Mathematical analysis
Lecture 4

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Topics of the 4th lecture

- Series
- Convergence of series
- Powertrains
- DevelopmentTaylor

Series

The concept of a series refers to the 'sum' of an infinite number of terms in a sequence of real numbers. a_n:

$$a_1 + a_2 + \ldots + a_n + \ldots$$

We define the sequence of partial sums of a_n as:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$

$$\ldots$$

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{I=1}^{n} a_I$$

Example

Let the sequence be $a_n = \frac{1}{n^2}$. The sequence of partial sums of a_n is:

$$S_1 = \frac{1}{1}$$

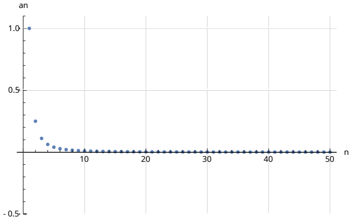
$$S_2 = \frac{1}{1} + \frac{1}{4}$$

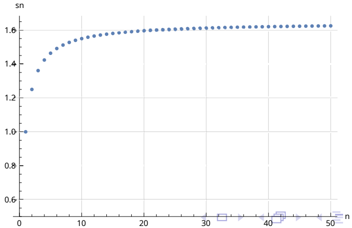
$$S_3 = \frac{1}{1} + \frac{1}{4} + \frac{1}{9}$$

$$S_4 = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$$

$$\dots$$

$$S_n = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{n^2} = \sum_{i=1}^{n} \frac{1}{i^2}$$





Series

The ordered pair $((a_n), S_n)$ is called a sequence of real numbers, and is symbolized by

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \text{or} \quad \sum_{n=1}^{\infty} a_n$$

The number a_n is called general or n-th term of the series and the sum S_n n-bone partial sum of the series.

We will say that the series $\sum_{n=1}^{\infty} a_n$ converges to a number $s \in \mathbb{R}$ if and only if the sequence of partial sums S_n converges to the number s. Then we write

$$\sum_{n=1}^{\infty} a_n = s \text{or} \lim_{n\to\infty} S_n = s$$

The number s is called the sum of the series.

If the sequence S_n does not converge to $\mathbb{R}$, then we say that the series diverges ($\sum_{n=1}^{\infty} a_n = \pm\infty$ or $\lim_{n\to\infty} S_n = \pm\infty$).

Examples of partial sums

- Sum n first terms of an arithmetic progression: $a_1 = a$, $a_2 = a + oh, a_3 = a + 2oh, \ldots, a_n = a + (n-1)oh$
  If we calculate the S_n we have:
  $S_n = a + a + oh + a + 2oh + \ldots + a + (n-1)oh = \left(\frac{a_1 + a_n}{2}\right) n = yes + \frac{n.n.1(-)}{2} oh$
  If $oh > 0$ then $\lim_{n\to\infty} S_n = +\infty$, so $\sum_{n=1}^{\infty} a_n = +\infty$
- Sum n first terms of geometric progression: $a_1 = a, a_2 = a\, l, a_3 = a\, l^2, \ldots a_n = a\, l^{n-1}$
  If we calculate the S_n we have:
  $S_n = a + a\, l + a\, l^2 + \ldots + a\, l^{n-1} = \frac{a(l^n - 1)}{\lambda - 1}, l \neq 1$
  If $|l| < 1$ then $\lim_{n\to\infty} S_n = \frac{a}{1-\lambda}$, so $\sum_{n=1}^{\infty} a_n = \frac{a}{1-\lambda}$

Non-convergence criterion

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(Equivalent sentence): If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ deviates.

Attention: If $\lim_{n\to\infty} a_n = 0$ does not imply that the series $\sum_{n=1}^{\infty} a_n$ converges.

Examples

- The series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ does not converge because $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.
- The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, even though $\lim_{n\to\infty} \frac{1}{n} = 0$ (it turns out with the convergence criterion *Cauchy* for the S_n - Lecture3 $|S_{2n} - S_n| = \frac{1}{n+1} + \cdots + \frac{1}{2n} > \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}$).
- The harmonic series r-class or series Dirichlet: $g(r) = \sum_{n=1}^{\infty} \frac{1}{n^r}$, converges if $p > 1$ while it deviates if $r \leq 1$.
- $\sum_{n=1}^{\infty} \frac{1}{n!} = e$
- $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{p^2}{6}$

Comparison Criterion

Let it be $a_n \geq 0, b_n \geq 0$ and $a_n \leq b_n$. Then:

- If the series $\sum_{n=1}^{\infty} b_n$ converges, then converges as well $\sum_{n=1}^{\infty} a_n$
- If the series $\sum_{n=1}^{\infty} a_n$ deviates, then the $\sum_{n=1}^{\infty} b_n$

Calculate whether the series converges. $\sum_{n=1}^{\infty} \frac{5}{3^n+1}$.

