Mathematical analysis
Lecture 2

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2nd lecture topics

- Functions, function image and inverse function
- Convexity of functions
- ϵ-areas
- Derivative of a function
- Differential

Set image via function f

Definition: When we have two sets X and Y, the function (function) from the X in Y is a rule that connects to each element of X, a single element of Y.

- The whole X is called the starting set or domain (domain) of function, the Y is called the arrival set (codomain) and the set of elements of Y (which may or may not be the entire set) Y) which are linked to the elements of X through the function is called a range of values (range) of the function.

- Using the symbol f for the rule by which the elements of the two sets, we can write the function as follows:

$$f: X \to Y, \text{with } y = f(x), x \in X$$

where the y often called an image (image) of x or price (value) of function f on the spot x.

Set image via function f

- The value range or image of X of a function can be displayed as set of images (image set):

$$f(X) = \{y \in Y : y = f(x), x \in X\}$$

- If $f(X) = Y$, we say that the f depicts the X on the Y or that the function f It is on.
- Can every x to have as its image a different element of Y, so the mapping is said to be one-to-one. To prove whether a function is one-to-one, it suffices to show that:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

or equivalent

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

Inverting functions

- Often we may want to invert the function $y = f(x)$ and to display it x as a function of y, that is $x = f^{-1}(y)$. This can only happen when the fit is one to one.
- If the inverse function is defined $x = f^{-1}(y)$ or equivalently the fit is a towards one, for a whole $B \subset Y$ the preimage of is defined $A = f^{-1}(B)$.

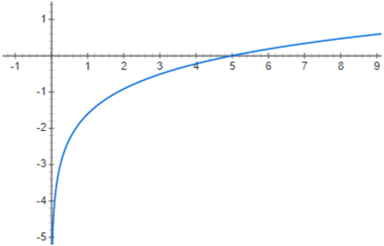
Example of a set preview

Suppose we want to find the preimage of the set $B = [1,2]$ for the function $f(x) = 5e^x$.

First we will find the inverse function of f: $y = 5e^x \Longleftrightarrow$

$$\frac{y}{5} = e^x \Longleftrightarrow x = \ln\left(\frac{y}{5}\right) \text{ or } f^{-1}(x) = \ln\left(\frac{x}{5}\right).$$

If we substitute the extreme values for the set B, since the f^{-1} it is genuinely increasing, we find that the pre-image of the set is $A = \left[f^{-1}(1), f^{-1}(2)\right] = \left[\ln\left(\frac{1}{5}\right), \ln\left(\frac{2}{5}\right)\right]$.



Shape: The graph of the function $\ln\left(\frac{x}{5}\right)$

Convex functions

The function f is convex (convex) for any two points of the field definition of x_1 and x_2 it is true that:

$$f(\bar{x}) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

where $\bar{x} = \lambda x_1 + (1-\lambda) x_2$ and $\lambda \in [0,1]$. It is strictly convex if:

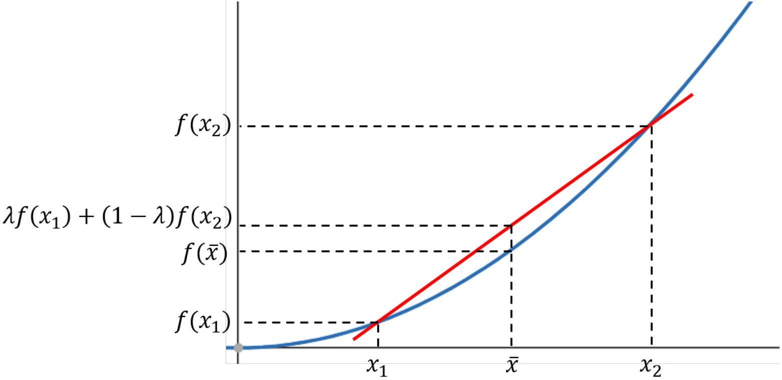
$$f(\bar{x}) < \lambda f(x_1) + (1-\lambda) f(x_2)$$

when $\lambda \in (0,1)$.

