ΕΛΛΗΝΙΚΗ ΔΗΜΟΚΡΑΤΙΑ
ΠΑΝΕΠΙΣΤΗΜΙΟ ΜΑΚΕΔΟΝΙΑΣ

Mathematical analysis
Lecture 6

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Topics of the 6th lecture

- TheoremYoung
- Maximum, Minimum and Inflection Points
- Major Minor
- Complex functions

Differentiable functions and classes

A function which has k derivatives are called k times of production- not. If the k-its derivative is also continuous, then we say that the f it is class C_k.

C_0 is the class of continuous functions.

His theoremYoung

For a function $y = f(x_1, x_2, \cdots, x_n)$ with continuous partial derivatives first and second order, the order of the derivative when calculating cross partial derivatives is unimportant.
That is, $f_{ij} = f_{yes}$ for any pair i, j with $i, j = 1, 2, \cdots, n$ and $I \neq j$ (obviously the equality also holds in the limiting case where $I = j$).

Example 1

Let the function $f(x, y) = x^2y^3 + x^4y + cary$.

Then

$$f_1 = 2xyz^3 + 4x^3y + e^y$$
$$f_2 = 3x^2y^2 + x^4 + cary$$

and

$$f_{12} = 6xyz^2 + 4x^3 + e^y$$
$$f_{21} = 6xyz^2 + 4x^3 + e^y$$

Therefore $f_{12} = f_{21}$.

Example 2

Let the function $f(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2+x_1x_2x_3$.

Then

$f_1=2x_1+x_2x_3$

$f_2=2x_2+x_1x_3$

$f_3=2x_3+x_1x_2$

and

$f_{12}=x_3$

$f_{21}=x_3$

$f_{13}=x_2$

$f_{31}=x_2$

$f_{23}=x_1$

$f_{32}=x_1$.

Therefore $f_{12}=f_{21}$, $f_{13}=f_{31}$ and $f_{23}=f_{32}$.

Function Optimization n variables

A stationary function value f certain in $\mathbb{R}^n$ is presented in a sign-minus $(x_1^*, \cdots, x_n^*)$ where the following equalities are simultaneously true:

$$f_1(x_1^*, \cdots, x_n^*) = 0$$
$$\cdots\cdots\cdots\cdots$$
$$f_n(x_1^*, \cdots, x_n^*) = 0$$

or

$$\nabla f(x_1^*, \cdots, x_n^*) = 0$$

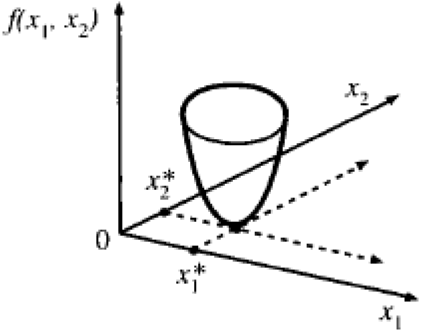
Function Optimization n variables

As in the case of functions with one variable, it is not necessary that all stationary points give extreme values, as they may be inflection points.

In the case of functions with n variables, there is also the possibility a stationary value to be a breakpoint (saddle point), where the function takes its maximum value in relation to price changes in some of the $x_1, \cdots, x_n$ and its minimum price compared to others.

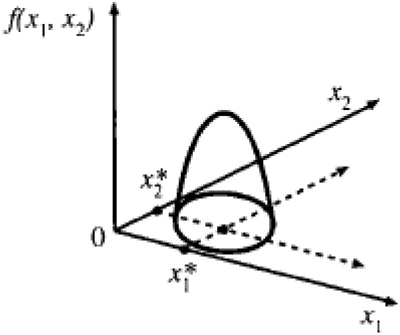
> Critical points of a function of several variables are called its stationary points as well as the points where at least one of its partial derivatives does not exist.

