ΕΛΛΗΝΙΚΗ ΔΗΜΟΚΡΑΤΙΑ
ΠΑΝΕΠΙΣΤΗΜΙΟ ΜΑΚΕΔΟΝΙΑΣ

Mathematical analysis
Lecture 1

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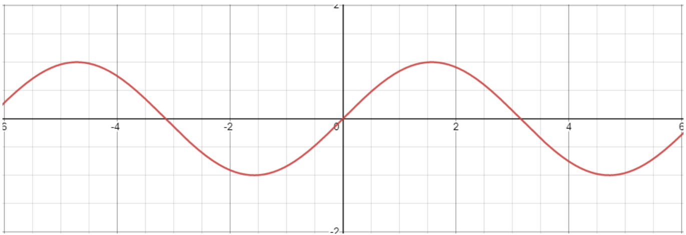
Topics of 1st lecture

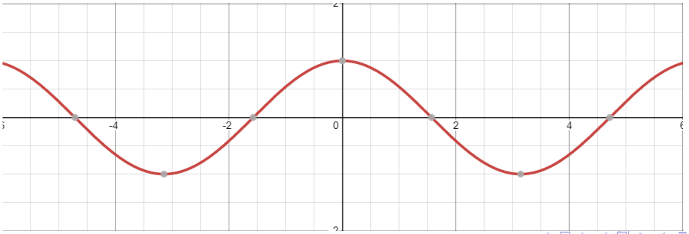
- Basic trigonometric functions
- Complex numbers
- Indefinite integrals
- Certain integrals

Trigonometric functions

- Sine ($\sin x$)

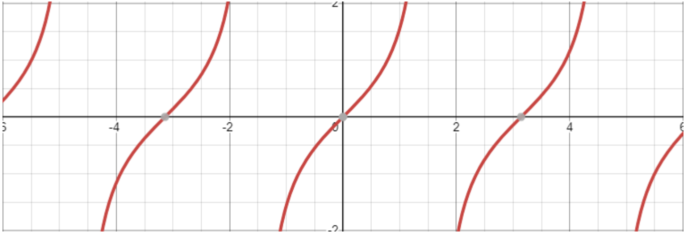


- Cosine ($\cos x$)

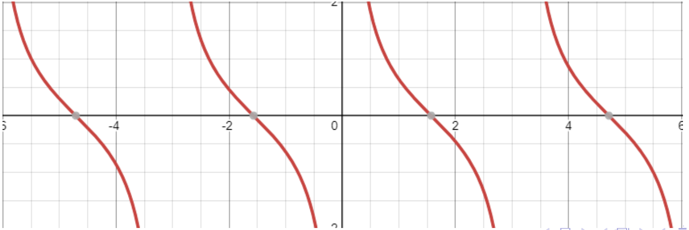


Trigonometric functions

- Tangent ($\tan x = \frac{\sin x}{\cos x}$)

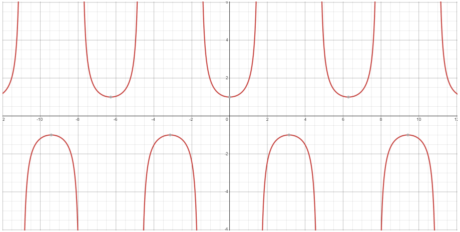


- Cotangent ($\cot x = \frac{\cos x}{\sin x}$)

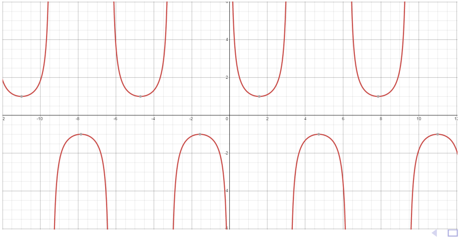


Trigonometric functions

- Cutting (seconds $x = \frac{1}{\cos x}$)

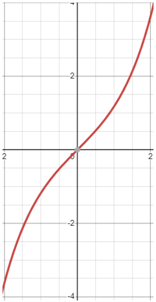


- Intersecting ($\csc x = \frac{1}{\sin x}$)

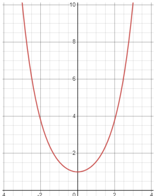


Inverse trigonometric functions

- Arc sine ($\arcsin x$)

