Mathematical analysis
Lecture 8

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Topics of the 8th lecture

- Definition of difference equations
- Classification of difference equations
- First-order differential equations
- Nonlinear first-order difference equations
- Phase diagram

Difference equations

A difference equation is an equation that involves the change of a variable.

Let it be $y(t)$ a real (or complex) function of the variable t. The difference or change Δ is defined as:

$$\mathrm{D}y(t) = y(t+1) - y(t)$$

or

$$\mathrm{D}y_t = y_{t+1} - y_t, t = 0, 1, 2, \ldots$$

Classification of difference equations

- Order: The order of a difference equation is determined by its highest order. difference that exists in the equation.

  For example, a first-order difference equation contains only the first difference of a variable, that is, the difference of the variable in two consecutive periods ($y_{t+1} - y_t$).

  A second-order difference equation also contains the second difference of a variable, that is, the difference of the variable observed when considering every second of the consecutive periods ($y_{t+2} - y_t$).

Classification of difference equations

- In practice this means that a first order difference equation contains variables that are only one period apart such as $y_{t+1} = 5y_t + 1$, while one A second-order difference equation contains variables that are at most two periods apart, such as $y_{t+2} = 5y_{t+1} + 4y_t + 1$ or equivalent $y_t = 5y_{t-1} + 4y_{t-2} + 1$
- Therefore a difference equation n-th order contains variables that are at most distant from each other n periods. We will only deal with first and second order differential equations

Classification of difference equations

- Autonomous: A difference equation is said to be autonomous if it does not depend on explicitly by time. The equation is called non-autonomous when the variable t appears directly as an independent variable, while autonomous when the variable t enters the equation only through y.

  For example, the $y_{t+1} = 4y_t + 5t$ is non-autonomous because it explicitly depends from the variable t, while the $y_{t+1} = 4y_t + 5$ is autonomous, because it does not depend explicitly from the variable t.

Classification of difference equations

- Linear or nonlinear. A difference equation is nonlinear if contains some nonlinear terms with respect to some of the y_t, y_{t+1}, y_{t+2} etc., while linear otherwise.

  For example, the $y_{t+1} = 4y_t^2 + 1$ and the $y_{t+1} = 4 in(y_t) + 1$ are nonlinear autonomous first-order equations, while the $y_{t+1} = 4y_t + t^2$ is a linear, non-autonomous difference equation.

Linear first-order difference equations

The general form of the linear, autonomous first-order difference equation order is given by:

$$y_{t+1} = ay_t + b, \quad t = 0, 1, 2, \ldots \tag{1}$$

- Solving a difference equation means finding the related function of time y_t from which it is created.
- If the y_0 is known, then when $t = 0$ the equation (1) implies that $y_1 = ay_0 + b$ where a and b are known constants. For $t = 1$: $y_2 = ay_1 + b = a(ay_0 + b) + b = a^2 y_0 + b(a + 1)$.
  For $t = 2$: $y_3 = ay_2 + b = a(a^2 y_0 + b(a + 1)) + b = a^3 y_0 + b(a^2 + a + 1)$
- We make the assumption that $y_t = a^t y_0 + b(a^{t-1} + a^{t-2} + \cdots + a + 1)$

Linear first-order difference equations

- We know that $1 + a + a^2 + \cdots + a^{t-1} = \begin{cases} \frac{1-a^t}{1-a}, & \text{if } a \neq 1 \\ t, & \text{if } a = 1 \end{cases}$
- Therefore, the solution we assumed for the difference equation can be expressed as:

$$y_t = \begin{cases} a^t y_0 + b\left(\frac{1-a^t}{1-a}\right), & \text{if } a \neq 1 \\ y_0 + bt, & \text{if } a = 1 \end{cases} \qquad t = 0, 1, 2 \cdots$$

Linear first-order difference equations

Theorem: The function y_t which is given by the equation

$$y_t = \begin{cases} a^t y_0 + b\left(\frac{1-a^t}{1-a}\right), & \text{if } a \neq 1 \\ y_0 + bt, & \text{if } a = 1 \end{cases} \qquad t = 0, 1, 2 \cdots$$

is the unique solution of the linear, autonomous first-order difference equation $y_{t+1} = a y_t + b$, where y_0 is the given initial condition.