If the function is twice differentiable then it is convex if $f''(x) \geq 0$ and strictly convex if $f''(x) > 0$ in the area we are examining.

Example of a convex function

$$f(\bar{x}) \leq \lambda f(x_1) + (1-\lambda) f(x_2),\ \bar{x} = l x_1 + (1-\lambda) x_2,\ l \in [0,1]$$



Shape:Example of a convex function

Concave functions

The function f is hollow (concave) if for any two points of scope of definition x_1 and x_2 it is true that:

$$f(\bar{x}) \geq \lambda f(x_1) + (1 - \lambda) f(x_2)$$

where $\bar{x} = \lambda x_1 + (1 - \lambda) x_2$ and $\lambda \in [0,1]$. It is strictly concave if:

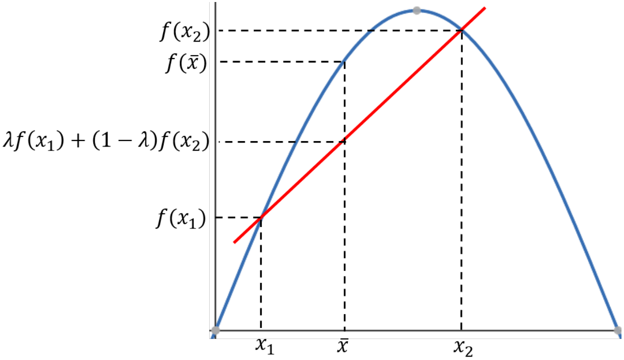
$$f(\bar{x}) > \lambda f(x_1) + (1 - \lambda) f(x_2)$$

when $\lambda \in (0,1)$.

If the function is twice differentiable then it is concave if $f''(x) \leq 0$ and strictly concave if $f''(x) < 0$ in the area we are examining.

Example of a concave function

$$f(\bar{x}) \geq \lambda f(x_1) + (1-\lambda) f(x_2),\ \bar{x} = l x_1 + (1-\lambda) x_2,\ l \in [0,1]$$



Shape:Example of a concave function

Example: Proof that absolute value is a convex function

We note that the second derivative criterion cannot be used because the absolute value function is not differentiable.

Let it be $x_1, x_2 \in \mathbb{R}$ and $a, b \in \mathbb{R}_{\geq 0}$ where $a + b = 1$. $(a = l,\ b = 1 - \lambda)(l \in [0,1])$. Then:

$f(ax_1 + bx_2) = |ax_1 + bx_2|$
$\leq |ax_1| + |bx_2|$ (from the triangular inequality for real numbers) $= |a||x_1|$
$+ |b||x_2| = a|x_1| + b|x_2| = af(x_1) + bf(x_2)$

Exercise

Show in two ways that the function $f(x) = x^2$ it is convex.

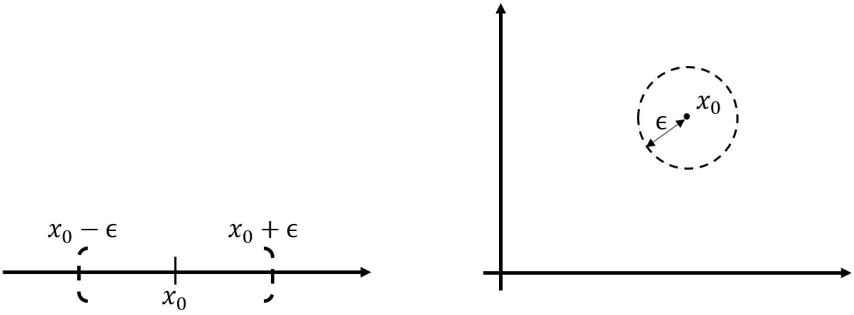
Solution

1st way: The function is twice differentiable with a second derivative $f''(x) = 2 > 0$ is therefore convex.

