ΕΛΛΗΝΙΚΗ ΔΗΜΟΚΡΑΤΙΑ
ΠΑΝΕΠΙΣΤΗΜΙΟ ΜΑΚΕΔΟΝΙΑΣ

Mathematical analysis
Lecture 12

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Topics of the 12th lecture

- Systems of first-order linear differential equations
- Homogeneous solution
- Complete solution
- Direct method
- Stability and phase diagrams

Systems of linear differential equations

Definition: A system of two autonomous linear differential equations first-order is defined as:

$$\begin{cases} \dot{y}_1 = a_{11} y_1 + a_{12} y_2 + b_1 \\ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 + b_2 \end{cases}, \quad a_{ij}, b_I \in \mathbb{R}$$

We divide the problem of finding the complete solutions into two parts. First we find the homogeneous solutions and then the partial solutions. The complete solutions are the sum of the homogeneous and the partial solutions. That is:

$$y_1 = y_{h\,1} + y_{p\,1}$$

$$y_2 = y_{h\,2} + y_{p\,2}$$

where y_I is the complete solution, $y_{h\,I}$ is the general homogeneous solution of y_I and $y_{p\,I}$ is the partial solution of y_I.

The general solution to homogeneous forms

Definition: The homogeneous form of the system of two linear differential equations first order equations are:

$$\begin{cases} \dot{y}_1 = a_{11} y_1 + a_{12} y_2 \\ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 \end{cases}$$

It is possible to transform this system of two first-order differential equations into a single second-order differential equation using a combination of derivatives and substitutions.

The general solution to homogeneous forms

We derive the first equation and we have:

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}\dot{y}_2$$

We use the second equation to replace the $\dot{y}_2$ This gives us:

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}(a_{21}y_1 + a_{22}y_2)$$

We solve the first equation in terms of y_2:

$$y_2 = \frac{\dot{y}_1 - a_{11}y_1}{a_{12}}$$

assuming that $a_{12} \neq 0$. Substituting this expression for y_2 we have:

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}\left(a_{21}y_1 + a_{22}\frac{\dot{y}_1 - a_{11}y_1}{a_{12}}\right)$$

By simplifications and rearranging the terms we get:

$$\ddot{y}_1 - (a_{11} + a_{22})\dot{y}_1 + (a_{11}a_{22} - a_{12}a_{21})y_1 = 0$$

Which is a linear homogeneous second-order differential equation with constant coefficients.

We find the solutions for y_1 and the solutions for y_2 arise from the $y_2 = \frac{\dot{y}_1 - a_{11}y_1}{a_{12}}$.

Example

Solve the following system of homogeneous differential equations:

$$\begin{cases} \dot{y}_1 = y_1 - 3y_2 \\ \dot{y}_2 = \frac{1}{4}y_1 + 3y_2 \end{cases}$$

We derive the first equation and we have:

$$\ddot{y}_1 = \dot{y}_1 - 3\dot{y}_2$$

We use the second equation to replace the $\dot{y}_2$ This is our

gives:

$$\ddot{y}_1 = \dot{y}_1 - 3\left(\frac{1}{4}y_1 + 3y_2\right)$$

We use the first equation to get an expression for y_2:

$$y_2 = \frac{y_1 - \dot{y}_1}{3}$$

Example

Therefore:

$$\ddot{y}_1 = \dot{y}_1 - 3\left(\frac{1}{4}y_1 + 3\,\frac{y_1 - \dot{y}_1}{3}\right)$$

Equivalents:

$$\ddot{y}_1 - 4\dot{y}_1 + \frac{15}{4}y_1 = 0$$

The characteristic equation is:

$$r^2 - 4r + \frac{15}{4} = 0$$

The distinguishing feature is $\Delta = 1$ and the roots $r_{1,2} = \frac{4 \pm 1}{2}$, that is $r_1 = 3/2$, $r_2 = 5/2$.
Therefore, the solution for y_1 is:

$$y_1(t) = C_1 e^{3t/2} + C_2 e^{5t/2}$$

Example

To find the y_2 based on $y_2 = \frac{y_1 - \dot{y}_1}{3}$, we calculate the $\dot{y}_1$:

