TOPIC 5

Optimisation Basics and Limited Resources Problems

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Time table

10:15-11:10 **-A- Introduction to optimisation**

- linkage to problem solving
- fundamental structure of optimisation models

11:15-12:00 **-B- Linear programming**

- formulate a simple optimisation model for a limited resources problem
- solve a simple optimisation model
  … manually
  … automatically using the Simplex Algorithm [Excel, Matlab]

13:15-15:00 **-C- Exercise 5 – Product Portfolio Problem**

[computer lab]

Goals

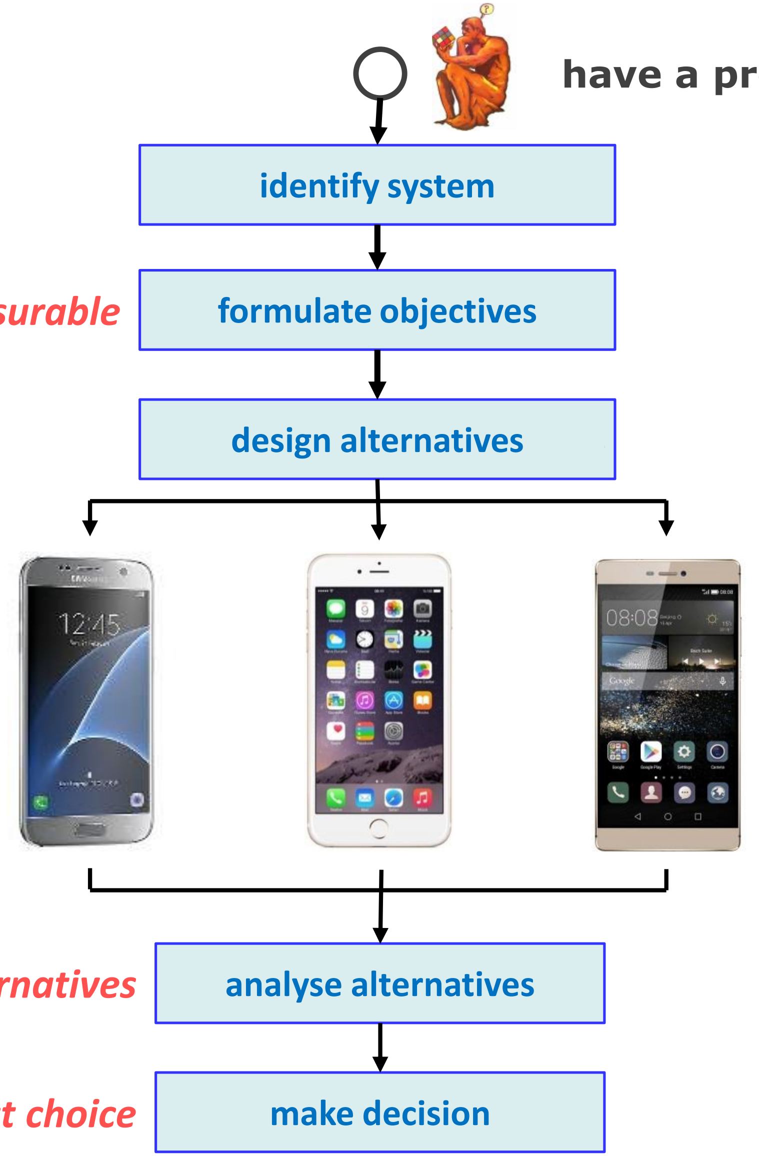
After completion of this topic you should be able to:

- Explain the fundamental structure of optimisation problems
- Explain the basics of the SIMPLEX ALGORITHM
- Formulate a simple optimisation model which includes limited resources
- Set up and solve a simple optimisation model in Excel and Matlab

- A -

INTRODUCTION TO MATHEMATICAL OPTIMIZATION

Analytic problem solving



have a problem….

«I need a new smartphone»