2nd way: $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2) \Leftrightarrow$
$(\lambda x_1 + (1-\lambda)x_2)^2 \le \lambda x_1^2 + (1-\lambda)x_2^2 \Leftrightarrow$
$\lambda^2 x_1^2 + 2x_1 x_2 \lambda(1-\lambda) + (1-\lambda)^2 x_2^2 \le \lambda x_1^2 + (1-\lambda)x_2^2 \Leftrightarrow$
$(\lambda - \lambda^2)x_1^2 - 2x_1 x_2 \lambda(1-\lambda) + (1-\lambda-1+2\lambda-\lambda^2)x_2^2 \ge 0 \Leftrightarrow$
$(\lambda - \lambda^2)x_1^2 - 2x_1 x_2 \lambda(1-\lambda) + (\lambda-\lambda^2)x_2^2 \ge 0 \Leftrightarrow \lambda(1-\lambda)(x_1^2 - 2x_1 x_2 + x_2^2) \ge 0 \Leftrightarrow$
$\lambda(1-\lambda)(x_1 - x_2)^2 \ge 0$ which is valid.

Region-ϵ

The area-ϵ (ϵ-neighborhood) of a point $x_0 \in \mathbb{R}^n$ is given by the set $N_\epsilon(x_0) = \{x \in \mathbb{R}^n : d(x_0, x) < \epsilon\}$. More simply, $N_\epsilon(x_0)$ is the set of points that are at a distance ϵ from the x_0.



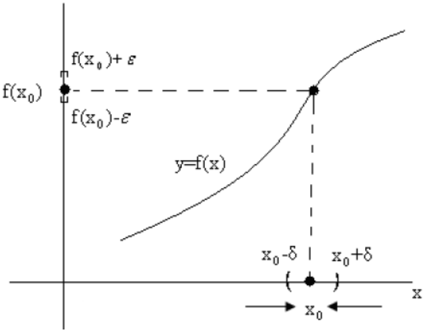
Shape: Areas-ϵ in $\mathbb{R}$ and $\mathbb{R}^2$

Open set

A set $X \subset \mathbb{R}^n$ is open (open) if, for each $x \in X$ there is one ϵ so that $N_\epsilon(x) \subset X$.

Continuous function

A function $f(x)$ defined in an open space to which it belongs the point $x = x_0$ is continuous at this point, if for any $\epsilon > 0$ is there any $d > 0$ so that it is true $|f(x) - f(x_0)| < \epsilon$, whenever $|x - x_0| < d$.



Shape: Function continuation

Theorem

> Let the function $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}$. The f is continuous at all points of A if and only if for every open $V \subset \mathbb{R}$, the preimage $f^{-1}(V)$ of V is an open set.

Evidence:
1) $``\Rightarrow"$ Let f continuous in A and $V \subset \mathbb{R}$ open. We will show that $f^{-1}(V)$ open.

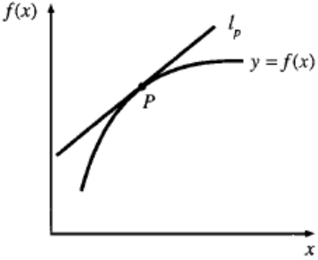
For each point $c \in f^{-1}(V)$ we have (by definition) that $f(c) \in V$.
Because the V is open, there is $\epsilon > 0$ so that $N_\epsilon(f(c)) \subset V$. Because the f it is continuous in c there is $d > 0$ such that $|x - c| < d \Rightarrow |f(x) - f(c)| < \epsilon$. That is, if $x \in N_d(c)$ then $f(x) \in N_\epsilon(f(c)) \subset V$.
But the fact that all the points of $N_d(c)$ are matched by the f within V means that the whole $N_d(c)$ is contained in the pre-image $f^{-1}(V)$ of V. So for each point c of $f^{-1}(V)$, we found an area-d which is contained in $f^{-1}(V)$ which means that the $f^{-1}(V)$ is open.

