Mathematical analysis
Lecture 9

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Topics of 9th lecture

- Linear second-order difference equations
- Homogeneous form
- General solution of second order difference equation
- Steady state and convergence
- Linear second-order difference equations with a variable term

Linear second-order difference equations

> Definition: The general form of the linear, autonomous second-order difference equation is:
>
> $$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b, t = 0, 1, 2, \cdots$$

This equation is:

- Linear because the terms y_t, y_{t+1} and y_{t+2} are raised to the first power.
- Second order because the largest difference that appears in the equation is the difference of two periods.
- Autonomous because it has fixed coefficients, a_1 and a_2 and a fixed term, him b.

If the coefficients or the term b changed along with the t then the equation would be non-autonomous.

Definition: The homogeneous form of the linear, autonomous difference equation second order is:

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0, t = 0, 1, 2, \cdots$$

Definition: The characteristic equation of the linear difference equation second order with constant coefficients is:

$$r^2 + a_1 r + a_2 = 0$$

The prices of r that satisfy the characteristic equation are called roots or eigenvalues or characteristic roots of the characteristic equation.

Theorem

For a linear, autonomous second-order difference equation of the form

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b,$$

if the y_p is a partial solution (like the steady-state solution) and y_h is the general solution of the homogeneous form of the equation, then the general solution of the complete difference equation is given by:

$$y_t = y_h + y_p,$$

where for convenience we have omitted the indices t in y_p and y_h.

Theorem

Theorem: The general solution of the homogeneous form of the linear, autonomous second order difference equation is given by the solution of the characteristic equation $r^2 + a_1 r + a_2 = 0$ as follows (where $\Delta = a_1^2 - 4a_2$):

- If $\Delta > 0$ (uneven real roots) then:

$$y_h = C_1 r_1^t + C_2 r_2^t$$

- If $\Delta = 0$ (equal real roots) then:

$$y_h = C_1 r^t + C_2 t r^t$$

where C_1 and C_2 are constants (the values of which will be determined by the initial conditions if given) and r_1 and r_2 are given by the $r_{1,2} = \frac{-a_1 \pm \sqrt{\Delta}}{2}$, while in the case where $\Delta = 0$ we have $r = \frac{-a_1}{2}$.

Theorem (Continued)

- If $D < 0$ (complex roots), then

$$y_h = a_2^{\frac{t}{2}}\left(C_1\cos(\theta t) + C_2\sin(\theta t)\right)$$

the C_1 and C_2 are constants (the values of which will be determined by the initial conditions if given) and the θ can be determined by the relationships:

$$\cos\theta = \frac{-a_1}{2\sqrt{a_2}}, \quad \text{or} \quad \sin\theta = \frac{\sqrt{4a_2 - a_1^2}}{2\sqrt{a_2}}.$$

Theorem: If the complex root is of the form $h \pm iv$, then the measure of the roots it is $R = \sqrt{h^2 + v^2}$ while the θ can be calculated using the two following relationships:

$$\cos(\theta) = \frac{h}{R}, \quad \sin(\theta) = \frac{v}{R}$$

Evidence

We test the format $y_t = Ar^t$ where A is a constant. Then we have $y_{t+1} = Ar^{t+1}$ and $y_{t+2} = Ar^{t+2}$ Substituting into the homogeneous form of the equation we have:

$$Ar^{t+2} + a_1 Ar^{t+1} + a_2 Ar^t = 0$$

Factoring we get:

$$(r^2 + a_1 r + a_2) Ar^t = 0$$

The proposed solution will verify the equation if we choose values for r where satisfy the quadratic equation $r^2 + a_1 r + a_2 = 0$ (after excluding the zero solutions $r = 0$ and $A = 0$). This is the characteristic equation of homogeneous difference equation.