identify system

measurable

formulate objectives

design alternatives

evaluate performance of alternatives

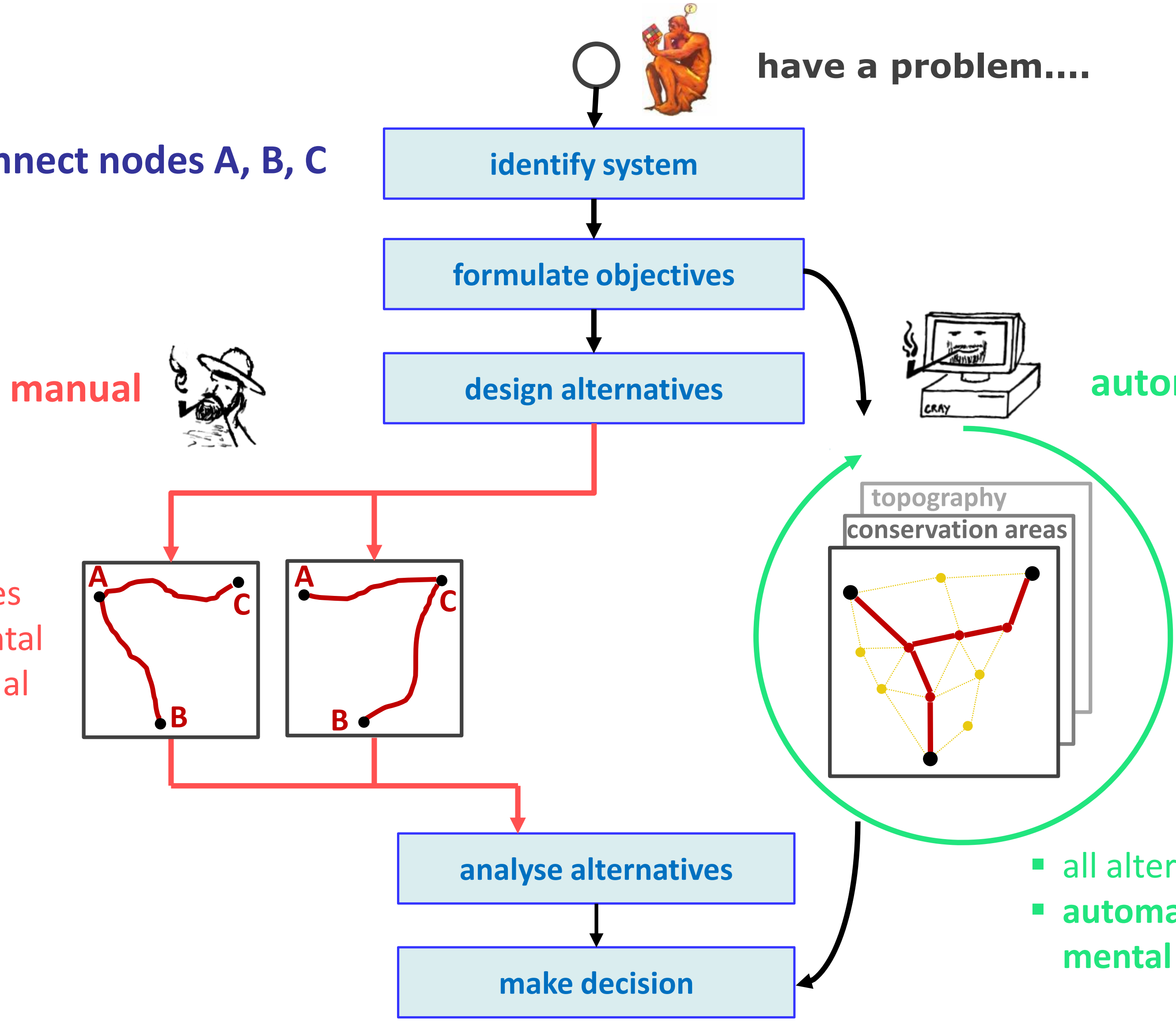
analyse alternatives

make the best choice

make decision

Analytic problem solving – many alternatives

«I want to connect nodes A, B, C via a road»



have a problem....

identify system

formulate objectives

design alternatives

manual

automated

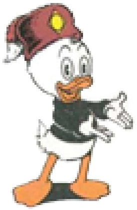
- few alternatives
- expensive mental work for manual
  … design
  … analysis

analyse alternatives

make decision

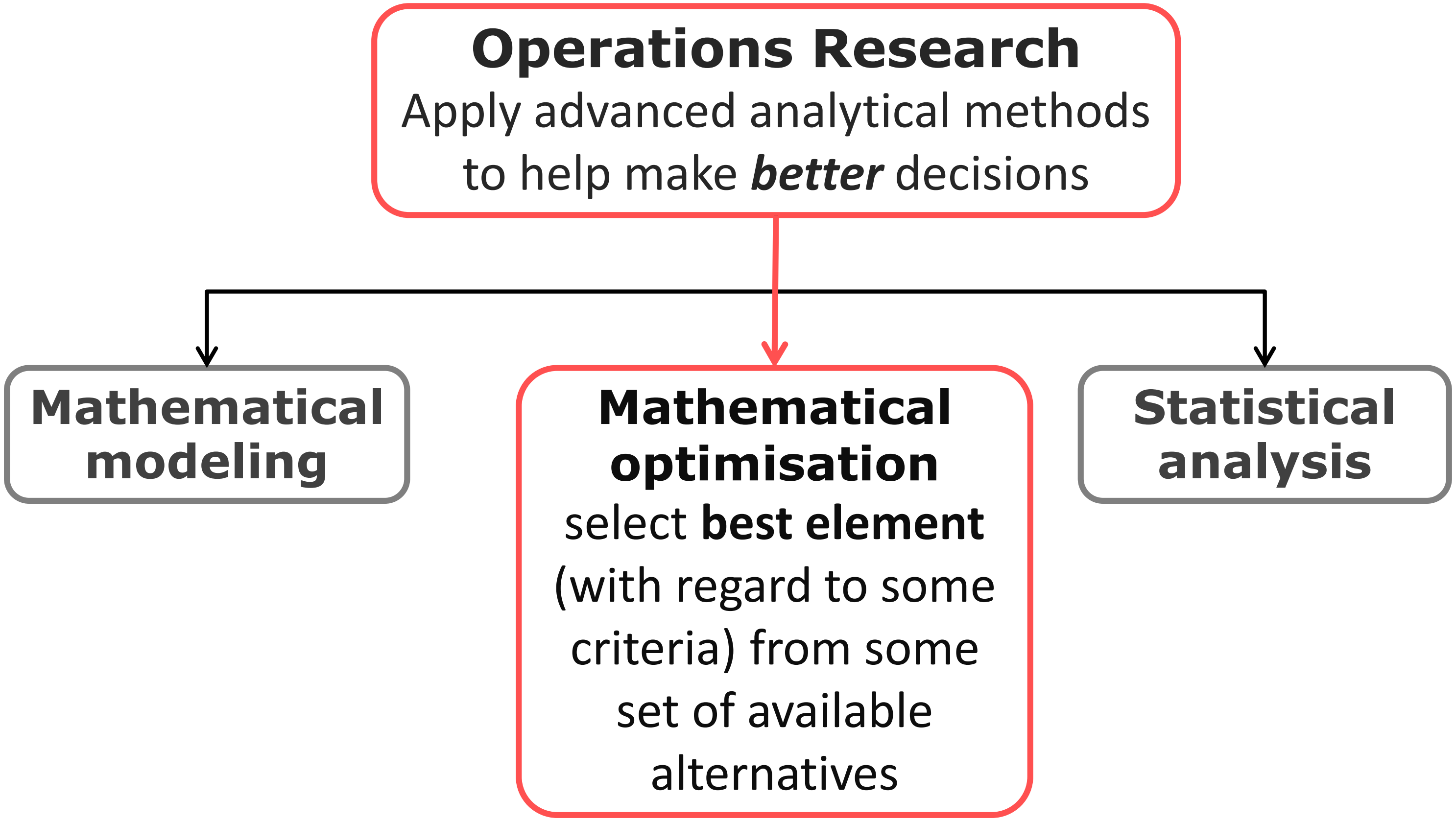
- all alternatives
- **automated mental work**

We need a method that enables the identification of the best choice when numerous alternatives are available and comparing them is laborious.



Mathematical optimisation!

Science of the better



Operations Research
Apply advanced analytical methods to help make ***better*** decisions