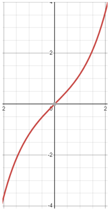


- Arc cosine ($\arccos x$)

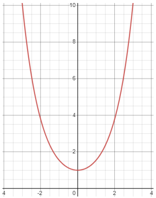


Hyperbolic trigonometric functions

- Hyperbolic sine ($\sinh x = \frac{e^x - e^{-x}}{2}$)



- Hyperbolic cosine ($\cosh x = \frac{e^x + e^{-x}}{2}$)



Definition of complex numbers

A complex number can be defined as an ordered pair:

$$z = (x, y), \text{where } x, y \in \mathbb{R}$$

with the operations of addition:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and multiplication:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

The real numbers in the expression $z = (x, y)$ are known as the real and fantastic part of it z, and are denoted as:

$$\mathrm{Re}(z) = x, \mathrm{I}(z) = y$$

Symbolization of complex numbers

If we symbolize a real number x as $(x, 0)$ and the imaginary number $(0,1)$ with I, then a complex number can be written as:

$$(x, y) = x + iy$$

Furthermore, based on the definitions, the following applies:

$$I^2 = I \cdot I = (0,1) \cdot (0,1) = (-1,0)$$

that is,

$$I^2 = -1 \Rightarrow I = \sqrt{-1}$$

Based on this notation, the addition and multiplication of two complex numbers are formulated as:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + I(y_1 + y_2)$$

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + I(x_1 y_2 + y_1 x_2)$$

Example - Quadratic equation

To solve the quadratic equation $x^2 - x + 1 = 0$, we observe that the discriminant is negative:

$$\Delta = 1 - 4 = -3$$

There is no solution in the set of real numbers, but in the set of complex numbers there are solutions:

$$r_{1,2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\,i}{2}$$

Algebraic properties

- Commutative

$$z_1+z_2=z_2+z_1, z_1 z_2=z_2 z_1$$

- Pre-cooperative

$$z_1 + (z_2+z_3) = (z_1+z_2) + z_3, z_1(z_2 z_3) = (z_1 z_2) z_3$$

- Distributive

$$z_1(z_2+z_3) = z_1 z_2+z_1 z_3$$

- Additional identifier $0 = (0,0)$ and multiplicative unit $1 = (1,0)$

$$z+0 = z,\ z\cdot 1 = z$$

- Additive inverse element $-z = (-x, -y)$

$$z + (-z) = 0$$

- Multiplicative inverse element $z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ for $z = x+iy \neq 0$

$$z z^{-1} = 1$$

Exercise

If $z = x + iy (\neq 0)$, prove that the multiplicative inverse of z is the $z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$.

Solution

To find the multiplicative inverse of $z = x + iy$, we should look for two real numbers, let *you* and v so that:

$$(x + iy)(you + iv) = 1 + I0$$

Analyzing the left-hand side of the equation, we have:

$$(x + iy)(you + iv) = (xu - yv) + I(xv + you) = 1 + I0$$

For two complex numbers to be equal, their real and imaginary parts must be equal. Therefore:

$$xu - yv = 1 \text{ and } xv + you = 0$$

If we solve this linear system (unknowns are u, v) then we get:

$$you = \frac{x}{x^2 + y^2}, \; v = \frac{-y}{x^2 + y^2}$$

Division of complex numbers

Division by a non-zero complex number is defined as:

$$\frac{z_1}{z_2} = z_1 z_2^{-1}, \quad z_2 \neq 0$$

or (if $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$):

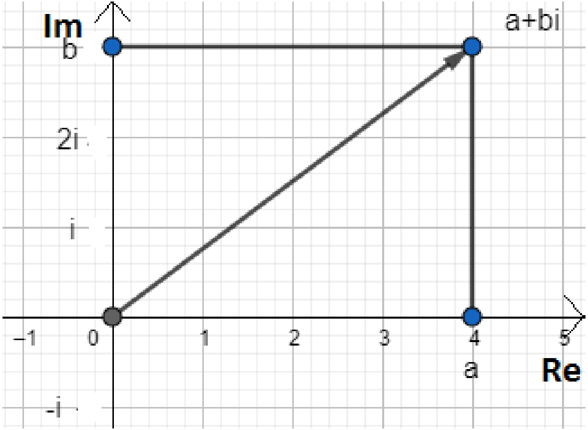
$$\frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + I\left(\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}\right)$$

