Mathematical analysis
Lecture 5

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Topics of the 5th lecture

- Monotonicity of functions, stationary and critical points
- Functions of several variables
- Partial derivatives of the first and second order
- Hessian matrix
- SeriesTaylor of multivariable functions

Monotonicity of a function

Let be a function $f(x)$. This is called:

Genuinely increasing if for any x_1, x_2 in its scope of definition: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Genuinely decreasing if for any x_1, x_2 in its scope of definition: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Ascending if for any x_1, x_2 in its scope of definition: $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

Descending if for any x_1, x_2 in its scope of definition: $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

Monotonicity of a function

If the function $f(x)$ is continuous, then:

- In the intervals of the domain of definition where $f(x) > 0$, the function is truly ascending
- In the intervals of the domain of definition where $f(x) < 0$, the function is genuinely declining

> Find the monotonic intervals of the function $f(x) = 2x^3 + 3x^2 - 12x + 1$.

The function is continuous (why?) so we calculate its first derivative: $f(x) = 2 \cdot 3x^2 + 3 \cdot 2x - 12 = 6x^2 + 6x - 12 = 6(x+2)(x-1)$

x	$-\infty$		-2		1		$+\infty$
$f(x)$		$+$	0	$-$	0	$+$	
$f(x)$		$\nearrow$		$\searrow$		$\nearrow$	

Maximum of a function

At one point $\hat{x}$ total maximum applies:

$$f(\hat{x}) \geq f(x) \text{ for all } x \text{ in its scope,}$$

while at one point x_* local maximum, it is valid:

$$f(x_*) \geq f(x), x_* - \epsilon \leq x \leq x_* + \epsilon, \ \epsilon > 0$$

for the x located in a space, possibly very small, around the x_*.

If the function has a local maximum at the point x_* then for f derivable, will the first derivative of the function must be zero at $x = x_*$, that is

$$f(x_*) = 0$$

First-order condition

We call this condition a first-order condition. To understand why it must hold, we find its differential $y = f(x)$ in x_*:

$$day = f(x) dx$$

If the function exhibits a local maximum at x_* it must be impossible to increase its price with small changes dx in one direction or the other of x_* This could not be true if $f(x_*) \neq 0$. Why if $f(x_*) > 0$, then choosing $dx > 0$, we get $dy > 0$ and the value of the function increases. If $f(x_*) < 0$, then if we choose $dx < 0$ again we get $dy > 0$ and the value increases of the function. Only if $f(x_*) = 0$, any one $dx \neq 0$ gives $day = 0$, so that the function cannot be increased.

Necessary condition

TheoremFermat

If the differentiable function f takes an extreme value at a point x_*, then $f(x_*) = 0$.

The first-order condition $f(x_*) = 0$ is a necessary condition to give the x_* an extreme value in the function. This condition is not sufficient for a extreme value, since there is another group of points, the so-called points inflection point, where the derivative can be zeroed, that is $f(x_*) = 0$. For example for the function

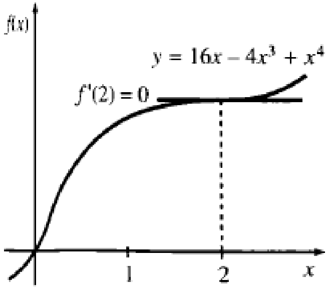
$$y = 16x - 4x^3 + x^4$$

We have:

$$\frac{day}{dx} = 16 - 12x^2 + 4x^3 = f(x)$$

and for $x = 2$ we get $f(2) = 0$.

Graph



Shape: Function with inflection point

The x=2 does not lead to an extreme value of the function. It happens that the tangent of the function at this point to be horizontal (y=16).

Stagnant and critical points

For a differentiable function f, the point x_* where $f(x_*) = 0$, char- is considered a stationary value of the function. At such stationary values the function may exhibit extreme values or inflection points. Every extreme value of a function occurs at a stationary value, but we do not necessarily have an extreme value at every stationary value.

Critical points of a function are called the stationary points of the system. of the curve together with the points for which their derivative is not defined.

Second-Class Conditions

If the $f(x)$ is strictly concave in the region of x_* and twice derivative, then the curvature of the function is negative or otherwise $f''(x_*) < 0$ as it turns the hollows down.

> If $f'(x_*) = 0$ and $f''(x_*) < 0$, then the f exhibits a local maximum at x_*.

