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Mathematical analysis
Lecture 11

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Topics of the 11th lecture

- Linear second-order differential equations
- Solution of a homogeneous equation
- Complete solution
- Convergence of second-order equations
- Non-autonomous second-order differential equations

Linear second-order differential equations

Definition: The linear, autonomous second-order differential equation is expressed as follows:

$$\ddot{y} + a_1 \dot{y} + a_2 y = b$$

Taking advantage of the fact that the complete solution of a linear differential equation is equal to the sum of the solution of its homogeneous form and a partial solution of the complete equation, we have:

$$y = y_h + y_p$$

Definition: The homogeneous form of the linear second order differential equation order with constant coefficients is

$$\ddot{y} + a_1 \dot{y} + a_2 y = 0$$

To solve the homogeneous equation, we will use what we know about the solution of linear, homogeneous first-order differential equations with constant coefficients. We have seen that the solutions of equations of this type have the form:

$$y(t) = Ae^{rt}$$

where the prices for A and the r are determined by the initial conditions for the coefficient of the equation. A logical assumption is that the solutions of second-order equations have the same form.

If our assumption is correct, then this equation must satisfy the homogeneous equation. Substituting the assumed solution ($y(t) = Ae^{rt}$) and its derivatives in the homogeneous equation the left-hand side becomes:

$$\ddot{y} + a_1 \dot{y} + a_2 y = r^2 A e^{rt} + a_1 r A e^{rt} + a_2 A e^{rt} =$$

$$A e^{rt} (r^2 + a_1 r + a_2)$$

If we exclude the special case where $A = 0$, our hypothesis is correct if the expression inside the parentheses is zero, because then the solution we assumed satisfies the homogeneous equation. If we choose the r to satisfy:

$$r^2 + a_1 r + a_2 = 0$$

then the equation is indeed a solution of the homogeneous equation.

Definition: The characteristic equation of the linear differential equation second order with constant coefficients is:

$$r^2 + a_1 r + a_2 = 0$$

The prices of r that solve the characteristic equation are known as characteristic roots of the characteristic equation.

Theorem: Let y_1 and y_2 are two different roots of the homophone equation. If c_1 and c_2 are any two constants, then the function $y = c_1 y_1 + c_2 y_2$ is a solution of the homogeneous equation. Conversely, if y is any solution of the homogeneous equation, then there are unique constants, the c_1 and the c_2, for which it applies $y = c_1 y_1 + c_2 y_2$.

Evidence

Proof: If y_1 and y_2 are solutions of the homogeneous equation then it follows that $\ddot{y}_1+a_1\dot{y}_1+a_2y_1=\ddot{y}_2+a_1\dot{y}_2+a_2y_2=0.$
We assume that $y=c_1y_1+c_2y_2$ If the y is a solution, then you must it is true that:

$$\ddot{y}+a_1\dot{y}+a_2y=0$$

But $\dot{y}=c_1\dot{y}_1+c_2\dot{y}_2$ and $\ddot{y}=c_1\ddot{y}_1+c_2\ddot{y}_2$ After substitution we end up with:

$$\ddot{y}+a_1\dot{y}+a_2y=(c_1\ddot{y}_1+c_2\ddot{y}_2)+a_1(c_1\dot{y}_1+c_2\dot{y}_2)+a_2(c_1y_1+c_2y_2)=$$

$$c_1(\ddot{y}_1+a_1\dot{y}_1+a_2y_1)+c_2(\ddot{y}_2+a_1\dot{y}_2+a_2y_2)=0$$

Therefore the y is a solution of the homogeneous equation.
The second part of the theorem says that any solution of the differential equation can be expressed as a linear combination of y_1 and her y_2 via appropriate choice of constants c_1 and c_2 To do this, the y_1 and the y_2 be distinct, which means they must be linear independent.

Consequences of the theorem

The consequences of the theorem are that the general solution of the homogeneous form of the equation is:

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where we have two new constants $C_1 = c_1 A_1$ and $C_2 = c_2 A_2$.

