Mathematical analysis
Lecture 3

Konstantinos Giannoutakis
Assistant Professor

Spyros Chalkidis
E.D.I.P.

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Topics of the 3rd lecture

- Method of Mathematical Induction
- Sequences
- Blocked sequences and monotony
- Convergence of sequences

Mathematical induction

The method of mathematical induction is used to prove propositions which depend, in the simplest case, on an integer variable. $n \in \mathbb{N}$. We symbolize $P(n)$ this proposal.

In the method of mathematical induction we follow the following three steps:

- Basic step: We first show the proposal for some $n = n_0$ for which we prove to be true. That is, we show that the $P(n_0)$ is a true proposition.
- Inductive Hypothesis: We assume that the proposition is true for some $n = k$ with $k > n_0$. That is, we assume that the $P(k)$ is a true proposal.
- Inductive step: We prove, using the previous assumption that the $P(k+1)$ is a true sentence.

If the implication in the last step is true, then the proposition is true for all $k \geq n_0$.

Induction Example 1

Show that for $n \geq 1$:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- For $n = 1$ we have $1 = \frac{1 \cdot 2}{2}$ which is valid.
- Suppose that the relationship holds for $n = k$, that is: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.
- T.d.o. $1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$.
  From the hypothesis we have that the above proposition is equivalent to $\frac{k(k+1)}{2} + k+1 = \frac{(k+1)(k+2)}{2} \Leftrightarrow \frac{k^2+k}{2} + k+1 = \frac{k^2+3k+2}{2} \Leftrightarrow \frac{k^2+3k+2}{2} = \frac{k^2+3k+2}{2}$.
  Thus we have shown by induction that the relation holds for every $n \geq 1$.

Induction Example 2

To show that $2^n \geq n^3$, for $n \geq 10$.

- For $n = 10$, $2^n = 1024 \geq 1000 = 10^3$ therefore the original hypothesis is valid.
- Let it be true for $n = k > 10$, that is $2^k \geq k^3$.
- T.d.o. $2^{k+1} \geq (k+1)^3$ Multiplying the inductive hypothesis by 2 we have that $2^{k+1} \geq 2k^3$. So it suffices to show that

  $2k^3 \geq (k+1)^3 \Leftrightarrow \left(2^{\frac{1}{3}} k\right)^3 \geq (k+1)^3$

  $\Leftrightarrow 2^{\frac{1}{3}} k \geq k+1 \Leftrightarrow k \geq \dfrac{1}{2^{\frac{1}{3}} - 1} \approx 3.85$ which is valid since $k \geq 10$.

Thus we have shown that the relationship $2^n \geq n^3$ applies to $n \geq 10$.

Induction Example 3

> It is shown that the n-second derivative of the function $f(x) = \frac{x}{1-x}$ it is $f^{(n)}(x) = (-1)^{n-1} n!(x-1)^{-(n+1)}$.

- For $n=1$ we have: $f'(x) = (-1)^0 1!(x-1)^{-(1+1)} = (x-1)^{-2}$ which is valid since
  $f'(x) = \frac{1(1-x)-x(1-x)'}{(x-1)^2} = \frac{1-x-x(-1)}{(x-1)^2} = \frac{1}{(x-1)^2}$.
- Let it be true for $n=k$, that is $f^{(k)}(x) = (-1)^{k-1} k!(x-1)^{-(k+1)}$.
- We will show that it is true and [i for $n=k+1$. Thus we have
  $f^{(k+1)}(x) = f^{(k)}(x)' = \left((-1)^{k-1} k!(x-1)^{-(k+1)}\right)' =$
  $(-1)^{k-1} k!(-(k+1))(x-1)^{-(k+1)-1} = (-1)^k (k+1)!(x-1)^{-(k+2)}$

Thus we have shown that $f^{(n)}(x) = (-1)^{n-1} n!(x-1)^{-(n+1)}, n \in \mathbb{N}$.

Definition of sequence

An important family of functions is that consisting of functions with domain the set of natural numbers. $\mathbb{N} = \{0, 1, \cdots\}$ (or the $\mathbb{N}_r = \{p, p+1, r+2, \cdots\}$ for some positive integer r). Such functions are called sequences.