Evidence

Case $D > 0$: We assume that the two roots that verify the characteristic equation are distinct real numbers. Then we have essentially found two solutions that satisfy the homogeneous equation. These are:

$$y^{(1)}_t = A_1 r_1^t \text{ and } y^{(2)}_t = A_2 r_2^t$$

Let us confirm that the $y^{(1)}_t$ is a solution of the homogeneous equation (similar to $y^{(2)}_t$). Given the $y^{(1)}_t$ results:

$$y^{(1)}_{t+1} = A_1 r_1^{t+1} \text{ and } y^{(1)}_{t+2} = A_1 r_1^{t+2}$$

Substituting these values into the homogeneous equation yields

$$y^{(1)}_{t+2} + a_1 y^{(1)}_{t+1} + a_2 y^{(1)}_t =$$

$$A_1 r_1^{t+2} + a_1 A_1 r_1^{t+1} + a_2 A_1 r_1^t =$$

$$A_1 r_1^t (r_1^2 + a_1 r_1 + a_2) = 0$$

Evidence

The last equality arises because we know that the r_1 satisfies the characteristic equation. Therefore the $y_t^{(1)}$ satisfies the homogeneous equation and is one solution.

Case $\Delta = 0$: If $r_1 = r_2 = r$ the two different solutions are:

$$y_t^{(1)} = A_1 r^t \text{ and } y_t^{(2)} = A_2 t r^t$$

It is possible to verify that both of these are solutions of the homogeneous equation by substitution as we did earlier. We will do this for the second solution. Due to the fact that $y_t^{(2)} = A_2 t r^t$, we have

$$y_{t+1}^{(2)} = A_2 (t+1) r^{t+1} \text{ and } y_{t+2}^{(2)} = A_2 (t+2) r^{t+2}$$

Evidence

Substituting these values into the homogeneous equation we have:

$$y^{(2)}_{t+2} + a_1 y^{(2)}_{t+1} + a_2 y^{(2)}_t =$$

$$A_2(t+2)r^{t+2} + a_1 A_2(t+1)r^{t+1} + a_2 A_2 t r^t =$$

$$A_2 r^t((t+2)r^2 + a_1(t+1)r + a_2 t) =$$

$$A_2 r^t(t(r^2 + a_1 r + a_2) + r(2r + a_1))$$

Because the r verifies the characteristic equation we have $r^2 + a_1 r + a_2 = 0$, while because in the case where $\Delta = 0 \Rightarrow r = -a_1/2$ it follows that:

$$A_2 r^t(t(r^2 + a_1 r + a_2) + r(2r + a_1)) =$$

$$A_2 r^t(0 + 0) = 0$$

Therefore, the above equation gives zero.

Evidence

Case $D<0$: If the discriminant of the characteristic equation is negative ($a_1^2-4a_2<0$) then again we can find a solution. The solution of characteristic equation will have the form:

$$r_{1,2}=\frac{-a_1\pm\sqrt{(-1)(4a_2-a_1^2)}}{2}=\frac{-a_1\pm I\sqrt{4a_2-a_1^2}}{2}.$$

Using the concept of the imaginary unit, the roots can be written as conjugate complex numbers:

$$r_{1,2}=h\pm vi$$

where $h=\frac{-a_1}{2}$, $v=\frac{\sqrt{4a_2-a_1^2}}{2}$ The solution of the homogeneous difference equation takes the form form:

$$y_h=c_1(h+vi)^t+c_2(h-vi)^t$$

Evidence

To make the equation easier to interpret $y_h = c_1(h + vi)^t + c_2(h - vi)^t$ we use the fact that a complex number can be expressed in polar or trigonometric form as:

$$h \pm vi = R(\cos(i) \pm I\sin(i))$$

where $R = \sqrt{h^2 + v^2}$ is the measure or absolute value of the complex roots and $\cos(i) = h/R$ and $\sin(i) = v/R$. Then we use the theorem *de Moivre* to bring the equation into an expression that is more easily interpreted. According to this theorem:

$$(R(\cos(i) + I\sin(i)))^n = R^n(\cos(nth) + I\sin(nth))$$

Evidence

Therefore the equation is written

$$y_h = c_1 R^t(\cos(\theta t) + I\sin(\theta t)) + c_2 R^t(\cos(\theta t) - I\sin(\theta t))$$