If the $f(x)$ is strictly convex in the region of x_* and twice derivative, then the curvature of the function is positive or otherwise $f''(x_*) > 0$ as well turns the concaves upwards.

> If $f'(x_*) = 0$ and $f''(x_*) > 0$, then the f exhibits a local minimum at x_*.

In the case of an inflection point the curvature of the function changes from concave to convex or vice versa. In this case $f''(x_*) = 0$.

Second-Class Conditions

Second-order conditions are sufficient to give a local extremum but not necessary. That is, we can have a local extremum and still have $f''(x_*) = 0$.

For example, the function $f(x) = -x^4$ shows a maximum at $x_* = 0$, but

$$f''(0) = -12(0)^2 = 0$$

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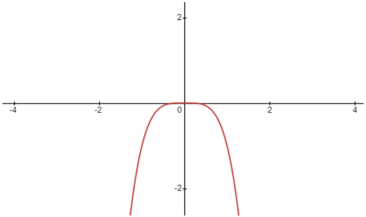
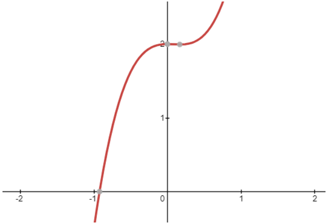


Example

To study the function in terms of extremes:

$$f(x) = y = 2x^3 - 0.5x^2 + 2$$

We have $\frac{dy}{dx} = 6x^2 - x$, therefore the stationary points are $x = 0$ and the $x = 1/6$.
$\frac{d^2y}{dx^2} = 12x - 1$.
In $x = 0$, $f''(x) = -1 < 0$, therefore we have a local maximum of the function. In $x = 1/6$, $f''(x) = 1 > 0$ therefore we have a local minimum of the function.



Exercise

Draw the graph of the function. $f(x) = x^4 - 2x^2$, finding its intervals of monotonicity, convexity, extremes and inflection points.

Solution

First we find where the function intersects the x-axis. $\sqrt{x}$:
$f(x)=0 \Rightarrow x^4-2x^2=0 \Rightarrow x^2(x^2-2)=0 \Rightarrow x=0, x=\sqrt{2}, x=-\sqrt{2}$.
Furthermore, we observe that the function is even, that is, it holds $f(-x)=f(x),\ \forall x \in \mathbb{R}$, which means that its graph is symmetric about the x-axis. y.

We calculate the roots of the first derivative:
$f'(x)=4x^3-4x=0 \Rightarrow 4x(x^2-1)=0 \Rightarrow x=0, x=1, x=-1$. For intervals monotonicity we construct the table:

x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$		$\searrow$		$\nearrow$		$\searrow$		$\nearrow$	

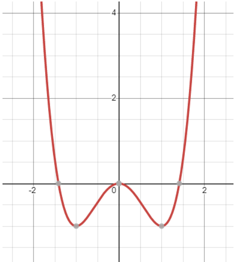
So the f has a local maximum at the point $x=0$ (with $f(0)=0$), and local minima at the points $x=-1$ and $x=1$ (with $f(1)=f(-1)=-1$).

Solution

We find where the second derivative vanishes: $f''(x) = 0 \Rightarrow 12x^2 - 4 = 0 \Rightarrow x = \frac{1}{\sqrt{3}}$ and $x = -\frac{1}{\sqrt{3}}$.

These points are turning points, so we expand the monotonicity matrix:

x	$-\infty$		-1		$-\frac{1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$		1		$+\infty$
$f''(x)$		$+$		$+$	0	$-$		$-$	0	$+$		$+$	
$f'(x)$		$-$	0	$+$		$+$	0	$-$		$-$	0	$+$	
$f(x)$													



Second-order conditions and series expansionTaylor

- When $f'(x_*) = 0$ is a useful method to study what happens to to formulate the second-order conditions is to take the expansion of the seriesTaylor.

- Suppose that $\hat{x}$ is any one x which belongs to a small space around from the x_* Then the expansion of the seriesTaylor in the form of our remainder gives:

$$f(\hat{x}) = f(x_*) + \frac{f'(x_*)(\hat{x} - x_*)}{1!} + \frac{f''(g)(\hat{x} - x_*)^2}{2!}$$

for some point g which is located between x_* and the $\hat{x}$.