Each function
$a : \mathbb{N} \to E,\ n \mapsto a(n) \in E$ (or $a : \mathbb{N}_r \to E,\ n \mapsto a(n) \in E$)
with domain the set of natural numbers $\mathbb{N}$ (or $\mathbb{N}_r$) and values in a set E, is called a sequence of elements of the set E in $\mathbb{N}$ (or in $\mathbb{N}_r$). In particular, if $E \subseteq \mathbb{R}$ the sequence is called a sequence of real numbers.

Sequences of real numbers

We will focus on the case of sequences of real numbers.

In the above correspondence the values of the sequence $a : \mathbb{N} \ni n \to a(n) \in \mathbb{R}$, are called terms of the sequence and the natural number n is called the index or class of term $a(n)$ which is also called n-th or general term of the sequence.

For the sake of brevity and simplicity, we will denote the sequence by $(a_n)_{n\in\mathbb{N}}$ or (a_n) and the $a(n)$ with a_n.

Representation forms of sequences of real numbers

We can represent sequences of real numbers either by giving the general term:

B.C. $a_n = \left(\frac{1}{2}\right)^n, n \in \mathbb{N}$

$$a_n = \begin{cases} 0, & n = 2r+1,\ r \in \mathbb{N} \\ 1, & n = 2^p,\ p \in \mathbb{N} \end{cases}$$

or by giving the recursive relation of the sequence and its initial value:
e.g. $a_{n+1} = 2a_n + 5$, $a_1 = 1$.

General form of reductive type (recursive relation)

When we give the recursive relation (reductive formula) of a sequence, the necessary first terms must be given and the recursive relation must allow us to find the next term. a_{n+1} of each term a_n from its predecessor, or more generally than some of its predecessors. Thus we have sequences of the form:

$a_1 = a (a \in \mathbb{R})$ and $a_{n+1} = f(a_n)$ or more

generally of the form:

$a_1 = a,\ a_2 = b (a, b \in \mathbb{R})$ and $a_{n+1} = f(a_n,\ a_{n-1})$

Blocked sequences

A sequence (a_n) we say that it is lower bounded if and only if there exists real number ϕ_k such that it is $a_n \geq \phi_k$ for each $n \in \mathbb{N}$.

Symbolically:

(a_n) bottom blocked $\Leftrightarrow \exists \phi_k \in \mathbb{R} : \forall n \in \mathbb{N},\ a_n \geq \phi_k$.

The number ϕ_k (as well as any other real numbers $s < \phi_k$) we say is a lower bound of the sequence.

Blocked sequences

A sequence (a_n) we say that it is upper bounded if and only if there exists real number ϕ_a such that it is $a_n \leq \phi_a$ for each $n \in \mathbb{N}$.

Symbolically:

(a_n) upper blocked $\Leftrightarrow \exists \phi_a \in \mathbb{R} : \forall n \in \mathbb{N},\ a_n \leq \phi_a$.

The number ϕ_a (as well as any other real number $s > \phi_a$) we say is an upper bound of the sequence.

Blocked sequences

A sequence (a_n) we say that it is bounded if and only if it is above and lower bounded, that is, if there are real numbers ϕ_k, ϕ_a ($\phi_k \leq \phi_a$) such that they are $\phi_k \leq a_n \leq \phi_a$ for each $n \in \mathbb{N}$.

Symbolically:

(a_n) blocked $\Leftrightarrow \exists \phi_k, \phi_a \in \mathbb{R} : \forall n \in \mathbb{N}, \phi_k \leq a_n \leq \phi_a$.

Blocked sequences

A sequence (a_n) we say that it is absolutely closed if and only if there exists real number ϕ, such that it holds $|a_n| \leq \phi$ for each $n \in \mathbb{N}$.

Symbolically:

(a_n) completely blocked $\Leftrightarrow \exists \phi \in \mathbb{R}^*_+ : \forall n \in \mathbb{N},\ |a_n| \leq \phi$.
The number ϕ (as well as any other real numbers $s > \phi$), we say that it is an absolute bound of the sequence. We can easily see that:
(a_n) blocked $\Leftrightarrow$ absolutely blocked (it is enough to consider $\phi = max\{|\phi_k|, |\varphi_a|\}$).

Blocked sequences

The minimum upper bound of an upper bounded sequence (a_n) is called the supremum of (a_n) and is denoted by $\sup a_n$.

The maximum lower bound of a lower bounded sequence (a_n) is called the infimum of (a_n) and is denoted by $\inf a_n$.

If a sequence (a_n) is not upper bounded, then we assume that $\sup a_n = +\infty$
Similarly, if a sequence (a_n) is not lower bounded, then we assume that $\inf a_n = -\infty$.

