# Mathematical analysis
# Lecture 10

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# Topics of 10th lecture

- Differential equations
- Classification of differential equations
- Linear first-order differential equations
- Separable variables method
- Non-autonomous first-order differential equations
- Differential equationsBernoulli and Riccati

# Basic concepts

An equation that contains the derivatives of one or more dependent variables, with respect to one or more independent variables, is called a Differential Equation (D.E.).

Examples:

- $\frac{dy}{dx} = y$
- $\left(\frac{dy}{dx}\right)^2 + y^2 = x$
- $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = \cos(x)$
- $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$
- $\left(\frac{\partial^3 y}{\partial x^3}\right)^2 + 3\sqrt{\frac{\partial y}{\partial x}} + y^2 = 0$

# Symbolisms

- Symbolism using tones

$$y' + 5y = e^{-x}$$

- SymbolismLeibniz

$$\frac{d^2x}{dt^2} + 10x = 0$$

- SymbolismNewton

$$\ddot{x} = -3$$

- Symbolization using subscripts

$$you_{xx} + you_{yyy} = 0$$

# Classification of differential equations (by type)

Differential equations are divided into:

- If the differential equation contains only derivatives of one or more functions with respect to a single independent variable is called Ordinary Differential Equation (D.E.). Examples:
    - $\frac{day}{dx}+6y=e^{-x}$
    - $\frac{d^2y}{dx^2}+\frac{day}{dx}-12y=0$
    - $\frac{dx}{dt}+\frac{day}{dt}=3x+2y$
- An equation that involves only a few derivatives of one or more functions of two or more independent variables is called Partial Differential Equation (PDE). Examples:
    - $\frac{\partial^2 you}{\partial x^2}+\frac{\partial^2 you}{\partial y^2}=0$
    - $\frac{\partial you}{\partial y}=-\frac{\partial v}{\partial x}$

# Classification of differential equations (in terms of order)

> Definition: The order of a differential equation (SDE or NDE) is the order of the higher derivative in the equation.

Examples:

- Second class: $\frac{d^2y}{dx^2}+5\left(\frac{day}{dx}\right)^4-y=e^x$
- Third grade: $\frac{\partial^3 f}{\partial x^3}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0$
- Fourth grade: $2\frac{\partial^4 you}{\partial x^4}+\frac{\partial^2 you}{\partial t^2}=0$

# Classification of differential equations (in terms of linearity)

A S.D.E. $n$ order is called linear with respect to the variable $y$, if it is of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where $g, a_0, a_1, \ldots, a_n$ continuous functions.

Two important special cases of linear equations are the first-order linear S.D.E. ($n = 1$):

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

and the second-order linear SDE ($n = 2$):

$$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

# Linear first-order differential equations

Definition: The general form of the linear, autonomous first-order differential equation with constant coefficients is

$$\dot{y} + ay = b$$

where $a$ and $b$ are known constants, while if $y = y(t)$ then $\dot{y} = \frac{dy}{dt}$.

If I $y_h$ symbolize the general solution of the homogeneous form (which is taken posing $b = 0$) and with $y_p$ symbolize the partial solution, then we can have the result

$$y = y_h + y_p$$

That is, the general solution $y$ of the complete equation is the sum of the general solution of the homogeneous form and a partial solution of the complete equation, such as the steady-state equilibrium solution.

Definition: The homogeneous form of the linear autonomous differential equation first class is:

$$\dot{y} + ay = 0, a \neq 0$$

If $a = 0$, the solution is easy to find by direct integration ($y(t) = C, C \in \mathbb{R}$). In the general case with $a \neq 0$ we can solve the homogeneous form with direct integration, after bringing it into a suitable form. We remove the $ay$ and from the two members of the equation and then divide by $y$. Thus we conclude:

$$\frac{\dot{y}}{y} = -a$$

In this form we can integrate each member with respect to the independent variable, even if $t$, that is