We have $3^n < 3^n + 1 \Rightarrow \frac{5}{3^n} > \frac{5}{3^n+1} \Rightarrow 5\left(\frac{1}{3}\right)^n > \frac{5}{3^n+1}$ But the series $\sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n = 5\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converges as a geometric function with a ratio $\frac{1}{3} < 1$, so it also converges original.

Finding

Let it be $a_n \geq 0, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = k$. Then:

1. If $0 < k < +\infty$ then the $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
2. If $k = 0$ then
   - If the $\sum_{n=1}^{\infty} b_n$ converges, then the will also converge $\sum_{n=1}^{\infty} a_n$
   - If the $\sum_{n=1}^{\infty} a_n$ deviates, then the will also deviate $\sum_{n=1}^{\infty} b_n$
3. If $k = \infty$ then
   - If the $\sum_{n=1}^{\infty} a_n$ converges, then the will also converge $\sum_{n=1}^{\infty} b_n$
   - If the $\sum_{n=1}^{\infty} b_n$ deviates, then the will also deviate $\sum_{n=1}^{\infty} a_n$.

Example

To examine the convergence of the series $\sum_{n=1}^{\infty} \frac{\ln n}{2n^3-1}$.

We consider the convergent series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

We calculate $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 \ln n}{2n^3-1} = \lim_{n\to\infty} \frac{\frac{1}{n}\ln n}{2-\frac{1}{n^3}} = 0$ (why?)

From the above conclusion (2), it follows that the series converges.

Reason Criterion (D'Alembert)

If $a_n > 0$, then:

- If $\frac{a_{n+1}}{a_n} \leq r < 1, \forall n > n_0$, then the sequence $\sum_{n=1}^{\infty} a_n$ converges.
- If $\frac{a_{n+1}}{a_n} \geq 1, \forall n > n_0$, then the sequence $\sum_{n=1}^{\infty} a_n$ deviates.

If there is $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$, then the series converges for $r < 1$ and deviates for > 1. For $r = 1$ we cannot decide.

Examples

To examine the series for convergence $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

We observe that

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2 (2n)!}{(2(n+1))!\,(n!)^2} = \frac{(n!(n+1))^2}{(2n)!(2n+1)(2n+2)} \frac{(2n)!}{(n!)^2} = \frac{n^2+2n+1}{4n^2+6n+2} \xrightarrow[n\to\infty]{} \frac{1}{4} < 1.$$

So the order converges.

To examine the series for convergence $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$.

We observe that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = \frac{2^{n+1} n^2}{2^n (n+1)^2} = 2\left(\frac{n}{n+1}\right)^2 \xrightarrow[n\to\infty]{} 2 > 1$$

so the series does not converges.

Root Criterion (Cauchy)

If $a_n > 0$, then:

- If $\sqrt[n]{a_n} \le r < 1, \forall n > n_0$, then the sequence $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sqrt[n]{a_n} \ge 1, \forall n > n_0$, then the sequence $\sum_{n=1}^{\infty} a_n$ deviates.

If there is $\lim_{n\to\infty} \sqrt[n]{a_n} = r$, then the series converges for $r < 1$ and deviates for > 1. For $r = 1$ we cannot decide.

Examples

> To examine the series for convergence $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$.

We observe that $\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \sqrt[n]{\left[\left(\frac{n}{n+1}\right)^{n}\right]^{n}} = \left(\frac{n}{n+1}\right)^{n} = \frac{1}{\left(\frac{n+1}{n}\right)^{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}} \xrightarrow{n\to\infty} \frac{1}{e} < 1$. So the order converges.

> To examine the series for convergence $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2}$.

We observe that $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{\sqrt[n]{n}\,\sqrt[n]{n}} \xrightarrow{n\to\infty} \frac{2}{1\cdot 1} = 2 > 1$ so the series diverges.

Absolute convergence

If the $\sum_{n=1}^{\infty} |a_n|$ converges, then converges as well $\sum_{n=1}^{\infty} a_n$.

Based on this theorem, we can counter the requirement $a_n > 0$ in the Ratio and Root criterion (using absolute value).