Mathematical modeling

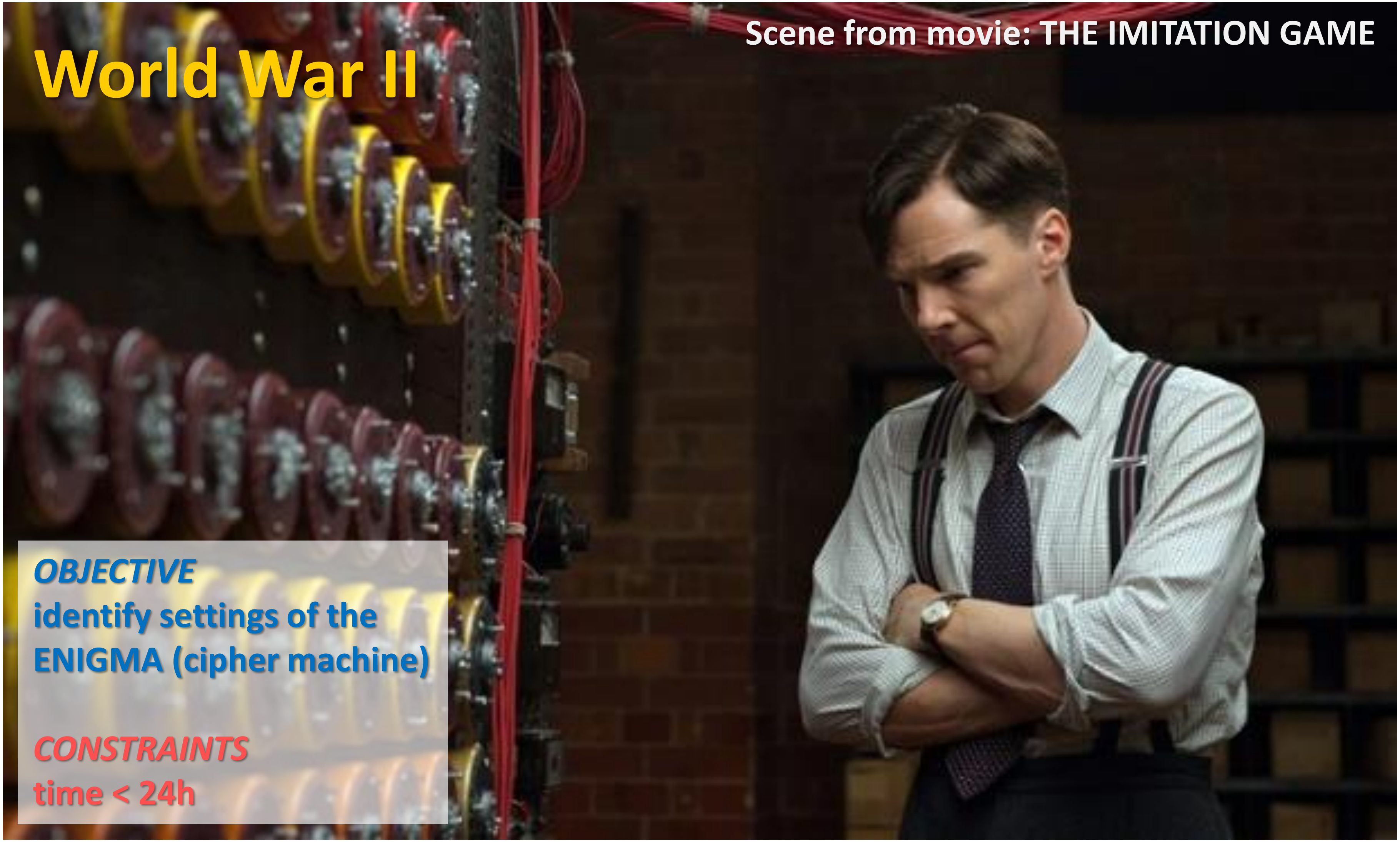
Mathematical optimisation
select **best element** (with regard to some criteria) from some set of available alternatives

Statistical analysis

https://www.informs.org/About-INFORMS/What-is-Operations-Research

http://en.wikipedia.org/wiki/Mathematical_optimization

Automation of mental work – The starting point



Scene from movie: THE IMITATION GAME

World War II

OBJECTIVE
identify settings of the ENIGMA (cipher machine)