Example of a blocked sequence

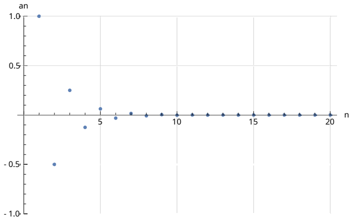
Prove that the sequence $a_n = 1 + (-\frac{1}{2}) + (-\frac{1}{2})^2 + \cdots + (-\frac{1}{2})^{n-1}$ it is blocked.

We have:

$$a_n = 1 + (-\tfrac{1}{2}) + (-\tfrac{1}{2})^2 + \cdots + (-\tfrac{1}{2})^{n-1} = \frac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})} = \frac{2}{3}\left(1-(-\tfrac{1}{2})^n\right).$$

so

$$\forall n \in \mathbb{N},\ |a_n| = \frac{2}{3}\left|1-(-\tfrac{1}{2})^n\right| \le \frac{2}{3}\left(1+\left|(-\tfrac{1}{2})^n\right|\right) = \frac{2}{3}\left(1+\frac{1}{2^n}\right) \le \frac{2}{3}\left(1+\frac{1}{2}\right) = 1.$$



Example of a blocked sequence

Prove that the sequences $a_n = \frac{3n^2-1}{2n^2+1}$ and $b_n = \frac{3n^2+1}{2n^2-1}$ they are blocked.

Both sequences are positive, so lower bounded by 0. For a_n we have:

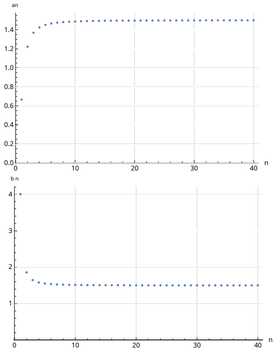
$$a_n = \frac{3n^2-1}{2n^2+1} \leq \frac{3n^2}{2n^2} = \frac{3}{2},$$

so it is blocked above by $\frac{3}{2}$.

For the b_n we have:

$$b_n = \frac{3n^2+1}{2n^2-1} \leq \frac{3n^2+n^2}{2n^2-n^2} = 4,$$

so it is upper bounded by 4.



Monotone sequences

A sequence (a_n) we say that it is increasing if and only if it holds $a_n \leq a_{n+1}$ for each $n \in \mathbb{N}$. Symbolically:
(a_n) ascending $\Leftrightarrow \forall n \in \mathbb{N},\ a_n \leq a_{n+1}$.

A sequence (a_n) we say that it is purely increasing if and only if it holds $a_n < a_{n+1}$ for each $n \in \mathbb{N}$. Symbolically:
(a_n) purely increasing $\Leftrightarrow \forall n \in \mathbb{N},\ a_n < a_{n+1}$.

Monotone sequences

A sequence (a_n) we say that it is decreasing if and only if it holds $a_n \geq a_{n+1}$ for each $n \in \mathbb{N}$. Symbolically:
(a_n) decreasing $\Leftrightarrow \forall n \in \mathbb{N},\ a_n \geq a_{n+1}$.

A sequence (a_n) we say that it is genuinely decreasing if and only if it holds $a_n > a_{n+1}$ for each $n \in \mathbb{N}$. Symbolically:
(a_n) genuinely decreasing $\Leftrightarrow \forall n \in \mathbb{N},\ a_n > a_{n+1}$.

Monotone sequences

A (purely) increasing or decreasing sequence is called a (purely) monotonic sequence.

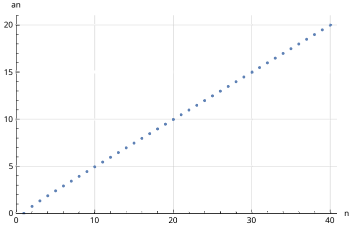
To test a sequence for monotonicity, we usually work with one of the following methods:
1) We examine the sign of the difference (of successive terms) $a_{n+1} - a_n$
2) We compare the ratio (of successive terms) $\frac{a_{n+1}}{a_n}$ (when the terms of the sequence retain sign).
3) Having an indication of monotonicity from the anisotropy relationship between the first terms of the sequence, we use the method of mathematical induction to show that this holds for every $n \in \mathbb{N}$.

Monotone sequences

Check the sequence for monotonicity $a_n = \frac{n^2-1}{2n}$.