Other properties of division that can easily be derived:

- $(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = 1$, for $z_1, z_2 \neq 0$
- $\frac{1}{z_1 z_2} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right)$, for $z_1, z_2 \neq 0$
- $\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}$; $\frac{z_1 z_2}{z_3 z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)$, for $z_3, z_4 \neq 0$

Isomorphism with $\mathbb{R}^2$

The set of complex numbers is denoted by $\mathbb{C}$.
The $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ ($\mathbb{C} = \mathbb{R} \times \mathbb{I}$). Thus we can understand intuitively better understand its meaning:



Shape: Isomorphism of $\mathbb{C}$ with $\mathbb{R}^2$

Complex number measure

As a measure or absolute value of a complex number $z = x + iy$ is defined as sity:

$$|z| = \sqrt{x^2 + y^2}$$

Geometrically, the measure expresses the distance of the point (x, y) from the beginning of axles. If $y = 0$, then the measure coincides with the usual absolute value of real numbers.
Triangular inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Polar form of a complex number

Based on the isomorphism of $\mathbb{C}$ with $\mathbb{R}^2$, we can write a complex number with coordinates (a, b) in polar form with coordinates (p, i).

$$i = \arctan(b/a)$$

$$r = \sqrt{a^2 + b^2}$$

where

$$a = r\cos(i)$$

and

$$b = r\sin(i)$$

and the polar form of the complex number z is:

$$z = r(\cos(i) + I\sin(i))$$

Conjugate Complex Numbers

If we have a complex number z, the corresponding complex number which has the same real part and opposite imaginary part is called its conjugate $\bar{z}$ and is symbolized $\bar{z}$. That is, if $z = a + bi$ then $\bar{z} = a - bi$.

Properties:

$z + \bar{z} = 2a = 2\mathrm{Re}(z)$. $z - \bar{z} = 2$
$bi = 2\mathrm{Im}(z)I$.
$z\bar{z} = a^2 - (bi)^2 = a^2 - b^2 I^2 = a^2 - b^2(-1) = a^2 + b^2$.

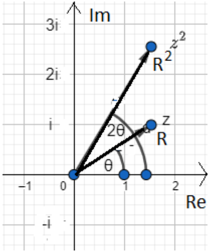
Type ofEuler

His typeEuler gives us that:

$$e^{ith}=\cos(i)+I\sin(i)$$

This formula allows us to easily raise a complex number to a power.

For example, instead of calculating $(a+bi)^n$ we convert the number $a+bi$ in polar form and calculate $(Re^{ith})^n=R^ne^{in}=R^n(\cos(nth)+I\sin(nth))$.

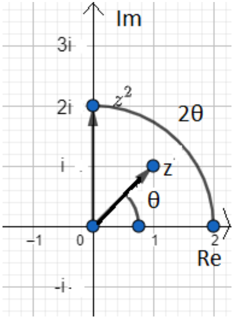


Shape:His typeEuler diagrams

Example of the type ofEuler

Let it be the complex one number $z = 1 + i$. Convert the number to polar form $R = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \arctan(1) = \frac{\pi}{4}$. Then, squaring,

we raise the modulus of the complex number to the square and the value doubles angle. That is, we have $z^2 = 2(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$.



Shape:His typeEuler for the given example

Indefinite integral

Let it be f a function defined on an interval D. Initial function or derivative or antiderivative of f in Δ is called any function F which is derivable in D and valid

$$F(x) = f(x), \text{ for each } x \in D.$$

Theorem: Let f a function defined on an interval D. If F it is one producer of f in D then:

- all functions of the form $G(x) = F(x) + c, c \in R$ are products of f in D and
- any other producer G her f in Δ takes the form: $G(x) = F(x) + c, c \in R$

Indefinite integral

Indefinite integral of the function $f(x)$ is called the set of parameters of taste functions of:

$$\int f(x)dx = \int F(x)dx = F(x) + c,\ c \in \mathbb{R}$$

For example:

$$\int x^2 dx = \int \left(\frac{x^3}{3}\right)' dx = \frac{x^3}{3} + c$$

because it is true:

$$\left(\frac{x^3}{3}\right)' = \frac{3x^2}{3} = x^2$$

while $\int e^{2x}dx = \int \left(\frac{e^{2x}}{2}\right)' dx = \frac{e^{2x}}{2} + c$, because: $\left(\frac{e^{2x}}{2}\right)' = \frac{2e^{2x}}{2} = e^{2x}$

