITION 1.1. *The lines $L=q+\mathbb{R}p$ and $L'=q'+\mathbb{R}p'$ are the same if and only if p and p' are multiples of each other and $q-q'$ lies on the line $\mathbb{R}p$; i.e., if and only if $\mathbb{R}p=\mathbb{R}p'$ and $q-q' \in \mathbb{R}p$.*

Proof.[2] $(\Rightarrow)$ Suppose the lines are the same, i.e., $L=L'$. Since q' is on the line L' it is equal to $q+\alpha p$ for some $\alpha \in \mathbb{R}$. Since q is on L, $q=q'+\beta p'$ for

1.7. Fear and loathing. I know most of you are pretty worried about this mysterious thing called $\mathbb{R}^n$. What does it look like? What properties does it have? How can I work with it if I can't picture it? What the heck does he mean when he talks about points and lines in $\mathbb{R}^n$?

Although we can't picture $\mathbb{R}^n$ we can ask questions about it.

This is an important point. Prior to 1800 or so, mathematicians and scientists only asked questions about things they could "see" or "touch". For example, $\mathbb{R}^4$ wasn't really considered in a serious way until general relativity tied together space and time. Going further back, there was a time when negative numbers were considered absurd. Indeed, there still exist primitive cultures that have no concept of negative numbers. If you tried to introduce them to negative numbers they would think you were some kind of nut. Imaginary numbers were called that because they weren't really numbers. Even when they held that status there were a few brave souls who calculated with them by following the rules of arithmetic that applied to *real* numbers.

Because we have a *definition* of $\mathbb{R}^n$ we can grapple with it, explore it, and ask and answer questions about it. That is what we have been doing in the last few sections. In olden days, mathematicians rarely defined things. Because the things they studied were "real" it was "obvious" what the words meant. The great counterexample to this statement is Euclid's books on geometry. Euclid took care to define everything carefully and built geometry on that rock. As mathematics became increasingly sophisticated the need for precise definitions became more apparent. We now live in a sophisticated mathematical world where *everything* is defined.

So, take a deep breath, gather your courage, and plunge into the bracing waters. I will be there to help you if you start sinking. But I can't be everywhere at once, and I won't always recognize whether you are waving or drowning. Shout "help" if you are sinking. I'm not telepathic.

1.8. Basic properties of $\mathbb{R}^n$. Here are some things we will prove:

(1) If p and q are different points in $\mathbb{R}^n$ there is one, and only one, line through p and q. We denote it by $\overline{pq}$.
(2) If L is a line through the origin in $\mathbb{R}^n$, and q and q' are points in $\mathbb{R}^n$, then either $q + L = q' + L$ or $(q + L) \cap (q' + L) = \varnothing$.
(3) If L is a line through the origin in $\mathbb{R}^n$, and q and q' are points in $\mathbb{R}^n$, then $q + L = q' + L$ if and only if $q - q' \in L$.
(4) If p and q are different points in $\mathbb{R}^n$, then $\overline{pq}$ is the line through p in the direction $q - p$.
(5) If L and L' are lines in $\mathbb{R}^n$ such that $L \subseteq L'$, then $L = L'$. This is a consequence of (2).

If L is a line through the origin in $\mathbb{R}^n$, and q and q' are points in $\mathbb{R}^n$, we say that the lines $q + L$ and $q' + L$ are parallel.

Don't just accept these as facts to be memorized. Learning is more than knowing—learning involves understanding. To understand why the above facts are true you will need to look at how they are proved.

Proposition 1.2. *If p and q are different points in $\mathbb{R}^n$ there is one, and only one, line through p and q, namely the line through p in the direction $q - p$.*

Proof. The line through p in the direction $q - p$ is $p + \mathbb{R}(q - p)$. The line $p + \mathbb{R}(q-p)$ contains p because $p = p + 0 \times (q - p)$. It also contains q because $q = p + 1 \times (q - p)$. We have shown that the line through p in the direction $q - p$ passes through p and q.

It remains to show that this is the *only* line that passes through both p and q. To that end, let L be any line in $\mathbb{R}^n$ that passes though p and q. We will use Proposition 1.1 to show that L is equal to $p + \mathbb{R}(q - p)$.

By our definition of the word line, $L = q' + \mathbb{R}p'$ for some points $q' \in \mathbb{R}$ and $p' \in \mathbb{R} - \{\underline{0}\}$. By Proposition 1.1, $q' + \mathbb{R}p'$ is equal to $p + \mathbb{R}(q - p)$ if $\mathbb{R}p' = \mathbb{R}(q - p)$ and $q' - p \in \mathbb{R}p'$.

We will now show that $\mathbb{R}p' = \mathbb{R}(q - p)$ and $q' - p \in \mathbb{R}p'$.

Since $p \in L$ and $q \in L$, there are numbers λ and μ such that $p = q' + \lambda p'$, which implies that $q' - p \in \mathbb{R}p'$, and $q = q' + \mu p'$. Thus $p - q = q' + \lambda p - q' - \mu q' = (\lambda - \mu)q'$. Because $p \neq q$, $p - q \neq \underline{0}$; i.e., $(\lambda - \mu)p' \neq \underline{0}$. Hence $\lambda - \mu \neq 0$. Therefore $p' = (\lambda - \mu)^{-1}(p - q)$. It follows that every multiple of p' is a multiple of $p - q$ and every multiple of $p - q$ is a multiple of p'. Thus $\mathbb{R}p' = \mathbb{R}(q - p)$. We already observed that $q' - p \in \mathbb{R}p'$ so Proposition 1.1 tells us that $q' + \mathbb{R}p' = p + \mathbb{R}(q - p)$, i.e., $L = p + \mathbb{R}(q - p)$. $\square$

1.8.1. *Notation.* We will write $\overline{pq}$ for the unique line in $\mathbb{R}^n$ that passes through the points p and q (when $p \neq q$). Proposition 1.2 tells us that

$$\overline{pq} = p + \mathbb{R}(q - p).$$

Proposition 1.1 tells us that $\overline{pq}$ has infinitely many other similar descriptions. For example, $\overline{pq} = q + \mathbb{R}(p - q)$ and $\overline{pq} = q + \mathbb{R}(q - p)$.