Evidence:

1. For $a = 1$, we have by the method of mathematical induction:
   - For $t = 0$: $y_0 + b \cdot 0 = y_0 + 0 = y_0$.
   - Let it be true for $t = k$: $y_k = y_0 + bk$.
   - We will show that it is true for $t = k+1$: T.d.o. $y_{k+1} = y_0 + b(k+1)$. Getting started from the relationship $y_{t+1} = y_t + b$ we have $y_{k+1} = y_k + b$. We replace the y_k from the case and we have $y_{k+1} = y_0 + bk + b = y_0 + b(k+1)$.

Linear first-order difference equations

2. For $a \neq 1$, I.D.O. $y_t = a^t y_0 + b\left(\frac{1-a^t}{1-a}\right)$. We have by the method of mathematical induction:

- For $t = 0$: $a^0 y_0 + b\left(\frac{1-a^0}{1-a}\right) = 1 \cdot y_0 + b\left(\frac{0}{1-a}\right) = y_0$ which is valid.
- Let it be true for $t = k$: $y_k = a^k y_0 + b\left(\frac{1-a^k}{1-a}\right)$.
- For $t = k+1$: T.d.o. $y_{k+1} = a^{k+1} y_0 + b\left(\frac{1-a^{k+1}}{1-a}\right)$. We add and subtract the term $ab(1-a^k)/(1-a)$ on the right-hand side and we get:
  $y_{k+1} = a^{k+1} y_0 + \frac{ab(1-a^k)}{1-a} - \frac{ab(1-a^k)}{1-a} + b\left(\frac{1-a^{k+1}}{1-a}\right) =$
  $a\left(a^k y_0 + b\left(\frac{1-a^k}{1-a}\right)\right) + \frac{b}{1-a}\left(-a + a^{k+1} + 1 - a^{k+1}\right)$.
  We replace the y_k from the case and we have:
  $y_{k+1} = a y_k + \frac{b}{1-a}(1-a) = a y_k + b$ which is valid.

Linear first-order difference equations

Theorem: There is a constant C such that every solution of the linear, of an autonomous first-order differential equation can be formulated as follows:

$$y_t = \begin{cases} Ca^t + b\left(\frac{1}{1-a}\right), & \text{if } a \neq 1 \\ C + bt, & \text{if } a = 1 \end{cases} \qquad t = 0, 1, 2 \cdots$$

Example

Solve the difference equation $y_{t+1} = 1/2\, y_t + 10$, with $y_0 = 1$.

Based on the theorem we have:

$$y_t = C(1/2)^t + 10\left(\frac{1-(1/2)^t}{1-1/2}\right)$$

If $y_0 = 1$ then for $t = 0$ we have $1 = C + 10 \cdot 0 \Leftrightarrow C = 1$.
So, to satisfy the given initial condition, the solution has the following form:

$$y_t = (1/2)^t + 10\left(\frac{1-(1/2)^t}{1-1/2}\right)$$

Example

Solve the difference equation $y_{t+1} = 5y_t - 3$, with $y_0 = 0$.

Based on the theorem we have:

$$y_t = C5^t - 3\left(\frac{1-5^t}{1-5}\right)$$

If $y_0 = 0$ then for $t = 0$ we have $0 = C - 3 \cdot 0 \Leftrightarrow C = 0$.
So, to satisfy the given initial condition, the solution has the following form:

$$y_t = 0 \cdot 5^t - 3\left(\frac{1-5^t}{1-5}\right) = -3\left(\frac{1-5^t}{1-5}\right)$$

Steady state difference equation

Definition: The steady state or stationary value in a linear, autonomous non-first order difference equation is defined as the value of y in which the system ceases to change, that is, it is true that $y_{t+1} = y_t$.

To find the steady-state value of y, which we will call $\bar{y}$, we set $y_{t+1} = y_t \equiv \bar{y}$ in the difference equation. This leads us to the relationship:

$$\bar{y} = yes + b.$$

Solving in terms of $\bar{y}$ we get:

$$\bar{y} = \frac{b}{1-a}, a \neq 1.$$

If $a = 1$, there is no steady-state solution (why?).

Steady state difference equation

If it y equals at some point its steady-state value, it will remain at this value for all successive time periods. But the important question is: If the y starting from an arbitrary value, it will always converge to its value steady state?