$$\int \frac{\dot{y}}{y} dt = -\int a dt$$

The integral of the right-hand side is $-at + C_1$ where $C_1$ is a constant

The integral of the left-hand side is written as

$$\int \frac{\dot{y}}{y} dt$$

Given that $\dot{y} = \frac{day}{dt}$, this is happening

$$\int \frac{dy/dt}{y} dt$$

and after removing the terms $dt$, takes the form

$$\int \frac{1}{y} day$$

The integral of $1/y$ it is in $|y| + C_2$, where $C_2$ is a constant of integration. Now we have found the integral of both members and we arrive at the relationship:

$$\text{in}\,|y| + C_2 = -at + C_1$$

so:

$$|y| = e^{-at+C_1-C_2} =$$

$$e^{-at}e^{C_1-C_2} =$$

$$Yes^{-at}$$

> Theorem: The general solution of the homogeneous form of the linear, autonomous first order differential equation is:
>
> $$y_h(t) = Yes^{-at}$$

# Example

Solve the homogeneous form of the differential equation $\dot{y} = 3y + 2$.

The homogeneous form is:

$$\dot{y} - 3y = 0 \Leftrightarrow \frac{\dot{y}}{y} = 3 \Leftrightarrow$$

$$\text{in}\ /y/ + C_2 = 3t + C_1 \Leftrightarrow y_h(t) = Yes^{3t}$$

> Definition: A steady state equilibrium value is determined by the treaty $\dot{y}=0$. It is the value of $y$, in which it is stationary. We will denote it by $\bar{y}$.

Setting $\dot{y}=0$ we have

$$0 + yes = b \Leftrightarrow \bar{y} = \frac{b}{a}$$

Therefore $y_p = \bar{y}$. Replacing it $\bar{y}$ in the complete differential equation we have

$$0 + yes = b$$

which is valid.

Theorem: The solution of any linear autonomous differential equation is of a first-order equation is equal to the sum of the homogeneous solution and any partial solution of the complete differential equation:

$$y = y_h + y_p$$

Proof: Suppose that $y_1$ and $y_2$ are any two solutions of the complete differential equation and we define it as $z = y_1 - y_2$ the difference between these two solutions. We can show that the $z$ is a solution of the homogeneous differential equation. This is done as follows:

$$\dot{z} = \dot{y}_1 - \dot{y}_2 = (-ay_1 + b) - (-ay_2 + b) = -a(y_1 - y_2) = -az$$

Therefore

$$\dot{z} + az = 0$$

which means that the $z$ satisfies the homogeneous form of the differential equation and therefore it is a solution.

Now suppose that $y$ is a general solution of the differential equation and let $y_p$ that is a partial solution. Since the $y$ and the $y_p$ are two solutions of the complete equation, then, as we have proven, the $z = y - y_p$ will be a solution of its homogeneous form. Because it is a solution of the homogeneous equation, we call it $y_h$ Therefore $y_h = y - y_p \Leftrightarrow y = y_h + y_p$

Theorem: The general solution of the complete, autonomous differential equation first class is:

$$y(t) = Yes^{-at} + \frac{b}{a}$$

# Example

Solve the differential equation:

$$\dot{y} + 2y = 8$$

The homogeneous form is $\dot{y} = -2y$.

Therefore the solution of the homogeneous form is $y_h(t) = Yes^{-2t}$

The partial solution is obtained from the steady-state value of $y$ in general form of the differential equation by setting $\dot{y} = 0$. Therefore:

$$0 + 2\bar{y} = 8 \Leftrightarrow \bar{y} = 4$$

Therefore, the general solution of the differential equation is:

$$y = Yes^{-2t} + 4$$

# The initial price problem

Solve the differential equation

$$\dot{y} = 0.1y - 1$$

to satisfy the initial condition $y(0) = 5$.