1.8.2. Although we can't picture $\mathbb{R}^8$ we can ask questions about it. For example, does the point (1,1,1,1,1,1,1,1) lie on the line through the points (8,7,6,5,4,3,2,1) and (1,2,3,4,5,6,7,8)? Why? If you can answer this you understand what is going on. If you can't you don't, and should ask a question.

1.9. Parametric description of lines. You already know that lines in $\mathbb{R}^3$ can be described by a pair of equations or parametrically. For example, the line given by the equations

$$\begin{cases} x + y + z & = 4 \\ x + 3y + 2z & = 9 \end{cases} \tag{1-1}$$

is the set of points of the form $(t, 1 + t, 3 - 2t)$ as t ranges over all real numbers; in set-theoretic notation, the line is

$$\{(t, 1 + t, 3 - 2t) \mid t \in \mathbb{R}\}; \tag{1-2}$$

we call t a *parameter*; we might say that t parametrizes the line or that the line is parametrized by t.

The next result shows that every line in $\mathbb{R}^n$ can be described parametrically.

PROPOSITION 1.3. *If p and q are distinct points in $\mathbb{R}^n$, then*

$$\overline{pq} \;=\; \{tp + (1 - t)q \mid t \in \mathbb{R}\}. \tag{1-3}$$

Proof. We must show that the sets $\overline{pq}$ and $\{tp + (1 - t)q \mid t \in \mathbb{R}\}$ are the same. We do this by showing each set contains the other one.

Let p' be a point on $\overline{pq}$. Since $\overline{pq} = p + \mathbb{R}(q - p)$, $p' = p + \lambda(q - p)$ for some $\lambda \in \mathbb{R}$. Thus, if $t = 1 - \lambda$, then

$$p' \;=\; (1 - \lambda)p + \lambda q \;=\; tp + (1 - t)q.$$

Therefore $\overline{pq} \;\subseteq\; \{tp + (1 - t)q \mid t \in \mathbb{R}\}$. If t is any number, then

$$tp + (1 - t)q \;=\; q + t(p - q) \;\in\; q + \mathbb{R}(p - q) \;=\; \overline{pq}$$

so $\{tp + (1 - t)q \mid t \in \mathbb{R}\} \;\subseteq\; \overline{pq}$. Thus, $\overline{pq} = \{tp + (1 - t)q \mid t \in \mathbb{R}\}$. □

If you think of t as denoting *time*, then the parametrization in (1-3) can be thought of as giving the position of a moving point at time t; for example, at $t = 0$ the moving point is at q and at time $t = 1$, the moving point is at p; at time $t = \frac{1}{2}$ the moving point is exactly half-way between p and q. This perspective should help you answer the question in §1.8.2.

1.9.1. *Don't forget this remark.* One of the key ideas in understanding systems of linear equations is moving back and forth between the parametric description of linear subsets of $\mathbb{R}^n$ and the description of those linear subsets as solution sets to systems of linear equations. For example, (1-2) is a parametric description of the set of solutions to the system (1-1) of two linear equations in the unknowns x, y, and z. In other words, every solution to (1-1) is obtained by choosing a number t and then taking $x = t$, $y = 1+t$, and $z = 3 - 2t$.

1.9.2. Each line L in $\mathbb{R}^n$ has infinitely many parametric descriptions.

1.9.3. *Parametric descriptions of higher dimensional linear subsets of* $\mathbb{R}^n$.

1.10. Linear subsets of $\mathbb{R}^n$. Let L be a subset of $\mathbb{R}^n$. We say that L is linear if

(1) it is a point or
(2) whenever p and q are different points on L every point on the line $\overline{pq}$ lies on L.

1.11. What is a plane? Actually, it would be better if you thought about this question. How would you define a plane in $\mathbb{R}^n$? Look at the definition of a line for inspiration. We defined a line parametrically, not by a collection of equations. Did we really need to define a line in two steps, i.e., first defining a line through the origin, then defining a general line?

1.12. Hyperplanes. The dot product of two points $\underline{u} = (u_1, \dots, u_n)$ and $\underline{v} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$ is the number

$$\underline{u} \cdot \underline{v} = u_1v_1 + \cdots + u_nv_n.$$

Notice that $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$. If $\underline{u}$ and $\underline{v}$ are non-zero and $\underline{u} \cdot \underline{v} = 0$ we say $\underline{u}$ and $\underline{v}$ are orthogonal. If n is 2 or 3, the condition that $\underline{u} \cdot \underline{v} = 0$ is equivalent to the condition that the line $\overline{0\underline{u}}$ is perpendicular to the line $\overline{0\underline{v}}$.

Let $\underline{u}$ be a non-zero point in $\mathbb{R}^n$ and c any number. The set

$$H := \{\underline{v} \in \mathbb{R}^n \mid \underline{u} \cdot \underline{v} = c\}$$

is called a hyperplane in $\mathbb{R}^n$.

PROPOSITION 1.4. *If p and q are different points on a hyperplane H, then all the points on the line $\overline{pq}$ lie on H.*

Proof. Since H is a hyperplane there is a point $\underline{u} \in \mathbb{R}^n - \{\underline{0}\}$ and a number c such that $H = \{\underline{v} \in \mathbb{R}^n \mid \underline{u} \cdot \underline{v} = c\}$.