Proof for the double root case

If $r_1 = r_2 = r$, the two distinct roots of the homogeneous equation are given by:

$$y_1 = A_1 e^{rt} \text{and} y_2 = tA_2 e^{rt}$$

These solutions are distinct because they are linearly independent. It is also possible to verify that the second solution satisfies the homogeneous equation. Then we derive the y_2 to get:

$$\dot{y}_2 = A_2 e^{rt} + rtA_2 e^{rt}$$

Subtracting again we get:

$$\ddot{y}_2 = rA_2 e^{rt} + rA_2 e^{rt} + r^2 tA_2 e^{rt}$$

Proof for the double root case

The left-hand side of the homogeneous equation takes the form:

$$\ddot{y} + a_1\dot{y} + a_2 y = 2rA_2 e^{rt} + r^2 tA_2 e^{rt} + a_1(A_2 e^{rt} + rtA_2 e^{rt}) + a_2(tA_2 e^{rt})$$

$$= A_2 e^{rt}(t(r^2 + a_1 r + a_2) + 2r + a_1)$$

But $r = -a_1/2$ therefore the above expression is equal to:

$$A_2 e^{rt}(t(a_1^2/4 - a_1^2/2 + a_2) - a_1 + a_1))$$

$$= A_2 e^{rt}\left(\frac{t}{4}(4a_2 - a_1^2)\right)$$

Because the discriminant is equal to zero, this expression is equal to zero.

The general solution of the homogeneous equation

If $r_1 = r_2 = r$ the two distinct roots of the homogeneous equation are given by:

$$y_1 = A_1 e^{rt} \text{ and } y_2 = tA_2 e^{rt}$$

Theorem: The solution of the homogeneous form of the linear differential equation second order with constant coefficients, since the roots of the characteristic equation r_1, r_2 are real numbers are:

$$y_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \text{ if } r_1 \neq r_2$$

$$y_h(t) = C_1 e^{rt} + C_2 you^{rt} \text{ if } r_1 = r_2 = r$$

Example

Solve the following homogeneous differential equation:

$$4\ddot{y} - 8\dot{y} + 3y = 0$$

After dividing both sides by 4, the characteristic equation is

$$r^2 - 2r + \frac{3}{4} = 0$$

$\Delta = 4 - 3 = 1$ and the roots are $r_1 = 1/2$ and $r_2 = 3/2$. Based on the previous theorem the solution of the differential equation is:

$$y_h(t) = C_1 e^{t/2} + C_2 e^{3t/2}$$

Example 2

Solve the following homogeneous differential equation:

$$\ddot{y} - 4\dot{y} + 4y = 0$$

The characteristic equation is $r^2 - 4r + 4 = 0$ with double root $r_1 = r_2 = 2$. Therefore according to the theorem:

$$y_h(t) = C_1 e^{2t} + C_2 you^{2t}$$

The general solution of the homogeneous equation

Theorem: If the roots of the characteristic equation are complex numbers $z_{1,2} = h \pm vi$, the solution of the homogeneous form of the second order differential equation with constant coefficients can be expressed as:

$$y_h = A_1 e^{ht} \cos(vt) + A_2 e^{ht} \sin(vt)$$

Complex roots

If $\Delta = a_1^2 - 4a_2 < 0$ then the characteristic roots are complex numbers. In
In this case we write the solution as:

$$r_{1,2} = \frac{-a_1 \pm I\sqrt{4a_2 - a_1^2}}{2}$$

Therefore $r_{1,2} = h \pm vi$ where $h = -\frac{a_1}{2}$ and $v = \pm\frac{\sqrt{4a_2 - a_1^2}}{2}$.
Now we can formulate the solution as follows:
$y_h = C_1 e^{(h+vi)t} + C_2 e^{(h-vi)t} = e^{ht}(C_1 e^{vit} + C_2 e^{-vit})$
From the type of Euler we have:

$$e^{I(vt)} = \cos(vt) + I\sin(vt)$$

and

$$e^{-i(vt)} = \cos(vt) - I\sin(vt)$$

Complex roots

Therefore

$$y_h = e^{ht}(C_1(\cos(vt) + I\sin(vt)) + C_2(\cos(vt) - I\sin(vt)))$$

or

$$y_h = e^{ht}(C_1 + C_2)\cos(vt) + \mathit{that\ is}_{ht}(C_1 - C_2)\sin(vt)$$

After $(C_1 + C_2)$ and $(C_1 - C_2)I$ are stable, we can rename them to A_1 and A_2

$$A_1 = C_1 + C_2$$

and

$$A_2 = (C_1 - C_2)I$$

The A_1 and A_2 are real numbers. The reason is that the C_1 and C_2 it is conjugate complex numbers, like roots. But the sum of conjugate complex numbers is always a real number. The product of I and the difference of conjugates complex is again a real number. Therefore we get a real solution of the differential equation even when the roots are complex numbers.

Complex roots

Theorem: If the roots of the characteristic equation are complex numbers, the solution of the homogeneous form of the second-order linear differential equation with constant coefficients can be expressed as:

$$y_h = A_1 e^{ht}\cos(vt) + A_2 e^{ht}\sin(vt)$$

where $h = -\frac{a_1}{2}$ and $v = \frac{\sqrt{4a_2 - a_1^2}}{2}$

Example

Solve the following homogeneous differential equation:

$$\ddot{y}+2\dot{y}+5y=0$$

The characteristic equation is:

$$r^2+2r+5 = 0$$

$\Delta = -16, r_{1,2} = \frac{-2 \pm 4I}{2} = -1 \pm 2I$ Therefore, based on the theorem we have:

$$y_h(t) = A_1 e^{-t} \cos(2t) + A_2 e^{-t} \sin(2t)$$

The complete solution

The partial solution is found by setting $\ddot{y} = \dot{y} = 0$. This gives us for $a_2 \neq 0$: $y_p = \bar{y} = \frac{b}{a_2}$.

The complete solution of a second-order differential equation is the sum of the homogeneous solution and the partial solution.

$$y = y_h + y_p$$

Theorem: The complete solution of the linear autonomous differential equation second order (with constant coefficients and a constant term) is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{b}{a_2} \quad \text{if } r_1 \neq r_2$$

$$y(t) = C_1 e^{rt} + t C_2 e^{rt} + \frac{b}{a_2} \quad \text{if } r_1 = r_2 = r$$

$$y(t) = e^{ht}(A_1 \cos(vt) + A_2 \sin(vt)) + \frac{b}{a_2} \quad \text{if the roots of the characteristic equation are complex numbers.}$$

Example

Solve the following linear differential equation:

$$\ddot{y}+2\dot{y}+5y=10$$

The partial solution is $y_p=2$. Therefore, based on the solution of the homogeneous differential equation we found previously, the general solution is:

$$y(t) = A_1 e^{-t}\cos(2t) + A_2 e^{-t}\sin(2t) + 2$$

Balance and convergence

Theorem: The solution of a second-order linear differential equation with constant coefficients and a constant term converges to the steady state of stable equilibrium if and only if the real parts of the roots of its characteristic equation are negative.

Balance and convergence

Case 1: The roots are real and unequal. The complete solution to this problem is case is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{b}{a_2}$$

Therefore:

$$\lim_{t\to\infty} y(t) = C_1 \lim_{t\to\infty} (e^{r_1 t}) + C_2 \lim_{t\to\infty} (e^{r_2 t}) + \frac{b}{a_2}$$

If both roots are negative, the two exponential terms converge to 0 as a limit. and therefore the $y(t)$ converges to b/a_2. If both roots are positive, then both terms that include the t diverge to infinity, so that the $y(t)$ deviates towards the $+\infty$ or at $-\infty$.
If one root is positive and the other negative, the term with the negative root converges to zero, but the other term diverges to infinity except when the corresponding constant is zero. As a result, $y(t)$ deviates, except in this special case.

