ΕΛΛΗΝΙΚΗ ΔΗΜΟΚΡΑΤΙΑ
ΠΑΝΕΠΙΣΤΗΜΙΟ ΜΑΚΕΔΟΝΙΑΣ

Mathematical analysis
Lecture 7

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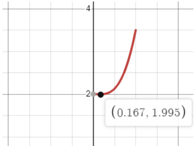
Topics of the 7th lecture

- Optimization of functions of one variable with interval constraints
- Optimization of functions of many variables with constraints space
- Multiplier techniqueLagrange

Optimization on an interval - Example 1

To solve the problem minutes $y=2x^3-\frac{1}{2}x^2+2$ under the constraint $0\leq x\leq 1$.

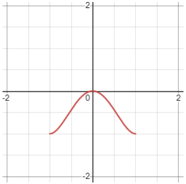
The first derivative is: $y'=6x^2-x$.
The stationary points are $x=0$ and $x=1/6$ which belong to the period in which is given to us.
The second derivative is: $y''=12x-1$ In $x=0$, where $y''=-1$ we have a local maximum. In $x=1/6$, where $y''=1$ we have a local minimum.(The prices of y at the stationary points is $(0,2)$ and $\left(\frac{1}{6},1.995\right)$.
We also check the edge of the interval for $x=1$, that is, the point $(1,7/2)$.
Therefore we have a global minimum in $x=1/6$.



Optimization on an interval - Example 2

To solve the problem $\max y = x^4 - 2x^2$ under the restriction $-1 \leq x \leq 1$.

The first derivative is: $y' = 4x^3 - 4x$
The stationary points are $x = 0, x = -1$ and $x = 1$ which belong to space which is given to us.
The second derivative is: $y'' = 12x^2 - 4$ In $x = 0$, where $y'' = -4$ we have a local maximum. In $x = -1$, where $y'' = 8$ we have a local minimum. In $x = 1$, where $y'' = 8$ we have a local minimum. Therefore we have a total maximum at $x = 0$ where $f(0) = 0$.



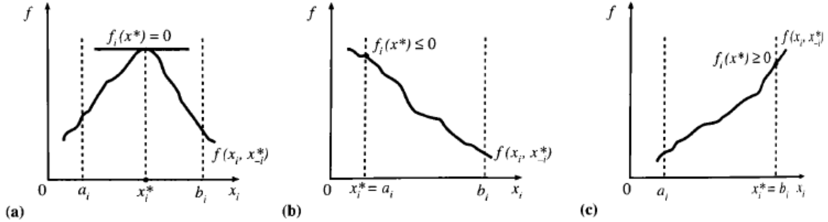
Direct Constraints on Variables

Let's assume we have a function with n variables and let's assume that each variable is limited to an interval $a_I \leq x_I \leq b_I, I = 1, \cdots, n$.

For some I can it x_I not be blocked from above or below, but we assume that for some at least I, the a_I and/or the b_I are finite.

Direct Constraints on Variables

Let's assume that the point x^* gives a maximum of the function with the constraint that every price x_i is within a given interval.
For each x_i therefore we must have one of three possible cases which are presented in the figure below, where with x^*_{-i} we denote the vector of fixed prices $(x^*_1, \cdots, x^*_{i-1}, x^*_{i+1}, \cdots, x^*_n)$.



Shape: Possible solutions when someone x_i must be within a range

Direct Constraints on Variables

Case1 $_{or}$: $a_I < x_*$ $_I < b_I$.
In this case it must be true $f_I(x_*) = 0$. To verify this we get the component $f_I(x_*)dx_I$ of the total differential df which corresponds to in the x_I:

$$day(x_*) = f_1(x_*)dx_1 + \ldots + f_I(x_*)dx_I + \ldots + f_n(x_*)dx_n$$

If $f_I(x_*) \neq 0$ then we can find a suitably small dx_I with the appropriate sign, so that $f_I(x_*)dx_I > 0$. This will increase the price of function, rejecting the original hypothesis that it is at a maximum. Therefore, it must hold that $f_I(x_*) = 0$.

Direct Constraints on Variables

Case 2 or: $a_I = x_I^*$.