$\square$

1.13. Solutions to systems of equations: an example. A system of equations is just a collection of equations. For example, taken together,

$$x^2 + y^2 = z^2 \qquad \text{and} \qquad x + y + z = 1 \tag{1-4}$$

form a system of two equations. We call x, y, and z, *unknowns*. A solution to the system (1-4) consists of three numbers a, b, c such that $a^2 + b^2 = c^2$ *and* $a+b+c = 1$. Such a solution corresponds to the point (a, b, c) in 3-space, $\mathbb{R}^3$. For example, $(0, \frac{1}{2}, \frac{1}{2})$ is a solution to this system of equations because

$$(0)^2 + \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \qquad \text{and} \qquad 0 + \frac{1}{2} + \frac{1}{2} = 1.$$

Similarly, $(\frac{1}{4}, \frac{1}{3}, \frac{5}{12})$ is a solution to the system of equations because

$$\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)^2 = \left(\frac{5}{12}\right)^2 \qquad \text{and} \qquad \frac{1}{4} + \frac{1}{3} + \frac{5}{12} = 1.$$

Thus, we think of solutions to the system (1-4) as points in $\mathbb{R}^3$. These points form a geometric object. We use our visual sense to perceive them, and our visual sense organize those individual points into a single organized geometric object.

If you are really interested in math you might try finding all solutions to the system (1-4). If not all, several. How did I find the second solution? Can you find another solution?

1.14. Solutions to systems of equations. A solution to a system of equations in n unknowns $x_1, \ldots, x_n$ is a set of numbers $s_1, \ldots, s_n$ such that every equation is a true equality when each x_i is replaced by the corresponding number s_i. This is the famous "plugging in" procedure. We will think of $s_1, \ldots, s_n$ as the coordinates of a point in n-space, $\mathbb{R}^n$. Thus, each solution is a point $p = (s_1, \ldots, s_n)$ in $\mathbb{R}^n$.

1.15. Systems of linear equations. The distinguishing feature of a system of *linear* equations is that

> if p and q are different solutions to a system of linear equations, then every point on the line through p and q is a solution to the system.

It is this linearity feature that gives rise to the terminology *linear algebra*.

It is convenient to write $\overline{pq}$ for the line through p and q.

Suppose p, q, and r, are solutions to a system of linear equations and r is not on the line through p and q. Then every point on the three lines $\overline{pq}$, $\overline{pr}$, and $\overline{qr}$, is a solution to the system. Now take any point p' on $\overline{pq}$ and any point q' on $\overline{pr}$; since p' and q' are solutions to the linear system so is every point on the line $\overline{p'q'}$. Repeating this operation we can draw more and more lines. You should be able to convince yourself that *all* the lines we obtain in this way eventually fill up a plane, the plane in $\mathbb{R}^n$ that contains the points p, q, and r. We denote that plane by $\overline{pqr}$.

[1]I like the notation $\mathbb{R}p$ for the set of all multiples of p because it is similar to the notation $2p$: when x and y are numbers we denote their product by xy, the juxtaposition of x and y. So $\mathbb{R}p$ consists of all products λp where λ ranges over the set of all real numbers.

[2]To show two sets X and X' are the same one must show that every element of X is in X', which is written as $X \subseteq X'$, and that every element of X' is in X.

To prove an "if and only if" statement one must prove each statement implies the other; i.e., the statement "A if and only if B" is true if the truth of A implies the truth of B *and* the truth of B implies the truth of A. Most of the time, when I prove a result of the form "A if and only if B" I will break the proof into two parts: I will begin the proof that A implies B by writing the symbol $(\Rightarrow)$ and will begin the proof that B implies A by writing the symbol $(\Leftarrow)$

2. Lines in $\mathbb{R}^2$

2.1. An ordered pair of real numbers consists of two real numbers where their order matters. For example, $(0,0)$, $(0,1)$, $(1,1)$, $(-2,7)$, and $(-3,-5)$, are ordered pairs of real numbers, and they differ from the ordered pairs $(1,0)$, $(7,-2)$, and $(-5,-3)$.

$\mathbb{R}^2$ denotes the set of ordered pairs of real numbers. For example, $(0,0)$, $(0,1)$, $(1,1)$, $(-2,7)$, and $(-3,-5)$, are points in $\mathbb{R}^2$. We use the word *ordered* because $(1,2)$ and $(2,1)$ are different points of $\mathbb{R}^2$. A formal definition is

$$\mathbb{R}^2 := \{(a,b) \mid a,b \in \mathbb{R}\}.$$

At high school you learned about x- and y-axes and labelled points of in the plane by ordered pairs of real numbers. The point labelled $(3,7)$ is the point obtained by starting at the origin, going 3 units of distance in the x-direction, then 7 units of distance in the y-direction. Thus, we have a geometric view of the algebraically defined object $\mathbb{R}^2$.

Linear algebra involves a constant interplay of algebra and geometry. To master linear algebra one must keep in mind both the algebraic and geometric features. A simple example of this interplay is illustrated by the fact that the unique solution to the pair of equations

$$\begin{aligned} 2x+3y &= 4 \\ 2x-3y &= -8 \end{aligned}$$

is the point where the lines $2x+3y=4$ and $2x-3y=-8$ intersect. This pair of equations is a system of two linear equations.

2.2. Notation. If p and q are two different points in the plane $\mathbb{R}^2$ or in 3-space, $\mathbb{R}^3$, we write $\overline{pq}$ for the line through them.

3. Points, lines, and planes in $\mathbb{R}^3$

3.1. $\mathbb{R}^3$ denotes the set of ordered triples of real numbers. Formally,

$$\mathbb{R}^3 := \{(a,b,c) \mid a,b,c \in \mathbb{R}\}.$$

The corresponding geometric picture is 3-space. Elements of $\mathbb{R}^3$, i.e., ordered triples (a,b,c) label points in 3-space. We call a the x-coordinate of the point $p=(a,b,c)$, b the y-coordinate of p, and c the z-coordinate of p.

3.2. Notation. If p and q are two different points in $\mathbb{R}^3$ we write $\overline{pq}$ for the line through them. If p, q, and r, are three points in $\mathbb{R}^3$ that do not lie on a line, there is a unique plane that contains them. That plane will be labelled $\overline{pqr}$.

3.3. The first difference from $\mathbb{R}^2$ is that a single linear equation, $2x+y-3z=4$ for example, "gives" a plane in $\mathbb{R}^3$. By "gives" I mean the following. Every solution to the equation is an ordered triple of numbers, e.g., $(2,6,2)$ is a solution to the equation $2x+y-3z=4$ and it represents (is the label for) a point in $\mathbb{R}^3$. We say that the point $(2,6,2)$ is a solution to the equation. The collection (formally, the set) of all solutions to $2x+y-3z=4$ forms a plane in $\mathbb{R}^3$.