This can be simplified even further if we consider that

$$R = (h^2 + v^2)^{1/2} = \left(\frac{a_1^2}{4} + \frac{4a_2 - a_1^2}{4} \right)^{1/2} = a_2^{1/2}$$

By reducing like terms and defining new constants C_1, C_2 where replace the c_1, c_2 of the above equation, we find the solution for homogeneous form of the first-order linear difference equation in the case of complex roots:

$$y_h = a_2^{t/2}(C_1\cos(\theta t) + C_2\sin(\theta t))$$

Example 1

Solve the following difference equation.

$$y_{t+2} - 6y_{t+1} + 8y_t = 0, t = 0, 1, 2, \cdots$$

The characteristic equation is $r^2 - 6r + 8 = 0$. The discriminant is $\Delta = 36 - 32 = 4 > 0$ and the roots 4 and 2. According to the theorem the general solution is:

$$y_t = C_1 2^t + C_2 4^t$$

Substituting into the original equation we have: $y_{t+2} - 6y_{t+1} + 8y_t =$
$C_1(2^{t+2} - 6(2^{t+1}) + 8(2^t)) + C_2(4^{t+2} - 6(4^{t+1}) + 8(4^t)) = C_1 2^t(2^2 - 6(2) + 8) + C_2 4^t(4^2 - 6(4) + 8) = 0.$
Therefore, the equation we found is a solution to the equation.

Example 2

Solve the difference equation

$$y_{t+2} - 4y_{t+1} + 4y_t = 0$$

The characteristic equation is $r^2 - 4r + 4 = 0$ with $\Delta = 16 - 16 = 0$.
So the characteristic root is $r = \frac{4}{2} = 2$.
According to the theorem, the general solution is:

$$y_t = C_1 2^t + C_2 t 2^t$$

Substituting into the original equation we have:
$y_{t+2} - 4y_{t+1} + 4y_t =$
$C_1 2^{t+2} + C_2(t+2)2^{t+2} - 4\left(C_1 2^{t+1} + C_2(t+1)2^{t+1}\right) + 4(C_1 2^t + C_2 t 2^t) =$
$2^t(4C_1 + 4tC_2 + 8C_2 - 8C_1 - 8tC_2 - 8C_2 + 4C_1 + 4tC_2) = 0$ Therefore, the equation we found is a solution to the equation.

Example 3

Solve the difference equation

$$y_{t+2} - 2y_{t+1} + 2y_t = 0$$

The characteristic equation is $r^2 - 2r + 2 = 0$ with $\Delta = 4 - 8 = -4 < 0$.
So the characteristic roots are $r_{1,2} = \frac{2 \pm 2I}{2} = 1 \pm I$.
The measure of the roots is $R = \sqrt{2}$ and the angle $cos(i) = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{p}{4}$. Therefore:

$$y_t = 2^{t/2}\left(C_1 \cos\left(\frac{p}{4}t\right) + C_2 \sin\left(\frac{p}{4}t\right)\right)$$

The complete solution

The complete solution is obtained if we add to the general solution of the homogeneous form a partial solution of the difference equation.

For autonomous difference equations the partial solution we seek is the steady-state value of y. This arises when the y_t become stagnant, thing which means that $y_{t+2} = y_{t+1} = y_t$, which as before we denote by $\bar{y}$. Setting $y_{t+2} = y_{t+1} = y_t = \bar{y}$ we have:

$$\bar{y} + a_1 \bar{y} + a_2 \bar{y} = b$$

Solving we have:

$$\bar{y} = \frac{b}{1 + a_1 + a_2}, \quad a_1 + a_2 \neq -1$$

If $a_1 + a_2 = -1$ such value does not exist. In this case you should find an alternative partial solution and find the general solution. The solution we will use in this case is $y_p = At$, where A it is one consistently using a method that we will define below.