In this case it should be true $f_I(x^*) \leq 0$. To find out this, we assume that $f_I(x^*) > 0$. We can choose any $dx_I > 0$, since so it x_I is maintained within the feasible interval, and we then have $f_I(x^*)dx_I > 0$ which contradicts the original assumption that the function is at a maximum. Therefore, we can rule out the possibility that $f_I(x^*) > 0$.

If $f_I(x^*)dx_I > 0$ only some $dx_I < 0$ could increase its price function, but this violates the constraint and therefore the value of the function cannot be increased. Similarly, if $f_I(x^*) = 0$ the value of function cannot be increased by small fluctuations in x_I.

Direct Constraints on Variables

Case3$_{or}$: x_{*} $_{I} = b_{I}$.
In this case it should be true $f_I(x_*) \geq 0$. To determine this let's assume that $f_I(x_*) < 0$. We can choose $dx_I < 0$ so that $f_I(x_*)dx_I > 0$ and therefore, without violating the constraint, the value of function can increase. So we have to rule out this case.

If $f_I(x_*)dx_I > 0$ only some $dx_I > 0$ could increase its price function, but this violates the constraint and therefore the value of the function cannot be increased. Similarly, if $f_I(x_*) = 0$ the value of function cannot be increased by small fluctuations in x_I.

Direct Constraints on Variables

Theorem: If $\mathbf{x}^*$ is a solution to the problem of maximizing $f(\mathbf{x})$, that is,

$\max f(\mathbf{x})$ under the restriction $a_i \le x_i \le b_i$ with $i = 1, \cdots, n$,
then one or both of the following conditions must apply:

1. $f_i(\mathbf{x}^*) \le 0$ and $(x_i^* - a_i) f_i(\mathbf{x}^*) = 0$
2. $f_i(\mathbf{x}^*) \ge 0$ and $(b_i - x_i^*) f_i(\mathbf{x}^*) = 0$

for all $i = 1, \cdots, n$.

Direct Constraints on Variables

Theorem: If $\mathbf{x}_*$ is a solution to the problem of minimizing $f(\mathbf{x})$ that is,

minutes $f(\mathbf{x})$ under the restriction $a_I \leq x_I \leq b_I$ with $I = 1, \cdots, n$,
then one or both of the following conditions must apply:

1. $f_I(\mathbf{x}_*) \geq 0$ and $(x_{*I} - a_I) f_I(\mathbf{x}_*) = 0$
2. $f_I(\mathbf{x}_*) \leq 0$ and $(b_I - x_{*I}) f_I(\mathbf{x}_*) = 0$

for all $I = 1, \cdots, n$.

Example 1

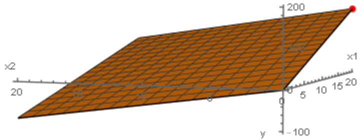
To solve the problem: $\max f(x) = y = 10x_1 - 5x_2$
subject to restrictions $0 \leq x_1 \leq 20$, $0 \leq x_2 \leq 20$.

This function is linear and increasing with respect to x_1 and linear and decreasing in relation to x_2. When there is no constraint interval there is no solution (why?). We can see that the solution is at the upper bound of x_1 and at the lower limit of x_2: $x^*_1 = 20, x^*_2 = 0$

This point satisfies the necessary conditions for a maximum since:

$$f_1 = 10 \geq 0,\ (20 - x^*_1)10 = 0$$
$$f_2 = -5 \leq 0,\ (x^*_2 - 0)(-5) = 0$$

in (20,0).



Example 2

To solve the problem: $\max f(\mathbf{x}) = y = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ subject to restrictions $0 \le x_1 \le 10$, $0 \le x_2 \le 10$.

$f_1(x_1, x_2) = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} > 0$ in the given interval, therefore it is purely increasing in this regard x_1.

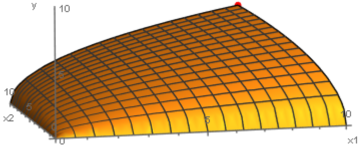
$f_2(x_1, x_2) = x_1^{\frac{1}{2}} \frac{1}{2} x_2^{-\frac{1}{2}} > 0$ in the given interval, therefore it is purely increasing in this regard x_2.

So we can see that the solution lies in the upper bounds of intervals: $x^*_1 = 10, x^*_2 = 10$.