We have that $a_{n+1} - a_n = \frac{(n+1)^2-1}{2(n+1)} - \frac{n^2-1}{2n} = \frac{n^2+n+1}{2n(n+1)} > 0, \forall n \in \mathbb{N}$. So the a_n it is truly ascending.

Alternatively, we have: $a_n = \frac{n^2-1}{2n} = \frac{n}{2} - \frac{1}{2n} < \frac{n+1}{2} - \frac{1}{2(n+1)} = a_{n+1}$.



Subsequences

Let the sequence of real numbers (a_n) and a purely increasing a- sequence of natural numbers (s_n). We can then define the sequence (b_n) with $b_n = a_{s_n}, n \in \mathbb{N}$. This is the sequence with terms:

$$a_{s_1}, a_{s_2}, \cdots, a_{s_n}, \cdots$$

and which is called a subsequence of (a_n).

Example: If we consider the sequence (a_n) with $a_n = \frac{(-1)^n}{n}$, then the subsequence which results if $s_n = 2n$ is the $b_n = a_{2n} = \frac{1}{2n}$.

The concept of limit

We say that the sequence (a_n) converges to $a \in \mathbb{R}$, if and only if for each $\epsilon > 0$ there is a natural number $n_0 = n_0(\epsilon)$ such that it is $|a_n - a| < \epsilon$ for each $n \geq n_0$
Symbolically:

$$\lim a_n = a \Leftrightarrow \forall \epsilon > 0, \exists n_0 = n_0(\epsilon) : |a_n - a| < \epsilon, \forall n \geq n_0$$

The concept of the limit of a sequence geometrically means that, if a sequence converges to a real number a then, any ϵ-area of a and if we choose, after some term of the sequence all the following ones will be in this region, no matter how small it is.

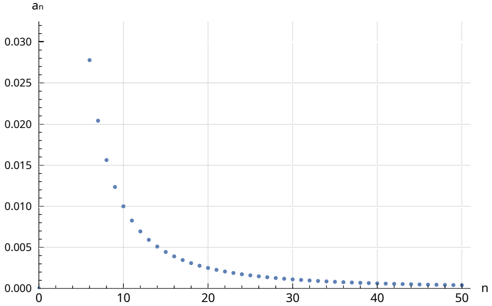
Limit proof example

The sequence (a_n) with $(a_n) = \frac{1}{n^2}$ has a limit 0 (is zero). Based on the definition of the limit we have:

$$\left|a_n - 0\right| = \frac{1}{n^2} < \epsilon \Leftrightarrow n^2 > \frac{1}{\epsilon} \Leftrightarrow n > \sqrt{\frac{1}{\epsilon}} \Leftrightarrow \ldots$$

$$n \geq n_0, \quad n_0 := \left\lfloor \sqrt{\frac{1}{\epsilon}} \right\rfloor + 1$$

Therefore $\forall \epsilon > 0, \exists n_0 = \left\lfloor \sqrt{\frac{1}{\epsilon}} \right\rfloor + 1 : \left|\frac{1}{n^2} - 0\right| < \epsilon, \forall n \geq n_0$



Properties of convergent sequences

We observe that the algebra of limits of convergent sequences is identical to the properties of the algebra of real numbers. That is, if (a_n) and (b_n) is convergent sequences with limits a and b respectively then:

(a) $\lim ca_n = ca$

(b) $\lim (a_n \pm \beta_n) = a \pm \beta$ (c)

$\lim (a_n)(b_n) = ab$

(d) $\lim (a_n/b_n) = a/b$ provided that $b \neq 0$

Properties of convergent sequences

The limit of a convergent sequence is unique.

$$\left.\begin{array}{l} a_n \to a \\ a_n \to b \end{array}\right\} \Rightarrow a = b$$

Any subsequence of a convergent sequence has the same limit as the sequence. In summary:

$$a_n \to a \Rightarrow a_{s_n} \to a$$

Properties of convergent sequences

If $k \in \mathbb{N}$ and $a \in \mathbb{R}$ then the equivalence holds:

$$a_n \to a \Leftrightarrow a_{n+k} \to a$$

Every convergent sequence is bounded. In summary:

$$a_n \to a \Rightarrow (a_n) \text{ blocked.}$$

The product of a zero sequence times a bounded sequence is a zero a-sequence. In summary:

$$\left.\begin{array}{r} a_n \to 0 \\ (b_n) \text{ blocked} \end{array}\right\} \Rightarrow a_n b_n \to 0$$