Be careful, if the $\sum_{n=1}^{\infty} a_n$ converges then it does not imply that it converges absolutely.

Completion Criterion

Let's assume that the $f(x)$ is a continuous, positive and decreasing function in space $[k, \infty)$ and $f(n) = a_n$. Then:

(1) If the integral $\int_k^\infty f(x)\,dx$ converges then the same is true for $\sum_{n=k}^{\infty} a_n$.

(2) If the integral $\int_k^\infty f(x)\,dx$ deviates then the same applies to the $\sum_{n=k}^{\infty} a_n$.

Example

Show that the series $\sum_{n=1}^{\infty} no^{-n^2}$ converges.

The function $f(x) = car^{-x^2}$ has a derivative $f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = e^{-x^2}(1 - 2x^2) < 0$ for $x \geq 1$. Therefore the $f(x)$ is descending to $[1, +\infty]$.

$$\int_1^{\infty} car^{-x^2} dx = -\frac{1}{2} \lim_{t \to +\infty} [e^{-x^2}]_1^t = \frac{1}{2e}.$$

Thus, according to the integral criterion, since the integral of $f(x)$ with $f(n) = no^{-n^2}$ converges, the corresponding series will also converge.

Alternating series

An alternating series is called one whose terms alternate their sign continuously, that is, it is of the form:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

Criterion of Leibniz

An alternating series converges when the following holds:

- $a_n > 0$
- is decreasing ($a_n \geq a_{n+1}, \forall n \in \mathbb{N}$)
- $\lim_{n \to \infty} a_n = 0$

Examples

Consider whether the sequence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ converges.

- $a_n = \frac{1}{2n-1} > 0, \forall n \in \mathbb{N}$
- a_n decreasing since $a_{n+1} = \frac{1}{2(n+1)-1} = \frac{1}{2n+1} \leq \frac{1}{2n-1} = a_n$
- $\lim_{n\to\infty} \frac{1}{2n-1} = 0$

Consider whether the sequence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$, with $p > 0$ converges.

- $a_n = \frac{1}{n^p} > 0, \forall n \in \mathbb{N}$
- a_n decreasing since $a_{n+1} = \frac{1}{(n+1)^p} \leq \frac{1}{n^p} = a_n$
- $\lim_{n\to\infty} \frac{1}{n^p} = 0$

Telescopic series

A series, $\sum_{n=1}^{\infty} a_n$, is called telescopic if it can be written as $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n - b_{n+1})$, where a_n and b_n sequences. A telescopic series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence b_n converges, in which case it is also true that $\sum_{n=1}^{\infty} a_n = b_1 - \lim_{n\to\infty} b_n$.

The convergence check in a telescopic series is carried out as follows: $s_n = a_1 + a_2 + a_2 + \cdots + a_{n-1} + a_n = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (b_{n-1} - b_n) + (b_n - b_{n+1}) = b_1 - b_{n+1}$, so the series will converge if the limit exists $\lim_{n\to\infty} b_{n+1}$ and its sum will be

$$\sum_{n=1}^{\infty} a_n = b_1 - \lim_{n\to\infty} b_n.$$

Example

Consider whether the sequence $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ converges.

The series is telescopic since $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$. Thus we have

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2} - \lim_{n\to\infty} \frac{1}{n+2} = \frac{1}{2} - 0 = \frac{1}{2}.$$

Consider whether the sequence $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ converges.

The series is telescopic since $\sum \frac{1}{4n^2-1} = \frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1}$. Thus we have

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2} - \lim_{n\to\infty} \frac{\frac{1}{2}}{2n+1} = \frac{1}{2} - 0 = \frac{1}{2}.$$

Powertrains

A power series is a series of the form:

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$$

where $c \in \mathbb{R}$ constant.

The x varies around the c, and for this reason we say that the series has a center c or that it is a power series around the point c.

Powertrains

The polynomial function:

$$S_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n, \forall n \in \mathbb{N}$$

is called the partial sum of the power series and the functions:

$$a_0, a_1(x - c), a_2(x - c)^2, \ldots, a_n(x - c)^n, \ldots$$

are called terms of the power series:

$$\sum_{n=0}^{\infty} a_n(x - c)^n$$

Convergence of power series

- A power series, although defined in $\mathbb{R}$, its convergence set does not is generally the whole $\mathbb{R}$.
- Every power series converges to its center, since for $x = c$ has a sum of a_0.
- If the c is not the only point where the power series converges, there will be a number r with $0 < r \leq \infty$, such that the power series converges when $|x - c| < r$ and deviate when $|x - c| > r$ The number r is called a radius convergence of the power series.