The solution of the homogeneous differential equation is: $y_h = Yes^{0.1t}$

The partial solution we use is the steady-state solution: $\bar{y} = 10$ Therefore the general solution is:

$$y = Yes^{0.1t} + 10$$

For the initial condition we have:

$$5 = C + 10 \Leftrightarrow C = -5$$

Therefore the general solution which satisfies the initial condition is

$$y = -5e^{0.1t} + 10$$

# The method of separable variables

If a first order differential equation is of the form:

$$\frac{day}{dx} = f(x)g(y)$$

then it is possible to separate the variables and thus the above equation can be written

$$\frac{1}{g(y)}\frac{day}{dx} = f(x)$$

Completing the two members in terms of $x$ we have:

$$\int \frac{1}{g(y)}\frac{day}{dx}dx = \int f(x)dx \Leftrightarrow$$

$$\int \frac{1}{g(y)} day = \int f(x)dx$$

# Example 1

Let the differential equation be:

$$x\frac{day}{dx} = y(y+1), x > 0$$

We separate the variables

$$\frac{day}{y(y+1)} = \frac{dx}{x}$$

Setting $\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$ we have equivalents $\frac{1}{y(y+1)} = \frac{Ay+A+By}{y(y+1)}$ Equivalents $A+B=0$ and $A=1$. Therefore $B=-1$.

Therefore $\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$

# Example 1

Therefore

$$\int \left(\frac{1}{y} - \frac{1}{y+1}\right) day = \int \frac{dx}{x} \Leftrightarrow$$

$$\text{in}\,|y| - \text{in}\,|y+1| = \text{in}\,x + C \Leftrightarrow$$

$\left|\frac{y}{y+1}\right| = e^{C}x \Rightarrow \frac{y}{y+1} = kx,$ where $k = e^{C}$. Equivalents

$y = kxy + kx \Leftrightarrow (1 - kx)y = kx \Leftrightarrow y = \frac{kx}{1-kx}$. Setting $A = \frac{1}{k}$ and multiplying numerator and denominator by $A$ we have:

$$y = \frac{x}{A - x}$$

# Example 2

Let the differential equation be:

$$y' = y^2 e^{-x}$$

We separate the variables (for $y \neq 0$, for $y = 0$ the equation is satisfied)

$$\frac{day}{y^2} = e^{-x} dx$$

Therefore we have

$$\int \frac{day}{y^2} = \int e^{-x} dx \Leftrightarrow$$

$$-\frac{1}{\ } = -e^{-x} + C \Leftrightarrow y$$

$$y = \frac{1}{e^{-x} - C}$$

# Example 3

Let the differential equation be:

$$\frac{dy}{dx} = \sqrt{x+y-2} - 1, \text{with } y(0) = 3$$

Because the separation of variables cannot be done directly, we will set $z(x) = x + y(x) - 2$, therefore:

$$\frac{dz}{dx} = 1 + \frac{dy}{dx} \Leftrightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$$

therefore our original differential equation becomes:

$$\frac{dz}{dx} - 1 = \sqrt{z} - 1 \Leftrightarrow \frac{dz}{dx} = \sqrt{z}$$

This S.D.E. is of separable variables, so:

$$\int \frac{dz}{\sqrt{z}} = \int dx \Leftrightarrow \int z^{-\frac{1}{2}} dz = \int dx \Leftrightarrow \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = x + c \Leftrightarrow 2\sqrt{z} = x + c$$

# Example 3

Using the initial condition $y(0) = 3$ in the relationship $z(x) = x + y(x) - 2$, we take

$$z = 0 + 3 - 2 = 1$$

so the $2\sqrt{z} = x + c$ for $t = 0$ gives

$$2\sqrt{1} = 0 + c \Leftrightarrow c = 2$$

So the relationship $2\sqrt{z} = x + c$ becomes:

$$2\sqrt{z} = x + 2 \Leftrightarrow 2\sqrt{x + y - 2} = x + 2 \Leftrightarrow \sqrt{x + y - 2} = \frac{x+2}{2}, \quad x + 2 \geq 0$$