- Because $f'(x_*) = 0$ if $f'(g) < 0$ then $f(\hat{x}) - f(x_*) < 0$ or $f(x_*) > f(\hat{x})$ and therefore the x_* gives a local maximum.
- If $f'(g) > 0$ then $f(\hat{x}) - f(x_*) > 0$ or $f(x_*) < f(\hat{x})$ and therefore the x_* gives a local minimum.

Second-order conditions and series expansionTaylor

- If $f''(g) = 0$ the next term in the series of the sequence is $f'''(x_*)(\hat{x} - x_*)^3/3!$ and this cannot lead us to a conclusion since the sign of $(\hat{x} - x_*)^3$ it can be positive or negative.
- If we assume that $f'''(x_*) = 0$ and this is true for all derivatives up to the derivative $(n-1)$-bone order which we denote by $f^{(n-1)}(x)$ then:

  $$f(\hat{x}) = f(x_*) + \frac{f^{(n)}(g)(\hat{x} - x_*)^n}{n!}$$

- If the n is even then $(\hat{x} - x_*)^n > 0$. Therefore, if $f^{(n)}(g) < 0$ then $f(x_*) > f(\hat{x})$ and therefore the x_* gives a local maximum. If $f^{(n)}(g) > 0$ then $f(x_*) < f(\hat{x})$ and therefore the x_* gives a local minimum. This is called Criterion n-bone derivative.

Example

We consider the function $f(x) = 10 - x^4$ We have:

$$f'(x) = -4x^3$$

$$f''(x) = -12x^2$$

$$f'''(x) = -24x$$

$$f^{(4)}(x) = -24$$

All derivatives up to the third derivative are zeroed at $x^* = 0$. This means that we must use the fourth derivative as the last term in the expansion of the series Taylor to explore her character function around the $x^* = 0$:

$$f(\hat{x}) = f(0) + \frac{f^{(4)}(g)(\hat{x})^4}{24}$$

Example

Given that $f^{(4)}(x) = -24$ for all x, we can formulate this as follows:

$f(x) = f(0) - x^4$ or $f(x) - f(0) < -x^4$
For any value of $x \neq 0$ we have $-x^4 < 0$ and therefore $f(x) - f(0) < 0$ or $f(0) > f(x)$. That is, the point $x_* = 0$ gives a local maximum of this function.

- In general, if all derivatives at a point up to one derivative are odd order is zero, while the next even order derivative is negative, then we have a local maximum at this point, while if it is positive then we have a local minimum at this point.

Example 2

We consider the function $f(x) = x^3$ We have:

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$f'''(x) = 6$$

With the $f'''(x)$ to be the first of the higher-order derivatives that are not is reset to zero x_* we must use the third derivative as the last term of the series expansionTaylor to investigate the character of the function around the $x_*=0$. That is,

$$f(\hat{x}) = f(0) + \frac{f'''(g)(\hat{x})^3}{6}$$

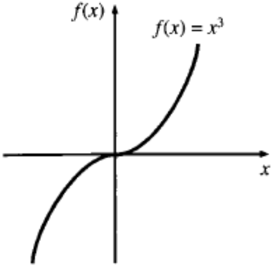
for g between x_* and the $\hat{x}$.

Example 2

Because $f'''(x) = 6$ for all x we can write this equation as:

$$f(\hat{x}) = f(0) + \hat{x}^3$$

For $\hat{x} > 0$ we get $f(\hat{x}) > f(0)$ while for $\hat{x} < 0$ we get $f(\hat{x}) < f(0)$. Therefore the $x_* = 0$ gives neither a maximum nor a minimum. It is a turning point, as shown in the diagram.



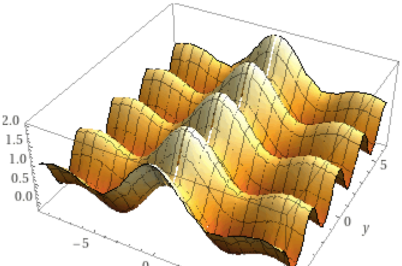
Shape: Graphic representation of $f(x) = x^3$

Functions of several variables

Function (scalar) n A correspondence that depicts variables is called every n -ada $(x_1, x_2, \ldots .x_n)$ of $\mathbb{R}^n$ (or any point of n-dimensional space) to a real number. The domain of such a function is a subset of $\mathbb{R}^n$.

$$f : X \to \mathbb{R}, \text{where } X \subseteq \mathbb{R}^n$$

Example: The function $f : \mathbb{R}_* \times \mathbb{R} \to \mathbb{R}$ with $f(x, y) = \frac{\sin(x) + \cos^2(y)}{x}$