Properties of convergent sequences

If the (b_n) is a null sequence and (a_n) sequence such that for each $n \geq n_0 \in \mathbb{N}$ to be $|a_n| \leq s|b_n|, > 0$, then the sequence (a_n) is zero. In summary:

$$\left.\begin{array}{r} a_n \leq s|b_n|, \forall n \geq n_0, > 0 \\ (b_n) \to 0 \end{array}\right\} \Rightarrow a_n \to 0$$

Property of isoconvergent sequences (interpolation criterion):

$$\left.\begin{array}{l} b_n \leq a_n \leq c_n, \forall n \geq n_0 \\ \lim b_n = \lim c_n = a \end{array}\right\} \Rightarrow a_n \to a$$

Properties of convergent sequences

If the sequences (a_n) and (b_n) are convergent and it holds $a_n < b_n$ for each $n \in \mathbb{N}$, then it will be $\lim a_n \leq \lim b_n$ In summary:

$$\left.\begin{array}{l} a_n \to a,\ b_n \to b\ a_n \\ < b_n, \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow a \leq b$$

For each sequence (a_n) is valid

$$\left.\begin{array}{r} a_{2n} \to a \\ a_{2n-1} \to a \end{array}\right\} \Leftrightarrow a_n \to a$$

Properties of convergent sequences

Convergence criterion of Cauchy

A sequence (a_n) converges if and only if for each $\epsilon > 0$ there is a natural number $n_0 = n_0(\epsilon)$ such that it is $|a_p - a_q| < \epsilon$ for each $p, q \geq n_0$ Symbolically:

$$\lim a_n = a \Leftrightarrow \forall \epsilon > 0, \exists n = n_0(\epsilon) : |a_p - a_q| < \epsilon, \forall p, q \geq n_0.$$

Remark: based on this criterion, we do not need to know the limit a in order to show that the sequence (a_n) converges.

Properties of convergent sequences

Every monotone and bounded sequence is convergent. In particular,

$$\left.\begin{array}{l}(a_n)\text{ ascending}\\ a_n \le \phi_a, \forall n \in \mathbb{N}\end{array}\right\} \Rightarrow \lim a_n = a \le \phi_a$$

$$\left.\begin{array}{l}(a_n)\text{ decreasing}\\ a_n \ge \phi_k, \forall n \in \mathbb{N}\end{array}\right\} \Rightarrow \lim a_n = a \ge \phi_k$$

In these cases the sequences converge to $\sup a_n$ and $\inf a_n$ respectively.

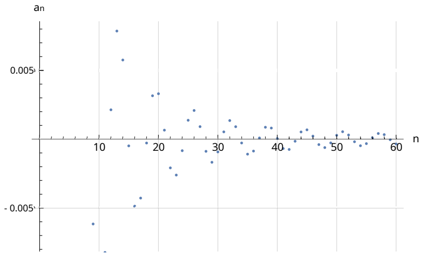
Example

Find the limit of the sequence $a_n = \frac{\sin(n)+\cos(n)}{n^2}$.

It is true that $-1 \leq \sin(n) \leq 1$ and $-1 \leq \cos(n) \leq 1$.

Therefore $-2 \leq \sin(n) + \cos(n) \leq 2$ and $\frac{-2}{n^2} \leq \frac{\sin(n)+\cos(n)}{n^2} \leq \frac{2}{n^2}$.

But $\lim \frac{-2}{n^2} = 0$ and $\lim \frac{2}{n^2} = 0$. Therefore, according to the interpolation criterion we have: $\lim \frac{\sin(n)+\cos(n)}{n^2} = 0$.



Finite sequences

We say that the sequence (a_n) is positively infinite or that the limit of (a_n) is the $+\infty$, if and only if for each $M > 0$ exists as a natural number $n_0 = n_0(M)$ (i.e. which depends on the M) such that it is $a_n > M$ for each $n \geq n_0$ Symbolically,

$$\lim a_n = +\infty \Leftrightarrow \forall M > 0, \exists n_0 = n_0(M) : a_n > M, \forall n \geq n_0$$