Two planes in $\mathbb{R}^3$ are said to be parallel if they do not intersect (meet).

3.4. Prerequisites for every linear algebra course. You *must* be able to do the following things:

- Fiind the equation of the plane $\overline{pqr}$ when p, q, and r, are three points in $\mathbb{R}^3$ that do not lie on a line. A relatively easy question of this type is to find the equation for the plane that contains $(0,0,0)$, $(2,3,7)$, and $(3,2,3)$. That equation will be of the form $ax+by+cz=d$ for some real numbers a,b,c,d. Since $(0,0,0)$ lies on the plane, $d=0$. Because $(2,3,7)$ and $(3,2,3)$ lie on the plane $ax+by+cz=0$ the numbers a,b,c must have the property that
$$\begin{aligned} &2a+3b+7c=0 \qquad \text{and} \\ &3a+2b+3c=0. \end{aligned}$$
This is a system of 2 linear equations in 3 unknowns. There are many ways you might solve this system but all are essentially the same. The idea is to multiply each equation by some number and add or subtract the two equations to get a simpler equation that a, b, and c, must satisfy.
- Find the equation of the line through two given points.
- Give a parametric form for a line that is given in the form $ax+by=c$. There are infinitely many different parametric forms for a given line. For example, the line $x=y$ is the line consisting of all points (t,t) as t ranges over $\mathbb{R}$, the set of real numbers. The same line is given by the parametrization $(1+2t,1+2t)$, $t\in\mathbb{R}$. And so on.
- Give a parametric form for the line through two given points. If $p=(a,b)$ and $q=(c,d)$ are two different points in the plane the set of points $t(a,b)+(1-t)(c,d)$, or, equivalently, $tp+(1-t)q$, gives all points on $\overline{pq}$ as t ranges over all real numbers. For example, $t=0$ gives the point q, $t=1$ gives the point p, $t=\frac{1}{2}$ gives the point

halfway between p and q. One gets all points between p and q by letting t range over the closed interval $[0,1]$. More formally, the set of points between p and q is $\{tp+(1-t)q \mid t \in [0,1]\}$. Similarly, $\overline{pq} = \{tp+(1-t)q \mid t \in \mathbb{R}\}$.

Notice that $t(a,b)+(1-t)(c,d) = (ta+(1-t)c, tb+(1-t)d)$.

- Decide whether two lines are parallel.
- Decide whether two lines are perpendicular.
- Given a line L and a point not on L, find the line parallel to L that passes through the given point.
- Given a line L and a point not on L, find the line perpendicular to L that passes through the given point.
- Find the point at which two lines intersect if such a point exists.

As t ranges over all real numbers the points $(1,2,3)+t(1,1,1)$ form a line in $\mathbb{R}^3$. The line passes through the point $(1,2,3)$ (just take $t=0$). We sometimes describe this line by saying *"it is the line through* $(1,2,3)$ *in the direction* $(1,1,1)$."

9.1 Lines and Planes in 3-Space

This section gives examples of most of the major kinds of problems one can ask about lines and planes in 3-space, and how to solve them. The subsections are labeled in general order of increasing difficulty.

- Space coordinates are $\{x, y, z\}$, equation coefficients $\{a, b, c, d\}$, lines $\{l\}$, points $\{P, Q\}$, arbitrary vectors $\{\vec{v}, \vec{w}\}$, normal vectors $\{\vec{n}\}$, line-parameters $\{s, t\}$, planes $\{Pl\}$, angles $\{\alpha, \theta, \varphi\}$.
- Recall: For vectors $\vec{v}_1$ and $\vec{v}_2$ forming an angle θ, we have $\boxed{\vec{v}_1 \cdot \vec{v}_2 = ||\vec{v}_1||\ ||\vec{v}_2|| \cos(\theta)}$.
- Recall: The cross product $\vec{v}_1 \times \vec{v}_2$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$.
- Recall: The projection of $\vec{w}$ onto $\vec{v}$ is given by $\mathrm{Proj}_{\vec{v}}(\vec{w}) = \left(\dfrac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}}\right) \vec{v}$; it gives the "piece" of $\vec{w}$ in the direction of $\vec{v}$.
- Recall: The line through the points $\vec{P}_1$ and $\vec{P}_2$ is given by $\boxed{l : \langle x, y, z\rangle = \vec{P}_1 + t\,(\vec{P}_2 - \vec{P}_1)}$.
  - The direction vector for this line is $\vec{v} = \vec{P}_2 - \vec{P}_1$. The direction vector is unique up to multiplication by a nonzero scalar.
  - Two lines are parallel if their direction vectors are (nonzero) scalar multiples of one another.
- Recall: The plane with equation $ax + by + cz = d$ has normal vector $\boxed{\vec{n} = \langle a, b, c\rangle}$.
  - The normal vector is orthogonal to every line that lies in the plane. It is unique up to multiplication by a nonzero scalar.

9.1.1 Line Through Point P With Direction Vector $\vec{v}$

- Idea: The line l with direction vector $\vec{v}$ which passes through point P has equation $\boxed{l : \langle x, y, z\rangle = \vec{P} + t\,\vec{v}}$.
- Example: Find the line in the direction of $\vec{v} = \langle 1, 2, 1\rangle$ passing through the point $P = (-1, 2, 2)$.
    - By the formula we see the line is parametrized by $\langle x, y, z\rangle = \langle -1, 2, 2\rangle + t\,\langle 1, 2, 1\rangle = \boxed{\langle t-1, 2t+2, t+2\rangle}$.
- Example: Find the line in the direction of $\vec{v} = \langle 6, 6, 6\rangle$ passing through the point $P = (4, -1, 100)$.
    - By the formula we see the line is parametrized by $\boxed{\langle x, y, z\rangle = \langle 6t+4, 6t-1, 6t+100\rangle}$.
- Example: Find the line l_1 passing through the point $(-2, 2, 3)$ which is parallel to the line $l_2 : \langle x, y, z\rangle = \langle -1, t-1, -5t+2\rangle$.
    - The direction vector of l_2 is $\langle 0, 1, -5\rangle$. Since l_1 and l_2 are parallel we see that the direction vector for l_1 is also $\langle 0, 1, -5\rangle$. By the formula we get $\boxed{l_2 : \langle x, y, z\rangle = \langle -2, t+2, -5t+3\rangle}$.