$$\dot{y}_1(t) = \frac{3}{2} C_1 e^{3t/2} + \frac{5}{2} C_2 e^{5t/2}$$

Based on the $y_2 = \frac{y_1 - \dot{y}_1}{3}$ we have:

$$y_2(t) = \frac{1}{3}\left(C_1(1 - 3/2) e^{3t/2} + C_2(1 - 5/2) e^{5t/2} \right)$$

$$= -\frac{1}{6} C_1 e^{3t/2} - \frac{1}{2} C_2 e^{5t/2}$$

The steady state equilibrium solution

Definition: The steady-state equilibrium solution of a system differential equations is the pair of values $\bar{y}_1$ and $\bar{y}_2$ where the $\dot{y}_1$ and $\dot{y}_2$ it is equal to zero.

The complete solutions - Example

Find the complete solution to the following system of differential equations:

$$\begin{cases} \dot{y}_1 = y_1 - 3y_2 - 5 \\ \dot{y}_2 = \frac{1}{4}y_1 + 3y_2 - 5 \end{cases}$$

First we formulate the system in its homogeneous form:

$$\begin{cases} \dot{y}_1 = y_1 - 3y_2 \\ \dot{y}_2 = \frac{1}{4}y_1 + 3y_2 \end{cases}$$

We previously found that:

$$y_{1h}(t) = C_1 e^{3t/2} + C_2 e^{5t/2}$$

$$y_{2h}(t) = -\frac{1}{6}C_1 e^{3t/2} - \frac{1}{2}C_2 e^{5t/2}$$

The complete solutions - Example

To find the equilibrium solutions we set $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$. Therefore we have:

$$\bar{y}_1 - 3\bar{y}_2 - 5 = 0$$

$$\frac{1}{4}\bar{y}_1 + 3\bar{y}_2 - 5 = 0$$

Therefore $\bar{y}_1 = 3\bar{y}_2 + 5$ and therefore:

$$\frac{1}{4}(3\bar{y}_2 + 5) + 3\bar{y}_2 - 5 = 0 \Leftrightarrow$$

$$\frac{15}{4}\bar{y}_2 = \frac{15}{4} \Leftrightarrow$$

$$\bar{y}_2 = 1$$

and $\bar{y}_1 = 8$.

The complete solutions - Example

Thus, the complete solutions are:

$$y_1(t) = C_1 e^{3t/2} + C_2 e^{5t/2} + 8$$

$$y_2(t) = -\frac{1}{6} C_1 e^{3t/2} - \frac{1}{2} C_2 e^{5t/2} + 1$$

The complete solutions - Example

Find the integration constants in the previous example so that the solutions satisfy the initial conditions $y_1(0) = 1$ and $y_2(0) = 3$.

Substituting we have:

$$1 = C_1 + C_2 + 8$$

$$3 = -\frac{1}{6}C_1 - \frac{1}{2}C_2 + 1$$

We have $C_1 = 1 - C_2 - 8$ and replacing:

$$3 = -\frac{1}{6}(1 - C_2 - 8) - \frac{1}{2}C_2 + 1 \Leftrightarrow$$

$$3 = \frac{1}{6}(C_2 + 7) - \frac{1}{2}C_2 + 1 \Leftrightarrow$$

$$\frac{5}{6} = -\frac{2}{6}C_2 \Leftrightarrow C_2 = -\frac{5}{2}$$

and $C_1 = -9/2$.

The complete solutions - Example

Therefore, the complete solutions are given by the relations:

$$y_1(t) = -\frac{9}{2}e^{3t/2} - \frac{5}{2}e^{5t/2} + 8$$

$$y_2(t) = \frac{9}{12}e^{3t/2} + \frac{5}{4}e^{5t/2} + 1$$

$$= \frac{3}{4}e^{3t/2} + \frac{5}{4}e^{5t/2} + 1$$

The direct method

Definition: A linear system with n autonomous differential equations is expressed in matrix form as follows:

$$\dot{y} = Ay + b$$

where A it is a uterus $n \times n$ fixed rates b is a vector fixed terms y is a vector n variables and $\dot{y}$ is a vector n derivatives.