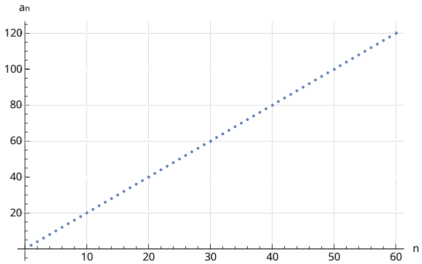
We say that the sequence (a_n) is negatively infinite or that the limit of (a_n) is the $-\infty$, if and only if for each $M > 0$ exists as a natural number $n_0 = n_0(M)$ (i.e. which depends on the M) such that it is $a_n < -M$ for each $n \geq n_0$
Symbolically,

$$\lim a_n = -\infty \Leftrightarrow \forall M > 0, \exists n_0 = n_0(M) : a_n < -M, \forall n \geq n_0$$

Example

Show that the sequence $a_n = 2n$ is positively experienced.

We need to show that for any value of $M > 0$ exists $n_0 = n_0(M)$ suchCome on. $a_n > M, \forall n > n_0$, equivalent $2n > M, \forall n > n_0$ We select the $n_0 = \frac{M}{2} + 1$.



Theorem

We assume that (a_n) is a convergent sequence with limit a, that the (b_n) is positively inferred. It is true that:

(a) $\lim ab_n = +\infty$ for $a > 0$ and $-\infty$ for $a < 0$. (b) $\lim (a_n + b_n) = +\infty$ (c) $\lim (a_n - b_n) = -\infty$

(d) $\lim (a_n)(b_n) = +\infty$ for $a > 0$ and $-\infty$ for $a < 0$ (e) $\lim (a_n / b_n) = 0$

Relationship between limits of functions and limits of sequences

If $\lim_{x \to +\infty} f(x) = L$ and $f(n) = a_n$ where $n \in \mathbb{N}$, then $\lim a_n = L$ ($L \in \mathbb{R}$ or $\pm\infty$).

If $\lim a_n = a$ and the function f is continuous in $x = a$, then

$$\lim f(a_n) = f(a)$$

Exercise

Show that $\lim_{n \to \infty} \frac{\ln(\ln(n))}{\ln(n)} = 0$.

Solution

We calculate the limit

$$\lim_{x\to\infty} \frac{\ln(\ln(x))}{\ln(x)} \overset{L'Hospital\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to\infty} \frac{(\ln(\ln(x)))'}{(\ln(x))'} = \lim_{x\to\infty} \frac{\frac{1}{\ln(x)}\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\ln(x)} = 0$$

Application

Show that the sequence $a_{n+1}=\frac{2}{5}a_n+5$, $a_0=0$ is convergent.

The tax rate is increasing.

For $n=0$ it is true that $a_0=0$ and $a_1=5>a_0$.

We proceed with induction trying to show that it is increasing:

For $n=k$ we assume that $a_{k+1}\geq a_k \Leftrightarrow \frac{2}{5}a_k+5\geq a_k \Leftrightarrow \frac{3}{5}a_k\leq 5 \Leftrightarrow a_k\leq \frac{25}{3}$.

For $n=k+1$ T.d.o. $a_{k+2}\geq a_{k+1}$ But, $a_{k+2}\geq a_{k+1} \Leftrightarrow \frac{2}{5}a_{k+1}+5\geq a_{k+1} \Leftrightarrow \frac{2}{5}\left(\frac{2}{5}a_k+5\right)+5\geq \frac{2}{5}a_k+5 \Leftrightarrow \frac{4}{25}a_k+7\geq \frac{2}{5}a_k+5 \Leftrightarrow 2\geq \frac{6}{25}a_k \Leftrightarrow a_k\leq \frac{25}{3}$ where is valid from the inductive hypothesis.

Application

The heating system is blocked from above.

Let it be s an upper dam of a_n. Then
$a_n < s \Leftrightarrow \frac{2}{5}a_n < \frac{2}{5}s \Leftrightarrow \frac{2}{5}a_n + 5 < \frac{2}{5}s + 5 \Leftrightarrow a_{n+1} < \frac{2}{5}s + 5$. If we choose $s \geq \frac{2}{5}s + 5 \Leftrightarrow \frac{3}{5}s \geq 5 \Leftrightarrow \sigma \geq \frac{25}{3}$ (Alternatively, we could directly solve the equation $x = \frac{2}{5}x + 5 \Leftrightarrow \frac{3}{5}x = 5 \Leftrightarrow x = \frac{25}{3}$).