Maximum, Minimum and Inflection Points



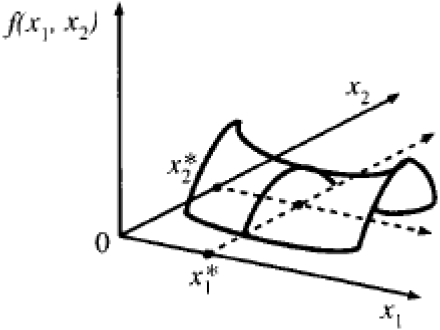
Shape: Minimum in x_1 address, minimum in x_2 address

Maximum, Minimum and Inflection Points



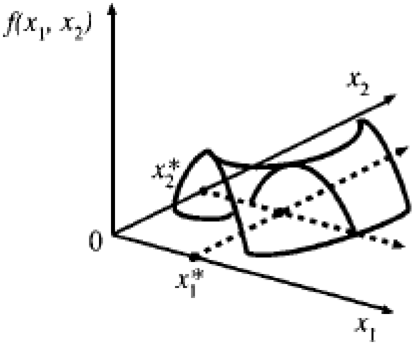
Shape: Maximum in x_1 address, maximum at x_2 address

Maximum, Minimum and Inflection Points



Shape: Minimum in x_1 address, maximum at x_2 address

Maximum, Minimum and Inflection Points



Shape: Maximum in x_1 address, minimum in x_2 address

Example

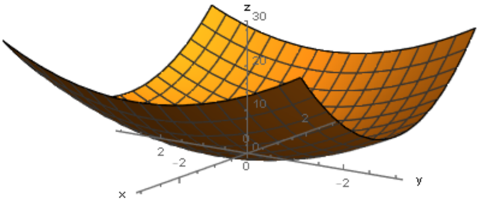
Find the stationary points of the function:

$$f(x, y) = 2x^2 + y^2$$

The first-order conditions are:

$$f_x = 4x = 0, \quad f_y = 2y = 0$$

These are satisfied only when $x = y = 0$. Therefore (0,0) is a stationary point.



Example

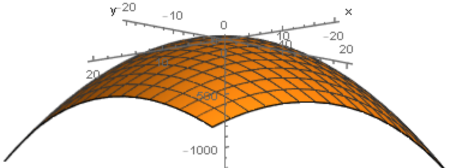
Find the stationary points of the function:

$$f(x, y) = 4x + 2y - x^2 - y^2 + xyz$$

The first-order conditions are:

$$f_x = 4 - 2x + y = 0 \quad f_y = 2 - 2y + x = 0$$

From the second equation we have $x = 2y - 2$. Substituting into the first one we have $4 - 4y + 4 + y = 0$ or equivalently $y = \frac{8}{3}$. Substituting into the first equation we have $x = \frac{10}{3}$.



Example

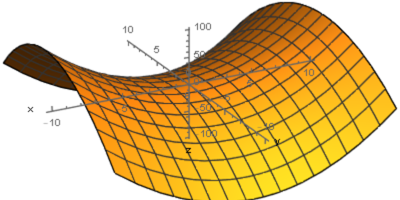
Find the stationary points of the function:

$$f(x, y) = x^2 - y^2$$

The first-order conditions are:

$$f_x = 2x = 0$$
$$f_y = -2y = 0$$

These are satisfied only when $x = y = 0$. Therefore (0,0) is a stationary point.



Second-Class Conditions

A sufficient condition to give the x^* a local maximum of C^2 function $f(x)$ is:

$$f_I(x^*) = 0, I = 1, \cdots, n$$

and the square shape

$$d^2 y(x^*) = \sum_I \sum_j f_{Ij} dx_I dx_j < 0$$

That is, the $d^2 y$ is negative definite, or the Hessian matrix H are negative certain because $d^2 y = dx^T HDX$.

Convex and concave functions

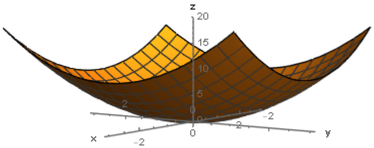
The function f:$\mathrm{R}_n \to \mathrm{R}$ is convex (convex) if for any two points of the scope of definition *you*= (*you*$_1$, . . . ,*you*$_n$) and *v*= (v_1, . . . ,v_n) it is true that:

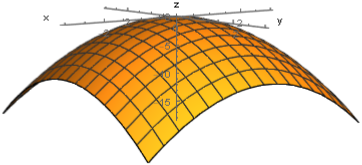
$$f(lou + (1-\lambda)v) \leq \lambda f(you) + (1-\lambda)f(v),$$

while it is hollow (concave) when:

$$f(lou + (1-\lambda)v) \geq \lambda f(you) + (1-\lambda)f(v),$$

where $/ \in [0,1]$.