9.1.2 Line Through Two Points P_1 and P_2

- Idea: If the points are P_1 and P_2, compute the direction vector $\vec{v} = \vec{P_2} - \vec{P_1}$, and apply the formula to get $l : \langle x, y, z\rangle = \vec{P_1} + t\,\vec{v}$.
- Example: Find the line through $(0, 0, 0)$ and $(1, 2, 3)$.
    - We compute $\vec{v} = \langle 1-0, 2-0, 3-0\rangle = \langle 1, 2, 3\rangle$. Then the formula gives $\boxed{l : \langle x, y, z\rangle = \langle t, 2t, 3t\rangle}$.
- Example: Find the line through $(1, -1, 3)$ and $(2, -2, 2)$.
    - We compute $\vec{v} = \langle 2-1, -2-(-1), 2-3\rangle = \langle 1, -1, -1\rangle$. Then the formula gives $\boxed{l : \langle x, y, z\rangle = \langle t+1, -t-1, -t+3\rangle}$.

9.1.3 Intersection Point of Line l and Plane Pl

- Idea: Take the parametrization of l, plug it into the equation for the plane, and solve for the parameter.
    - Note 1: If the resulting equation has no solution, the line and plane do not intersect. Equivalently, the line is parallel to the plane.
    - Note 2: If the resulting equation reduces to an identity (i.e., is always true) then every point on the line lies inside the plane. Equivalently, the line itself lies inside the plane.
- Example: Find the intersection of the line $l : \langle x, y, z\rangle = \langle -2t+1, t-1, -t\rangle$ and the plane $x + 2y + 3z = 5$.
    - The line tells us $x = -2t+1$, $y = t-1$, $z = -t$.
    - Plugging into the plane gives $5 = (-2t+1) + 2(t-1) + 3(-t) = -3t - 1$ from which $t = -2$.
    - Hence the point of intersection is at $t = -2$, which is just $\boxed{(x, y, z) = (5, -3, 2)}$.
        - For a sanity check, this does lie on the plane, since $5 + 2(-3) + 3(-2) = 5$.
- Example: Find the intersection of the line $l : \langle x, y, z\rangle = \langle 3t-1, 2t+2, -4t\rangle$ and the plane $2y + z = 1$.
    - The line tells us $x = 3t-1$, $y = 2t+2$, $z = -4t$.
    - Plugging into the plane gives $1 = 2(2t+2) + (-4t) = 4$ from which we get $1 = 4$, which does not have any solution.
    - Hence the line and plane do not intersect, which is to say, the line is parallel to the plane.
- Example: Find the intersection of the line $l : \langle x, y, z\rangle = \langle 3t-1, 2t+2, -4t\rangle$ and the plane $2y + z = 4$.
    - The line tells us $x = 3t-1$, $y = 2t+2$, $z = -4t$.
    - Plugging into the plane gives $4 = 2(2t+2) + (-4t) = 4$ from which we get $4 = 4$, which is an identity.
    - Hence every point on the line lies in the plane, which is to say, the line lies inside the plane.

9.1.4 Intersection Point of Two Lines l_1 and l_2 (if it exists)

- Idea: Given parametrizations for the two lines $l_1 : \langle x, y, z\rangle = \vec{P}_1 + t\,\vec{v}_1$ and $l_2 : \langle x, y, z\rangle = \vec{P}_2 + s\,\vec{v}_2$, set the coordinates equal to get 3 equations in the two variables s and t. [Make sure to use DIFFERENT labels for the parameters for the two lines.] Solve this system of 3 equations in the 2 variables s and t – if there is a solution, it will give the intersection point. If there is no solution, the lines do not intersect.

- Example: Find the intersection point of the two lines $l_1 : \langle x, y, z\rangle = \langle t, 2t, 3t\rangle$ and $l_2 : \langle x, y, z\rangle = \langle -s+2, s+1, 2s+$

  - We solve the system $\langle t, 2t, 3t\rangle = \langle -s+2, s+1, 2s+1\rangle$, which is just the 3 equations $t = -s+2$, $2t = s+1$, and $3t = 2s+1$.
  - Plugging the first equation into the second one shows $-2s+4 = s+1$ so that $s = 1$.
  - Back-subbing gives $t = 1$, and we check that $\{s = 1, t = 1\}$ satisfies all three equations. Having $t = 1$ gives the point $\langle 1, 2, 3\rangle$ on l_1.
  - Therefore the lines $\boxed{\text{intersect at } (1, 2, 3)}$.

- Example: Find the intersection point of the two lines $l_1 : \langle x, y, z\rangle = \langle -2t+1, t-1, -t\rangle$ and $l_2 : \langle x, y, z\rangle = \langle -s+2, s+1, 2s+1\rangle$.

  - We solve the system $\langle -2t+1, t-1, -t\rangle = \langle -s+2, s+1, 2s+1\rangle$, which is just the 3 equations $-2t+1 = -s+2$, $t-1 = s+1$, $-t = 2s+1$.
  - The first equation gives $s = 2t+1$, and plugging into the second yields $t-1 = 2t+2$ hence $t = 3$.
  - Back-subbing gives $s = 7$. However, $\{s = 7, t = 3\}$ does not satisfy the third equation.
  - Therefore the lines $\boxed{\text{do not intersect}}$.

9.1.5 Plane With Normal Vector $\vec{n}$ Through Point P

- Idea: Given $\vec{n} = \langle a, b, c\rangle$ we have one equation of the plane of the form $ax + by + cz = \square$ for some value of $\square$. To find $\square$ we plug in the point P to the left-hand expression.