Proof of Theorem

2) "$\Leftarrow$" We now assume that $f^{-1}(V)$ open to everyone V open in the price range, and we will show that the f is continuous in every $c \in A$.
For $c \in A$ and $\epsilon >$ We know that the area-ϵ $N_\epsilon(f(c))$ is an open set in range of values. So (according to our hypothesis) the pre-image of $f^{-1}(N_\epsilon(f(c))$ is an open set which of course contains c.
Because $c \in f^{-1}(N_\epsilon(f(c))$ there is $d > 0$ such that $N_d(c) \subset f^{-1}(N_\epsilon(f(c))$ why the pre-image is an open set according to our initial assumption.
The last sentence can also be written as $|x - c| < d \Longrightarrow |f(x) - f(c)| < \epsilon$ which is equivalent to the proposition that the f is continuous in c.

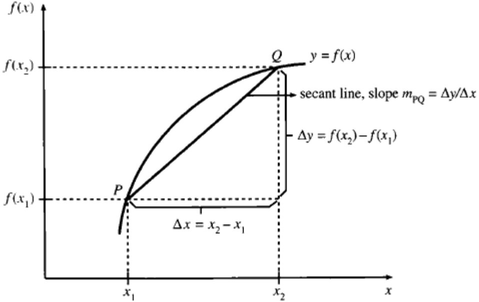
Tangent of a curve

The tangent (tangent) of a curve is a straight line which is exactly tangent to the curve at a given point.



Shape:The tangent of a curve at the point P

Curve intersection



Shape: The secant of a curve

The process of determining the rate of change Dy/Dx is done by taking successively smaller values of Dx The reason Dy/Dx as $Dx \to 0$ is the instantaneous rate of change of the function. When we take this limit the secant is essentially the same as the tangent. The slope of the secant between the points P and Q is symbolized as m_{PQ}.

Definition of derivative

The derivative (derivative) of a function $y = f(x)$ at the point $P = (x_1, f(x_1))$ is the slope of the tangent at this point:

$$f(x_1) = \lim_{\mathrm{D}x \to 0} m_{PQ} = \lim_{\mathrm{D}x \to 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

where $\mathrm{D}x = x_2 - x_1$ We can also write:

$$f(x_1) = \lim_{\mathrm{D}x \to 0} m_{PQ} = \lim_{\mathrm{D}x \to 0} \frac{f(x_1 + \mathrm{D}x) - f(x_1)}{\mathrm{D}x}$$

The derivative of a function $f(x)$ is also written as $\frac{day}{dx}$ Intuitively, the *day* and the *dx* reflect the meaning of the changes in y and his x, such as $\mathrm{D}y$ and the $\mathrm{D}x$ respectively. The expression $day = f(x)dx$ is known as its differential function $f(x)$.

Total differential at a point

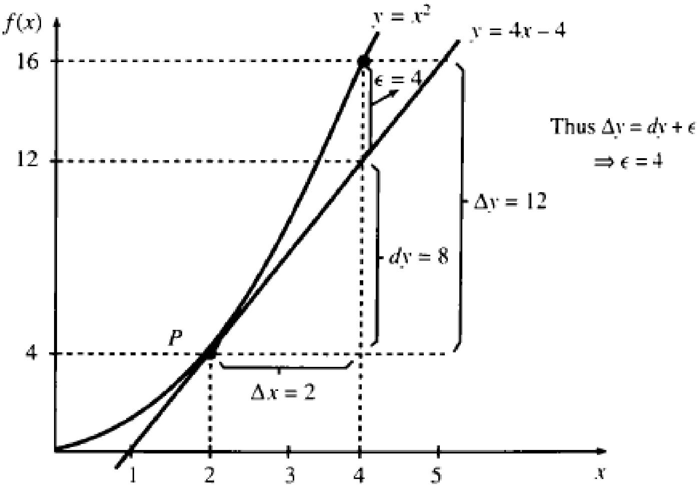
If $f(x_0)$ is the derivative of the function $y = f(x)$ at the point x_0, then the total differential at the point is:

$$day = df(x_0, dx) = f(x_0)dx$$

Therefore the differential is a function of x and his dx.