To answer this question, we rearrange the solution given by the equation:

$$y_t = \begin{cases} a^t y_0 + b\left(\frac{1-a^t}{1-a}\right), & \text{if } a \neq 1 \\ y_0 + bt, & \text{if } a = 1 \end{cases} \qquad t = 0,1,2\cdots$$

to get:

$$y_t = a^t\left(y_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}, \quad \text{if } a \neq 1, t = 0,1,2,\cdots$$

Steady state difference equation

Examining this expression, we see that the issue of convergence or deviation is determined exclusively by the term a^t, since this is the only one that contains the t.

If this term converges to zero as the t tends to infinity, then the y_t converges towards $\frac{b}{1-a}$. On the contrary, if this term diverges towards infinity as the t tends towards infinity, then it will also diverge y_t.

We can consider the term a^t with $t = 0, 1, 2, \cdots$ as a sequence of numbers:

$$\{a^t\} = 1, a, a^2, a^3, \cdots, a^t, \cdots$$

Then we know that a sequence like this converges to zero as $t \to \infty$ if $|a| < 1$ and diverges if $|a| \geq 1$.

Theorem: In the case of a linear, autonomous difference equation first class, the y_t converges to the steady-state value $b/(1-a)$ if and only if $|a| < 1$.

Steady state difference equation

When $|a| < 1$, while convergence is the certain path followed over time by y_t is very different depending on the sign of a.

If $0 < a < 1$, then the y_t will converge monotonically to $b/(1-a)$. This is because each term of the sequence $\{a_t\}$ is smaller than the previous one. For example if $a = 1/2$ the sequence is

$$\left\{\left(\frac{1}{2}\right)^t\right\} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$$

But if $-1 < a < 0$, the y_t will converge towards $b/(1-a)$ deleting one oscillating path. We know this because each term of the sequence $\{a\}_t\}$ will have the opposite sign of the previous term. For example, if $a = -1/2$ the sequence is:

$$\left\{\left(-\frac{1}{2}\right)^t\right\} = 1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \dots$$

Steady state difference equation

There are three more cases that need to be considered separately:
(a) If $a=0$, we see from the equation

$$y_t = a^t\left(y_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}, \quad a \neq 1, t = 0,1,2,\cdots$$

that the y_t is constant over time and equal to b
(b) If $a=1$, we see from equation

$$y_t = y_0 + bt, \; t = 0,1,2\cdots$$

that the y_t converges to $+\infty$ if >0 and in $-\infty$ if $b<0$. (c) If $a=-1$ we see from the equation

$$y_t = a^t\left(y_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}, \quad t = 0,1,2,\cdots$$

that the y_t toggles between prices y_0 and $b-y_0$ ($y_t = (-1)^t(y_0 - b/2) + b/2$).

Example

Suppose that the y_t symbolizes the number of fish in a fish population. Suppose that the dynamic behavior of the fish population is governed by the difference equation:

$$y_{t+1} = ay_t + 10.$$

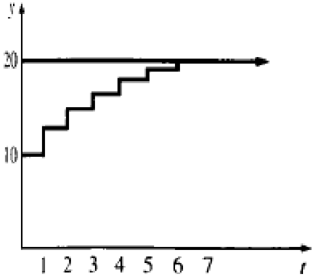
Find the steady state fish population and construct a graphical representation of it. y_t, initially for the case $a = 0.5$ and then for the case $a = -0.5$.

The stationary value of y is found posing $y_{t+1} = y_t = \bar{y}$ This gives us $\bar{y} = \frac{10}{1-a}$. The solution of the difference equation can be expressed as:

$$y_t = a^t\left(y_0 - \frac{10}{1-a}\right) + \frac{10}{1-a}$$

Example

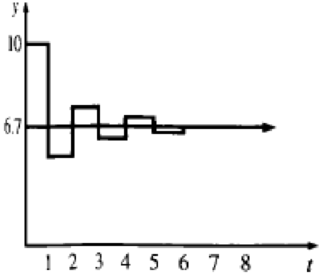
If $|a| < 1$, then the y_t converges towards $10/(1-a)$ as the t tends towards infinity. Therefore if $a = 0.5$, then the y_t approaches the steady state value $\bar{y} = 20$ monotonous.



Shape: Approach route for $a = 0.5$

Example

If $a = -0.5$, then the y_t approaches the steady state value $\bar{y} = 10/1.5$ with oscillations.



