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201914046

Assignment MATH-247

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Question - 1

Define Laplace transform:

The Laplace transform of a function f(t), defined for all real number $t>0$ is the function F(s) which is a unilated transform define by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

The meaning of the integal depands on type of functions of interest. A necessary condition for existence of the integal is that f must be ~~beatt~~ locally integrable that decay at infinity. For locally integrable function that decay at infinity, the integral can be understood or are of exponetial type, the integral can be a proper.

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basic function of Laplace transform:

(I) $\mathcal{L}\{1\}$

$\therefore \mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$

$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$

$\therefore \boxed{\mathcal{L}\{1\} = \frac{1}{s}}$

(II) $\mathcal{L}\{t\}$

So, $\mathcal{L}\{t\} = \int_0^\infty e^{-st} \, t \, dt$

$= \left[t \, \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 1 \, dt$

$= \left[\frac{t e^{-st}}{-s} + \frac{1}{s} \cdot \frac{e^{-st}}{-s}\right]_0^\infty$

$= -\left[\frac{0}{s} - \frac{1}{s^2}\right] = \frac{1}{s^2}$

$\therefore \boxed{\mathcal{L}\{t\} = \frac{1}{s^2}}$

(III) $\mathcal{L}\{t^n\}$

Now, $\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \, t^n \, dt$

$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \, t^n \, dt$

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Now, let

$x = st$

$t = x/s \Rightarrow dt = \frac{dx}{s}$

and, $t^n = \frac{x^n}{s^n}$

$\therefore \mathcal{L}\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$

$= \frac{1}{s^{n+1}} \left[\int_0^\infty e^{-x} x^n \, dx \right]$ —— (I)

We know, gamma function, $\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt = \int_0^\infty e^{-x} x^{n-1} dt$ —— (II)

So, eqn (I) is,

$\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}} \left[\int_0^\infty e^{-x} x^{(n+1)-1} dx \right]$

[getting eqn (II)]

$\Rightarrow \mathcal{L}\{t^n\} = \frac{1}{s^{n+1}} \cdot \Gamma(n+1)$

$= \frac{n!}{s^{n+1}}$

$$\therefore \boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$$

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(4)

(iv) $\mathcal{L}\{e^{at}\}$

$\Rightarrow \mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st}\, dt = \int_0^\infty e^{-t(s-a)}\, dt$

$\Rightarrow \mathcal{L}(e^{at}) = \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty = \frac{e^{-\infty}}{-(s-a)} - \frac{e^0}{-(s-a)}$

$\therefore \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$\therefore \boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$

(v) $\mathcal{L}\{\sin at\}$

$\therefore \mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$

let, $I = \int e^{-st} \sin at\, dt$

$\Rightarrow I = e^{-st} \frac{-\cos at}{a} - \int \left\{-se^{-st} \cdot \frac{-\cos at}{a}\right\} dt$

$\Rightarrow I = -\frac{e^{-st}\cos at}{a} - \frac{s}{a}\int e^{-st} \cos at\, dt$

$\Rightarrow I = \frac{-e^{-st}\cos at}{a} - \frac{s}{a}\left[\frac{e^{-st}\sin at}{a} - \int -se^{-st} \cdot \frac{\sin at}{a}\, dt\right]$

$\Rightarrow I = -\frac{e^{-st}\cos at}{a} - \frac{se^{-st}\sin at}{a} - \frac{s^2}{a^2} I$

$\Rightarrow (a^2+s^2) I = -e^{-st}\cos at - e^{-st}\sin at$

$\Rightarrow I = -\frac{e^{-st}(\sin at + \cos at)}{(a^2+s^2)}$

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(5)

$\therefore$ so, $\int_0^\infty e^{-st} \sin at \, dt = \left[\frac{-e^{-st}(a \cos at + s \cdot \sin at)}{(a^2+s^2)} \right]_0^\infty$