Total extremes

We assume that the $y = f(x)$ is a strictly concave function defined for $x \in R^n$ If in $x = x^*$ all first derivatives are zero, i.e. $f_i(x^*) = 0, I = 1, 2, \cdots, n$, then the x^* gives a unique overall maximum.

We assume that the $y = f(x)$ is a strictly convex function defined by for $x \in R^n$ If in $x = x^*$ all first derivatives are zero, i.e. $f_i(x^*) = 0, I = 1, 2, \cdots, n$, then the x^* gives a unique overall minimum.

Leading major minors

For a function $y = f(x), x \in \mathbb{R}^n$ which is twice continuously differentiable with Hessian matrix H, the leading major minors are:

$$|H_1| = |f_{11}|,$$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix},$$

...

$$|H_n| = |H| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}.$$

Positive/Negative Definite Matrix

Let it be H the Hessian matrix associated with a function $y = f(\mathrm{x})$, two times continuously differentiable where $\mathrm{x} \in \mathbb{R}^n$ It is valid that:

1. The uterus H is positively definite in $\mathbb{R}^n$ if and only if the leading its major and minor are positive. That is,
   $|H_1| > 0, \ |H_2| > 0, \cdots, \ |H_n| = |H| > 0$ for $\mathrm{x} \in \mathbb{R}^n$ In this case the f is strictly convex.
2. The uterus H is negatively definite in $\mathbb{R}^n$ if and only if the leading its major and minor keys have alternating signs starting with a negative sign for $k = 1$. That is:

   $$|H_1| < 0, \ |H_2| > 0, \cdots, \ |H_n| = |H| = \begin{cases} > 0 & \text{if } n \text{ is even if } n \text{ it is} \\ < 0 & \text{unnecessary} \end{cases}$$

   In this case the f is strictly concave.

Positive/Negative Definite Matrix

Let it be H the Hessian matrix associated with a function $y = f(x)$, twice continuously differentiable where $x \in R^n$ It is valid that:

1. OR H is positive definite if and only if all its eigenvalues are positive. In this case the f is strictly convex.
2. OR H is negative definite if and only if all its eigenvalues are negative. In this case the f is strictly concave.

Positive/Negative Definite Matrix: Example

Let the uterus $H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Then its leading major minors are:

$|H_1| = 2 > 0, |H_2| = |H| = \begin{vmatrix} 2 & 1 \\ 1 & \end{vmatrix} = 3 > 0$. Therefore the matrix is positive 2 certain.

Using the|of eigenvalues:|

$|\lambda I - H| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \end{vmatrix} = (\lambda - 2)^2 - 1 = \lambda^2 + 4 - 4\lambda - 1 = \lambda^2 - 4\lambda + 3$ $\lambda - 2|$

$|\lambda I - H| = 0 \Longleftrightarrow \lambda^2 - 4\lambda + 3 = 0$

$\Delta = 16 - 12 = 4, \lambda_1 = \frac{4+2}{2} = 3 > 0, \lambda_2 = \frac{4-2}{2} = 1 > 0.$

Since both eigenvalues are positive, the matrix is positive definite.

Example

Find and characterize the stationary points of the function:

$$f(x_1, x_2) = 2x_1^2 + x_2^2$$

$$f_1 = 4x_1, f_2 = 2x_2$$

$$f_{11} = 4, f_{12} = 0, f_{21} = 0, f_{22} = 2 \text{ for each } x \in \mathbb{R}^n$$

$$\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 > 0 \text{ for each } x \in \mathbb{R}^n$$

We conclude that the stationary point (0,0) is the unique global minimum of function.

Example 2

Find and characterize the stationary points of the function:

$$f(x_1,x_2) = 4x_1+2x_2-x_1^2-x_2^2+x_1x_2$$

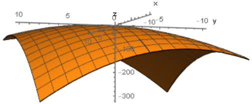
We have previously found the stationary point $(10/3, 8/3)$.

$$f_1 = 4-2x_1+x_2,\quad f_2 = 2-2x_2+x_1$$

$$f_{11} = -2,\ f_{12} = 1,\ f_{21} = 1,\ f_{22} = -2 \text{ for each } x \in \mathbb{R}^n$$

$$H = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4-1 = 3 > 0 \text{ for each } x \in \mathbb{R}^n$$

Leading major minors change sign starting from negative ($|H_1| = -2$, $|H_2| = 3$), so the Hessian matrix is negative definite and therefore the function is strictly concave and the stationary point is a global maximum.