Ratio Criterion for power series

Let it be $a_n \neq 0$, for each $n \in \mathbb{N}$, and r the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$. Then -

$$r = \begin{cases} +\infty, & \text{if } \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 \\ 0, & \text{if } \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty \\ 1/\ell, & \text{if } \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ell \in \mathbb{R} \end{cases}$$

and the power series respectively:

- converges for every $x \in \mathbb{R}$
- deviates for each $x \in \mathbb{R} - \{c\}$
- converges with convergence interval $(c-r, c+r)$. In this case, we should check the convergence at the edges of the interval (replacing $x = c+r$ and $x = c-r$, and checking the resulting series for convergence).

Root Criterion for power series

Let it be $a_n \neq 0$, for each $n \in \mathbb{N}$, and r the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$. Then -

$$r = \begin{cases} +\infty, & \text{if } \lim_{n\to+\infty} \sqrt[n]{|a_n|} = 0 \\ 0, & \text{if } \lim_{n\to+\infty} \sqrt[n]{|a_n|} = +\infty \\ 1/\ell, & \text{if } \lim_{n\to+\infty} \sqrt[n]{|a_n|} = \ell \in \mathbb{R} \end{cases}$$

and the power series respectively:

- converges for every $x \in \mathbb{R}$
- deviates for each $x \in \mathbb{R} - \{c\}$
- converges with convergence interval $(c-r, c+r)$. In this case, we should check the convergence at the edges of the interval (replacing $x = c + r$ and $x = c - r$, and checking the resulting series for convergence).

Example

Find the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$.

We have $\displaystyle\lim_{n\to\infty}\left|\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}\right| = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\lim_{n\to\infty}\frac{n}{n+1}} = 1$ (alternatively $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|\frac{1}{\sqrt{n}}\right|} = \lim_{n\to\infty}\sqrt{\frac{1}{\sqrt[n]{n}}} = \sqrt{1} = 1$)

Therefore the radius of convergence is $r = \frac{1}{1} = 1$ and the convergence interval is $(c-r, c+r) = (1-1, 1+1) = (0,2)$. We should check both ends of interval:

- For $x=0$ the power series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ which converges (why?)
- For $x=2$ the power series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which deviates (why?)

So the convergence interval is [0,2).

Example

Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n}(x+2)^n$.

We have $\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)^2 3^{n+1}}}{\frac{(-1)^n}{n^2 3^n}} \right| = \lim_{n\to\infty} \frac{n^2}{3(n+1)^2} = \frac{1}{3}$.

Therefore the radius of convergence is $r = \frac{1}{1/3} = 3$ and the convergence interval is $(c-r, c+r) = (-2-3, -2+3) = (-5, 1)$. We should check both ends of space:

- For $x = -5$ the power series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n}(-3)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges (Why;)
- For $x = 1$ the power series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n} 3^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges (Why;)

So the convergence interval is $[-5, 1]$.

Geometric interpretation of power series convergence

We have shown that the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(x+2)^n$ converges in space [-5.1].

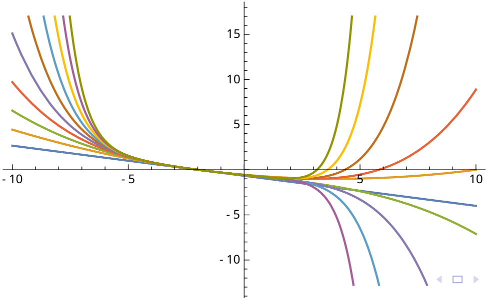
If we calculate the partial sums of the power series, we will have:

$S_1 = -\frac{1}{3}(x+2)$

$S_2 = -\frac{1}{3}(x+2) + \frac{1}{36}(2+x)^2$

$S_3 = -\frac{1}{3}(x+2) + \frac{1}{36}(2+x)^2 - \frac{1}{243}(2+x)^3$

...



DevelopmentTaylor

If a function f is infinitely differentiable, with continuous derivatives in the range of a real number x_0, then the function can be written as an infinite series:

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \ldots = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$

which is called a seriesTaylor of the function with center x_0.

If $x_0 = 0$, then the expansion is called a Maclaurin series expansion:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

Example 1

To find the orderMaclaurin for the function $f(x) = e^x$.