The general solution of the complete difference equation

The general solution of the complete difference equation

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b,$$

when $a_1 + a_2 \neq -1$, is as follows:

- If $D > 0$ (real and distinct roots):
  $y_t = C_1 r_1^t + C_2 r_2^t + \frac{b}{1 + a_1 + a_2}$
- If $\Delta = 0$ (real and equal roots):
  $y_t = C_1 r^t + C_2 t r^t + \frac{b}{1 + a_1 + a_2}$
- If $D < 0$ (complex roots):
  $y_t = R^t (C_1 \cos(\theta t) + C_2 \sin(\theta t)) + \frac{b}{1 + a_1 + a_2}$

where C_1 and C_2 are arbitrary constants, the r_1, r_2 the roots of the characteristic stic equation and the R, θ the measure and angle of the complex number that arises in the case of complex roots.

Example 1

Solve the difference equation:

$$2y_{t+2}+8y_{t+1}+6y_t=32.$$

with initial values $y_0=1$ and $y_1=2$.

We formulate the difference equation in its usual form

$$y_{t+2}+4y_{t+1}+3y_t=16$$

The homogeneous form of this difference equation is

$$y_{t+2}+4y_{t+1}+3y_t=0$$

The characteristic equation is:

$$r^2+4r+3 = 0$$

distinguishing me $\Delta = 16-12 = 4>0$ and roots $r_{1,2}=\frac{-4\pm 2}{2}$ therefore $r_1=-1$ and $r_2=-3$.

Example 1

The partial equilibrium solution is obtained by solving:

$$\bar{y}+4\bar{y}+3\bar{y}=16$$

which gives $\bar{y}=2$.

Therefore, the general solution of the equation is:

$$y_t=C_1(-1)^t+C_2(-3)^t+2$$

For $t=0$ the solution becomes $y_0=C_1+C_2+2=1 \Rightarrow C_1+C_2=-1$, while for $t=1$ we have $y_1=-C_1-3C_2+2=2 \Rightarrow -C_1-3C_2=0$. Solving this linear system we get $C_1=-\frac{3}{2}$ and $C_2=\frac{1}{2}$, therefore the solution of the difference equation is:

$$y_t=-\frac{3}{2}(-1)^t+\frac{1}{2}(-3)^t+2$$

Example 2

Solve the difference equation:

$$y_{t+2} - 2y_{t+1} + 2y_t = 10$$

The characteristic equation of the corresponding homogeneous difference equation is:

$$r^2 - 2r + 2 = 0$$

$\Delta = 4 - 8 = -4$ and $r_{1,2} = \frac{2 \pm 2I}{2} = 1 \pm I$

We calculate $R = \sqrt{2}$ and $\theta = \frac{\pi}{4}$ The equilibrium solution is obtained by solving:

$$\bar{y} - 2\bar{y} + 2\bar{y} = 10$$

which gives $\bar{y} = 10$. Therefore, the general solution of the complete equation is:

$$y_t = 2^{t/2}\left(C_1 \cos\left(\frac{\pi}{4}t\right) + C_2 \sin\left(\frac{\pi}{4}t\right)\right) + 10$$

Example 2

In the previous equation, determine the constants so that the solutions satisfy the initial conditions $y_0 = 1$ and $y_1 = 2$.

- For $t = 0$ we have $1 = C_1 + 10$ from which it follows $C_1 = -9$.
- For $t = 1$ we have $2 = \sqrt{2}\left(C_1 \cos(\pi/4) + C_2 \sin(\pi/4)\right) + 10 \Rightarrow$ $2 = \sqrt{2}\left(\frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}}\right) + 10 \Rightarrow -8 = -9 + C_2 \Rightarrow C_2 = 1.$

Therefore, the solution takes the form:

$$y_t = 2^{t/2}\left(-9\cos\left(\frac{\pi}{4}t\right) + \sin\left(\frac{\pi}{4}t\right)\right) + 10$$

Steady state and convergence

Theorem: The path of y_t in a linear, autonomous difference equation second-order converges to the steady-state value $\bar{y}$ from any initial value, where

$$\bar{y} = \frac{b}{1 + a_1 + a_2}$$

if $a_1 + a_2 \neq -1$ if and only if the absolute values of both roots are smaller than unity.