This point satisfies the necessary condition of the maximum theorem, since:

$f_1(10,10) = \frac{1}{2} 10^{-\frac{1}{2}} 10^{\frac{1}{2}} \ge 0$, $(10-10) f_1 = 0$

$f_2(10,10) = \frac{1}{2} 10^{\frac{1}{2}} 10^{-\frac{1}{2}} \ge 0$, $(10-10) f_2 = 0$



Example 3

> To solve the problem: $\max f(x) = y = 4x_1 + 2x_2 - x_1^2 - x_2^2 + x_1 x_2$ underlying to the restrictions $0 \le x_1 \le 10$, $0 \le x_2 \le 10$.

We have $f_1 = 4 - 2x_1 + x_2$, $f_2 = 2 - 2x_2 + x_1$

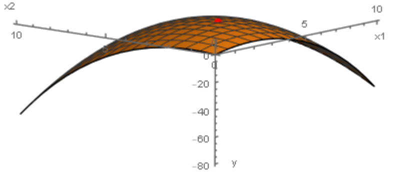
From the first-order conditions and the second equation we have that $x_1 = 2x_2 - 2$. Substituting into the first equation we have

$4 - 4x_2 + 4 + x_2 = 0 \iff 8 - 3x_2 = 0 \iff x_2 = \frac{8}{3}$. Substituting we have $x_1 = \frac{10}{3}$. The point is internal to both intervals and we have at the point $(\frac{10}{3}, \frac{8}{3})$:

$f_1 = 0$, $(\frac{10}{3} - 0) f_1 = (10 - \frac{10}{3}) f_1 = 0$

$f_2 = 0$, $(\frac{8}{3} - 0) f_2 = (10 - \frac{8}{3}) f_2 = 0$

So the conditions for a maximum are satisfied.



Example 4

To solve the problem: $\max f(x) = y = 4x_1 + 2x_2 - x_1^2 - x_2^2 + x_1 x_2$ underlying to the restrictions $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq \frac{8}{3}$

In this case we have the same function as before, but the intervals are different.

For the x_1, the given interval excludes the previous optimal solution. For x_2 the previous optimal solution is the upper bound of the interval and continues to be available.
But caution is needed because even if it continues to be feasible, this price of x_2 is not necessarily optimal for the new one (due to changing constraints) problem.

Example 4

We can test the upper bounds of the two intervals, the point $(1, \frac{8}{3})$. The partial derivatives of the function are:

$$f_1 = 4 - 2x_1 + x_2, f_2 = 2 - 2x_2 + x_1$$

Therefore in $(1, \frac{8}{3})$ we have:

$$f_1(1, \tfrac{8}{3}) = \tfrac{14}{3} > 0, f_2(1, \tfrac{8}{3}) = -\tfrac{7}{3} < 0$$

We conclude that this point cannot be a maximum according to the theorem, because we need $f_2 \geq 0$, when the x_2 is at the upper limit of of its space.

Example 4

We can find the possible solution by first noting that for all x_1 within space [0,1] and for all x_2 in $0, \frac{8}{3}$ valid $f_1 > 0$. This means that the function is increasing with respect to x_1 Therefore, it makes sense to put the x_1 at its upper limit $x_1 = 1$. As we saw a moment ago in $1, \frac{8}{3}$ the partial derivative $f_2 < 0$ fact which means we can increase the value of the function reducing the x_2 But until when? We can find the answer if we ask $x_1 [= 1]$ in the function y and maximize it in terms of x_2 in space $0, \frac{8}{3}$ That is, our problem is:

$$\max y = 3 + 3x_2 - x_2^2 \text{ under the restriction } 0 \leq x_2 \leq \frac{8}{3}$$

From the first-order condition we have:

$$3 - 2x_2 = 0$$

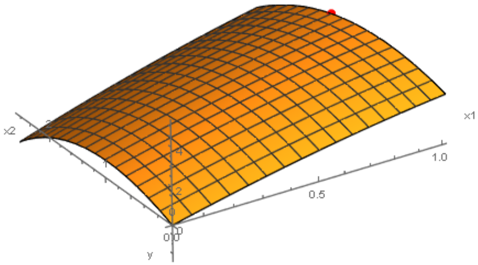
which means that $x_2^* = 1.5$

Example 4

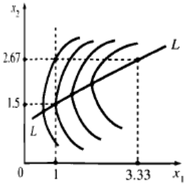
To check if the point (1,1.5) satisfies the necessary conditions for maximum we have:

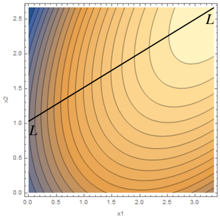
$$f_1 = 4 - 2(1) + 1.5 = 3.5 > 0, f_1(1 - x_1^*) = 0$$
$$f_2 = 2 - 2(1.5) + 1 = 0, f_2\left(\frac{8}{3} - 1.5\right) = 0$$

and therefore the conditions apply.