Basic formula of indefinite integrals

1. $\int dx = x + c$
2. $\int x^n dx = \frac{1}{n+1} x^{n+1} + c,\ n \in \mathbb{N}^*$
3. $\int x^a dx = \frac{1}{a+1} x^{a+1} + c,\ a \in \mathbb{R} \setminus \{-1\}$
4. $\int \frac{1}{x} dx = \ln|x| + c$
5. $\int \sin x\, dx = -\cos x + c$
6. $\int \cos x\, dx = \sin x + c$
7. $\int \frac{1}{\cos^2 x} dx = \tan x + c$
8. $\int e^x dx = e^x + c$
9. $\int a^x dx = \frac{1}{\ln a} a^x + c, \quad 0 < a \neq 1$
10. $\int \frac{1}{x^2+1} \mathrm{d}x = \arctan x + c, \quad \int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \arcsin x + c, \quad \int \frac{-1}{\sqrt{1-x^2}} \mathrm{d}x = \arccos x + c$
11. $\int \sinh x\, dx = \cosh x + c, \quad \int \cosh x\, dx = \sinh x + c$

Linearity integral

$$\int (c_1 f(x) + c_2 g(x))\, dx = c_1 \int f(x)\, dx + c_2 \int g(x)\, dx$$

Examples

- $\int \sqrt{3x}\,dx = \sqrt{3}\int \sqrt{x}\,dx = \sqrt{3}\int x^{\frac{1}{2}}\,dx = \sqrt{3}\,\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \sqrt{3}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2\sqrt{3}}{3}\sqrt{x^3} + c$
- $\int \frac{2}{\sqrt[4]{x^3}}\,dx = \int \frac{2}{x^{\frac{3}{4}}}\,dx = 2\int x^{-\frac{3}{4}}\,dx = 2\,\frac{x^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + c = 2\,\frac{x^{\frac{1}{4}}}{\frac{1}{4}} + c = 4 \cdot 2\sqrt[4]{x} + c$
- $\int \frac{2x^2-x+5}{x^2}\,dx = \int \left(2 - \frac{1}{x} + \frac{5}{x^2}\right)dx = \int 2\,dx - \int \frac{1}{x}\,dx + \int \frac{5}{x^2}\,dx =$
  $2x - \ln|x| - 5x^{-1} + c = 2x - \ln|x| - \frac{5}{x} + c$
- $\int \frac{1-\sin^2(x)}{\cos(x)}\,dx = \int \frac{\cos^2(x)}{\cos(x)}\,dx = \int \cos(x)\,dx = \sin(x) + c$

Exercise

Calculate the integral: $$\int \frac{4}{\sin^2(2x)}\,dx$$

Solution

Using the trigonometric identities:

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\sin^2(x) + \cos^2(x) = 1$$

we have:

$$\int \frac{4}{\sin^2(2x)}\,dx = \int \frac{4}{(2\sin(x)\cos(x))^2}\,dx = \int \frac{4}{4\sin^2(x)\cos^2(x)}\,dx =$$

$$= \int \frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)\cos^2(x)} = \int \frac{\sin^2(x)}{\sin^2(x)\cos^2(x)}\,dx + \int \frac{\cos^2(x)}{\sin^2(x)\cos^2(x)}\,dx =$$

$$= \int \frac{1}{\cos^2(x)}\,dx + \int \frac{1}{\sin^2(x)}\,dx = \tan(x) - \cot(x) + c$$

Factor integration method

$$\int f(x) g'(x)\, dx = f(x) g(x) - \int f'(x) g(x)\, dx$$

Examples:

- Calculate the integral $\int x^2 e^x dx$:
  $\int x^2 e^x dx = \int x^2 (e^x)' dx = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2\int x (e^x)' dx = x^2 e^x - 2xe^x + 2\int e^x dx = x^2 e^x - 2xe^x + 2e^x + c$
- Calculate the integral $\int x\cos(axe) dx$:
  $\int x\cos(axe) dx = \int x \left(\frac{\sin(axe)}{a}\right)' dx = \frac{x\sin(axe)}{a} - \frac{1}{a}\int x' \sin(axe) dx = \frac{x\sin(axe)}{a} - \frac{1}{a}\int \sin(axe) dx = \frac{x\sin(axe)}{a} + \frac{\cos(axe)}{a^2} + c$

Completion by replacement

$$\int f(g(x))g'(x)\,dx = \int f(you)\,you$$

where $you = g(x)$ and $you' = g'(x)\,dx$

Examples:

- $\int 2x\sqrt{x^2+1}\,dx$ We set $you = x^2+1$ so $you' = (x^2+1)'dx = 2x\,dx$.

  Therefore $\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{u}\,du = \int you^{\frac{1}{2}}\,you = \frac{2}{3}you^{\frac{3}{2}} + c = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + c = \frac{2}{3}\sqrt{(x^2+1)^3} + c$

- $\int \frac{x^2}{\sqrt{1-x^6}}\,dx$ We set $you = x^3$ so $you' = (x^3)' = 3x^2\,dx$.

  Therefore $\int \frac{x^2}{\sqrt{1-x^6}}\,dx = \frac{1}{3}\int \frac{(x^3)'}{\sqrt{1-(x^3)^2}}\,dx = \frac{1}{3}\int \frac{you(x)'}{\sqrt{1-(you(x))^2}}\,dx = \frac{1}{3}\arcsin(you(x)) + c = \frac{1}{3}\arcsin(x^3) + c$

Integration of rational functions

$$\int \frac{p(x)}{(q(x))^k}\,dx$$

Where $p(x), q(x)$ are polynomials for which $p(x) = q'(x)$ and $k \in \mathbb{N}$. The replacement we make is $you = q(x)$.

Example: $\int \frac{6x-1}{(3x^2-x-12)^2}dx$ We set $you = 3x^2 - x - 12 \Leftrightarrow \frac{you}{dx} = 6x - 1 \Leftrightarrow you = (6x-1)dx$ Therefore

$$\int \frac{6x-1}{(3x^2-x-12)^2}dx = \int \frac{you}{you^2} = \frac{you^{-2+1}}{-2+1} + c = -\frac{1}{you} + c = -\frac{1}{3x^2-x-12} + c$$

Integration of rational functions

$$\int \frac{p(x)}{q(x)}\,dx$$

An explicit function with a numerator of a first degree polynomial and a denominator of a second degree polynomial that has no real roots, so it is not factorable.

$$\int \frac{8x+4}{x^2-2x+5}\,dx = \int \frac{8x+4}{x^2-2x+1+4}\,dx = \int \frac{8x+4}{(x-1)^2+4}\,dx \underset{\substack{you=x-1\\ you=dx}}{=} \int \frac{8(you+1)+4}{you^2+4}\,you =$$

$$\int \frac{8you}{you^2+4}\,you + \int \frac{12}{you^2+2^2}\,d\,you = 4\int \frac{2you\,d}{you^2+4}\,you + 6\int \frac{2}{you^2+2^2}\,you \underset{\substack{v=you^2+4\\ (dv=2you)you}}{=} 4\int \frac{dv}{v} + 6\int \frac{2}{you^2+2^2}\,you =$$

$$4\ \mathrm{in}\,/v/ + 6\arctan\left(\frac{you}{2}\right) + c = 4(\ln/you^2) + 4/ + 6\arctan\frac{x-1}{2} + c =$$

$$4\ln(x^2-2x+5) + 6\arctan\frac{x-1}{2} + c$$

Integration of rational functions

Case where the numerator has a degree lower than the denominator and the denominator has simple roots:

$$f(x) = \frac{n(x)}{d(x)} = \frac{n(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_n}{(x-a_n)}$$

Example: $\int \frac{x+2}{x^2+2x-8}$ We analyze into factors:

$$\frac{x+2}{x^2+2x-8} = \frac{x+2}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2} = \frac{A(x-2)+B(x+4)}{(x+4)(x-2)} = \frac{(A+B)x-2A+4B}{(x+4)(x-2)}$$