CONSTRAINTS
time < 24h

source: http://www.theguardian.com

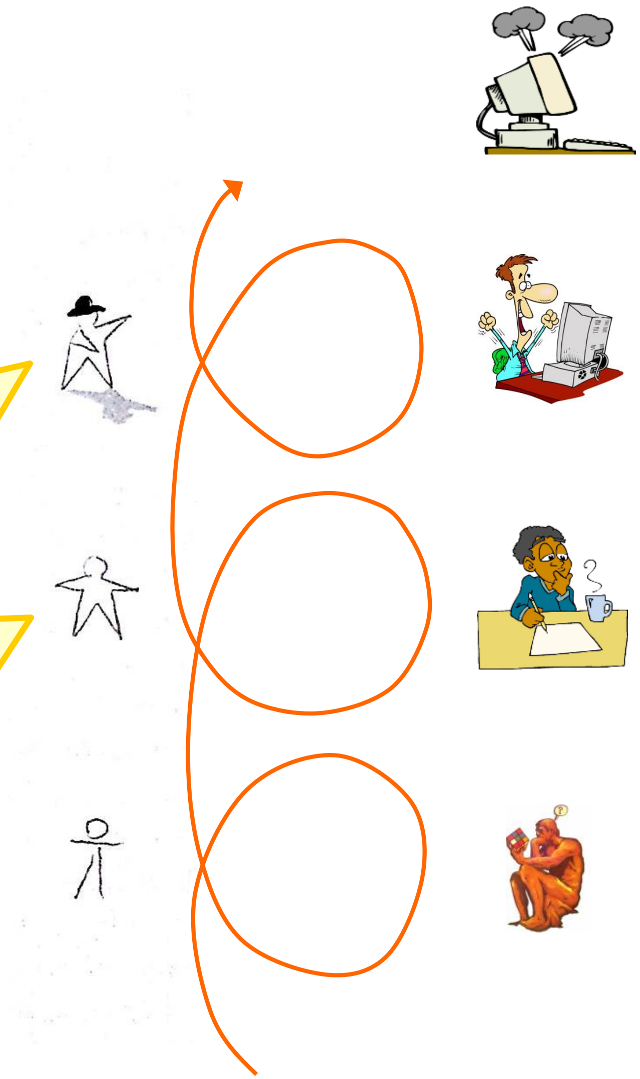
Learning concept and goals

[III] Improve problem solving skills

[IV] combine spatial information with optimisation models

[I] Fundamental concepts of mathematical optimisation

[II] Representation of time and space



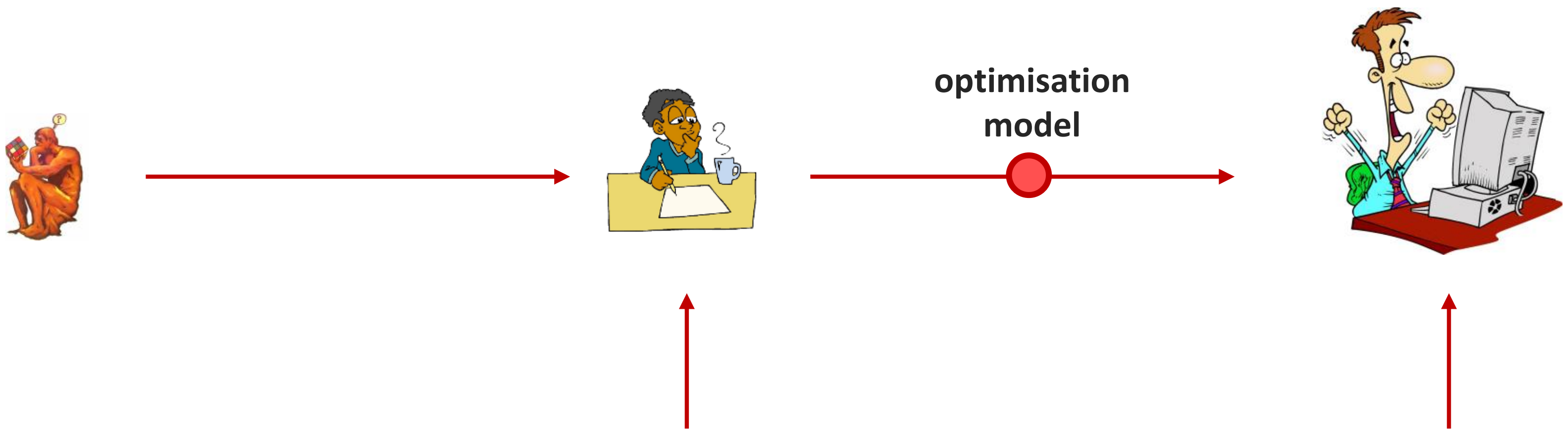
solve

implement

conceptualize & represent

have a problem....

Workflow for solving optimisation problems



Conceptualize & Represent

Design a concept on how to
analytically solve a problem

Identify a suitable representation of
decisions and objectives

Implement

Set-up optimisation model
in a solver software
Excel, Matlab, ArcGIS

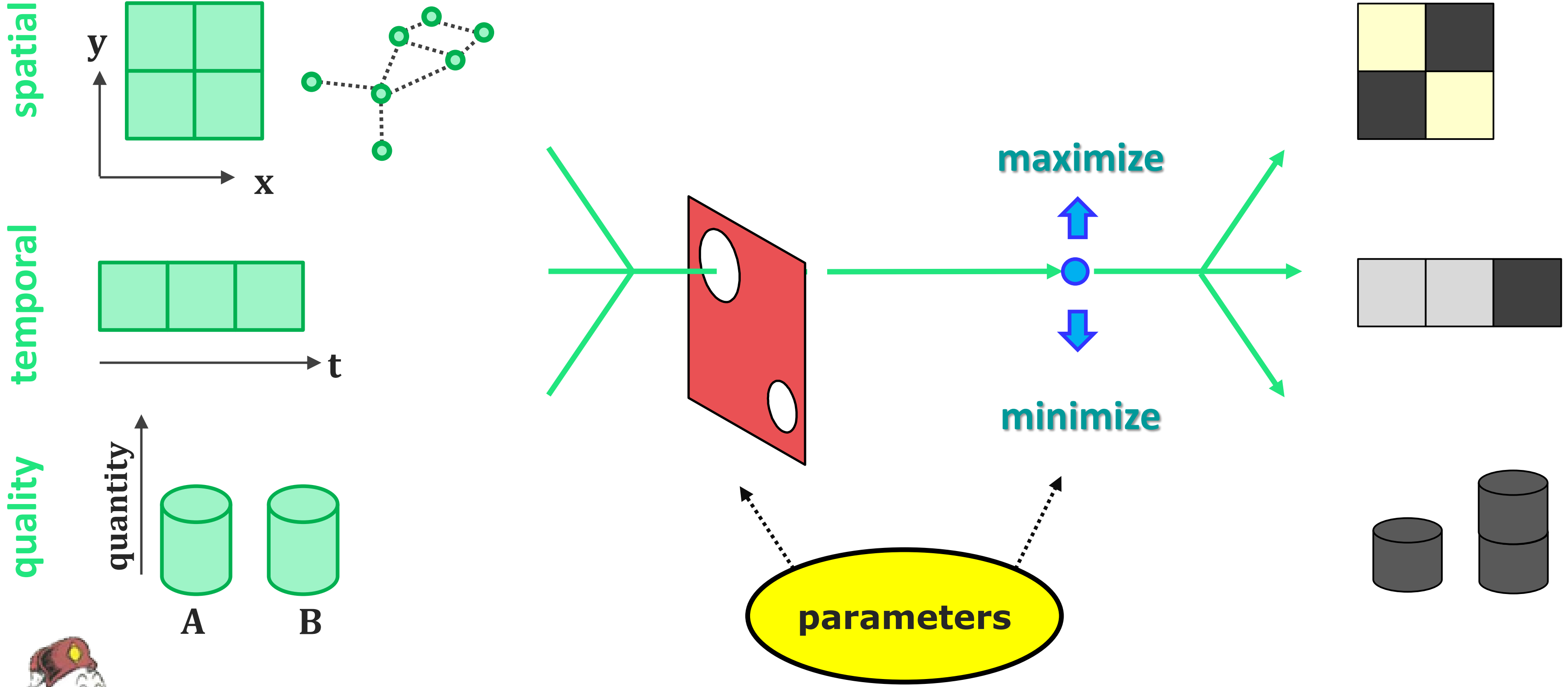
Fundamental structure of optimisation models

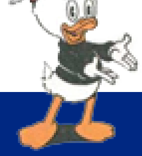
decision variables
- elements to build a solution

constraints
- limited resources
- model rules for…
  … spatial relationship
  … dynamic model