The solution to the complete system of equations is obtained by summing the homogeneous solutions and the partial solutions. We begin by writing the complete system in its homogeneous form:

$$\dot{y} = Ay$$

The direct method

We continue by 'assuming' that the homogeneous solutions are of the form:

$$\mathbf{y} = \mathbf{k}\, e^{rt}$$

where $\mathbf{k}$ is a vector n dimensions with constants and r a scalar. To To determine if this assumption we made is correct, we check whether the conjectured solution and its first derivative satisfy the system of differential equations. The derivative of the solution we are testing is:

$$\dot{\mathbf{y}} = r\mathbf{k}\, e^{rt}$$

By substituting these derivatives and the assumed solutions into the original system of equations we obtain:

$$r\mathbf{k}\, e^{rt} = A\mathbf{k}\, e^{rt}$$

equivalents:

$$(A - rI)\mathbf{k} = \mathbf{0}$$

where I is the unit matrix and $\mathbf{0}$ is the zero vector.

The direct method

The previous system has a non-zero solution if and only if the determinant of the matrix $[A - rI]$ is equal to zero. Therefore the values of the solutions for r are obtained by solving:

$$|A - rI| = 0$$

which is a polynomial equation of degree n towards the unknown r This is known as the characteristic equation of the matrix. A and its solutions are called characteristic roots or eigenvalues of the matrix A. A non-zero vector k_1 the which is a solution of the equation $[A - rI]k = 0$ for a specific eigenvalue r_1 is called the eigenvector of the matrix A that corresponds to the eigenvalue r_1.

Example

Solve the following $2x2$ system of differential equations using the direct of the method:

$$\dot{y} = \begin{bmatrix} 4 & -1 \\ -4 & 4 \end{bmatrix} y$$

The characteristic equation is:

$$|A - rI| = \begin{vmatrix} 4-r & -1 \\ -4 & 4-r \end{vmatrix} = 0$$

which is done: $(4-r)^2 - 4 = 0 \Longleftrightarrow 16 + r^2 - 8r - 4 = 0 \Longleftrightarrow r^2 - 8r + 12 = 0$,
$\Delta = 64 - 48 = 16$, $r_{1,2} = \frac{8 \pm 4}{2}$ therefore $r_1 = 2$ and $r_2 = 6$.

Example

For $r_1 = 2$ we want to calculate non-zero solutions for the eigenvectors:

$$\begin{bmatrix} 4-2 & -1 \\ -4 & 4-2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

This gives us $2k_1 - k_2 = 0$. We set $k_1 = 1$ that gives $k_2 = 2$. Therefore, the first solution set is:

$$\mathbf{y}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

Example

For $r_2 = 6$ eigenvectors are solutions of the equation:

$$\begin{bmatrix} 4-6 & -1 \\ -4 & 4-6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

which gives $-2k_1 - k_2 = 0$. With $k_1 = 1$ we have $k_2 = -2$ Therefore, the second set solutions are:

$$y_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{6t}$$

Since the two solutions are linearly independent, the general solution is:

$$y(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{6t}$$

Theorem

Theorem: If a homogeneous linear system of differential equations has as a root of the characteristic equation a complex number r, then the roots of the system are the vector of the real part and the vector of the imaginary part resulting from the eigenvectors-solutions of $[A - rI]\mathrm{k} = 0$.

Example 2

Solve the homogeneous system of differential equations:

$$\dot{y} = Ay, \text{ where } A = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$$

$$|A - rI| = 0 \Leftrightarrow \begin{vmatrix} 2-r & -5 \\ 2 & -4-r \end{vmatrix} = 0 \Leftrightarrow -8 + 2r + r^2 + 10 = 0 \Leftrightarrow r^2 + 2r + 2 = 0$$

$\Delta = 4 - 8 = -4$, $r_{1,2} = \frac{-2 \pm 2I}{2}$ Therefore $r_1 = -1 + I$ and $r_2 = -1 - I$.