Balance and convergence

Case 2: The roots are real and equal. The complete solution to this problem is case is:

$$y(t) = C_1 e^{rt} + C_2 \, you^{rt} + \frac{b}{a_2}$$

Taking the limits on both sides we have:

$$\lim_{t \to \infty} y(t) = \frac{b}{a_2} + C_1 \lim_{t \to \infty} (e^{rt}) + C_2 \lim_{t \to \infty} (you^{rt})$$

If the double root r is positive, then the $y(t)$ will deviate towards the positive or negative infinity. If the root is negative, then the $y(t)$ will converge to $\frac{b}{a_2}$. Then the term you^{rt} takes the form $(\infty \cdot 0)$. We can put it in the form (∞/∞) writing it as t/e^{-rt}. Then, we can apply the rule L' Hospital and differentiating the numerator and denominator we get $(-1/r)e^{rt}$, whose limit is zero for $r < 0$.

Balance and convergence

Case 3: Complex roots. The complete solution in this case is

$$y(t) = e^{ht}(A_1 \cos(vt) + A_2 \sin(vt)) + \frac{b}{a_2}$$

The term in parentheses is an oscillating function that is bounded as $t \to \infty$
This term is multiplied by e^{ht} and will increases indefinitely if $h > 0$.

If $h < 0$, then the e^{ht} converges to zero.

Therefore, the $y(t)$ diverges with increasing oscillations if $h > 0$, while it converges to b/a_2 with ever-shrinking oscillations if $h < 0$. The h but it is real part of the complex root ($h = -a_1/2$) and we conclude that the $y(t)$ converges towards b/a_2 if the real part of the complex roots is negative.

Example

Examine the differential equation for convergence:

$$\ddot{y} + 3\dot{y} + 4y = 10$$

We have $\Delta = -7$, so $r_{1,2} = \frac{-3 \pm I\sqrt{7}}{2}$.
The real part of the roots is negative ($-3/2$) therefore the solution converge to the fixed point $\bar{y} = 5/2$.

The second-order linear equation with one variable term

Case 1: If the term $b(t)$ is a polynomial n-bone degree as to t, even if it $p_n(t)$ then we assume that the partial solution is also a polynomial. That is, we assume that:

$$y_p = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$$

where A_i are the constants, the values of which are determined by replacing the of a presumed partial solution to the differential equation and then equating the coefficients of like terms.

Case 2: If the term $b(t)$ is of the form $e^{at} p_n(t)$, where $p_n(t)$ is a polynomial in terms of t and if a is a known constant, then we assume that the A partial solution is given by:

$$y_p = e^{at}(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0)$$

The second-order linear equation with one variable term

Case 3: If the term $b(t)$ is of the form $e^{kt}(p_1(t)\cos(mt) + p_2(t)\sin(mt))$, where $p_1(t), p_2(t)$ are polynomials of degree n_1 and n_2 respectively, then we are looking for partial solution of the form:

$$y_p = e^{kt}(Q_n(t)\cos(mt) + R_n(t)\sin(mt))$$

where $Q_n(t)$ and $R_n(t)$ are polynomials of degree n, where n the maximum of n_1 and n_2.

In any case, if any term of the conjectured solution is also a term of y_h, then the assumed solution must be modified as follows: We multiply the assumed solution by t^k, where k is the smallest positive integer so that common terms are eliminated.