We show the desired result by induction:

- For $n = 0$, $a_0 = 0 < \frac{25}{3}$.
- For $n = k$, even if $a_k \leq \frac{25}{3}$.
- For $n = k+1$, T.D.O. $a_{k+1} \leq \frac{25}{3}$.
  From the inductive hypothesis we have
  $a_k \leq \frac{25}{3} \Leftrightarrow \frac{2}{5}a_k \leq \frac{2}{5}\frac{25}{3} \Leftrightarrow \frac{2}{5}a_k + 5 \leq \frac{10}{3} + 5 \Leftrightarrow a_{k+1} \leq \frac{25}{3}$.

Therefore, the sequence is convergent as increasing and upper bounded.

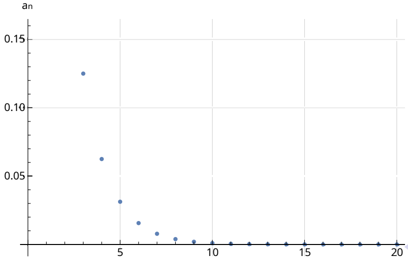
Example

Show that the sequence $a_n = \frac{1}{2^n}, n = 1, 2, 3, \cdots$ is convergent.

The sequence $a_n = \frac{1}{2^n}$ is decreasing. To see this, we take

considering that:

$a_{n+1} = \frac{1}{2^{n+1}} = \frac{1}{2}\frac{1}{2^n} = \frac{1}{2}a_n$ and therefore $a_{n+1} < a_n, \forall n \in \mathbb{N}$. Also the sequence
This is blocked below since: $0 < \frac{1}{2^n} < 1, \forall n \in \mathbb{N}$. Therefore the sequence is convergent.



Useful known limits

- $\lim_{n\to\infty} \frac{1}{n} = 0$
- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} \sqrt[n]{a} = 1, a > 0$
- $\lim_{n\to\infty} a^n = \begin{cases} 0 & |a| < 1 \\ 1 & a = 1 \\ +\infty & a > 1 \\ \text{there is no} & a \leq -1 \end{cases}$

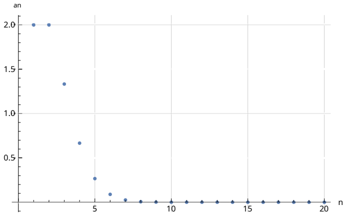
Reason criterion

When $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = b$, with $a_n \neq 0$ and $0 \le b < 1$ then $a_n \to 0$.
If >1 then $a_n \to \infty$, while if $b = 1$ we cannot decide.

Example:
Let it be $a_n = \frac{2^n}{n!}$. Then based on the ratio criterion we have:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0.$$

Therefore the sequence converges to 0.



Root criterion

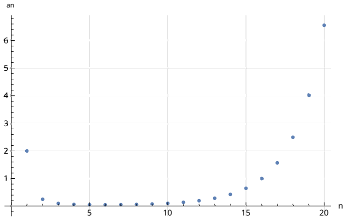
When $\lim_{n\to\infty} \sqrt[n]{a_n} = b$, with $a_n \geq 0$ and $0 \leq b < 1$ then $a_n \to 0$.
If >1 then $a_n \to \infty$, while if $b = 1$ we cannot decide.

Example:

Let it be $a_n = \frac{2^n}{n^4}$. Then based on the root criterion we have:

$$\sqrt[n]{\frac{2^n}{n^4}} = \frac{2}{(\sqrt[n]{n})^4} \xrightarrow[n\to\infty]{} \frac{2}{1^4} = 2 > 1$$

Therefore the sequence diverges at $+\infty$.



Examples

Examine the sequence for convergence: $a_n = \frac{n^2}{n^4+n^3+1}$

We observe that $0 \le$ we $\frac{n^2}{n^4+n^3+1} \le \frac{n^2}{n^3} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$, so a by the interference criterion conclude that $a_n \to 0$.