The equality of the numerators leads us to the system $\begin{cases} A+B=1 \\ -2A+4B=2 \end{cases}$ which has unique solution $A=1/3$ and $B=2/3$. Therefore:

$$\int \frac{x+2}{x^2+2x-8}\, dx = \int \left(\frac{1}{3(x+4)} + \frac{2}{3(x-2)} \right) dx = \frac{1}{3}\ln|x+4| + \frac{2}{3}\ln|x-2| + c$$

Integration of rational functions

Cases where the numerator has a degree lower than the denominator and the denominator has multiple roots:

$$f(x) = \frac{p(x)}{(x+a_1)^k(x+a_2)^m\cdots(x+a_\ell)^n} = \frac{A_1}{x+a_1} + \frac{A_2}{(x+a_1)^2} + \cdots + \frac{A_k}{(x+a_1)^k} + \frac{B_1}{x+a_2} + \frac{B_2}{(x+a_2)^2} + \cdots + \frac{B_m}{(x+a_2)^m} + \cdots + \frac{C_1}{x+a_\ell} + \frac{C_2}{(x+a_\ell)^2} + \cdots \frac{C_k}{(x+a_\ell)^n}$$

or

$$f(x) = \frac{p(x)}{(x+a)^k(x^2+bx+c)^n} = \frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_k}{(x+a)^k} + \frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n}$$

Integration of rational functions

Cases where the numerator has a degree greater than the denominator

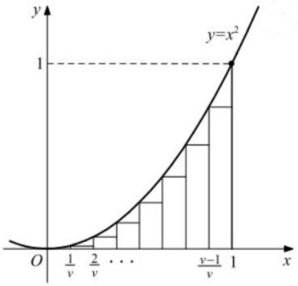
$f(x) = \frac{p(x)}{q(x)} = A(x) + \frac{p_1(x)}{q(x)}$ (division of polynomials), then:

$$\int \left(A(x) + \frac{p_1(x)}{q(x)}\right)dx = \int A(x)\,dx + \int \frac{p_1(x)}{q(x)}\,dx, \text{ with degree } p_1(x) < \text{degree } q(x)$$

Example:

$$\int \frac{x^4 - x^3 + 2x - 3}{x^2 - 1}\,dx = \int \left((x^2 - x + 1) + \frac{x-2}{x^2-1}\right)dx = \int (x^2 - x + 1)\,dx + \int \frac{x-2}{x^2-1}\,dx$$

Area of a parabolic village



Shape: Approximate area of $f(x) = x^2$ 'below'

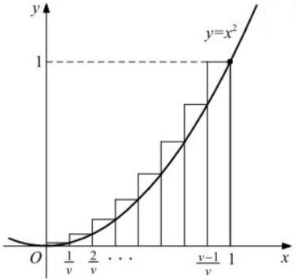
$$\epsilon_n = f(0)\frac{1}{n} + f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots f\left(\frac{n-1}{n}\right)\frac{1}{n} =$$

$$\frac{1}{n}\left(0^2 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right) = \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right) =$$

$$\frac{1}{n^3}\frac{(n-1)n(2n-1)}{6} = \frac{2n^2 - 3n + 1}{6n^2}$$

(using the property $1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$)

Area of a parabolic village



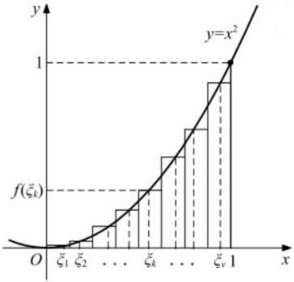
Shape: Approximate area of $f(x) = x^2$ 'over'

$$E_n = f(\tfrac{1}{n})\tfrac{1}{n} + f(\tfrac{2}{n})\tfrac{1}{n} + \cdots + f(\tfrac{n}{n})\tfrac{1}{n} = \tfrac{1}{n}\left((\tfrac{1}{n})^2 + (\tfrac{2}{n})^2 + \cdots + (\tfrac{n}{n})^2\right) =$$

$$\frac{1}{n^3}(1^2+2^2+\cdots+n^2) = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{2n^2+3n+1}{6n^2}$$

But for the area E valid $\lim_{n\to+\infty} \epsilon_n \le E \le \lim_{n\to+\infty} E_n$ Therefore $E = \frac{1}{3}$.