Example 3

Find and characterize the stationary points of the function $f(x, y) = x^2 - y^2$.

The stationary points of f are calculated by setting:

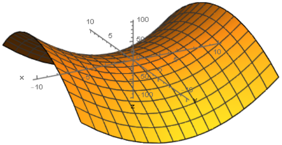
$$\nabla f(x, y) = (2x, -2y) = (0,0).$$

So the only stationary point is (0,0]).

Her Hessian womb $f$it is $\nabla^2 f = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

Its leading major minors are: $|H_1| = 2 > 0$, $|H_2| = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$.

Therefore, the Hessian matrix is neither positive nor negative definite. The point (0,0) is a sagmatic point.



Major minors

The determinants for all major minors (not just the leading ones) for $n=3$ are given below:

$$|H_1^*| = f_{11}, f_{22}, f_{33}$$

$$|H_2^*| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, \begin{vmatrix} f_{11} & f_{13} \\ f_{31} & f_{33} \end{vmatrix}, \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix}$$

$$|H_3^*| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

Positive/Negative semi-definite matrix

The Hessian uterus H is positive semi-definite in R^n if and only if all the main its minors are positive or zero. That is:
$|H_1^*| \geq 0,\ |H_2^*| \geq 0,\ |H_3^*| \geq 0, \cdots |H_n^*| = |H| \geq 0$ for $x \in R^n$.

The Hessian uterus H is negatively semi-definite in R^n if and only if all the its major and minors alternate in sign or are zeroed starting from a negative or zero value for $k=1$. That is:
$|H_1^*| \leq 0,$
$|H_2^*| \geq 0,$
...
$|H_n^*| = |H| = \begin{cases} \geq 0, \text{if the } n \text{ is even} \\ \leq 0, \text{if the } n \text{ it is unnecessary} \end{cases}$

First Order Total Differential for a Function of Two Variables

The first-order total differential for the function $y = f(x_1, x_2)$ is:

$$day = f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2$$

Derivation of a Complex Function

A complex function is a function of the form $F(x, y) = 0$.

A simple example is $3y + 9x - 12 = 0$. With a few mathematical operations we can define it in detail y as a function of x with the $y = -3x + 4$ and we see that the derivative is $dy/dx = -3$.

However, finding an analytical solution for the y from an equation involving and the x and the y is not always so easy. Therefore, it is useful to have a process for finding the dy/dx when it y is defined indirectly.

For example, in $e^{x^2+y} - 7 = 0$ a given value of x implies some specific y so that equality is satisfied.

Derivation of a Complex Function

Instead of solving analytically in terms of y, the discovery of dy/dx it can be done through the process of complex derivation. First, we assume for the moment that the above equation means that the y we can do it we formulate as follows:

$$e^{x^2+f(x)} - 7 = 0$$

By differentiation with respect to the variable x (using the chain rule) we have:

$$\left(\frac{d}{dx}\left[e^{x^2+f(x)}\right]\right) = \left[2x + f'(x)\right] e^{x^2+f(x)} = 0$$

and then dividing by $e^{x^2+f(x)}$ which cannot be zero, we have that $2x + f'(x) = 0 \Longrightarrow f'(x) = -2x$ which gives us the desired result.

Derivation of a Complex Function

It turns out that in this case we can verify the result quite easily by first solving directly for y, considering it as a function of x. If we take the natural logarithm of both sides of the equation $e^{x^2+y}=7$ we end up with:

$$(x^2+y)\ \mathrm{in}\, e = \mathrm{in}\ 7$$

But $\mathrm{in}\, e = 1$, so:

$$y = \mathrm{in}\ 7 - x^2$$

and therefore $dy/dx = -2x$.