For $r_1 = -1 + I$ (in the case of complex eigenvalues we can choose either of the two) the eigenvectors are the solutions of:

$$\begin{bmatrix} 3-I & -5 \\ 2 & -3-I \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

Example 2

From the first line we have $(3-I)k_1 - 5k_2 = 0$ or $k_2 = (3-I)k_1/5$. Setting $k_1 = 5$ we have $k_2 = 3-I$ Therefore the first solution is:

$$y_1(t) = \begin{bmatrix} 5 \\ 3-I \end{bmatrix} e^{(-1+I)t}$$

So the eigenvector corresponding to $r_1 = -1 + I$ is:

$$k = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -i \end{bmatrix}$$

The resulting complex solution is:

$$y(t) = e^{(-1+I)t} \cdot k = e^{-t}(\cos(t) + I\sin(t)) \left(\begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -i \end{bmatrix} \right) =$$

$$e^{-t}(\cos(t) + I\sin(t)) \left(\begin{bmatrix} 5 \\ 3 \end{bmatrix} + I \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

Example 2

We separate the real and imaginary parts and we have the two basic solutions [of the] system:

$$y^{(1)} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} e^{-t}\cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-t}\sin(t) = e^{-t}\begin{bmatrix} 5\cos(t) \\ 3\cos(t)+\sin(t) \end{bmatrix}$$

$$y^{(2)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-t}\cos(t) + \begin{bmatrix} 5 \\ 3 \end{bmatrix} e^{-t}\sin(t) = e^{-t}\begin{bmatrix} 5\sin(t) \\ -\cos(t)+3\sin(t) \end{bmatrix}$$

Finally, the general solution of the system is:

$$y(t) = C_1 y^{(1)} + C_2 y^{(2)} =$$

$$e^{-t}\left(C_1\begin{bmatrix} 5\cos(t) \\ 3\cos(t)+\sin(t) \end{bmatrix} + C_2\begin{bmatrix} 5\sin(t) \\ -\cos(t)+3\sin(t) \end{bmatrix}\right)$$

The partial solutions

The steady-state equilibrium solutions give us the partial solutions. We set $\dot{y} = 0$ in the complete system of differential equations. This gives us:

$$A\bar{y} + b = 0$$

for which the solution is:

$$\bar{y} = -A^{-1}b$$

provided that the inverse matrix A^{-1} there is.

Example

Let the system of differential equations be:

$$\dot{\mathrm{y}} = A\mathrm{y} + \mathrm{b}, \text{ where } A = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We previously found the homogeneous solution of the system:

$$y_h(t) = e^{-t}\left(C_1 \begin{bmatrix} 5\cos(t) \\ 3\cos(t) + \sin(t) \end{bmatrix} + C_2 \begin{bmatrix} 5\sin(t) \\ -\cos(t) + 3\sin(t) \end{bmatrix}\right)$$

The partial solution is calculated as

$$y_p(t) = \bar{\mathrm{y}} = -A^{-1}\mathrm{b} = -\begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} -2 & 5/2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9/2 \\ 2 \end{bmatrix}.$$

The complete solution is: $y(t) = y_h(t) + y_p(t) =$

$$e^{-t}\left(C_1 \begin{bmatrix} 5\cos(t) \\ 3\cos(t) + \sin(t) \end{bmatrix} + C_2 \begin{bmatrix} 5\sin(t) \\ -\cos(t) + 3\sin(t) \end{bmatrix}\right) + \begin{bmatrix} 9/2 \\ 2 \end{bmatrix}$$

Stability analysis and linear phase diagrams

The steady-state equilibrium solutions for an autonomous system of differential equations are said to be stable if the system converges to these solutions.

> Theorem: The steady-state equilibrium solution of a linear system is of complex, autonomous differential equations is asymptotically stable if and only if the characteristic roots are negative (in the case of complex roots if the real part is negative).

Sagmatic point balance

Theorem: If one of the characteristic roots is positive and the other is negative, the equilibrium state is called a sagmatic equilibrium point. It is unstable. But the $y_1(t)$ and the $y_2(t)$ converge to the solutions their steady state equilibrium if the initial conditions for the y_1 and the y_2 satisfy the following equation:

$$y_2 = \frac{r_1 - a_{11}}{a_{12}}(y_1 - \bar{y}_1) + \bar{y}_2$$

where r_1 is the negative root. The locus of the points (y_1, y_2) which defined by this equation is known as the saddle path.