$= -\left[\frac{e^0 (a \cos 0 + s \cdot \sin 0)}{a^2+s^2} \right]$

$= \frac{a}{a^2+s^2}$

$\therefore \boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}}$

(vi) $\mathcal{L}\{\cos at\}$

$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt.$

let, $I = \int_0^\infty e^{-st} \cos at \, dt$

$\Rightarrow I = \frac{e^{-st} \sin at}{a} - \int \{-s \cdot e^{-st} \frac{\sin at}{a}\} dt$

$\Rightarrow I = \frac{e^{-st} \sin at}{a} + \frac{s}{a} \left[\int e^{-st} \sin at \, dt \right]$

$\Rightarrow I = \frac{e^{-st} \sin at}{a} + \frac{s}{a} \left[e^{-st} \frac{-\cos at}{a} - \int \{-s e^{-st} \frac{(-\cos at)}{a}\} dt \right]$

$\Rightarrow I = \frac{e^{-st} \sin at}{a} - \frac{s}{a^2} e^{-st} \cos at - \frac{s^2}{a^2} \int e^{-st} \cos \, dt$

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$$\Rightarrow (a^2+s^2) I = \frac{e^{-st}\sin at}{a} - \frac{se^{-st}\cos at}{a^2}$$

$$\Rightarrow I = \frac{e^{-st}(a\sin at - s\cos at)}{(a^2+s^2)}$$

$$\therefore \int_0^\infty e^{-st}\cos at\,dt = \left[\frac{e^{-st}(a\sin at - s\cos at)}{(a^2+s^2)}\right]_0^\infty$$

$$= -\left[\frac{e^0\{a\sin 0 - s.\cos 0\}}{a^2+s^2}\right]$$

$$= \frac{s}{a^2+s^2}$$

$$\therefore \boxed{\mathcal{L}\{\cos at\} = \frac{s}{a^2+s^2}}$$

(VII) $\mathcal{L}\{\sinh at\}$

$$\therefore \mathcal{L}\{\sinh at\} = \int_0^\infty e^{-st}\sinh at\,dt$$

$$= \int_0^\infty e^{-st}\frac{e^{at}-e^{-at}}{2}\,dt$$

$$= \int_0^\infty \frac{e^{-t(s-a)}}{2}\,dt + \int_0^\infty \frac{e^{-t(s+a)}}{2}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-t(s-a)}}{-(s-a)} - \frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^0}{-(s-a)} - \frac{e^0}{-(s+a)}\right]$$

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$$= \frac{1}{2} \times \frac{2a}{s^2-a^2}$$

$$= \frac{a}{s^2-a^2}$$

$$\therefore \boxed{\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}}$$

(viii) $\mathcal{L}\{\cosh at\}$

$$\mathcal{L}\{\cosh at\} = \int_0^\infty \cosh at \, dt$$

$$= \int_0^\infty e^{-st} \frac{e^{at}+e^{-at}}{2} dt$$

$$= \frac{1}{2}\left[\int_0^\infty \frac{e^{-t(s-a)}}{1} dt + \int_0^\infty e^{-t(s+a)} dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-t(s-a)}}{-(s-a)} + \frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-0(s-a)}}{(s-a)} + \frac{e^{-0(s+a)}}{s+a}\right]$$

$$= \frac{1}{2} \times \frac{s+a+s-a}{s^2-a^2}$$

$$= \frac{s}{s^2-a^2}$$

$$\boxed{\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}}$$

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Question - 2

We Know. Lineare property is $\mathcal{L}\{F_1(t)\}$

$$= f_1(s) = \int_0^\infty e^{-st} F_1(t)\,dt \quad \& \quad \mathcal{L}\{F_2(t)\} = f_2(s)$$

$$= \int e^{-st} F_2(t)\,dt$$

if c_1 and c_2 are two function then

$$\mathcal{L}\{c_1 F_1(t) \pm c_2 F_2(t)\} = c_1 f_1(s) \pm c_2 f_2(s)$$

so, $$\mathcal{L}\{c_1 F_1(t) \pm c_2 F_2(t)\} = \int_0^\infty e^{-st}\{c_1 F_1(t) \pm c_2 F_2(t)\}\,dt$$

$$= \int_0^\infty e^{-st} c_1 F_1(t)\,dt \pm \int_0^\infty e^{-st} c_2 F_2(t)\,dt$$