Equivalent sets

Equilibrium set (level set) of the function $y = f(x_1, x_2, \cdots, x_n)$ is the total

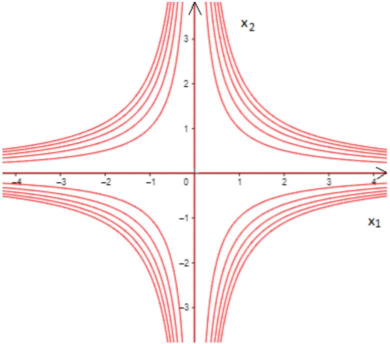
$$L = \{(x_1, \cdots, x_n)\} \in \mathbb{R}^n : f(x_1, x_2, \cdots, x_n) = c\}$$

for a given number $c \in \mathbb{R}$.

Example of Equivalent Sets

Suppose we want to find the equivalence sets of the function $y = f(x_1, x_2) = x_1^2 x_2^2$.

We set $x_1^2 x_2^2 = c \Longleftrightarrow x_2 = \pm\sqrt{\frac{c}{x_1^2}}$ The equivalence sets of the function are illustrated in the diagram below:



Shape: Equivalence sets of the function $y = x_1^2 x_2^2$

Applications of Equilibrium Curves

In Economics, we encounter equilibrium sets:

- in consumer theory (where they are referred to as indifference curves), where any point on the same curve has the same utility.
- in producer theory (where they are referred to as curves isoproduction), where any point on the same curve corresponds to the same level of production.

Derivation of multivariable functions

The partial derivative of a function $y = f(x_1, x_2, \cdots, x_n)$ (can be written- eat and as $y = f(x)$ where $x \in \mathbb{R}^n$) with respect to the variable x_I is:

$$\frac{\partial f}{\partial x_I} = \lim_{\Delta x_I \to 0} \frac{f(x_1, \cdots, x_I + \Delta x_I, \cdots, x_n) - f(x_1, \cdots, x_I, \cdots, x_n)}{\Delta x_I}$$

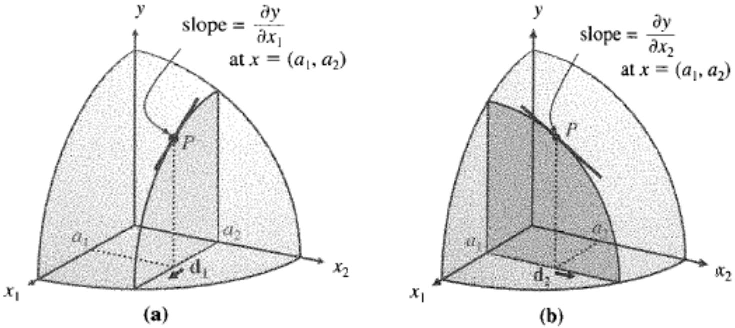
The symbols are used interchangeably $\frac{\partial f}{\partial x_I}$ or $f_I(\mathrm{x})$ or simply f_I.

For a function $z = f(x, y)$, the partial derivatives with respect to x and y is:

$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$ and $\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta x) - f(x,y)}{\Delta x}$

Geometric interpretation

The partial derivative expresses the rate of change of a function with respect to one variable, when all other variables remain constant.



Shape:Some Derivatives

Derivation of multivariable functions

Instead of calculating partial derivatives based on the principles introduced by their definition, we can find them using the rules of differentiation, as we did for univariate functions. Because when we calculate the $\frac{\partial f}{\partial x_I}$ we hold all variables constant except for x_I we can consider all the terms of the function f(x) which does not depend on the x_I as a constant c and then use- with the rules of derivation for functions of one variable.

Properties

If there are partial derivatives with respect to the variable x of functions f, g, then the following applies:

- $\frac{\partial(f \pm g)}{\partial x} = \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}$
- $\frac{\partial(\lambda f)}{\partial x} = \lambda \frac{\partial f}{\partial x}, \lambda \in \mathbb{R}$
- $\frac{\partial(fg)}{\partial x} = g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}$
- $\frac{\partial(f/g)}{\partial x} = \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}$

Example

Let it be $y = f(x_1, x_2) = x_1^2 x_2$ The variable x_2 is held constant when calculating the $\frac{\partial f}{\partial x_1}$ If we put $x_2 = c$, where c a constant, then the function takes the form:

$$y = cx_1^2$$

and therefore:

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{d[cx_1^2]}{dx_1} = 2cx_1$$

We replace the c with x_2 and we have

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = 2x_1x_2$$

Exercise

If $f(x, y, z) = 3x^2y^3 - 2xyz^2 + 4y^4 + z^2$, calculate the partial derivatives of f as for x, y and z.