Phase diagram for two negative roots (Stable node)

Let the differential equations be:

$$\begin{cases} \dot{y}_1 = -2y_1 + 2 \\ \dot{y}_2 = -3y_2 + 6 \end{cases}$$

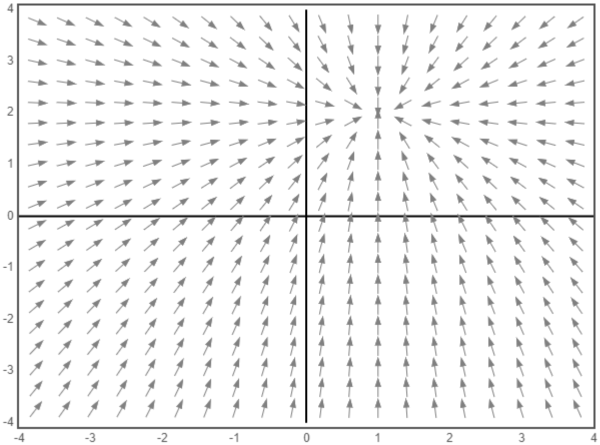
The differential equations are independent of each other and the solutions are

$$y_1(t) = C_1 e^{-2t} + 1$$

$$y_2(t) = C_2 e^{-3t} + 2$$

Phase diagram for two negative roots (Stable node)

The phase diagram is:



Shape:Phase diagram for stable node

Phase diagram for two positive roots (Unstable node)

Let the differential equations be:

$$\begin{cases} \dot{y}_1 = 2y_1 - 2 \\ \dot{y}_2 = 3y_2 - 6 \end{cases}$$

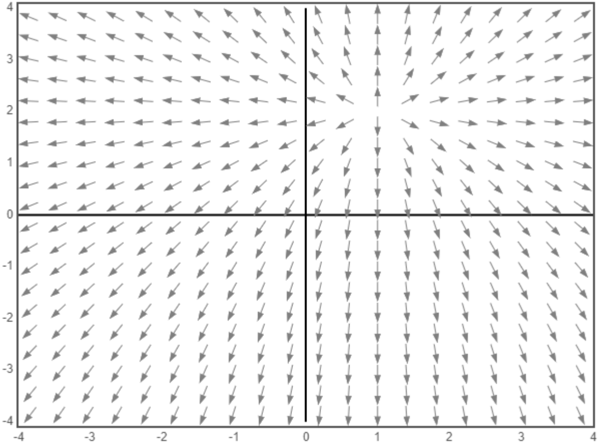
The differential equations are independent of each other and the solutions are

$$y_1(t) = C_1 e^{2t} + 1$$

$$y_2(t) = C_2 e^{3t} + 2$$

Phase diagram for two positive roots (Unstable node)

The phase diagram is:



Shape: Phase diagram for unstable node

Phase diagram for radicals with opposite signs (Sagmatic point)

Let the differential equations be:

$$\begin{cases} \dot{y}_1 = y_2 - 2 \\ \dot{y}_2 = y\,\frac{1}{4} - 2 \quad \frac{1}{} \end{cases}$$

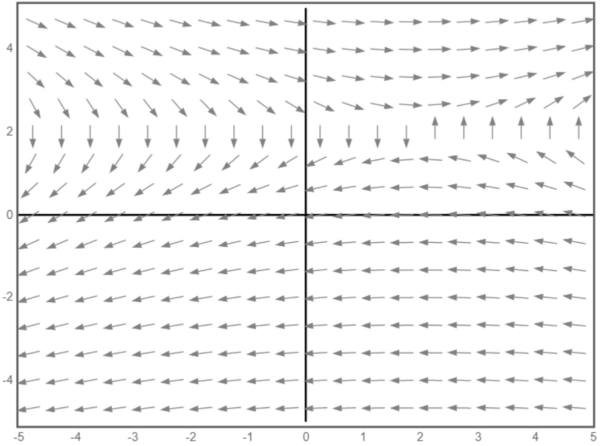
The characteristic equation is:

$$|A - rI| = \begin{vmatrix} 0 - r & 1 \\ 1/4 & 0 - r \end{vmatrix}$$

with roots $r_1 = -1/2$ and $r_2 = 1/2$. Since the roots have opposite signs, the A steady-state solution is a breakpoint equilibrium.