The differential provides us with a method of estimating the impact it has on y a change of x equal to D x The D y is the exact change of y while the day is the approximate change. Based on the definition of the derivative, this is equivalent to using the tangent of a function to estimate the effect of a change in x on the y.

Total differential approach



Shape:OR$dy = f'(x)dx$ as an approximation of a change in y

Derivation rules

1st rule: Derivative of a constant function

If $f(x) = c$, where c is a constant, then $f'(x) = 0$.

2nd rule: Derivative of a linear function

If $f(x) = mx + b$, where m and b are stable, then $f'(x) = m$.

3rd rule: Derivative of a power function

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Derivation rules

4th rule: Derivative of the product of a constant and a function

If $g(x) = cf(x)$, with c a constant, then $g(x) = cf(x)$.

5th rule: Derivative of the sum or difference of two functions

If $h(x) = g(x) + f(x)$ then $h(x) = g(x) + f(x)$. if $h(x) = g(x) - f(x)$ then $h(x) = g(x) - f(x)$.

6th rule: Derivative of a sum of a finite number of functions

If $h(x) = \sum_{i=1}^{n} g_i(x)$ then $h(x) = \sum_{i=1}^{n} g'_i(x)$.

Derivation rules

7th rule: Derivative of the product of two functions

If $h(x) = f(x)g(x)$, then $h(x) = f(x)g(x) + f(x)g(x)$.

8th rule: Derivative of the quotient of two functions

If $h(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$, then $h(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.

9th rule: Derivative of a complex function - chain rule

If $y=f(you)$ and $you=g(x)$, that is $y=f(g(x)) = h(x)$, then $h(x) = f(you)g(x)$ or

$$\frac{day}{dx} = \frac{you\;you}{right}$$

Derivation rules

10th rule: Derivative of the inverse of a function

If the $y=f(x)$ has as its inverse function the $x=g(y)$, that is, if $g(y)=f^{-1}(y)$ and $f \neq 0$ then:

$\frac{dx}{dy} = \frac{1}{dy/dx}$ or $g(y) = \frac{1}{f(x)}$ where $y=f(x)$.

11th rule: Derivative of the exponential function

If $y=e^x$, then $dy/dx=e^x$.

12th rule: Derivative of the logarithmic function

If $y=\mathrm{in}\,x$, then $dy/dx=1/x$.

Examples

- $\frac{d}{dx}\left[\frac{x^2-2x+1}{x^2-5x+6}\right] = \frac{(x^2-5x+6)\frac{d}{dx}(x^2-2x+1)-(x^2-2x+1)\frac{d}{dx}(x^2-5x+6)}{(x^2-5x+6)^2} = \frac{(x^2-5x+6)(2x-2)-(x^2-2x+1)(2x-5)}{(x^2-5x+6)^2}$

- $\frac{d}{dx}\left[(3x+1)^2\right] = [2(3x+1)]\frac{d}{dx}[3x+1] = 2(3x+1)3 = 6(3x+1)$