Derivation of a Complex Function

However, following the above steps to derive a complex function is often a laborious process. A more convenient method is to first formulate the relationship between x and the y with the convoluted function $F(x, y) = 0$ and then taking the total differential of this expression we arrive at the relationship:

$$F_x dx + F_y day = 0$$

Rearranging its terms we end up with:

$$\frac{day}{dx} = -\frac{F_x}{F_y}$$

Derivation of a Complex Function

Convoluted function theorem (for two variables): Let the $F(x, y) = 0$ is a convoluted function with continuous first derivatives, which is satisfied at some point (x_0, y_0) and is defined in a region of this point. If $F_y \neq 0$ at this point, then there is a function $y = f(x)$ defined in an area of $x = x_0$ and which corresponds to the relation which is defined by the $F(x, y) = 0$, so that:

(1) $y_0 = f(x_0)$, and
(2) $f(x_0) = -F_x/F_y$

This theorem determines the conditions under which it is possible to conclude that the entangled function $F(x, y) = 0$ implies the existence a clear functional relationship $y = f(x)$ as well as the method of calculating it its derivative. The central point is that $F_y \neq 0$.

Example

> Interpret the entangled function theorem using the
>
> $$F(x, y) = x^2 + y^2 - 25 = 0.$$

This is the equation of a circle in $\mathbb{R}^2$ centered at the origin of the axes (0,0) and radius 5. Because we can write this equation either as:

$$y^2 = 25 - x^2$$

either as:

$$x^2 = 25 - y^2$$

it follows that $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$ because the left-hand sides as squares are positive numbers.

Example

We choose $x_0=3, y_0=4$, a specific point that satisfies the $F(x, y) = 0$. Because $F_y(x, y) = 2y$, implies that in $y_0=4$ is valid $F_y \neq 0$ and therefore the basic condition is satisfied. Therefore in the region of (3,4) we can let's think of one y as a function of x, and we have:

$$f(x) = \frac{day}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y}$$

Now, at the point (3,4) we have:

$$\frac{day}{dx} = -\frac{6}{8} = -\frac{3}{4}$$

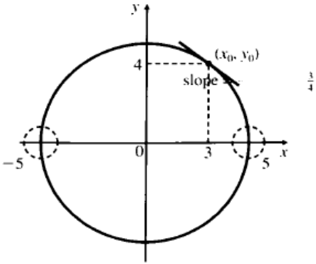
Example

For this example it is easy √ Let's solve this immediately regarding $y = f(x)$ in the area of the point (3,4) with $y = \sqrt{25 - x^2}$ and check its price derivative with the following steps:

$$\frac{day}{dx} = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x) \Longleftrightarrow \frac{day}{dx} = -\frac{x}{\sqrt{25 - x^2}}$$

and in $x_0 = 3$ we get $dy/dx = -3/4$.

Chart



Shape:Example of a convoluted function

The results are presented in the figure above. We observe in this figure that at the points (5,0) and (-5,0) it is not possible to see the relationship between x and the y as a function $y = f(x$ The entangled function theorem identifies this difficulty since $F_y = 2y = 0$ in prices $x = 5$ or $x = -5$ (since $y = 0$ in each such case).

Entangled function theorem: Let the $F(x_1, x_2, \cdots, x_n, y) = 0$ is a complex function with continuous first derivatives, which can be is fixed at some point $(x_1^0, x_2^0, \cdots, x_n^0, y^0)$ and is defined in some area of this point. If $F_y \neq 0$ at this point, then there is a function $y = f(x_1, x_2, \cdots, x_n)$ defined in a region of $\mathrm{x} = \mathrm{x}_0 = (x_1^0, x_2^0, \cdots, x_n^0)$ so that:

1. $y_0 = f(\mathrm{x}_0)$, and
2. $f_I(\mathrm{x}_0) = -F_{x_I}/F_y$

Example

Use the interleaved derivative and find the derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ of the function determined by the relation:

$$F(x_1,x_2,y) = 5x_1x_2+2x_2y^2+x_1^2x_2^2y-5 = 0$$

$F_{x_1}=5x_2+2x_1x_2^2y$, $F_{x_2}=5x_1+2y^2+2x_1^2x_2y$ and $F_y=4x_2y+x_1^2x_2^2$

Therefore:

$$\frac{\partial y}{\partial x_1}=-\frac{F_{x_1}}{F_y}=-\frac{5x_2+2x_1x_2^2y}{4x_2y+x_1^2x_2^2} \quad \text{and} \quad \frac{\partial y}{\partial x_2}=-\frac{F_{x_2}}{F_y}=-\frac{5x_1+2y^2+2x_1^2x_2y}{4x_2y+x_1^2x_2^2}$$