$$= c_1 \int_0^\infty e^{-st} F_1(t)\,dt \pm c_2 \int_0^\infty e^{-st} F_2(t)\,dt$$

$$\therefore \mathcal{L}\{F_1(t)\} = \int_0^\infty e^{-st} F_1(t) = f_1(s)$$

and, $$\mathcal{L}\{F_2(t)\} = \int_0^\infty e^{-st} F_2(t) = f_2(s)$$

$$\therefore \text{so, } \mathcal{L}\{c_1 F_1(t) \pm c_2 F_2(t)\} = c_1 f_1(s) \pm c_2 f_2(s)$$

(proved)

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Again,

$$\mathcal{L}\{e^{at} F(t)\} = \int_0^\infty e^{at} e^{-st} F(t)\, dt$$

$$\Rightarrow \mathcal{L}\{e^{at} F(t)\} = \int_0^\infty e^{-t(s-a)} F(t)\, dt \quad \text{--- (1)}$$

let, $f(s) = \int_0^\infty e^{-st} F(t)\, dt$

so, $f(s-a) = \int_0^\infty e^{-t(s-a)} F(t)\, dt$

Now, form eqn (1) we get,

$$\mathcal{L}\{e^{at} F(t)\} = f(s-a)$$

So, the 1st shifting property $\mathcal{L}\{F(t)\} = f(s)$ then we can say that,

$$\mathcal{L}\{e^{at} F(t)\} = f(s-a)$$

(proved)

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scale property:

if $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F(at)\} = \frac{1}{a} f(s/a)$

so $\mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at)\, dt$

let, $m = at$

$\Rightarrow dm = a\, dt$

$\Rightarrow dt = \frac{dm}{a}$

$t:$	0	∞
$m:$	0	∞

so. $\mathcal{L}\{F(at)\} = \int_0^\infty e^{-s \cdot \frac{m}{a}} \cdot F(m) \frac{dm}{a}$

$= \frac{1}{a} \int_0^\infty e^{-m\frac{s}{a}} F(m)\, dm$

$\because f(s) = \int_0^\infty e^{-st} F(t)\, dt$

$\Rightarrow f(s/a) = \int_0^\infty e^{-s/a\, t} F(t)\, dt = \int_0^\infty e^{-s/a\, m} F(m)\, dm$

so, $\mathcal{L}\{F(at)\} = \frac{1}{a} f(s/a)$

(proved)

(11)

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Question - 3

Given that,

$$L\{f(x)\} = f(s) \quad \text{. than } \quad \mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s).$$

→ libniz law.

$$\frac{d}{ds}\int F(s)\, ds = \int \frac{\partial}{\partial s} F(s)\, ds$$

so, $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$ —— (1)

differenciation in eq^n (1)

$$\Rightarrow \frac{d}{ds}\int_0^\infty e^{-st} F(t)\, dt = \frac{d}{ds}\{f(s)\}$$

$$\Rightarrow \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) F(t)\, dt = \frac{d}{ds}\{f(s)\}$$

$$\Rightarrow \int_0^\infty (-t) e^{-st} F(t)\, dt = \frac{d}{ds}\{f(s)\}$$

$$\Rightarrow -\int_0^\infty e^{-st}\, t\, F(t)\, dt = \frac{d}{ds}\{f(s)\}$$

$$\Rightarrow \int_0^\infty e^{-st}\, t\{F(t)\}\, dt = -\frac{d}{ds}\{f(s)\}$$

$$\Rightarrow \mathcal{L}\{t' F(t)\} = (-1)' \frac{d'}{ds'}\{f(s)\}$$ —— (11)

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putting $n=K$ in eqn (ii) we get,

$$\mathcal{L}\{t^K F(t)\} = (-1)^K \frac{d^K}{ds^K} f(s)$$

$$\Rightarrow \int_0^\infty e^{-st} t^K F(t) \cdot dt = (-1)^K \frac{d^K}{ds^K} f(s) \quad \text{--- (iii)}$$