Examine the sequence for convergence: $a_n = \frac{4\cdot 10^n - 3\cdot 10^{2n}}{3\cdot 10^{n-1} - 2\cdot 10^{2n-1}}$

We divide the numerator and denominator by 10^{2n-1} and we have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4\frac{1}{10^{n-1}} - 3\cdot 10}{3\cdot\frac{1}{10^n} - 2} = \frac{4\cdot 0 - 3\cdot 10}{3\cdot 0 - 2} = 15$$

Examples

Examine the sequence for convergence: $a_n = \frac{n^2 + 3^n - 1}{4^n + n}$

We divide the numerator and denominator by 4^n and we have:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{n^2}{2^n} + \left(\frac{3}{4}\right)^n - \frac{1}{4^n}}{1 + \frac{n}{4^n}}$. We will first calculate the limit $\lim_{n\to\infty} \frac{n^2}{2^n}$ using it criterion of speech: $\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{2} = \frac{1}{2} < 1$

therefore $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$. In a similar way we calculate that $\lim_{n\to\infty} \frac{n}{4^n} = 0$. Cat we know that $\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = \lim_{n\to\infty} \frac{1}{4^n} = 0$. Finally, so finally: $\lim_{n\to\infty} a_n = \frac{0+0-0}{1+0} = 0$.

Examples

Examine the sequence for convergence: $a_n = \frac{1+3^n}{1+\lambda^n+3^n}$ where $\lambda \geq 0$.

We divide by 3^n and we have: $a_n = \frac{(\frac{1}{3})^n + 1}{(\frac{1}{3})^n + (\frac{\lambda}{3})^n + 1}$ We distinguish 3 cases:

- $0 \leq \lambda < 3$ where applicable $\lim\limits_{n\to\infty} (\lambda/3)^n = 0$, so
$$\lim_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty} (1/3)^n + 1}{\lim\limits_{n\to\infty} (1/3)^n + \lim\limits_{n\to\infty} (\lambda/3)^n + 1} = 1.$$
- $\lambda = 3$ where applicable $\lim\limits_{n\to\infty} (3/3)^n = 1$, so $\lim\limits_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty} (1/3)^n + 1}{\lim\limits_{n\to\infty} (1/3)^n + \lim\limits_{n\to\infty} (3/3)^n + 1} = \frac{1}{2}.$
- $\lambda > 3$ where applicable $\lim\limits_{n\to\infty} (\lambda/3)^n = +\infty$, so
$$\lim_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty} (1/3)^n + 1}{\lim\limits_{n\to\infty} (1/3)^n + \lim\limits_{n\to\infty} (\lambda/3)^n + 1} = \frac{0+1}{1+\infty+1} = 0.$$

Examples

Examine the sequence for convergence. $a_n = \sqrt{2 + a_{n-1}}$ with $a_1 = \sqrt{2}$.

We will first show that it is closed. Let $s > 0$ an upper dam of a_n, then we will have equivalents $a_{n-1} < s \Leftrightarrow 2 + a_{n-1} < 2 + s \Leftrightarrow a_n = \sqrt{2 + a_{n-1}} < \sqrt{2 + s}$. Enough let's choose the s so that $s \geq \sqrt{2 + s} \Leftrightarrow s^2 \geq 2 + s \Leftrightarrow s^2 - s - 2 \geq 0 \Leftrightarrow (s+1)(s-2) \geq 0$. Because $s > 0$ then that's enough $s \geq 2$, so we choose $s = 2$.

Continuing with induction:

- For $n = 1$ we have $a_1 = \sqrt{2} < 2$.
- Let it be true for $n = k$, that is $a_k < 2$.
- Then $a_k < 2 \Leftrightarrow a_k + 2 < 2 + 2 \Leftrightarrow \sqrt{a_k + 2} < \sqrt{4} \Leftrightarrow a_{k+1} < 2$, therefore it is valid.

We will show that it is increasing. We have:

$a_n^2 - a_{n-1}^2 = 2 + a_{n-1} - a_{n-1}^2 = -(a_{n-1}^2 - a_{n-1} - 2) = (2 - a_{n-1})(a_{n-1} + 1) > 0$ because it is true $0 < a_{n-1} < 2$. Therefore $a_n^2 > a_{n-1}^2 \Leftrightarrow a_n > a_{n-1}$ (since the terms of sequence are positive). Therefore the sequence is purely increasing.

Exercise

To what number does the converge? $a_n = \sqrt{2 + a_{n-1}}$ with $a_1 = \sqrt{2}$;

Solution

Let it be $x = \lim_{x\to\infty} a_n \geq 0$. Then, $x = \lim_{x\to\infty} \sqrt{2 + a_{n-1}} = \sqrt{2 + \lim_{x\to\infty} a_{n-1}} = \sqrt{2 + x}$.

We square both sides: $x^2 - x - 2 = 0 \Leftrightarrow (x-2)(x+1) = 0 \Leftrightarrow x = 2$ (after $x \geq 0$).