Logarithmic derivative

Logarithmic differentiation is the technique in which the calculation of the derivative of a function is $f(x)$ is done through the derivative of $\ln(f(x))$, taking advantage of the property $\ln(xyz) = \ln(x) + \ln(y)$.
For example, suppose we want to calculate the first derivative of

$$f(x) = \sqrt[3]{x^2}\sqrt{\frac{1-x}{1+x^2}}\sin^3(x)\cos^2(x)$$

then we have equivalents:

$$y = \sqrt[3]{x^2}\sqrt{\frac{1-x}{1+x^2}}\sin^3(x)\cos^2(x) \Leftrightarrow \ln(y) = \ln\left(\sqrt[3]{x^2}\sqrt{\frac{1-x}{1+x^2}}\sin^3(x)\cos^2(x)\right) \Leftrightarrow$$

$$\ln(y) = \ln\left(\sqrt[3]{x^2}\right) + \ln\left(\sqrt{\frac{1-x}{1+x^2}}\right) + \ln(\sin^3(x)) + \ln(\cos^2(x))$$

Logarithmic derivative (continued)

Generating both sides of the equation, we have the following equation:

$$(\ln(y))' = \left(\ln \sqrt[3]{x^2} + \ln\left(\frac{1-x}{1+x^2}\right) + \ln(\sin^3(x)) + \ln(\cos^2(x))\right)' \Leftrightarrow$$

$$\frac{y'}{y} = \frac{2}{3x} + \frac{1}{1-x}(1-x)' - \frac{1}{1+x^2}(1+x^2)' + 3\frac{1}{\sin(x)}(\sin(x))' + 2\frac{1}{\cos(x)}(\cos(x))' \Leftrightarrow$$

$$\frac{y'}{y} = \frac{2}{3x} - \frac{1}{1-x} - \frac{2x}{1+x^2} + 3\frac{\cos(x)}{\sin(x)} - 2\frac{\sin(x)}{\cos(x)} \Leftrightarrow$$

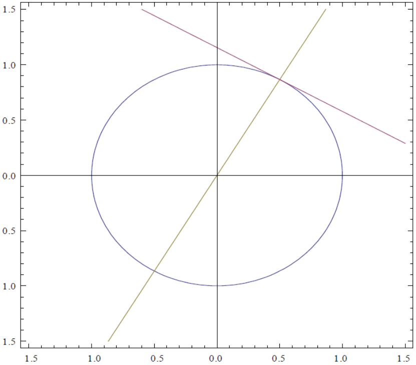
$$\frac{y'}{y} = \frac{2}{3x} - \frac{1}{1-x} - \frac{2x}{1+x^2} + 3\,\mathrm{cots}(x) - 2\tan(x) \Leftrightarrow$$

$$y' = y\left(\frac{2}{3x} - \frac{1}{1-x} - \frac{2x}{1+x^2} + 3\,\mathrm{cots}(x) - 2\tan(x)\right) \Leftrightarrow$$

$$y' = \sqrt[3]{x^2}\,\frac{1-x}{1+x^2}\sin^3(x)\cos^2(x)\left(\frac{2}{3x} - \frac{1}{1-x} - \frac{2x}{1+x^2} + 3\cot(x) - 2\tan(x)\right)$$

Exercise

To find the (equation) the tangent of the circle with center (0,0) and radius1, on the spot $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ as well as the equation of the perpendicular tangent to this point.



Solution

The equation of the circle is described by $x^2+y^2=1$. Producing and two members as to x we have:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) \Leftrightarrow 2x\frac{d}{dx}(x) + 2y\frac{dy}{dx} = \frac{d}{dx}(1) \Leftrightarrow 2x+2y\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Therefore, the direction coefficient of the tangent at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ it is

$$\left.\frac{dy}{dx}\right|_{(1/2,\ \sqrt{3}/2)} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Therefore, the equation of the tangent is: $y = -\frac{\sqrt{3}}{3}\left(x-\frac{1}{2}\right)+\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}$

The equation of the perpendicular to the tangent line passing through the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, has slope $\lambda = -\frac{1}{-\frac{\sqrt{3}}{3}} = \sqrt{3}$.

Therefore, its equation is: $y = \sqrt{3}\left(x-\frac{1}{2}\right)+\frac{\sqrt{3}}{2} = \sqrt{3}x$