- Example: Find the plane passing through $(1, 1, 1)$ whose normal vector is $\vec{n} = \langle 1, 2, -1\rangle$.

  - By the formula, an equation for the plane is $x + 2y - z = d$ for some value of d.
  - We plug in $(1, 1, 1)$ to see $1 + 2 - 1 = d$ so $d = 3$, and the equation is $\boxed{x + 2y - z = 3}$.

- Example: Find the plane passing through $(2, -3, 2)$ whose normal vector is $\vec{n} = \langle 6, 6, 6\rangle$.

  - By the formula, an equation for the plane is $6x + 6y + 6z = d$ for some value of d.
  - We plug in $(2, -3, 2)$ to see $12 - 18 + 12 = d$ so $d = 6$, and the equation is $\boxed{6x + 6y + 6z = 6}$.
    - * Note that we could divide through by 6 to get the simpler equation $x + y + z = 1$.

9.1.6 Plane Containing Two Vectors $\vec{v}_1$ and $\vec{v}_2$ (or Parallel to Lines l_1 and l_2), Through Point P

- Idea: The normal vector to the plane is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$ and is unique up to a scalar. The vector $\vec{v}_1 \times \vec{v}_2$ is also orthogonal to both $\vec{v}_1$ and $\vec{v}_2$, and so we can just take $\vec{n} = \vec{v}_1 \times \vec{v}_2$.

  - Variation 1: If instead the plane is asked to contain be parallel to two arbitrary lines, we can reduce to the above. We need the plane's normal vector to be orthogonal to both lines, and so we take $\vec{n} = \vec{v}_1 \times \vec{v}_2$ where $\vec{v}_1$ and $\vec{v}_2$ are the direction vectors for l_1 and l_2.
    - * Note: This setup includes the special case where the lines l_1 and l_2 intersect. In this case we will be finding a plane parallel to the plane containing l_1 and l_2.
  - Variation 2: If instead we ask for a plane containing one line l_1 and parallel to another line l_2, we again compute $\vec{n} = \vec{v}_1 \times \vec{v}_2$, and then use as the point P any arbitrary point on the line l_1.

- Example: Find an equation for the plane containing the vectors $\vec{v}_1 = \langle 1, 1, 1 \rangle$ and $\vec{v}_2 = \langle 1, 2, 3 \rangle$ and passing through $(0, 0, 0)$.
  - We compute $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \vec{k} = \langle 1, -2, 1 \rangle$.
    - For a sanity check, we compute $\vec{n} \cdot \vec{v}_1 = (1)(1) + (-2)(1) + (1)(1) = 0$ and $\vec{n} \cdot \vec{v}_2 = (1)(1) + (-2)(2) + (1)(3) = 0$.
    - This assures us that the normal vector we found really is orthogonal to both vectors, which we know it's supposed to be.
  - Hence the equation for the plane is $x - 2y + z = d$ for some d.
  - We compute d by plugging in the point $(x, y, z) = (0, 0, 0)$, which just gives $d = 0$.
  - Hence the plane is $\boxed{x - 2y + z = 0}$.

- Example: Find an equation for the plane passing through $(1, -1, 2)$ which is parallel to the lines $l_1 : \langle x, y, z \rangle = \langle -2t + 1, t - 1, -t \rangle$ and $l_2 : \langle x, y, z \rangle = \langle -s + 2, s + 1, 2s + 1 \rangle$.
  - We have $\vec{v}_1 = \langle -2, 1, -1 \rangle$ and $\vec{v}_2 = \langle -1, 1, 2 \rangle$.
  - We compute $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} \vec{k} = \langle 3, 5, -1 \rangle$.
    - Sanity check: $\vec{n} \cdot \vec{v}_1 = (3)(-2) + (5)(1) + (-1)(-1) = 0$ and $\vec{n} \cdot \vec{v}_2 = (3)(-1) + (5)(1) + (-1)(2) = 0$.
  - Hence the equation for the plane is $3x + 5y - z = d$ for some d.
  - We compute d by plugging in the point $(1, -1, 2)$, so that $d = 3(1) + 5(-1) - (2) = -4$.
  - Hence the plane is $\boxed{3x + 5y - z = -4}$.

- Example: Find an equation for the plane containing $l_1 : \langle x, y, z \rangle = \langle -2t + 1, t - 1, -t \rangle$ and parallel to the line $l_2 : \langle x, y, z \rangle = \langle -s + 2, s + 1, 2s + 1 \rangle$. Also find an equation for the plane containing l_2 and parallel to l_1. Use your planes to explain why l_1 and l_2 cannot intersect.
  - We computed the cross product of the direction vectors of l_1 and l_2 in the previous example; we obtained $\langle 3, 5, -1 \rangle$.
  - Hence the equation for either of the desired planes is $3x + 5y - z = d$, for some d.
  - To get the plane containing l_1, we pick an easy point on l_1, with $t = 0$. This gives the point $(1, -1, 0)$.
  - We plug in this point to get $d = 3(1) + 5(-1) - 0 = -2$.
  - Hence the plane containing l_1 parallel to l_2 is $\boxed{3x + 5y - z = -2}$.
  - For the plane containing l_2, we need a point on l_2: take $s = 0$ to get $(2, 1, 1)$.
  - Plugging in gives $d = 3(2) + 5(1) - 1 = 10$.
  - Hence the plane containing l_2 parallel to l_1 is $\boxed{3x + 5y - z = 10}$.
  - We see in particular that the lines l_1 and l_2 cannot intersect, because they lie in different parallel planes.