Proof of the theorem for steady state and convergence

We consider three cases:
Real and unequal roots: Then the solution is:

$$y_t = C_1 r_1^t + C_2 r_2^t + \bar{y}$$

In this case, because the r_1 and the r_2 rise in the t, as well as $t \to +\infty$ the solution converges to the steady state $\bar{y}$ if and only if the absolute values and of the two roots are less than unity. In this case the terms r_1^t and r_2^t converge to zero. Otherwise, as $t \to +\infty$ or y_t is becoming more and more limited.

Proof of the theorem for steady state and convergence

Real and equal roots: Then the solution is:

$$y_t = C_1 r^t + C_2 t r^t + \bar{y}$$

If the root r is in absolute value greater than 1 it is clear that the root diverges. If the root r is in absolute value less than 1 then the term $C_1 r^t$ converges to zero. To calculate the limit of $t r^t$ we convert the term into $\frac{t}{r^{-t}}$ and so we have the form (∞/∞) when $|r| < 1$. Using the rule of the derivation $\frac{d a^x}{dx} = a^x \ln(a)$ and applying the L'Hospital rule we take $\frac{1}{-r^{-t}\ln(r)}$ which converges to zero.

Proof of the theorem for steady state and convergence

Complex roots: The solution in this case is:

$$y_t = R^t(C_1 \cos(\theta t) + C_2 \sin(\theta t)) + \bar{y}$$

In this case the functions $C_1 \cos(\theta t)$ and $C_2 \sin(\theta t)$ are blocked in absolute value from the C_1 and C_2 respectively. Therefore, convergence depends on exclusively from the term R^t. If for the measure of the two complex roots it holds $|R| < 1$, then the y_t converges to $\bar{y}$, otherwise it deviates.

Sufficient and necessary conditions for convergence

Theorem: The absolute value of the roots of the characteristic equation (for the linear, autonomous 2nd order difference equation) is less than 1, if and only if the three conditions are satisfied:

i) $1 + a_1 + a_2 > 0$

(ii) $1 - a_1 + a_2 > 0$

(iii) $a_2 < 1$

Example: For the equation $y_{t+2} - 2y_{t+1} + 2y_t = 10$ we have

i) $1 + a_1 + a_2 = 1 - 2 + 2 = 1 > 0$

(ii) $1 - a_1 + a_2 = 1 + 2 + 2 = 5 > 0$

(iii) $a_2 = 2 > 1$

therefore the absolute value of the roots is not less than 1 ($|1 \pm i| = \sqrt{2} > 1$).

The second-order linear difference equation with a variable term

When the term b is not constant and is a function of t (we will symbolize it with b_t), then the second-order linear difference equation is non-autonomous. Even when the b is stable, there is no steady-state solution if $1 + a_1 + a_2 = 0$.

There is an alternative technique for finding a stable solution. When the term b_t is not constant, we use the method of undetermined coefficients. This method relies on one's ability to 'guess' the form of the partial solution.

The second-order linear difference equation with a variable term

Case 1: If the b_t is a polynomial of degree n as for t, then we assume that the partial solution is also a polynomial. That is, we assume that:

$$y_p = A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$$

where the A_i are constants that we determine.

Case 2: If the b_t is of the form k^t where k is a constant, then we assume that:

$$y_p = A k^t$$

where A a constant that we define.