Shape: Approach route for $a = -0.5$

Summary of the Convergence Analysis

For the equation of differences

$$y_{t+1} = a y_t + b$$

the solution is $y_t = \begin{cases} a^t(y_0 - \bar{y}) + Yes, & \text{if } a \neq 1 \\ y_0 + bt, & \text{if } a = 1 \end{cases} \quad t = 0, 1, 2 \cdots$

where

$$\bar{y} = \frac{b}{1-a} \text{ if } a \neq 1$$

is the stable equilibrium state that exists when $a \neq 1$.
The steady state of equilibrium is stable (i.e. the y_t converges towards $\bar{y}$), if and only if:

$$-1 < a < 1.$$

His route y_t as it approaches the $\bar{y}$ is called the approach path and it is

- Monotonous, if a is positive (and smaller than 1)
- Talented, if it a is negative (and greater than -1)

Summary of the Convergence Analysis

Furthermore, if $a \geq 1$, then the y_t deviates from the $\bar{y}$ monotonously.

If $a < -1$, then the y_t deviates from the $\bar{y}$ with oscillations that are constantly increasing.

If $a = -1$, then the y_t never approaches the $\bar{y}$, but its value alternates between y_0 and the $b - y_0$.

If $a = 0$, then the y_t is constant and equal to b.

Nonlinear first-order difference equations

- We saw that linear first-order difference equations can have analytical solution.
- The same applies, as we will see, to linear difference equations. second class.
- In contrast, nonlinear difference equations generally cannot be have an analytical solution.
- However, it is possible to obtain qualitative information about the solution, analyzing a nonlinear difference equation with the help of a phase diagram.

Nonlinear first-order difference equations

The general form of the first-order nonlinear difference equation is as follows:

$$y_t = g(y_t, t), t = 0, 1, 2, \cdots$$

However, we will only study autonomous, nonlinear difference equations, that is, difference equations that do not depend directly on time.

Nonlinear first-order difference equations

Definition: The nonlinear, autonomous first-order difference equation has the following form:

$$y_{t+1} = f(y_t), t = 0, 1, 2, \cdots$$

If there is a stable equilibrium (or equilibria if there is more than one), it is usually found by setting $y_{t+1} = y_t = \bar{y}$, where $\bar{y}$ is a constant value of y. More generally, this leads us to the following relationship:

$$\bar{y} = f(\bar{y})$$

Nonlinear first-order difference equations

Our main concern when performing a qualitative analysis of a nonlinear difference equation is to verify whether the y_t converges or not towards a stable balance.

- If it does converge, then regardless of the starting value y_0, the course of y_t will eventually lead to the price $\bar{y}$. Then, even when we cannot solve the problem analytically, y_t as a function of t, we can see where its path always leads.
- But if it does not converge, then we can verify whether the y_t deviates from infinity or if it cycles back and forth between specific values or if it exhibits chaotic behavior.

Example

Let the following nonlinear difference equation be:

$$y_{t+1} = y_t^a,\quad a > 0,\ t = 0, 1, 2, \cdots$$

The steady-state values (the stationary values) of y they are posing $y_{t+1} = y_t = \bar{y}$. In this way and by rearranging the terms we are led to the relationship:

$$\bar{y}(\bar{y}^{a-1} - 1) = 0.$$

Therefore $\bar{y} = 0$ and $\bar{y} = 1$ are the stationary values. So if the y_t equalize some moment with 0 or 1, it will remain at that value forever.

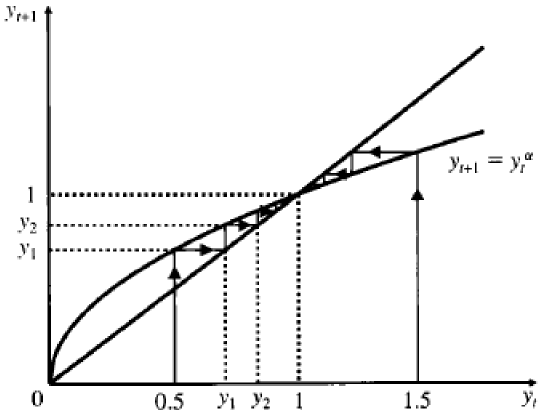
Phase diagram

It is useful to construct a phase diagram to see if the y_t tends to move towards or away from stable values.

The phase diagram for a difference equation is a diagram that depicts the y_{t+1} as for the y_t.