Therefore

$$x + y - 2 = \left(\frac{x}{2} + 1\right)^2 \Leftrightarrow x + y - 2 = \frac{x^2}{4} + 1 + x \Leftrightarrow y = \frac{x^2}{4} + 3, x + 2 \geq 0$$

# The stable equilibrium state and convergence

The general solution equation ($y(t) = Yes^{-at} + b/a$) for $t = 0$ gives $y(0) = y_0 = C + b/a \Leftrightarrow C = y_0 - (b/a)$. Because $\bar{y} = b/a$ we have $C = y_0 - \bar{y}$ and we can write

$$y(t) = (y_0 - \bar{y})e^{-at} + \bar{y}$$

Then

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} ((y_0 - \bar{y})e^{-at} + \bar{y})$$

If $a > 0$ then the $y(t)$ converges to $\bar{y}$ while if $a < 0$ then the $y(t)$ deviates.

**Theorem** The solution $y(t)$ in a linear autonomous first-order differential equation order converges towards the stable equilibrium state $\bar{y} = b/a$, regardless of the initial value $y_0$ if and only if the coefficient $a$ of the differential equation is positive.

# Example

Let the differential equation be

$$\dot{y} = 5y - 10$$

If at the time $t = 0$, $y(t) = 100$ to find if it converges to one equilibrium state:

$$y(t) = Yes^{5t} + 2$$

At the time $t = 0$ must satisfy the $y(0) = 100$. This means

$$100 = C + 2 \Leftrightarrow C = 98$$

The solution becomes:

$$y(t) = 98e^{5t} + 2$$

which deviates from the stable equilibrium state $\bar{y} = 2$ why it $e^{5t}$ tends to infinite as $t \to +\infty$.

# The case of $a = 0$

If $a = 0$ the steady state solution is not defined. In this case we have the differential equation

$$\dot{y} = b$$

can be completed immediately and we have $y(t) = bt + C$

## Non-autonomous equations

If the coefficient $a$ and/or the term $b$ in a linear differential equation is function of time then the equation is non-autonomous.

Definition: The general form of the first-order linear differential equation it is

$$\dot{y} + a(t)y = b(t)$$

where $a(t)$ and $b(t)$ are known continuous functions of $t$.

Theorem: The general solution of any linear differential equation first class is

$$y(t) = e^{-A(t)} \left( \int e^{A(t)} b(t)\,dt + C \right)$$

# Non-autonomous equations

In the theorem we use the term $A(t)$, which is defined as the integral of the coefficient $a(t)$ ($A(t) = \int a(t)dt$). To see how we can get the general solution, we differentiate the function:

$$e^{A(t)}y(t)$$

where it results:

$$e^{A(t)}\left(\frac{dA(t)}{dt}y(t) + \dot{y}\right)$$

After $a(t) = \frac{dA(t)}{dt}$ we have shown that

$$\frac{d}{dt}(e^{A(t)}y(t)) = e^{A(t)}(a(t)y(t) + \dot{y})$$

## Non-autonomous equations

The previous result shows that we can use the following technique to solve the differential equation: We multiply the entire equation by the term $e^{A(t)}$. Thus it follows:

$$e^{A(t)}(a(t)y(t)+\dot{y}) = e^{A(t)}b(t)$$

As we showed previously, the left-hand side is equal to $\frac{d}{dt}(e^{A(t)}y(t))$ so

$$\frac{d}{dt}(e^{A(t)}y(t)) = e^{A(t)}b(t)$$

Completing we get

$$e^{A(t)}y(t) = \int e^{A(t)}b(t)\,dt + C$$

Dividing by $e^{A(t)}$ eventually arises

$$y(t) = e^{-A(t)} \left( \int e^{A(t)} b(t)\, dt + C \right)$$

Theorem: The general form of the integrating factor for linear first order differential equation is

$$e^{A(t)}$$

where $A(t) = \int a(t)\, dt$.