It is true that $f^{(n)}(x) = e^x$ for $n = 1, 2, 3, \cdots$.
Therefore $f^{(n)}(0) = 1, \forall n \in \mathbb{N}$.
So the expansion of the seriesMaclaurin for the function $f(x) = e^x$ is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} e^0 = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 2

To find the orderMaclaurin for the function $f(x) = e^{-x}$.

$$f(x) = e^{-x} \text{ and } f(0) = 1$$
$$f'(x) = -e^{-x} \text{ and } f'(0) = -1$$
$$f''(x) = e^{-x} \text{ and } f''(0) = 1$$
$$f'''(x) = -e^{-x} \text{ and } f'''(0) = -1$$
$$\ldots$$
$$f^{(n)}(x) = (-1)^n e^{-x} \text{ and } f^{(n)}(0) = (-1)^n$$

Therefore:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Example 3

To find the orderTaylor for the function $f(x) = e^{-x}$ around the $x = -4$.

$f^{(n)}(x) = (-1)^n e^{-x}, f^{(n)}(-4) = (-1)^n e^4$

Therefore the seriesTaylor is:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n e^4}{n!}(x+4)^n$$

Example 4

To find the orderMaclaurin for the function $f(x) = cos(x)$

$f(x) = cos(x)$ and $f(0) = 1$ $f'(x) = -sin(x)$ and $f'(0) = 0$ $f''(x) = -cos(x)$ and $f''(0) = -1$
$f'''(x) = sin(x)$ and $f'''(0) = 0$ $f^{(4)}(x) = cos(x)$ and $f^{(4)}(0) = 1$ $f^{(5)}(x) = -sin(x)$ and $f^{(5)}(0) = 0$ $f^{(6)}(x) = -cos(x)$ and $f^{(6)}(0) = -1$

Therefore:

$$cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

or

$$cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

TheoremTaylor

If a function f it is $n+1$ times differentiable, with continuous derivatives in an open interval containing a real number x_0, then the function can be written as a series (power series):

$$f(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(x)}{(n+1)!}(x - x_0)^{n+1}$$ for some $x \in (x_0, x)$.

Then, we can approach the f as:

$$f(x) \approx f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

with remainder (error) of this polynomial approximation n degree: $R_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}(x - x_0)^{n+1}$

Example for seriesTaylor

Exercise Find it2nd order approximation with orderTaylor of the function $f(x) = e^x$ around the point $x_0 = 0$. Use the approach that you found to appreciate the $f(0.1)$, and give an upper bound for error that arises in this approach.

$$f(x) = f(x) = f'(x) = f''(x) = e^x$$

Around the $x_0 = 0$, $f(x) = 1 + x + \frac{x^2}{2!} = 1 + x + x^2/2$.
Therefore, the approach of $f(0.1)$ is $f(0.1) = 1.105$. The real price is $f(0.1) = 1.10517$ with a difference of 0.00017

Example for seriesTaylor

Regarding the estimation of the truncation error we have $R_2(x) = \frac{M|x|^3}{3!}$ where the M is an upper barrier for $f'''(x)$ in $x \in [0, 0.1]$. In the space which we are given we have $|x| \leq 0.1 \iff |x|^3 \leq 0.001$.

Also, $f'''(x) = e^x \leq e^{0.1}$ in the time given to us. Therefore $R_2(x) = \frac{0.001\, e^{0.1}}{6} = 0.000184$.

Therefore, the estimation of the error through the residualTaylor is slightly greater than the actual difference.

Example for seriesTaylor2

Find the 2nd order approximation in orderTaylor of the function $f(x) = cos(x)$ around the point $x_0=0$. Use the approach that you found to appreciate the $f(0.6)$, and give an upper bound for error that arises in this approach.

$$f(0) = 1$$
$$f'(x) = -sin(x) \text{ and } f'(0) = 0 \quad f''(x) = -cos(x) \text{ and } f''(0) = -1$$
$$f'''(x) = sin(x)$$

Around the $x_0=0$, $f(x) = 1 - \frac{x^2}{2}$

Therefore, the approach of $f(0.6)$ is $f(0.6) = 0.82$. The real price is $f(0.6) = 0.8253$ with a difference of 0.0053.

Example for seriesTaylor2

Regarding the estimation of the truncation error we have $R_2(x) = \frac{M|x|^3}{3!}$ where the M is an upper barrier for $f'''(x)$ in $x \in [0, 0.6]$. In the space which we are given we have $|x| \leq 0.6 \Longleftrightarrow |x|^3 \leq 0.216$.