Example 1

Solve the equation $\ddot{y}+3\dot{y}-4y=t^2$

First we solve the homogeneous equation

$$\ddot{y}+3\dot{y}-4y=0$$

$\Delta = 25$, $r_{1,2}=\frac{-3\pm 5}{2}$ with roots 1 and -4. So

$$y_h(t) = C_1 e^t + C_2 e^{-4t}$$

To find a partial solution, we observe that the term $b(t)$ is a polynomial second degree in relation to t. Therefore, we assume that

$$y_p = A_2 t^2 + A_1 t + A_0$$

Example 1

$$\dot{y}_p = 2A_2 t + A_1$$

$$\ddot{y}_p = 2A_2$$

We substitute the partial solution into the differential equation and we have:

$$2A_2 + 3(2A_2 t + A_1) - 4(A_2 t^2 + A_1 t + A_0) = t^2$$

Equivalents:

$$-(4A_2 + 1)t^2 + (6A_2 - 4A_1)t + (2A_2 + 3A_1 - 4A_0) = 0$$

Therefore: $A_2 = -\frac{1}{4}$, $-\frac{3}{2} - 4A_1 = 0 \Leftrightarrow A_1 = -\frac{3}{8}$, $2A_2 + 3A_1 - 4A_0 = 0 \Leftrightarrow -\frac{1}{2} - \frac{9}{8} - 4A_0 = 0 \Leftrightarrow A_0 = -\frac{13}{32}$ Therefore the general solution is:

$$y = C_1 e^t + C_2 e^{-4t} - \frac{1}{4}t^2 - \frac{3}{8}t - \frac{13}{32}$$

Example 2

Solve the equation: $\ddot{y}+3\dot{y}-4y=5e^{t}$.

The homogeneous solution is the same as in the previous example. The partial solution is found by assuming that

$$y_p=A_0e^{t}$$

But this has the same form as one of the terms of the solution of the homogeneous equation. Therefore, we multiply by t and we have $y_p=tA_0e^{t}$ Therefore $\dot{y}_p=A_0e^{t}+tA_0e^{t}$ and $\ddot{y}_p=2A_0e^{t}+tA_0e^{t}$. Substituting we have:

$$2A_0e^{t}+tA_0e^{t}+3A_0e^{t}+3tA_0e^{t}-4tA_0e^{t}=5e^{t}\Leftrightarrow 5A_0e^{t}=5e^{t}$$

Therefore $A_0=1$ and the solution is:

$$y(t)=you^{t}+C_1e^{t}+C_2e^{-4t}$$

Example 3

Solve the equation: $\ddot{y} - 4\dot{y} + 5y = \sin(3t)$.

The homogeneous equation is $\ddot{y} - 4\dot{y} + 5y = 0$, having characteristic polynomial $r^2 - 4r + 5 = 0$ with roots $r_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm I$.
The solution of the homonymous is $y_h = e^{2t}(C_1\cos(t) + C_2\sin(t))$.

The second term of the differential ($\sin(3t)$) is of the form of Case 3 with $k=0$, $m=3$ and $n=0$. Therefore, the partial solution will be of the form (the polynomials $Q_n(t), R_n(x)$ of zero degree will be constant):

$$y_p = a\sin(3t) + b\cos(3t)$$

Example 3

We calculate the derivatives:

$$\dot{y}_p = (a\sin(3t) + b\cos(3t))' = 3a\cos(3t) - 3b\sin(3t)$$

$$\ddot{y}_p = (3a\cos(3t) - 3b\sin(3t))' = -9a\sin(3t) - 9b\cos(3t)$$

Substituting into the differential equation:

$-9a\sin(3t) - 9b\cos(3t) - 4(3a\cos(3t) - 3b\sin(3t)) + 5(a\sin(3t) + b\cos(3t)) = \sin(3t) \Rightarrow (-4a + 12b)\sin(3t) + (-12a - 4b)\cos(3t) = \sin(3t) \Rightarrow$

$$-4a + 12b = 1 \text{ and } -12a - 4b = 0$$

Solving the linear system results in that $a = -\frac{1}{40}$ and $b = \frac{3}{40}$. Thus, the partial solution is:

$$y_p = -\frac{1}{40}\sin(3t) + \frac{3}{40}\cos(3t)$$

and the general solution:

$$y = e^{2t}(C_1\cos(t) + C_2\sin(t)) - \frac{1}{40}\sin(3t) + \frac{3}{40}\cos(3t)$$