objective function

solution
- best performing set of decision variable values

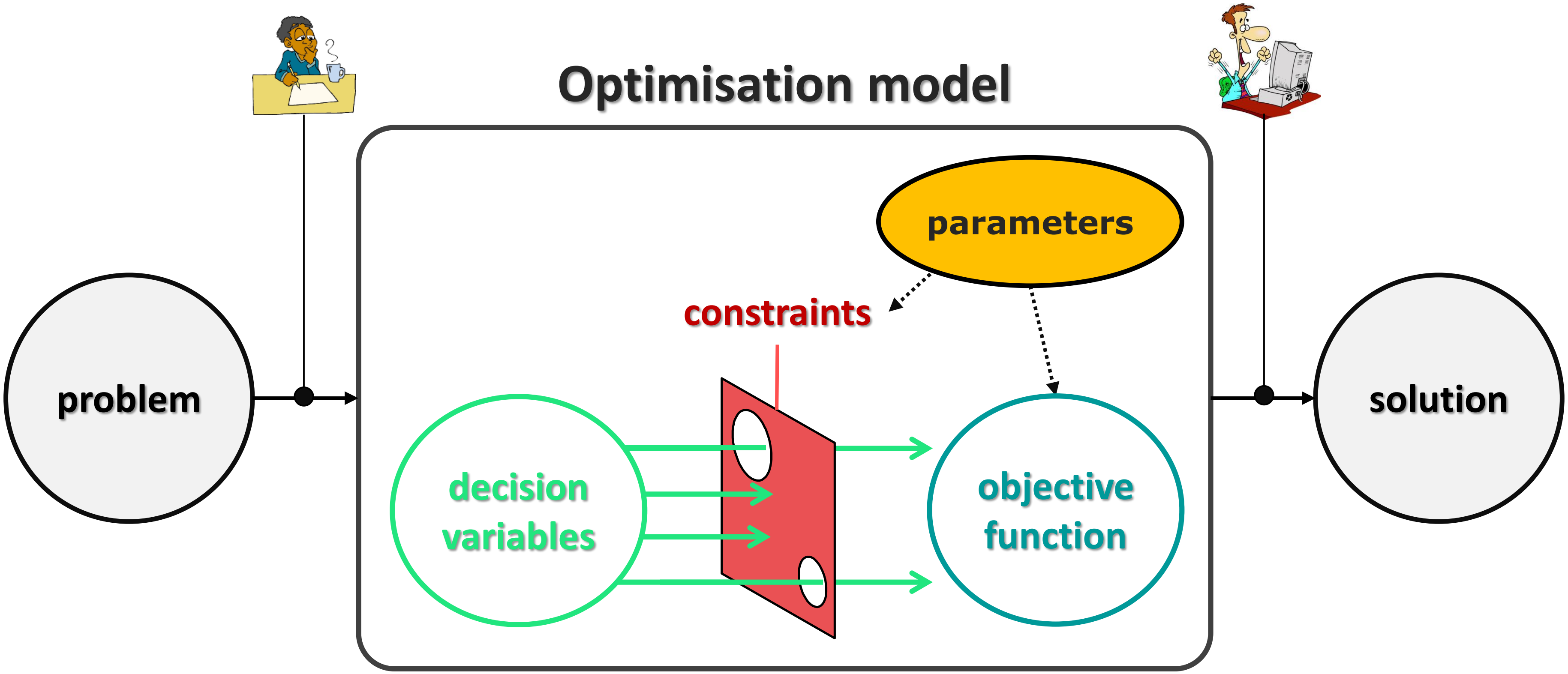




The solution of a problem is bound to the characterization of the decision variables

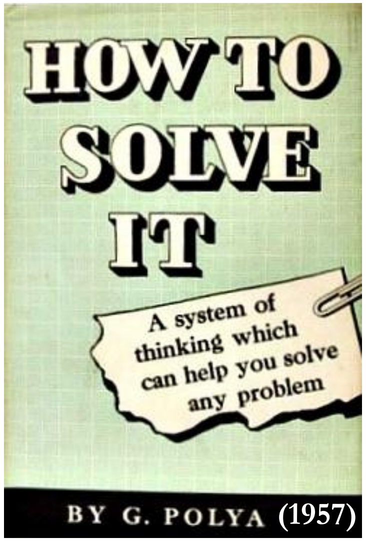
Constraints

FS - Definitions



Decision variables	representation of all alternatives
Objective function	performance of solution (*maximize* or *minimize*)
Constraints	control of the choice of alternatives
Parameters	values used to formulate objective function and constraints

Aim of the optimisation-related exercises



Analogy

"Can you find a problem analogous to your problem and solve that?"

This course provides examples of problem classes which can be adapted to solve many other problems

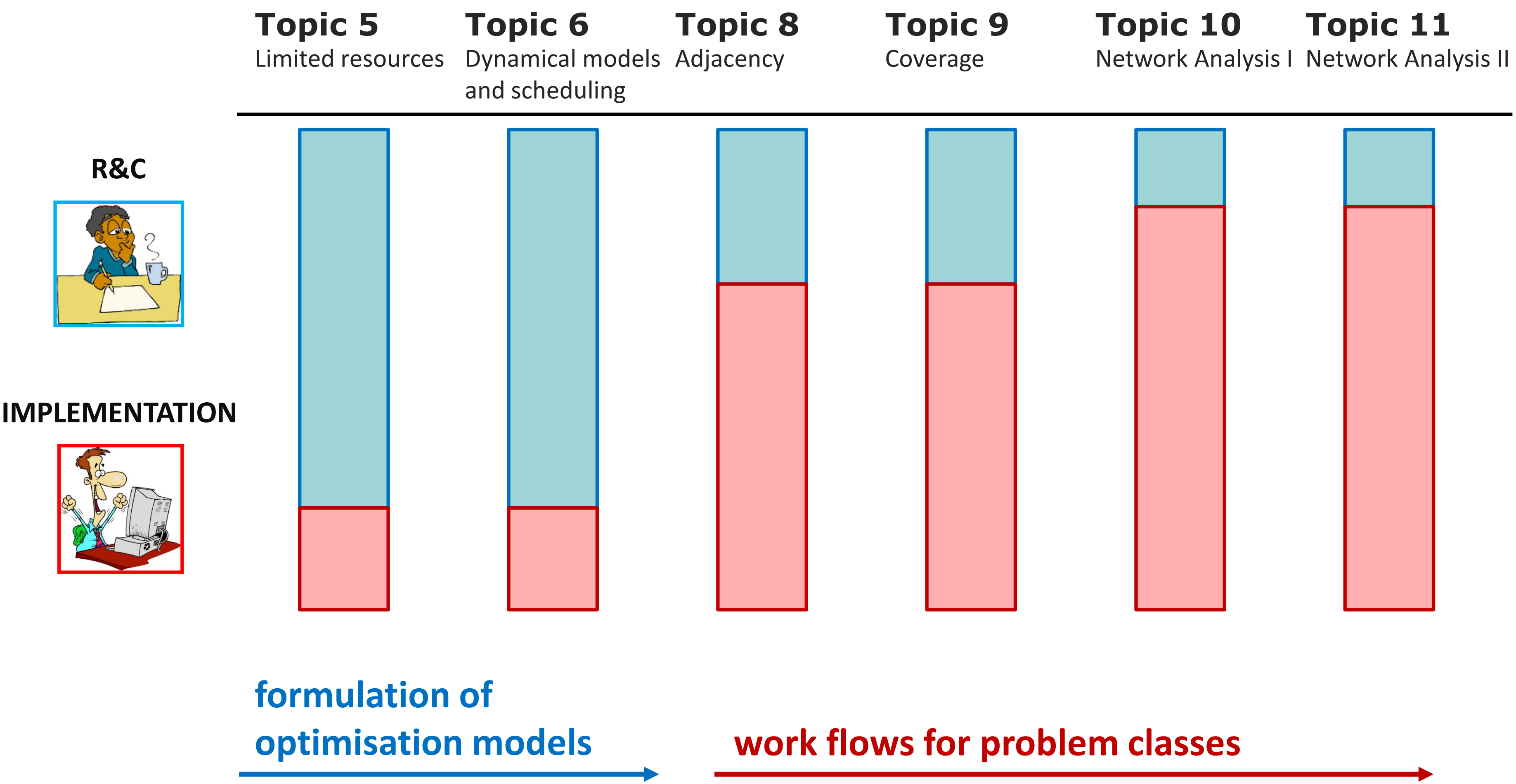
Problem classes discussed in this course