Solution

After $f(x, y, z) = 3x^2y^3 - 2xyz^2 + 4y^4 + z^2$, then we have:

$$\frac{\partial f}{\partial x} = 6xyz^3 - 2y^2$$

$$\frac{\partial f}{\partial y} = 9x^2y^2 - 4xyz + 16y^3$$

$$\frac{\partial f}{\partial z} = 2z$$

Partial second-order derivatives

Each second derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ is denoted as:

$$f_{ij} = \frac{\partial f_i(x)}{\partial x_j}, \quad i, j = 1, 2, \cdots, n$$

where the f_{ij} is found by first differentiating the function $f(x)$ as to the variable x_i and then deriving the result $f_i(x)$ as to the variable x_j.

Partial second-order derivatives

For a function $f(x, y)$, the second-order partial derivatives are the following:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yyy} = f_{y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xyz} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The last two are called mixed or cross derivatives.

Example

Let it be $f(x, y, z) = x^2y^3 + x^4y + cary$ Then the 2nd order partial derivatives are:

1. $\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}(2xyz^3 + 4x^3y + e^y) = 2y^3 + 12x^2y$
2. $\frac{\partial^2 f}{\partial y^2} = f_{yyy} = \frac{\partial}{\partial y}(3x^2y^2 + x^4 + cary) = 6x^2y + cary$
3. $\frac{\partial^2 f}{\partial x \partial y} = f_x = \frac{\partial}{\partial x}(3x^2y^2 + x^4 + cary) = 6xyz^2 + 4x^3 + e^y$
4. $\frac{\partial^2 f}{\partial y \partial x} = f_{xyz} = \frac{\partial}{\partial y}(2xyz^3 + 4x^3y + e^y) = 6xyz^2 + 4x^3 + e^y$

Mixed derivatives

Two mixed derivatives of order $k \geq 2$ of a function $f(x, y)$ with a domain of definition one open set are equal if:

- All partial derivatives up to order k are continuous
- If the total number of derivatives with respect to each variable is the same in both mixed derivatives

Exercise: Let the function $f(x, y) = \begin{cases} \frac{xyz(x^2-y^2)}{x^2+y^2}, & \text{for } (x, y) \neq (0,0) \\ 0 & \text{for } (x, y) = (0,0) \end{cases}$.

Calculate and compare the $f_x(0,0)$ and $f_{xyz}(0,0)$.

Mixed derivatives

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h\cdot 0\,\frac{h^2-0}{h^2+0} - 0}{h} = \lim_{h\to 0} 0 = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0\cdot h\,\frac{0-h^2}{0+h^2} - 0}{h} = \lim_{h\to 0} 0 = 0$$

$$f_x(x,y) = y\frac{x^2-y^2}{x^2+y^2} + xy\frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2} = \frac{y(x^4-y^4)}{(x^2+y^2)^2} + xy\frac{4xy^2}{(x^2+y^2)^2} = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

$$f_y(x,y) = x\frac{x^2-y^2}{x^2+y^2} + xy\frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{x(x^4-y^4)}{(x^2+y^2)^2} - \frac{4x^3y^2}{(x^2+y^2)^2} = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{\frac{-h^5}{h^4} - 0}{h} = -1$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^5}{h^4} - 0}{h} = 1$$

Gradient vector

It is common practice to place the first-order partial derivatives together in a column vector or a row vector called the gradient vector using the following symbols:

$$\nabla f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \text{ or } \nabla f^T = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}$$

Example

> Find the gradient vector for the function $f(x) = 5 - 2x_1 + 3x_2$.