Case 3: If the b_t is of the form $k^t p_n(t)$, then we assume that:

$$y_p = A k^t (A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n)$$

The second-order linear difference equation with a variable term

There is one important exception to these guidelines for assumptions regarding the form of solutions.

If any term of the assumed partial solution is also a term of the homogeneous solution (regardless of the constants by which it is multiplied), then the assumed solution must be modified as follows: We multiply the assumed solution by t^k, where k is the smallest positive integer, so that we have no common terms.

Example 1

Solve the equation

$$y_{t+2} - 3y_{t+1} + 2y_t = 10$$

The characteristic equation is $r^2 - 3r + 2 = 0$ and has roots 1 and 2. Therefore, the solution for the homogeneous form is:

$$y_h = C_1 2^t + C_2$$

We want to find a partial solution, but we notice that $1 + a_1 + a_2 = 1 - 3 + 2 = 0$. Therefore we will use the method of unspecified factors.
Because the b_t in this case it is a constant ($b_t = 10$), first we will let's try a partial solution of this form, that is $y_p = A$. However, this is similar to the term C_2 of the homogeneous solution, that's why we will try the solution $y_p = At$.

Example 1

The partial solution must satisfy the difference equation and we use this to solve for A.

$$A(t+2) - 3A(t+1) + 2At = 10$$

Solving we get $A(t+2-3t-3+2t) = 10 \Leftrightarrow A = -10$ Therefore the general solution of the complete equation is

$$y_t = C_1 2^t + C_2 - 10t$$

Example 2

Solve the equation

$$y_{t+2} - 3y_{t+1} + 2y_t = 1 + t$$

The homogeneous solution is the same as in the previous example ($y_h = C_1 2^t + C_2$ Our initial assumption for the partial solution is:

$$y_p = A_0 + A_1 t$$

However, the conjectured solution has a term in common with the homogeneous solution. Therefore, we multiply the first tentative solution by t to get:

$$y_p = A_0 t + A_1 t^2$$

Example 2

This tentative solution has no common terms with the homogeneous solution and therefore we can substitute it into the complete difference equation:

$$(A_0(t+2)+A_1(t+2)^2)-3(A_0(t+1)+A_1(t+1)^2)+2(A_0t+A_1t^2)=1+t$$

$$\Leftrightarrow (A_0t+2A_0+A_1t^2+4A_1+4A_1t)-3(A_0t+A_0+A_1t^2+A_1+2A_1t)+2(A_0t+A_1t^2)=1+t \Leftrightarrow (2A_0+4A_1-3A_0-3A_1)+t(A_0+4A_1-3A_0-6A_1+2A_0)+(A_1-3A_1+2A_1)t^2=1+t \Leftrightarrow (A_1-A_0)+t(-2A_1)+(0)t^2=1+t$$

Therefore $A_1=-\frac{1}{2}$ and $A_0=-\frac{3}{2}$.

Therefore the complete solution of the difference equation is:

$$y_t=C_1 2^t+C_2-\frac{3}{2}t-\frac{1}{2}t^2$$

Example 3

Solve the equation

$$y_{t+2} - \frac{5}{2} y_{t+1} + y_t = 3^t$$

.

We find the homogeneous solution: $\Delta = \frac{25}{4} - 4 = \frac{9}{4} > 0$, so the roots are 2 and 1/2. The homogeneous solution is

$$y_h = C_1 2^t + C_2 \left(\frac{1}{2}\right)^t$$

.

Our initial assumption for the partial solution is:

$$y_p = A 3^t$$

Example 3

This tentative solution has no common terms with the homogeneous solution and therefore we can substitute it into the complete difference equation:

$A3^{t+2} - \frac{5}{2}A3^{t+1} + A3^{t} = 3^{t} \Leftrightarrow A3^{2} - \frac{15}{2}A + A = 1 \Leftrightarrow A = \frac{2}{5}$

So the complete solution is:

$$y_t = C_1 2^t + C_2 \left(\frac{1}{2}\right)^t + \frac{2}{5}3^t$$