Topic 5 Limited resources	**Topic 6** Dynamical models and scheduling	**Topic 8** Adjacency	**Topic 9** Coverage	**Topic 10** Network Analysis I	**Topic 11** Network Analysis II
identify quantities	scheduling actions	scheduling actions in space	location of actions in space	location of actions in space	characterize connectivity
limited resources	simulate growth	spatial adjacency *polygon*	spatial coverage	spatial adjacency *point, line*	spatial adjacency *point*

manage/preprocess data using GIS

Particular problem types *Linear Programming*	Standard problem types ready-to-use algorithms

Course overview - Workload



Topic 5 Limited resources

Topic 6 Dynamical models and scheduling

Topic 8 Adjacency

Topic 9 Coverage

Topic 10 Network Analysis I

Topic 11 Network Analysis II

R&C

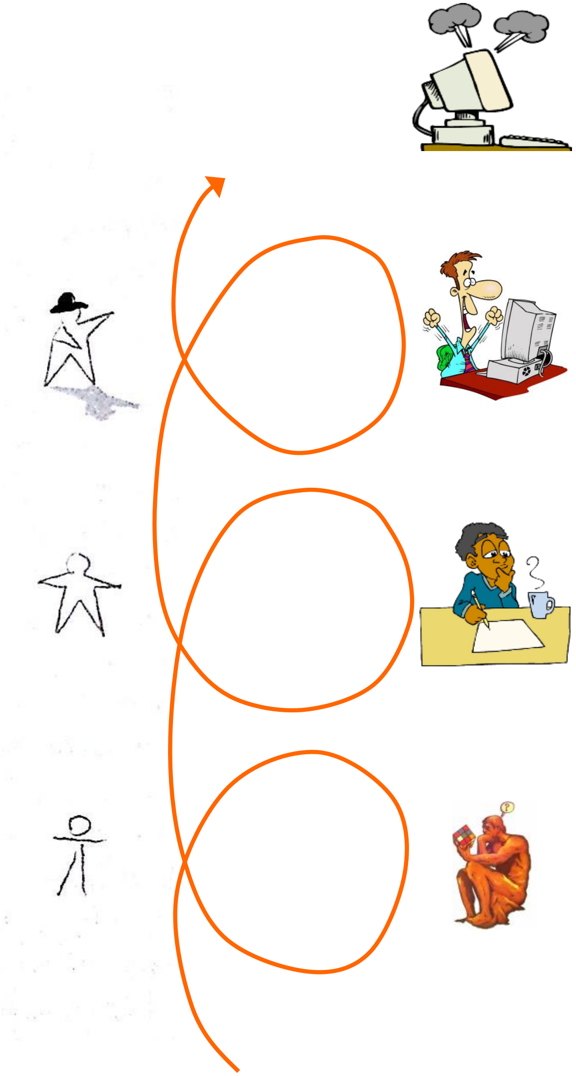
IMPLEMENTATION

formulation of optimisation models

work flows for problem classes

- B -

LINEAR PROGRAMMING



solve

implement

conceptualize & represent

have a problem....

FARMER'S CHOICE PROBLEM

A farmer owns 9 ha of land that he uses for growing carrots and grain. He wants to know the amount of area for each good which maximizes his revenues. Subsidieses are given up to an area threshold. Thus, he has no incentive to plant a larger area than the one which can be subsidised.

Latest price estimates

Carrots : 5'000 Fr./ha
Grain : 8'000 Fr./ha

Regulations

Area threshold for subsidies
Carrots : 7.0 ha
Grain : 4.5 ha

Problem characterization

Decision variables	area carrots area grain
Objective function	MAXIMIZE revenues
Parameters	profit carrots 5’000 Fr./ha profit grain 8’000 Fr./ha
Constraints	Available land limited Thresholds for subsidised area

Parameters

available area:	9ha
subsidised area of grain:	4.5ha
subsidised area of carrots:	7.0ha

Optimisation model formulation I

Decision variables

area carrots x_C ha

area grain x_G ha

Objective function

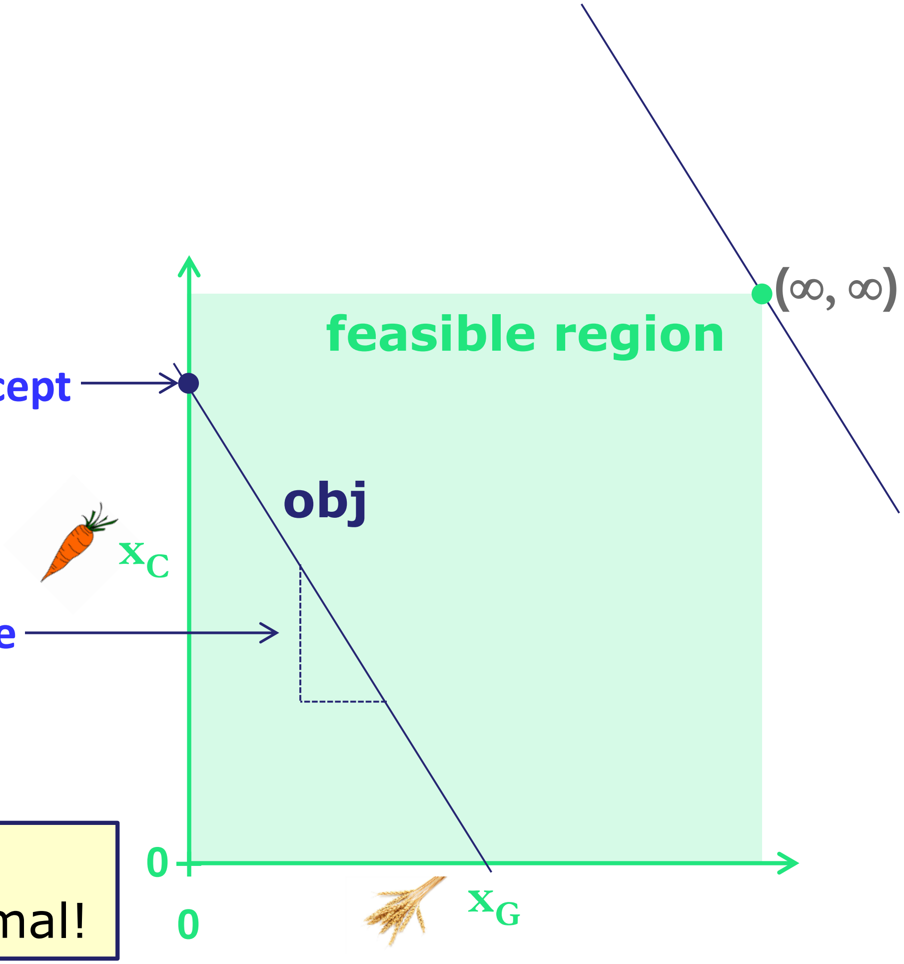
$$MAX\ \boldsymbol{F} = 5000 \cdot x_C + 8000 \cdot x_G$$

or, reformulated:

obj: $$x_C = -\frac{8000}{5000} \cdot x_G + \frac{\boldsymbol{F}}{5000}$$

y-intercept

slope



Find combination (x_C,x_G) such that y-intercept **F/5000** becomes maximal!