The first partial derivatives of the function are $f_1 = -2$ and $f_2 = 3$. Therefore the gradient vector is:

$$\nabla f = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Hessian matrix

To better track second-order derivatives, it is preferable to use a matrix. For a function of two variables $y = f(x_1, x_2)$ there are 4 second-order partial derivatives:

$f_{11} \equiv \frac{\partial f_1(x_1,x_2)}{\partial x_1}$, $f_{12} \equiv \frac{\partial f_1(x_1,x_2)}{\partial x_2}$, $f_{21} \equiv \frac{\partial f_2(x_1,x_2)}{\partial x_1}$, $f_{22} \equiv \frac{\partial f_2(x_1,x_2)}{\partial x_2}$

We can present these in matrix form as follows:

$$\nabla^2 F \equiv \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

A function of three variables $y = f(x_1, x_2, x_3)$ has 9 second order derivatives and so on. We can present these in matrix form as follows:

$$\nabla^2 F \equiv \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

Hessian matrix

$$\nabla^2 F \equiv \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}.$$

generally for functions n variables.
For the case with the n variables we can more simply write the matrix with the second-order partial derivatives as follows:

$$\nabla^2 F \equiv [f_{ij}]$$

with the class element i, j of the uterus $\nabla^2 F$ to represent the result of derivative of the function $f(x)$ first with respect to the variable x_i and then as to the variable x_j.

The matrix with the second-order partial derivatives of the function f is called Hessian matrix.

Example

Find the first and second order partial derivatives of the function s $f(x_1, x_2) = x_1^2 x_2$ and be presented in vector/matrix form respectively.

The first-order partial derivatives are:

$$f_1 = 2x_1 x_2, \quad f_2 = x_1^2$$

While the second-order partial derivatives are:

$$f_{11} = 2x_2, \quad f_{12} = 2x_1, \quad f_{21} = 2x_1, \quad f_{22} = 0$$

Placing these derivatives in a vector and matrix respectively, we have:

$$\nabla f = \begin{bmatrix} 2x_1 x_2 \\ x_1^2 \end{bmatrix} \quad \text{and} \quad \nabla^2 F = \begin{bmatrix} 2x_2 & 2x_1 \\ 2x_1 & 0 \end{bmatrix}$$

SeriesTaylor multivariate functions

For the seriesTaylor between points $x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix}$ Mix. $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$ for function of two variables we have:

$$dx = \hat{x} - x^{(0)} \begin{bmatrix} (\hat{x}_1 - x_1^{(0)}) \\ (\hat{x}_2 - x_2^{(0)}) \end{bmatrix}$$

and therefore for the first-order total differential we get:

$$day(x^{(0)}) = \begin{bmatrix} f_1(x_1^{(0)}, x_2^{(0)}), f_2(x_1^{(0)}, x_2^{(0)}) \end{bmatrix} \begin{bmatrix} (\hat{x}_1 - x_1^{(0)}) \\ (\hat{x}_2 - x_2^{(0)}) \end{bmatrix}$$

For the second-order total differential we obtain using the notation $H \equiv \nabla^2 F$

$$d^2 y(x_1, \xi_2) = dx^T H dx =$$

$$\begin{bmatrix} (\hat{x}_1 - x_1^{(0)}), (\hat{x}_2 - x_2^{(0)}) \end{bmatrix} \begin{bmatrix} f_{11}(x_1, \xi_2) & f_{12}(x_1, \xi_2) \\ f_{21}(x_1, \xi_2) & f_{22}(x_1, \xi_2) \end{bmatrix} \times \begin{bmatrix} (\hat{x}_1 - x_1^{(0)}) \\ (\hat{x}_2 - x_2^{(0)}) \end{bmatrix}$$

SeriesTaylor multivariable function

The remainder formula for the expansion of the seriesTaylor of a company of this $y = f(\mathrm{x})$ defined in $\mathbb{R}^n$, which develops around the point $\mathrm{x}_{(0)}$ and includes two terms:

$$f(\hat{\mathrm{x}}) = f(\mathrm{x}_{(0)}) + d\,y(\mathrm{x}_{(0)}) + \frac{1}{2} d^2 y(x)$$

where the x is between $\mathrm{x}_{(0)}$ and $\hat{\mathrm{x}}$.

Necessary and Sufficient Conditions for Optimal Multivariate Function

A sufficient condition to give the x_* a local optimum of twice differentiable function $y = f(x)$ is:

$$f_I(x_*) = 0, I = 1, \cdots, n \text{(necessary)}$$

and

$$\forall \delta \in R^n,\ d \neq 0,\ d^T H d > 0 \text{ for local minimum or } < 0 \text{ for local maximum}$$

That is, the Hessian uterus H is positive definite for a local minimum or negative defined for a local maximum.