The stationary points will be located where the $f(y_t)$ with the 45 line$^\circ$ because along this line the relationship holds $y_{t+1} = y_t$.

Phase diagram



Shape: Phase diagram for the equation $y_{t+1} = y_t^a$ when $a = 1/2$

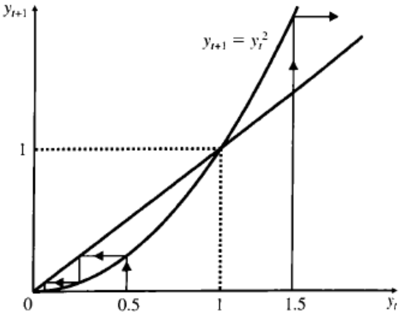
Phase diagram

From any starting point $y_0 > 0$ the course of y_t seems to converge towards $\bar{y} = 1$ highlighting the $\bar{y} = 1$ at a stable equilibrium point.

On the contrary, the point $\bar{y} = 0$ is a point of unstable equilibrium, because for $y > 0$ the y_t deviates from 0.

Phase diagram - Example

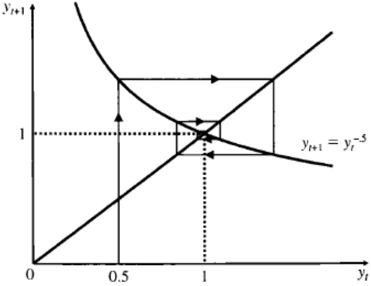
Construct the phase diagram and perform a qualitative analysis of the difference equation. $y_{t+1} = y_t^2$.



At the point $\bar{y} = 1$ an unstable equilibrium appears while in $\bar{y} = 0$ appears a locally stable equilibrium.

Phase diagram - Example 2

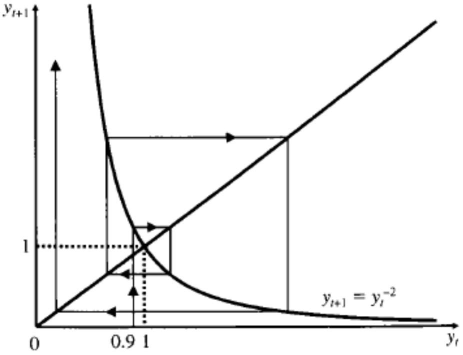
> Construct phase diagrams and perform a qualitative analysis of the difference equation $y_{t+1} = y_t^a$ when $a = -1/2$ and $a = -2$.



Shape: Phase diagram when $a = -1/2$

In $\bar{y} = 1$ we seem to have a stable equilibrium.

Phase diagram - Example 2



Shape: Phase diagram when $a = -2$

In $\bar{y} = 1$ we seem to have an unstable balance.

Theorem

Theorem: A stable equilibrium state at a stationary point of a of any autonomous, nonlinear first-order difference equation is locally stable if the absolute value of the derivative $f(\bar{y})$ is smaller from 1.

It is unstable if the absolute value of the derivative is greater than 1 at that point.

Example

Use the previous theorem to find the properties of the local stability of:

$$y_{t+1} = y_t^a,$$

for the various prices of a.

We have:

$$f'(y_t) = a y_t^{a-1}.$$

At the standstill $\bar{y} = 1$ we have:

$$f'(1) = a.$$

According to the theorem at point $\bar{y} = 1$ we have locally stable equilibrium only when $-1 < f'(\bar{y}) < 1 \Rightarrow -1 < a < 1$. For all other prices, the equilibrium at $\bar{y} = 1$ is unstable.

Example

For $a > 0$, we found another equilibrium point, the $\bar{y} = 0$. The derivative at this point is:

$$f'(0) = 0 \text{ if } a > 1$$
$$f'(0) \text{ not defined if } 0 < a < 1 \text{ (division by zero)}$$

If $a > 1$ the balance at the point $\bar{y} = 0$ is locally stable (because $f'(0) < 1$). It is not completely stable because it does not converge to 0 for any $y_t \geq 1$. When $0 < a < 1$ the $\bar{y} = 0$ is an unstable equilibrium point because the derivative becomes infinitely large (the a divided by 0).

Theorem

A first-order difference equation will lead to oscillations of y_t if the producer f' is negative for all $y_t > 0$, but the y_t will move monotonously if the derivative is positive for all $y_t > 0$.