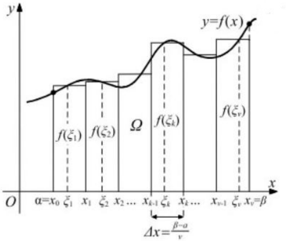
Area of a parabolic village



Shape: Approximate area of $f(x) = x^2$ 'with intermediate points'

$S_n = \frac{1}{n} f(x_1) + \frac{1}{n} f(x_2) + \cdots + \frac{1}{n} f(x_n)$. Because $f(x_{k-1}) \leq f(x_k) \leq f(x_k), k = 1, \cdots, n$ will be: $\frac{1}{n} f(x_{k-1}) \leq \frac{1}{n} f(x_k) \leq \frac{1}{n} f(x_k)$. Therefore $\epsilon_n \leq S_n \leq E_n$ Therefore $\lim_{n \to +\infty} \epsilon_n = \lim_{n \to +\infty} E_n = S_n$ and $S_n = \frac{1}{3}$.

Definition of area



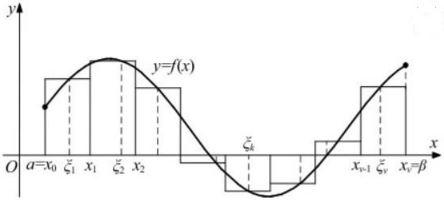
Shape:General definition of area

We divide the space [a, b] in n equal intervals of length $\Delta x = \frac{\beta-\alpha}{n}$ with $a = x_0 < x_1 < x_2 \cdots < x_n = b$

In each subinterval we arbitrarily choose a point x_k and we form the rectangles that have a base Dx and heights x_k

$S_n = f(x_1)Dx + f(x_2)Dx + \cdots + f(x_n)Dx = Dx(f(x_1) + f(x_2) + \cdots + f(x_n))$. We calculate the $\lim\limits_{n \to +\infty} S_n$.

Definite integral



Shape: Definite Integral

$$S_n = f(x_1)\mathrm{D}x + f(x_2)\mathrm{D}x + \cdots + f(x_n)\mathrm{D}x = \mathrm{D}x(f(x_1) + f(x_2) + \cdots f(x_n)) = \sum_{k=1}^{n} f(x_k)\mathrm{D}x.$$

The limit of the above sum when $n \to +\infty$ exists in R and is independent $\int$o from the selection of x_k.

It is written $_a^b f(x)dx$ and is read as the integral of f from the a in b.

Definite integral

It is true that:

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- If $f(x) \geq 0$, then $\int_a^b f(x)dx \geq 0$ for $a < b$

Theorem 1 *the*: Let f, g continuous functions on $[a, b]$ and $l, \mu \in \mathbb{R}$. Then apply:

- $\int_a^b l f(x)dx = l\int_a^b f(x)dx$
- $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b l f(x) + \mu g(x)dx = l\int_a^b f(x)dx + \mu\int_a^b g(x)dx$

Definite Integral

Theorem2*the*: If the f is continuous over an interval D and $a, b, c \in$ D, then applies:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Theorem3*the*: Let f a continuous function on an interval $[a, b]$. If $f(x) \geq 0$ for each $x \in [a, b]$ and the function f it is not zero everywhere in this time then

$$\int_a^b f(x)dx > 0$$

The function $F(x) = \int_a^x f(t)\,dt$

Theorem: If f is a continuous function on an interval D and a it is a point of Δ, then the function:

$$F(x) = \int_a^x f(t)\,dt, x \in D$$

is a derivative of f in D. That is, it is true:

$$\left(\int_a^x f(t)\,dt\right)' = f(x), \text{ for each } x \in D$$

Theorem: (Fundamental theorem of integral calculus)
Let it be f a continuous function on an interval $[a, b]$. If G is a pa- her/his voice f in $[a, b]$, then

$$\int_a^b f(t)\,dt = G(b) - G(a) = [G(x)]_a^b$$

The formulas of integration by factors and for certain integrals

The factorial integration formula for the definite integral takes the form

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

where f', g' are continuous functions on $[a, b]$.