Schematic representation of the problem





Shape:Constraint interval that varies the optimal values of both variables

The peak of the graph is at (3.33, 2.67), but in the constrained problem we are limited to [0,1] for the x_1. The point (1, 2.67) is not on the highest contour line we can achieve. The highest possible contour line is achieved by moving to (1, 1.50). We note that this is a point of contact (point of tangency) between the vertical restriction line and the highest possible contour line.

Schematic representation of the problem

The first stage of this process is equivalent to finding the locus of the iso-contour curves of this function with the vertical lines corresponding to the various values of x_1.

This locus is marked with LL in the figure. Therefore the intersection of this of the locus with the perpendicular line to $x_1 = 1$ gives the overall solution.

Multiplier TechniqueLagrange

Definition: The MethodLagrange to find a solution (x^*_1, x^*_2) in problem

$\max f(x_1, x_2)$ under the constraint $g(x_1, x_2) = 0$

consists of creating the following first-order conditions for finding the stationary point(s) of the functionLagrange. $L(x_1, x_2, l) = f(x_1, x_2) + l g(x_1, x_2)$ which is:

$$\frac{\partial L}{\partial x_1} = f_1(x^*_1, x^*_2) + l^* g_1(x^*_1, x^*_2) = 0$$

$$\frac{\partial L}{\partial x_2} = f_2(x^*_1, x^*_2) + l^* g_2(x^*_1, x^*_2) = 0$$

$$\frac{\partial L}{\partial l} = g(x^*_1, x^*_2) = 0$$

Example

> To solve the constrained maximization problem
> $\max y = x_1^2 x_2$ under the restriction $100 - x_1 - 2x_2 = 0$

$L(x_1, x_2, \lambda) = x_1^2 x_2 + \lambda(100 - x_1 - 2x_2) = 0$

The first-order conditions are:

$$\frac{\partial L}{\partial x_1} = 2x_1 x_2 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = x_1^2 - 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 100 - x_1 - 2x_2 = 0$$

We have $x_1 = 100 - 2x_2$ and $\lambda = 2x_1 x_2 = 2(100 - 2x_2)x_2$ and
$x_1^2 - 2\lambda = 0 \Leftrightarrow (100 - 2x_2)^2 - 4(100 - 2x_2)x_2 = 0 \Leftrightarrow (100 - 2x_2)(100 - 2x_2 - 4x_2) = 0$
$\Leftrightarrow (100 - 2x_2)(100 - 6x_2) = 0 \Leftrightarrow x_2 = 50$ or $x_2 = \frac{50}{3}$

Therefore the stationary points are $(0, 50)$ and the $(\frac{200}{3}, \frac{50}{3})$.

Example 2

Solve the following constrained minimization problem:

$$\text{minutes}\; y = x_1 + x_2 \;\text{under the restriction}\; 1 - x_1^{1/2} - x_2 = 0$$

The functionLagrange is

$$L = x_1 + x_2 + \lambda(1 - x_1^{1/2} - x_2)$$

The first-order conditions are:

$$1 - \left(\frac{\lambda}{2}\right) x_1^{-1/2} = 0$$

$$1 - \lambda = 0$$

$$1 - x_1^{1/2} - x_2 = 0$$

We have $\lambda = 1$ from the second equation and substituting into the first $\frac{1}{2} x_1^{-1/2} = 1 \iff x_1^{1/2} = \frac{1}{2} \iff x_1 = \frac{1}{4}$. Therefore $x_1 = \frac{1}{4}$. Substituting in third equation we have $x_2 = \frac{1}{2}$.