Optimisation model formulation II

Decision variables

$x_C, x_G \in \mathbb{R}_0^+$

Objective function

$$MAX\ \boldsymbol{F} = 5000 \cdot x_C + 8000 \cdot x_G$$

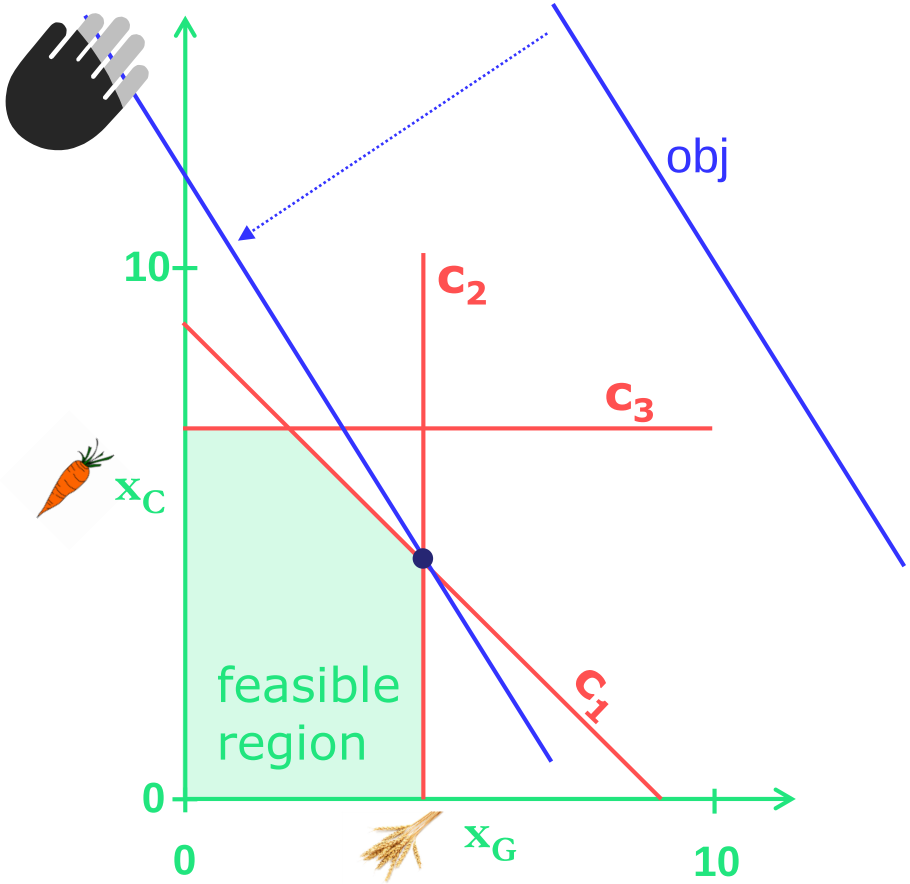
Constraints

(1) Farmer owns 9 ha of land
(2) Grain is subsidised up to 4.5 ha
(3) Carrot is subsidised up to 7.0 ha