9.1.7 Plane Through Three Points P_1, P_2, and P_3

- Idea: We need the normal vector $\vec{n}$. By the previous example we just need to find two vectors lying in the plane, and then take their cross product to get $\vec{n}$. We get the two vectors as $\vec{v}_1 = \vec{P_2} - \vec{P_1}$ and $\vec{v}_2 = \vec{P_3} - \vec{P_1}$.
- Example: Find an equation for the plane passing through $\vec{P_1} = (3, 0, -1)$, $\vec{P_2} = (1, 2, 2)$ and $\vec{P_3} = (-2, 1, 4)$.
  - We have $\vec{v}_1 = \vec{P_2} - \vec{P_1} = \langle -2, 2, 3 \rangle$ and $\vec{v}_2 = \vec{P_3} - \vec{P_1} = \langle -5, 1, 5 \rangle$.
  - Then $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \vec{i} - \begin{vmatrix} -2 & 3 \\ -5 & 5 \end{vmatrix} \vec{j} + \begin{vmatrix} -2 & 2 \\ -5 & 1 \end{vmatrix} \vec{k} = \langle 7, -5, 8 \rangle$.

    * Sanity check: $\vec{n} \cdot \vec{v}_1 = (7)(-5) + (-5)(1) + (8)(5) = 0$ and $\vec{n} \cdot \vec{v}_2 = (7)(-2) + (-5)(2) + (8)(3) = 0$.
  - Applying the formula gives the plane's equation as $7x - 5y + 8z = d$, for some d.
  - Finally we plug in the point $\vec{P_1}$ to see $d = (7)(3) - 5(0) + 8(-1) = 13$.
  - Therefore the equation of the plane is $\boxed{7x - 5y + 8z = 13}$.
    * For an extra error check, we verify that all three points do lie in this plane: we have $7(3) - 5(0) + 8(-1) = 13$, $7(1) - 5(2) + 8(2) = -13$, and $7(-2) - 5(1) + 8(4) = 13$.

9.1.8 Line of Intersection of Two Planes Pl_1 and Pl_2

- Idea: Two nonparallel planes Pl_1 and Pl_2 will intersect in a line l. The normal vectors $\vec{n}_1$ and $\vec{n}_2$ of the planes will both be orthogonal to l, so the direction vector for l will be given by the cross product $\vec{v} = \vec{n}_1 \times \vec{n}_2$. To find a point on l we need just to find a point (x, y, z) lying in both planes.
- Example: Parametrize the line of intersection of $x - y + 2z = 3$ and $2x + y - z = 0$.
  - We have $\vec{n}_1 = \langle 1, -1, 2 \rangle$ and $\vec{n}_2 = \langle 2, 1, -1 \rangle$.
  - The cross product is $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle -1, 5, 3 \rangle$.
    * Sanity check: $\vec{n} \cdot \vec{v}_1 = -1(1) + 5(-1) + 3(2) = 0$ and $\vec{n} \cdot \vec{v}_2 = -1(2) + 5(1) + 3(-1) = 0$.
  - We need a point in both planes. We try looking for one with $x = 0$: this requires $-y + 2z = 3$ and $y - z = 0$. Solving yields $y = z = 3$, so $(0, 3, 3)$ is in both planes hence on the line l.
  - Applying the line formula gives $\boxed{l : \langle x, y, z \rangle = \langle -t, 5t + 3, 3t + 3 \rangle}$.

9.1.9 Angle Between Two Planes Pl_1 and Pl_2

- Idea: Draw two intersecting normal lines N_1 and N_2 from the two planes. They form a quadrilateral with two right angles, one angle θ formed by the two planes, and a fourth angle φ formed by the two normals. Thus we have $\theta = \pi - \varphi$, where φ is the angle between the two normals. But φ is just the angle between $\vec{n}_1$ and $\vec{n}_2$ which we can find via the Dot Product Theorem: $\vec{n}_1 \cdot \vec{n}_2 = ||\vec{n}_1||\,||\vec{n}_2|| \cos(\varphi)$.
  - Note: Typically we want the acute angle α between the two planes. This will satisfy $\cos(\alpha) = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||\,||\vec{n}_2||}$ since cosine is nonnegative for $0 \le \alpha \le \dfrac{\pi}{2}$.
- <u>Example</u>: Find the angle formed by the planes $x - y + 2z = 3$ and $2x + y + z = 0$.
  - We have $\vec{n}_1 = \langle 1, -1, 2\rangle$ and $\vec{n}_2 = \langle 2, 1, 1\rangle$, with $\vec{n}_1 \cdot \vec{n}_2 = 3$ and $||\vec{n}_1|| = ||\vec{n}_2|| = \sqrt{6}$.
  - Thus the Dot Product Theorem gives $\cos(\alpha) = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||\,||\vec{n}_2||} = \dfrac{3}{\sqrt{6}\cdot\sqrt{6}} = \dfrac{1}{2}$.
  - Hence $\alpha = \dfrac{\pi}{6}$ and so the angle between the planes is $\boxed{\dfrac{\pi}{6} \text{ radians}}$.
- <u>Example</u>: Find the angle formed by the planes $-2x + y - 4z = 7$ and $3x - 4y + 5z = 2$.
  - We have $\vec{n}_1 = \langle -2, 1, -4\rangle$ and $\vec{n}_2 = \langle 3, -4, 5\rangle$, with $\vec{n}_1 \cdot \vec{n}_2 = -30$ along with $||\vec{n}_1|| = \sqrt{21}$ and $||\vec{n}_2|| = 5\sqrt{2}$.
  - Thus the Dot Product Theorem gives $\cos(\alpha) = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||\,||\vec{n}_2||} = \dfrac{30}{5\sqrt{42}} = \dfrac{\sqrt{42}}{7}$.
  - Hence the angle α between the planes is $\boxed{\cos^{-1}\left(\dfrac{\sqrt{42}}{7}\right) \text{ radians}}$.