Suitable conditions for optimal

The bounded Hessian matrix of the functionLagrange is:

$$H^{*} = \begin{vmatrix} f_{11}+\lambda^{*} g_{11} & f_{12}+\lambda^{*} g_{12} & g_1 \\ f_{21}+\lambda^{*} g_{21} & f_{22}+\lambda^{*} g_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix}$$

Theorem: If $(x_1^{*}, x_2^{*}, \lambda^{*})$ gives a stationary value of the Lagrange function $L = f(x_1, x_2) + \lambda g(x_1, x_2)$, then the point

- gives a maximum if the determinant of the bounded Hessian $|H^{*}| > 0$, and
- gives a minimum if the determinant of the bounded Hessian $|H^{*}| < 0$

Check for the 1st example

$$f(x_1,x_2) = x_1^2 x_2, \quad g(x_1,x_2) = 100 - x_1 - 2x_2 = 0$$

$f_1 = 2x_1x_2 - \lambda, f_{11} = 2x_2, f_{12} = 2x_1$

$f_2 = x_1^2 - 2\lambda, f_{21} = 2x_1, f_{22} = 0$

$g_1 = -1, g_2 = -2, g_{11} = 0, g_{12} = 0, g_{21} = 0, g_{22} = 0$

The area broken Hessian is:

$$|H^*| = \begin{vmatrix} 2x_2 & 2x_1 & -1 \\ 2x_1 & 0 & -2 \\ -1 & -2 & 0 \end{vmatrix} = -\begin{vmatrix} 2x_1 & -1 \\ 0 & -2 \end{vmatrix} + 2\begin{vmatrix} 2x_2 & -1 \\ 2x_1 & -2 \end{vmatrix} = 4x^*_1 + 2(-4x^*_2 + 2x^*_1) =$$

$8x^*_1 - 8x^*_2$.

For the $\left(\frac{200}{3}, \frac{50}{3}\right)$, $|H^*| = 400 > 0$ so we have a local maximum for $(0,50)$, $|H^*| = -400 < 0$ so we have a local minimum.

Check for the 2nd example

$$f(x_1,x_2) = x_1 + x_2, \quad g(x_1,x_2) = 1 - x_1^{1/2} - x_2 = 0$$

$f_1 = 1, f_2 = 1, f_{11} = 0, f_{12} = 0, f_{21} = 0, f_{22} = 0$

$g_1 = -\frac{1}{2}x_1^{-1/2}, \; g_2 = -1, g_{11} = \frac{1}{4}x_1^{-3/2}, \; g_{12} = 0, g_{21} = 0, g_{22} = 0$

$$|H_*| = \begin{vmatrix} \frac{1}{4}x_1^{-3/2} & 0 & -\frac{1}{2}x_1^{-1/2} \\ 0 & 0 & -1 \\ -\frac{1}{2}x_1^{-1/2} & -1 & 0 \end{vmatrix} = -\frac{1}{4}x_1^{-3/2}$$

which in $x_1^* = \frac{1}{4}$ gives $|H_*| = -\frac{1}{4}\left(\frac{1}{4}\right)^{-3/2} = -\left(\frac{1}{4}\right)^{-1/2} = -2 < 0.$

Therefore we do indeed have a local minimum at $x_1^* = \frac{1}{4}, x_2^* = \frac{1}{2}$.

Example 3

Find the dimensions of a closed cylindrical metal soft drink can so that its volume is maximized and its total surface area is equal to $24p$.

Let us denote by x_1 the radius of the base of the cylinder, and with x_2 the its height. Then its volume is equal to $px_1^2 x_2$ while its surface area equal to $2xx_1x_2+2xx_1^2$ The problem boils down to:

$$\max y = xx_1^2 x_2 \text{ under the restriction } 2xx_1x_2+2xx_1^2=24p$$

The function Lagrange is

$$L=px_1^2x_2+l(2xx_1x_2+2xx_1^2-24p)$$

Example 3

The first-order conditions are:

$$\frac{\partial L}{\partial x_1} = 2p\,x_1x_2 + 2\,etc.\,x_2 + 4\,etc.\,x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = p x_1^2 + 2\,etc.\,x_1 = 0$$