Also, $f'''(x) = sin(x)$ with $|sin(x)| \leq 1$. Therefore $|R_2(x)| \leq \frac{0.216}{6} = 0.036$.

Therefore, the estimation of the error through the residualTaylor is older from the real difference.

Application to the sequential approachTaylor

Many of the important uses of the type ofTaylor can be implemented using only two terms (n=2). In this case we get:

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{f''(x)(x_1 - x_0)^2}{2}$$

for x between x_0 and x_1.

If we transfer the $f(x_0)$ on the left-hand side of the relation and use the relationships $dx = (x_1 - x_0)$, $day = f'(x)dx$ and $Dy = f(x_1) - f(x_0)$ we end up with below relationship:

$$Dy = day + \frac{f''(x)(x_1 - x_0)^2}{2}$$

for x between x_0 and x_1.

Application to the sequential approachTaylor

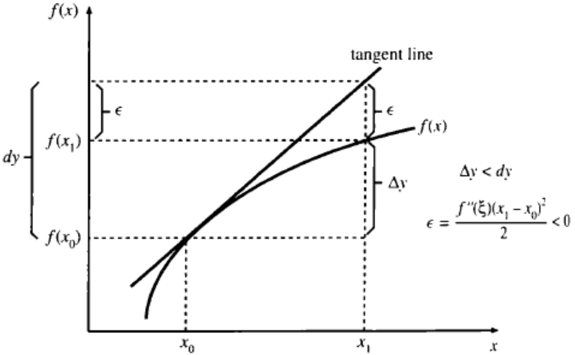
The error is essentially the remainder in the series formulaTaylor, that is

$\epsilon = \Delta y - dy = \frac{f''(x)(x_1 - x_0)^2}{2}$ or otherwise $\Delta y = day + \frac{f''(x)(x_1 - x_0)^2}{2}$.

If we now assume that the $f(x)$ is a strictly concave function (everywhere) so that $f''(x) < 0$ because $(x_1 - x_0)^2$ is positive for any value $x_1 \neq x_0$, the remainder will be negative and the *day* it will be an overestimation of Δy.

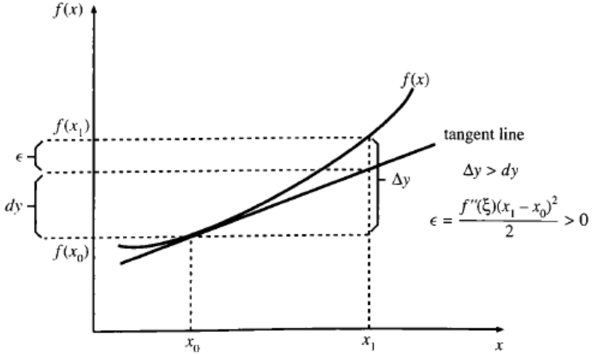
If we assume that $f(x)$ is a strictly convex function (everywhere) such that $f''(x) > 0$, then the remainder will be positive and the total differential *day* it will be one underestimation of Δy.

Application to the sequential approachTaylor



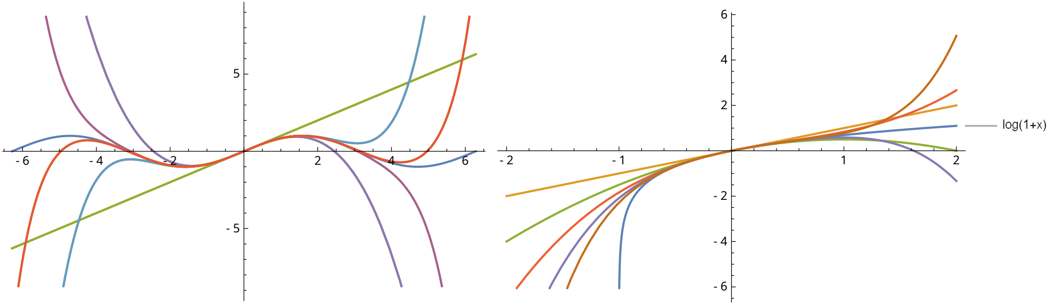
Shape:The total differential overestimates the change in the value of a concave function

Geometric interpretation of approximation by seriesTaylor



Shape:The total differential underestimates the change in the value of a convex function

Graphical representation of approximation with polynomialsTaylor



Shape:PolynomialsTaylor for the functions sin(x) and log(1 +x) at the point$x$=0