For example:

$$\int_0^{p/2} x\cos x dx = \int_0^{p/2} x(\sin x)' dx = [x\sin x]_0^{p/2} - \int_0^{p/2} \sin x dx =$$

$$[x\sin x]_0^{p/2} + [\cos x]_0^{p/2} = \frac{p}{2} - 1 = \frac{p-2}{2}$$

The formula for integration by change of variable for certain integrals

The formula for integration by change of variable for the definite integral takes the form

$$\int_a^b f(g(x))g'(x)dx = \int_{you_1}^{you_2} f(you)you$$

where f, g are continuous functions, $you = g(x)$, $you' = g'(x)dx$ and $you_1 = g(a)$, $you_2 = g(b)$.

For example $I = \int_1^e \frac{lnx}{x}dx = \int_1^e lnx(lnx)'dx$ We set $you = lnx$, so $you = (lnx)'dx$, $you_1 = in1 = 0$, $you_2 = line = 1$. Therefore

$$I = \int_0^1 udu = \left[\frac{you^2}{2}\right]_0^1 = \frac{1}{2}$$

.

Applications of integrals - Calculating areas

The area between two functions for which it applies $f_1(x) \geq f_2(x)$ for $a \leq x \leq b$ equals to

$$E = \int_a^b |f_1(x) - f_2(x)|\, dx$$

Example: Calculate the area of the enclosed space defined by functions $f_1(x) = \sqrt{x}$ and $f_2(x) = x^2$.

First we find the intersection points of the two curves: $\sqrt{x} = x^2 \Leftrightarrow x = x^4 \Leftrightarrow$ $x^4 - x = 0 \Leftrightarrow x(x^3 - 1) = 0 \Leftrightarrow x(x-1)(x^2 + x + 1) = 0 \Leftrightarrow x = 0$ or $x = 1$.

We observe that in the interval $(0,1)$ the function $\sqrt{x} - x^2$ takes positive values, so the area is:

$$E = \int_0^1 |\sqrt{x} - x^2|\, dx = \left|\int_0^1 (\sqrt{x} - x^2)\, dx\right| = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \left(\frac{2}{3}1^{\frac{3}{2}} - \frac{1^3}{3}\right) - \left(\frac{2}{3}0^{\frac{3}{2}} - \frac{0^3}{3}\right) = \frac{1}{3}$$

Applications of integrals - Calculating the length of a curve segment

The length of a curve segment of a function $y = f(x)$ which is para- pliable in space $[a, b]$ equals:

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx$$

Example: Find the length of a curve segment of $y = x^{\frac{3}{2}}$ which is defined between the straight lines $x = 0$ and $x = 4$. For this particular function we have $y' = \frac{3}{2} x^{\frac{3}{2} - 1} = \frac{3}{2} x^{\frac{1}{2}}$, therefore the length is:

$$L = \int_0^4 \sqrt{1 + \left(\frac{3}{2} x^{\frac{1}{2}}\right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4} x}\, dx = \frac{1}{2} \int_0^4 \sqrt{4 + 9x}\, dx$$

If we make a change change of $you = 4 + 9x$, then we can easily arrive at the result

$$L = \frac{8}{27}\left(10^{\frac{3}{2}} - 1\right)$$

Generalized integrals (first kind)

If in an integral at least one of the two integration ends is $\pm\infty$, then this is called a generalized integral of the first kind.
We can distinguish the following three cases with respect to space: completion:

- space $[a, \infty)$, then
$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^{b} f(x)\,dx$$
- space $(-\infty, a]$, then
$$\int_{-\infty}^{a} f(x)\,dx = \lim_{b\to-\infty} \int_b^{a} f(x)\,dx$$
- space $(-\infty, +\infty)$, then
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

If the limit exists and is a real number, then we say that the generalized integral converges, otherwise it diverges.

Examples

- $\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} (\ln b - \ln 1) = +\infty$
- $\int_0^\infty \cos(x)\,dx = \lim_{b\to\infty} \int_0^b \cos(x)\,dx = \lim_{b\to\infty} [\sin(x)]_0^b = \lim_{b\to\infty} (\sin(b) - \sin(0)).$

  But the limit $\lim_{b\to\infty} \sin(b)$ does not exist (why?), therefore the integral does not converges.
- $\int_2^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \left[\frac{x^{-2+1}}{-2+1}\right]_2^b = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_2^b = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{2}\right) = \frac{1}{2}.$