Phase diagram for radicals with opposite signs (Sagmatic point)

The phase diagram is:



Shape:Phase diagram for a critical point

Phase diagram for complex roots with negative real part (stable focus)

Let the differential equations be:

$$\begin{cases} \dot{y}_1 = -y_2 + 2 \\ \dot{y}_2 = y_1 - y_2 + 1 \end{cases}$$

Then:

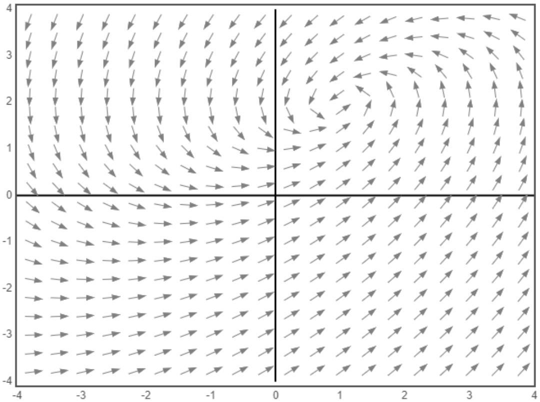
$$|A - rI| = \begin{vmatrix} -r & -1 \\ 1 & -1-r \end{vmatrix} = 0$$

The characteristic equation is $r^2 + r + 1 = 0$, $\Delta = 1 - 4 = -3$. The roots are $r_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} I$.

The steady-state solutions are $\bar{y}_1 = 1$ and $\bar{y}_2 = 2$.

Phase diagram for complex roots with negative real part (stable focus)

The phase diagram is:



Shape:Phase diagram for a stable focus

Phase diagram for complex roots with positive real part (unstable focus)

Let the differential equations be:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -2y_1 + y_2 \end{cases}$$

Then:

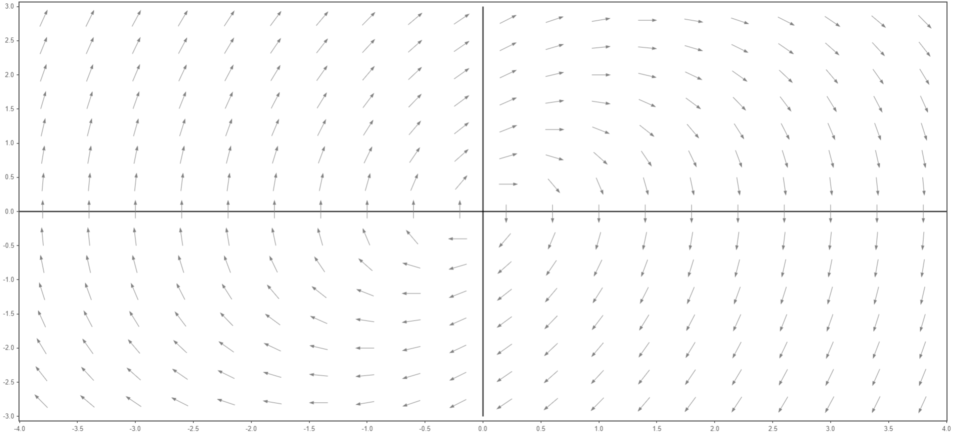
$$|A - rI| = \begin{vmatrix} -r & 1 \\ -2 & 1 - r \end{vmatrix} = 0$$

The character√The heuristic equation is $r^2 - r + 2 = 0$, $\Delta = 1 - 8 = -7$. The roots are $r_{1,2} = \frac{1 \pm \sqrt{7} I}{2}$.

The steady-state solutions are $\bar{y}_1 = 0$ and $\bar{y}_2 = 0$.

Phase diagram for complex roots with positive real part (unstable focus)

The phase diagram is:



Shape:Phase diagram for unstable focus

Types of balance

- If $/A/ < 0$
  - the r_1, r_2 heteroskedastic real then we have a sagmatic point
- If $/A/ > 0$
  - If $r_1, r_2 \in \mathbb{R}$
    - $r_1, r_2 < 0$ then we have a stable node
    - $r_1, r_2 > 0$ then we have an unstable node
    - $r_1 = r_2 < 0$ then we have a generalized stable node
    - $r_1 = r_2 > 0$ then we have a generalized unstable node
  - If $r_1, r_2 \in \mathbb{C}$ (with $\mathrm{Im}(r_1), \mathrm{Im}(r_2) \neq 0$)
    - $\mathrm{Re}(r_1), \mathrm{Re}(r_2) < 0$ then we have a stable focus
    - $\mathrm{Re}(r_1), \mathrm{Re}(r_2) > 0$ then we have an unstable focus
    - $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) = 0$ then we have a center (vortex)