9.1.10 Distance From Point P to Plane Pl

- Idea: The shortest vector connecting P to the plane will be in the same direction as the normal vector to the plane. In theory we could use this information to parametrize the line joining P to the plane, find the intersection of the line and plane, and finally compute the distance we seek. However, there is a less messy way using vector projections:
- Clever Idea:
  - Pick any point Q on the plane Pl.
  - Let $\vec{n}$ be the normal vector to the plane Pl.
  - Then the vector $\vec{v}$ connecting P to the plane is given by the vector projection of $\vec{w} = \vec{Q} - \vec{P}$ onto the normal $\vec{n}$. Explicitly, $\vec{v} = \mathbf{Proj}_{\vec{n}}(\vec{w}) = \left(\dfrac{\vec{n}\cdot\vec{w}}{\vec{n}\cdot\vec{n}}\right)\vec{n}$.
  - In particular, the length of $\vec{v}$ is $\boxed{||\vec{v}|| = \dfrac{|\vec{n}\cdot\vec{w}|}{||\vec{n}||}}$.
  - In fact we can actually write down a formula: if $\vec{n} = \langle a, b, c\rangle$ and $P = \langle x_0, y_0, z_0\rangle$ we compute $\vec{n}\cdot\vec{w} = \vec{n}\cdot\vec{P} - \vec{n}\cdot\vec{Q} = (ax_0 + by_0 + cz_0) - (d)$, and $||\vec{n}|| = \sqrt{a^2 + b^2 + c^2}$.
    - The fact that $\vec{n}\cdot\vec{Q} = d$ is just a restatement of the fact that $\vec{Q}$ lies in the plane Pl.
  - Putting it all together gives us the rather nice formula $\boxed{\dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}}$ for the distance from the point (x_0, y_0, z_0) to the plane $ax + by + cz = d$.
- <u>Example</u>: Find the distance from the plane $x + y + z = 1$ to the origin $(0, 0, 0)$.

- We have $\vec{n} = \langle 1, 1, 1 \rangle$, along with $P = \langle 0, 0, 0 \rangle$. We also grab any old point $Q = \langle 1, 0, 0 \rangle$ on the plane.
- We compute $\vec{w} = \langle 1, 0, 0 \rangle$, and then get $\vec{n} \cdot \vec{w} = 1$ and $||\vec{n}|| = \sqrt{3}$.
- Thus we obtain that the distance is $\boxed{||\vec{v}|| = \dfrac{1}{\sqrt{3}}}$.
- We check this with the formula: it gives $\dfrac{|1(0) + 1(0) + 1(0) - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \boxed{\dfrac{1}{\sqrt{3}}}$.

- Example: Find the distance from the plane $2x - y + 2z = 12$ to the point $(2, 1, 3)$.
  - We have $\vec{n} = \langle 2, -1, 2 \rangle$, along with $P = \langle 2, 1, 3 \rangle$. We also grab our favorite point $Q = \langle 6, 0, 0 \rangle$ on the plane.
  - We compute $\vec{w} = \langle 4, -1, -3 \rangle$, and then get $\vec{n} \cdot \vec{w} = 3$ and $||\vec{n}|| = 3$.
  - Thus we obtain that the distance is $\boxed{||\vec{v}|| = \dfrac{3}{3} = 1}$.
  - We check this with the formula: it gives $\dfrac{|2(2) - 1(1) + 2(3) - 12|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \boxed{\dfrac{3}{3} = 1}$.

9.1.11 Minimal Distance Between Two Lines l_1 and l_2

- Idea: Imagine the two lines l_1 and l_2 as sitting inside two parallel planes Pl_1 and Pl_2.
  - The distance between the lines will then be the same as the distance between the planes, which we can compute just by finding the distance of any arbitrary point P of Pl_1, to the plane Pl_2. [This we know how to do with the point-to-plane distance formula.]
  - In order to find these two parallel planes, we just need to find their (common) normal vector $\vec{n}$, which will be orthogonal to the direction vectors of both lines. [This we also know how to do using the cross product.]
    - To save us a little work, if we find the equations for the two planes as $Pl_1 : ax + by + cz = d_1$ and $Pl_2 : ax + by + cz = d_2$, we can compute using the point-to-plane formula that the distance will be $\boxed{\dfrac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}}$.

- Example: Find the smallest distance between a point on the line $l_1 : \langle x, y, z \rangle = \langle -2t + 1, t - 1, -t \rangle$ and a point on the line $l_2 : \langle x, y, z \rangle = \langle -s + 2, s + 1, 2s + 1 \rangle$.
  - We have $\vec{v}_1 = \langle -2, 1, -1 \rangle$ and $\vec{v}_2 = \langle -1, 1, 2 \rangle$, and can compute $\vec{n} = \langle 3, 5, -1 \rangle$.
    - Sanity check: $\vec{n} \cdot \vec{v}_1 = (3)(-2) + (5)(1) + (-1)(-1) = 0$ and $\vec{n} \cdot \vec{v}_2 = (3)(-1) + (5)(1) + (-1)(2) = 0$.
  - Hence the equations for the two plane Pl_1 and Pl_2 are $3x + 5y - z = d$ for some d.
  - To get the plane containing l_1, we pick an easy point on l_1, with $t = 0$. This gives the point $(1, -1, 0)$. We plug to get $d = 3(1) + 5(-1) - 0 = -2$, so we get $\boxed{Pl_1 : 3x + 5y - z = -2}$.
  - For the plane containing l_2, we need a point on l_2: take $s = 0$ to get $(2, 1, 1)$. Plugging in gives $d = 3(2) + 5(1) - 1 = 10$, so we get $\boxed{Pl_2 : 3x + 5y - z = 10}$.
  - Applying the formula above shows that the distance between these planes, and hence the lines, is $\dfrac{|10 - (-2)|}{\sqrt{3^2 + 5^2 + (-1)^2}} = \boxed{\dfrac{12}{\sqrt{35}}}$.
    - Remark: If we tried to find the minimum distance as a regular optimization problem, we would have to compute the minimum value of the square of the distance, which is $[(-s + 2) - (-2t + 1)]^2 + [(s + 1) - (t - 1)]^2 + [(2s + 1) - (-t)]^2$. It expands to the still not nice expression $6s^2 - 2st + 6t^2 + 6s + 2t + 6$. This requires either tremendous cleverness or multivariable calculus to minimize, and it's not easy either way! But our not-so-difficult vector computation shows that the minimum of this expression is $\dfrac{144}{35}$.