$$\frac{\partial L}{\partial l} = 2p\,x_1x_2 + 2x x_2\;{}_1 - 24p = 0$$

From the first condition we have: $l = -\frac{x_1x_2}{2x_1+x_2}$, while from the second: $x x_1(x_1+2l) = 0 \Rightarrow \lambda = -\frac{x_1}{2}$ (we can't have $x_1 = 0$ since it is natural quantity), therefore $-\frac{x_1}{2} = -\frac{x_1x_2}{2x_1+x_2} \Rightarrow 2x_2 = 2x_1 + x_2 \Rightarrow x_1 = \frac{x_2}{2}$. From the third condition replacing the x_1 we get: $2p\frac{x_2}{2}x_2 + 2p\left(\frac{x_2}{2}\right)^2 - 24p = 0 \Rightarrow x_2 = 4$.
Substituting in the remaining conditions, we get the stationary point:

$$x_1^* = 2,\ x_2^* = 4\ l^* = -1$$

Example 3

We check whether the point we found is a maximum of f.

$f_1 = 2xx_1x_2, f_2 = px_1^2, f_{11} = 2xx_2, f_{12} = 2xx_1, f_{21} = 2xx_1, f_{22} = 0$

$g_1 = 2xx_2 + 4xx_1, g_2 = 2xx_1, g_{11} = 4p, g_{12} = 2p, g_{21} = 2p, g_{22} = 0$

$$|H^*| = \begin{vmatrix} 2xx_2^* + \lambda^* 4p & 2xx_1^* + \lambda^* 2p & 2xx_2^* + 4xx_1^* \\ 2xx_1^* + \lambda^* 2p & 0 & 2xx_1^* \\ 2xx_2^* + 4xx_1^* & 2xx_1^* & 0 \end{vmatrix} = \begin{vmatrix} 4p & 2p & 16p \\ 2p & 0 & 4p \\ 16p & 4p & 0 \end{vmatrix} = 192p^3 > 0.$$

Therefore we have a local maximum at $x_1^* = 2, x_2^* = 4$, with maximum volume $f(x_1^*, x_2^*) = 16p$.

For many dimensions

Theorem: If the functionLagrange $f(x_1, \cdots, x_n) + lg(x_1, \cdots, x_n)$ has one stationary value at $(x^*_1, \cdots, x^*_n, l^*)$, then the point $(x^*_1, \cdots, x^*_n)$ is a solution of the following problem:

1. To maximize the function $f(x_1, \cdots, x_n)$ under the constraint $g(x_1, \cdots, x_n) = 0$ if the following consecutive major minors of determining $|H^*|$ have alternating sign as follows:

   $$\begin{vmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} > 0, \quad \begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22} & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} < 0, \cdots$$

   with the $|H^*|$ (whole) to take the sign of $(-1)^n$

2. To minimize the function $f(x_1, \cdots, x_n)$ subject to restriction $g(x_1, \cdots, x_n) = 0$ if the above leading major minors her $|H^*|$ are strictly negative.

Example

Find the maximum and minimum values of $f(x_1, x_2, x_3) = x_1^2 + x_1 + 2x_2^2 + 3x_3^2$ under the restriction $x_1^2 + x_2^2 + x_3^2 = 1$.

The functionLagrange is

$$L = x_1^2 + x_1 + 2x_2^2 + 3x_3^2 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1)$$

The first-order conditions are:

$$\frac{\partial L}{\partial x_1} = 2x_1 + 1 + 2\lambda x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 4x_2 + 2\lambda x_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 6x_3 + 2\lambda x_3 = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 + x_3^2 - 1 = 0$$

Example

From the 2nd equation we have $4x_2 + 2\lambda x_2 = 0 \Rightarrow 2x_2(2+\lambda) = 0$ so $x_2 = 0$ or $\lambda = -2$. From the 3rd equation we have $6x_3 + 2\lambda x_3 = 0 \Rightarrow 2x_3(3+\lambda)x_3 = 0$ so $x_3 = 0$ or $\lambda = -3$.