# Example

Solve the differential equation

$$\dot{y} - 2you = bt$$

In this case we have $a(t) = -2t$, therefore

$$A(t) = \int(-2t)\,dt = -t^2$$

Multiplying both sides of the differential equation by the integration factor we have:

$$e^{-t^2}(\dot{y} - 2you) = e^{-t^2}bt$$

Equivalents

$$\frac{d}{dt}(e^{-t^2}y) = e^{-t^2}bt$$

# Example

Completing both members we have:

$$e^{-t^2} y = -\frac{be^{-t^2}}{2} + C$$

Dividing by $e^{-t^2}$

$$y = e^{t^2}\left( -\frac{be^{-t^2}}{2} + C \right) = -\frac{b}{2} + Yes^{t^2}$$

# Example 2

Solve the differential equation

$$\cos(x)\frac{dy}{dx} + (\cos(x) + \sin(x))y = \sin(x)\cos^2(x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

We first divide by $\cos(x)$

$$\frac{dy}{dx} + (1 + \tan(x))y = \sin(x)\cos(x)$$

In this case we have $a(x) = 1 + \tan(x)$ therefore

$$A(x) = \int (1 + \tan(x))\,dx = x - \ln(\cos(x))$$

The integral factor is

$$e^{x - \ln(\cos(x))} = \frac{e^x}{\cos(x)}$$

# Example 2

Multiplying the original S.D.E. by the integral factor we have:

$$\frac{e^x}{\cos(x)}\frac{day}{dx} + \frac{e^x}{\cos(x)}(1 + \tan(x))y = e^x\sin(x)$$

The first member of this equation must be of the form

$$\frac{d}{dx}\left(\frac{e^x}{\cos(x)}y\right)$$

Thus we have (we can verify this)

$$\frac{d}{dx}\left(\frac{e^x}{cos(x)}y\right) = e^x\sin(x)$$

(Indeed $\frac{d}{dx}\left(\frac{e^x}{\cos(x)}y\right) = \frac{e^x}{\cos(x)}\frac{day}{dx} + \frac{e^x\cos(x) - e^x(-\sin(x))}{\cos^2(x)}y = \frac{e^x}{\cos(x)}\frac{day}{dx} + \frac{e^x}{\cos(x)}(1 + \tan(x))y$)

# Example 2

Completing both members we have:

$$\frac{e^x}{\cos(x)} y = \int e^x \sin(x)\,dx$$

$K = \int e^x \sin(x)\,dx$ which, factoring, It gives you $K = e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\,dx$ Therefore $2K = e^x(\sin(x) - \cos(x)) + C$ or $\int e^x \sin(x)\,dx = \frac{e^x}{2}(\sin(x) - \cos(x)) + C_1$, where $C_1 = C/2$.

Substituting we have

$$\frac{e^x}{\cos(x)} y = \frac{e^x}{2}(\sin(x) - \cos(x)) + C_1$$

or equivalent

$$y = \frac{\cos(x)}{2}(\sin(x) - \cos(x)) + C_1 \cos(x) e^{-x}$$

Differential equationsBernoulli is of the form

$$y' + g(x)y = f(x)y^n$$

These equations are transformed into linear differential equations by multiplying both sides by $y^{-n}$

$$y'y^{-n} + g(x)y^{1-n} = f(x)$$

Then we set $you(x) = y^{1-n}$, so $you'(x) = (1-n)y^{-n}y'(x)$, and we get

$$\frac{you'(x)}{1-n} + g(x)you(x) = f(x) \Leftrightarrow$$

$$you'(x) + (1-n)g(x)you(x) = (1-n)f(x)$$

which is a linear differential equation.