DWR to s in eqn (iii),

$$\frac{d}{ds}\int_0^\infty e^{-st} t^K F(t)\, dt = \frac{d}{ds}\left\{(-1)^K \frac{d^K}{dk} f(s)\right\}$$

$$\Rightarrow \int_0^\infty \frac{\partial}{\partial s}(e^{-st})\, t^K F(t)\, dt = (-1)^K \frac{d^{K+1}}{ds^{K+1}} f(s)$$

$$\Rightarrow \int_0^\infty (-t)\, e^{-st}\, t^K \cdot F(t)\, dt = (-1)^K \frac{d^{K+1}}{ds^{K+1}} f(s)$$

$$\Rightarrow -\int_0^\infty e^{-st}\, t^{K+1}\, F(t) \cdot dt = (-1)^K \frac{d^{K+1}}{ds^{K+1}} f(s)$$

$$\Rightarrow \int_0^\infty e^{-st}\, t^{K+1}\, F(t)\, dt = (-1)^K \frac{d^{K+1}}{ds^{K+1}} f(s)$$

$$\Rightarrow \mathcal{L}\{t^{K+1} F(t)\} = (-1)^{K+1} \frac{d^{K+1}}{ds^{K+1}} f(s)$$

So, for $n=1$, $n=K$ and $n=K+1$ the theorem is proved.

$$\therefore \mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

(proved)

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Question - 4

$$\mathcal{L}\{e^{at} F(t)\} = \int_0^\infty e^{-st} . e^{at} F(t)\, dt$$

$$= \int_0^\infty e^{-t(s-a)} F(t)\, dt .$$

$$= \int_0^\infty e^{-t(s-a)} F(t)\, dt$$

We know,

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

So, $f(s-a) = \int_0^\infty e^{-t(s-a)} F(t)\, dt$

$$\therefore \quad \mathcal{L}\{e^{at} F(t)\} = f(s-a)$$

(proved)

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Question - 5

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st}\, G(t)\, dt$$

$$\Rightarrow \mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} \cdot 0 \, dt + \int_0^\infty e^{-st}\, F(t-a)\, dt$$

$$\Rightarrow \mathcal{L}\{G(t)\} = \int_0^\infty e^{-st}\, F(t-a)\, dt$$

Now, let,

$$m = t-a$$

$$\Rightarrow dt = dm$$

t	a	α
u	0	α

so, $$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-s(m+a)}\, F(m)\, dm$$

$$\Rightarrow \mathcal{L}\{G(t)\} = \int_0^\infty e^{-sm} \cdot e^{-sa}\, F(m)\, dm$$

$$\Rightarrow \mathcal{L}\{G(t)\} = e^{-as} \int e^{-sm}\, F(m)\, dm$$

$$\Rightarrow \mathcal{L}\{G(t)\} = e^{-as} f(s) \quad \left[f(s) = \int_0^\infty e^{-st} F(t)\, dt \right]$$

$$\therefore \mathcal{L}\{G(t)\} = e^{-as} f(s)$$

(proved)

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Question - 06

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^{2\pi/3} e^{-st} \cdot 0 \, dt + \int_{2\pi/3}^\infty e^{-st} \cos(t - 2\pi/3)\, dt$$

As, $F(t) = \begin{cases} \cos(t - 2\pi/3) & , t > 2\pi/3 \\ 0 & , t < 2\pi/3 \end{cases}$

Now, let $t - 2\pi/3 = m$

$\Rightarrow dm = dt$

t	$2\pi/3$	∞
m	0	∞

So, $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-s(m + 2\pi/3)} \cos m \, dm$

$$= \int_0^\infty e^{-sm} \cdot e^{-2\pi/3} \cos m \, dm$$

$$= e^{-2\pi/3} \int_0^\infty e^{-sm} \cos m \, dm = e^{-2\pi/3} \cdot \frac{s}{s^2+1}$$

$$\therefore \mathcal{L}\{F(t)\} = \frac{s e^{-2\pi/3}}{s^2+1}$$

So, $\boxed{\mathcal{L}\{F(t)\} = \frac{s e^{-2\pi/3}}{s^2+1}}$

(Ans)

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