We distinguish the following cases:

a) $x_2 = x_3 = 0$ then from the 4th equation we have $x_1^2 + 0 + 0 - 1 = 0 \Rightarrow x_1 = \pm 1$ and $\lambda = -\frac{3}{2}, \lambda = -\frac{1}{2}$ respectively (for $x = 1$ and $x = -1$)

b) $x_2 = 0$, $\lambda = -3$ then from the 1st equation we have $2x_1 + 1 + 2(-3)x_1 = 0 \Rightarrow x_1 = \frac{1}{4}$, and from the 4th equation: $\left(\frac{1}{4}\right)^2 + 0 + x_3^2 - 1 = 0 \Rightarrow x_3 = \pm\frac{\sqrt{15}}{4}$

c) $\lambda = -2, x_3 = 0$ then from the 1st equation $2x_1 + 1 + 2(-2)x_1 = 0 \Rightarrow x_1 = \frac{1}{2}$ and from the 4th equation: $\left(\frac{1}{2}\right)^2 + x_2^2 + 0 - 1 = 0 \Rightarrow x_2 = \pm\frac{\sqrt{3}}{2}$

Therefore we have a total of stationary points $(x_1^*, x_2^*, x_3^*, \lambda^*)$: $\left(1, 0, 0, -\frac{3}{2}\right)$, $\left(-1, 0, 0, -\frac{1}{2}\right)$, $\left(\frac{1}{4}, 0, \frac{\sqrt{15}}{4}, -3\right)$, $\left(\frac{1}{4}, 0, -\frac{\sqrt{15}}{4}, -3\right)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, -2\right)$, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, -2\right)$.

Example

We will check whether the stationary points we found are maxima or minima of f.

$L_1 = 2x_1 + 1 + 2\lambda x_1$, $L_2 = 4x_2 + 2\lambda x_2$, $L_3 = 6x_3 + 2\lambda x_3$, $L_{11} = 2 + 2\lambda$, $L_{12} = 0$, $L_{13} = 0$, $L_{21} = 0$, $L_{22} = 4 + 2\lambda$, $L_{23} = 0$, $L_{31} = 0$, $L_{32} = 0$, $L_{33} = 6 + 2\lambda$, $g_1 = 2x_1$, $g_2 = 2x_2$, $g_3 = 2x_3$.

We calculate the successive major minors of the determinant $|H^*|$:

$$\begin{vmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = \begin{vmatrix} 2+2\lambda & 0 & 2x_1 \\ 0 & 4+2\lambda & 2x_2 \\ 2x_1 & 2x_2 & 0 \end{vmatrix} = -4x_1^2(2\lambda+4) - 4x_2^2(2\lambda+2), \text{ and}$$

$$\begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22} & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = \begin{vmatrix} 2+2\lambda & 0 & 0 & 2x_1 \\ 0 & 4+2\lambda & 0 & 2x_2 \\ 0 & 0 & 6+2\lambda & 2x_3 \\ 2x_1 & 2x_2 & 2x_3 & 0 \end{vmatrix} =$$

$$-16\left(x_3^2(\lambda+2)(\lambda+1) + (\lambda+3)\left(x_1^2(\lambda+2) + x_2^2(\lambda+1)\right)\right)$$

Example

For each point separately we have for its successive major and minor determining $|H_*|$:

$$\left(1,0,0,-\tfrac{3}{2}\right):\ \begin{vmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = -4 < 0\ ,\quad \begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22} & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = -12 < 0$$

$$\left(-1,0,0,-\tfrac{1}{2}\right):\ \begin{vmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = -12 < 0,\quad \begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22} & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = -60 < 0$$

Example

For each stationary point separately we have for the successive major minors of the determinant $|H_*|$:

$$\left(\frac{1}{4}, 0, \frac{\sqrt{15}}{4}, -3\right):\ \begin{vmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = \frac{1}{2} > 0,\ \begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22} & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = -30 < 0$$

$$\left(\frac{1}{4}, 0, -\frac{\sqrt{15}}{4}, -3\right):\ \begin{vmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = \frac{1}{2} > 0,\ \begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22} & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = -30 < 0$$

Example

For each point separately we have for the successive major minors of the determinant $|H_*|$:

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, -2\right): \begin{vmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = 6 > 0, \quad \begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22} & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = 12 > 0$$

$$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, -2\right): \begin{vmatrix} L_{11} & L_{12} & g_1 \\ L_{21} & L_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = 6 > 0, \quad \begin{vmatrix} L_{11} & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22} & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = 12 > 0$$

So the points $\left(1, 0, 0, -\frac{3}{2}\right)$ and $\left(-1, 0, 0, -\frac{1}{2}\right)$ are local minima, while the $\left(\frac{1}{4}, 0, \frac{\sqrt{15}}{4}, -3\right)$ and $\left(\frac{1}{4}, 0, -\frac{\sqrt{15}}{4}, -3\right)$ local maxima.