# Example

Solve the differential equation: $y' = \frac{3}{x} y + x^4 \sqrt[3]{y}$

The differential equation is also written as

$$y' - \frac{3}{x} y = x^4 y^{\frac{1}{3}}$$

so it is a differential equationBernoulli with $n = \frac{1}{3}$. We multiply both members with $y^{-\frac{1}{3}}$ and we get

$$y' y^{-\frac{1}{3}} - y^{\frac{2}{3}} \frac{3}{x} = x^4$$

We set $you(x) = y^{1-\frac{1}{3}} = y^{\frac{2}{3}}$, so $you'(x) = \frac{2}{3} y^{-\frac{1}{3}} y'(x) \Rightarrow y'(x) = \frac{3}{2} y^{\frac{1}{3}} you'(x)$, therefore we end up with the linear equation

$$\frac{3}{2} y^{\frac{1}{3}} you'(x) y^{-\frac{1}{3}} - \frac{3}{x} you(x) = x^4 \Rightarrow \frac{3}{2} you'(x) - \frac{3}{x} you(x) = x^4 \Rightarrow you'(x) - \frac{2}{x} you(x) = \frac{2}{3} x^4$$

# Example

The general solution of the linear equation is

$$y(x) = e^{-\int -\frac{2}{x}dx}\left(\int \frac{2}{3}x^4 e^{\int -\frac{2}{x}dx}\,dx + c\right)$$

$$= e^{\ln(x^2)}\left(\int \frac{2}{3}x^4 e^{\ln x^{-2}}\,dx + c\right)$$

$$= x^2\left(\int \frac{2}{3}x^2\,dx + c\right)$$

$$= x^2\left(\frac{2}{9}x^3 + c\right)$$

$$= \frac{2}{9}x^5 + cx^2$$

So finally

$$y^{\frac{2}{3}} = \frac{2}{9}x^5 + cx^2 \Leftrightarrow y = \left(cx^2 + \frac{2}{9}x^5\right)^{\frac{3}{2}}$$

# Differential equationsRiccati

Differential equationsRiccati is of the form

$$y' + f(x)y^2 + g(x)y + h(x) = 0$$

If we know a partial solution to them, even $y_1(x)$, then they are transformed into linear differential equations by the substitution

$$y(x) = y_1(x) + \frac{1}{you(x)}$$

# Example

Solve the differential equation:

$$y' + \frac{1-x}{2x^2}y^2 - \frac{y}{x} + \frac{x-1}{2} = 0$$

which has a partial solution at $y_1(x) = x$.

We set $y(x) = x + \frac{1}{you(x)}$, so $y'(x) = 1 - \frac{you'(x)}{you^2(x)}$ so our initial differential becomes

$$1 - \frac{you'}{you^2} + \frac{1-x}{2x^2}\left(x^2 + \frac{2}{you}x + \frac{1}{you^2}\right)\left(-\frac{1}{x}\right)\left(x + \frac{1}{you}\right) + \frac{x-1}{2} = 0 \Rightarrow$$

$$1 - \frac{you'}{you^2} + \frac{1-x}{2} + \frac{1-x}{xu} + \frac{1-x}{2x^2you^2} - 1 - \frac{1}{xu} + \frac{x-1}{2} = 0 \Rightarrow$$

$$-\frac{you'}{you^2} + \frac{1-x}{xu} + \frac{1-x}{2x^2you^2} - \frac{1}{xu} = 0 \Rightarrow$$

# Example

$$-2x^2 you' + 2xu(1-x) + (1-x) - 2xu = 0 \Rightarrow$$

$$-2x^2 you' - 2x^2 you + (1-x) = 0 \Rightarrow you' + you = \frac{1-x}{2x^2}$$

The solution of the linear differential equation $you' + you = \frac{1-x}{2x^2}$ it is

$$you(x) = e^{-\int dx}\left(\int \frac{1-x}{2x^2} e^{\int dx} dx + c\right) = e^{-x}\left(\int \frac{1-x}{2x^2} e^{x} dx + c\right)$$

$$= e^{-x}\left(\int \left(-\frac{e^{x}}{2x}\right)' dx + c\right) = e^{-x}\left(-\frac{e^{x}}{2x} + c\right) = -\frac{1}{2x} + this^{-x}$$

Therefore

$$y(x) = x + \frac{1}{you(x)} = x + \frac{1}{-\frac{1}{2x} + this^{-x}}$$