c₁: $x_C + x_G \leq 9$

c₂: $x_G \leq 4.5$

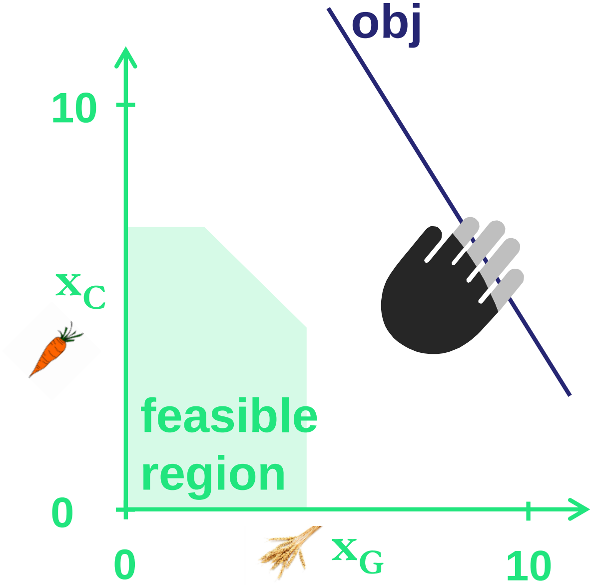
c₃: $x_C \leq 7$



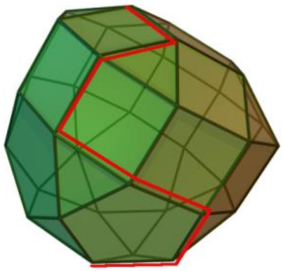
Graphical solution

Limitations of graphical identification of solution

2 Variables



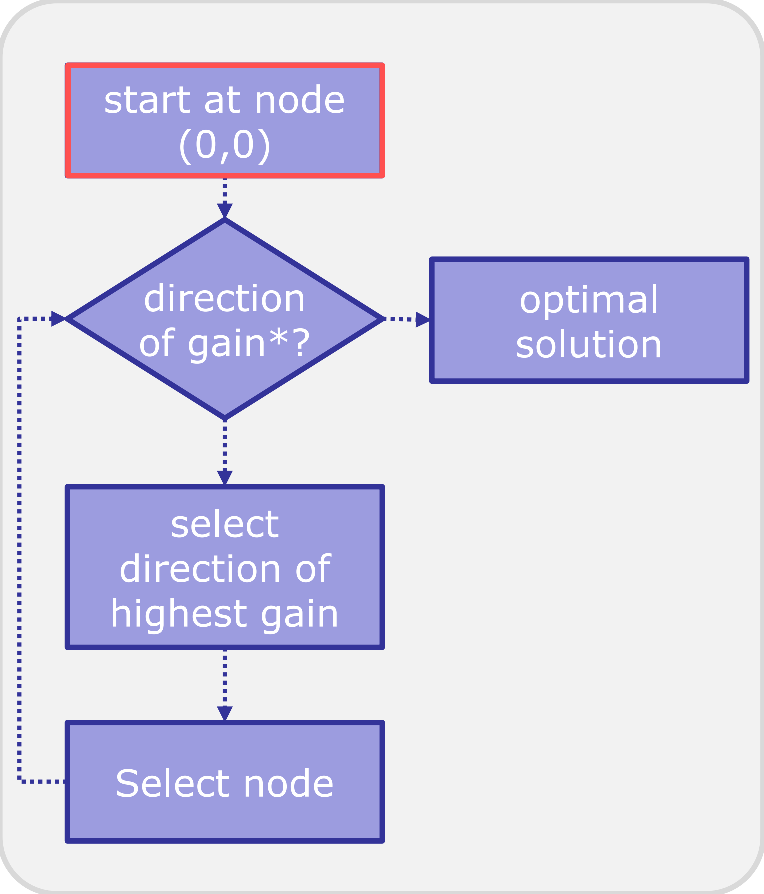
3 Variables



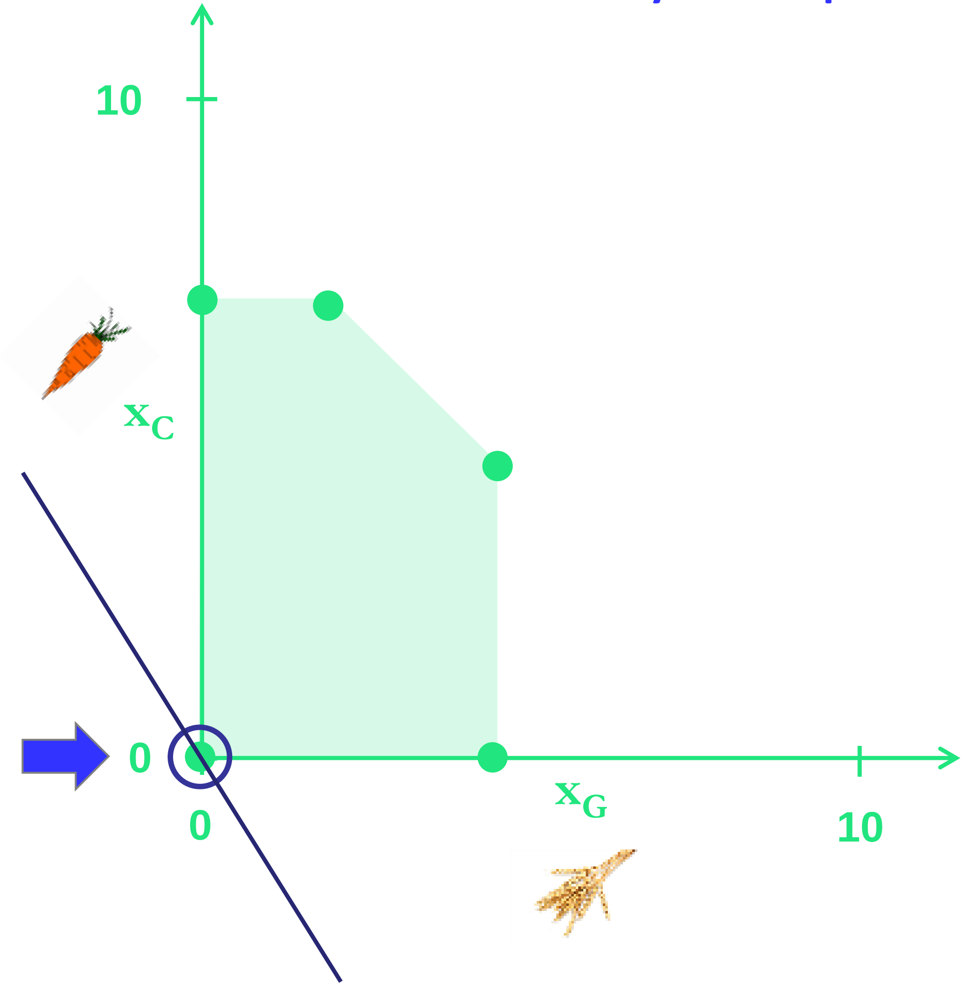
100 Variables ?

Simplex Algorithm

[G. Dantzig]

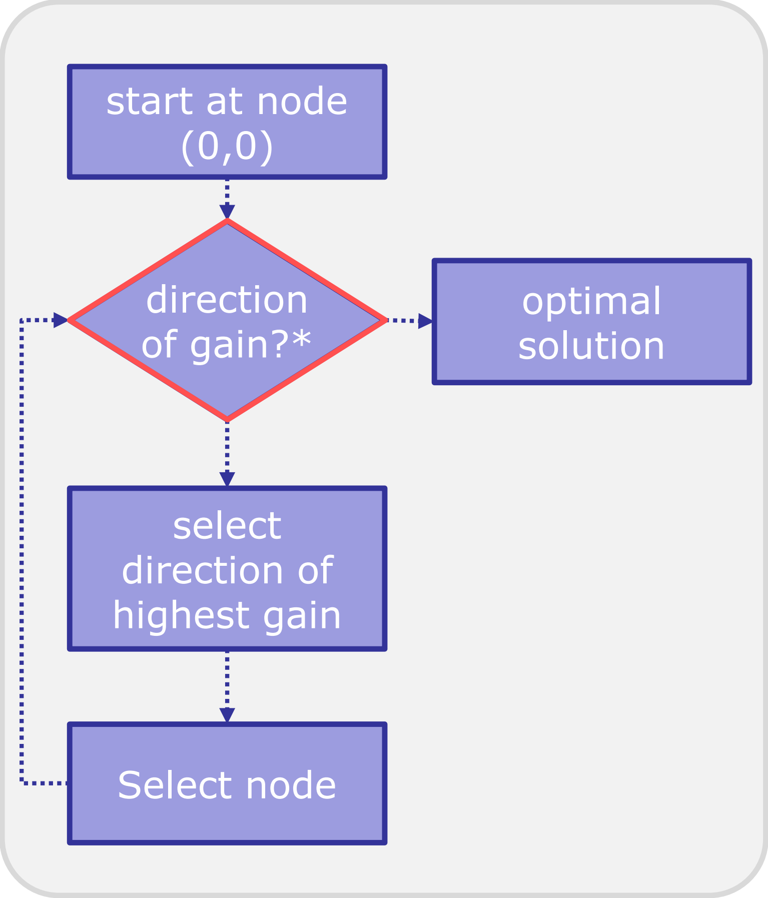


Maximize for y-intercept



Simplex Algorithm

[G. Dantzig]



start at node (0,0)

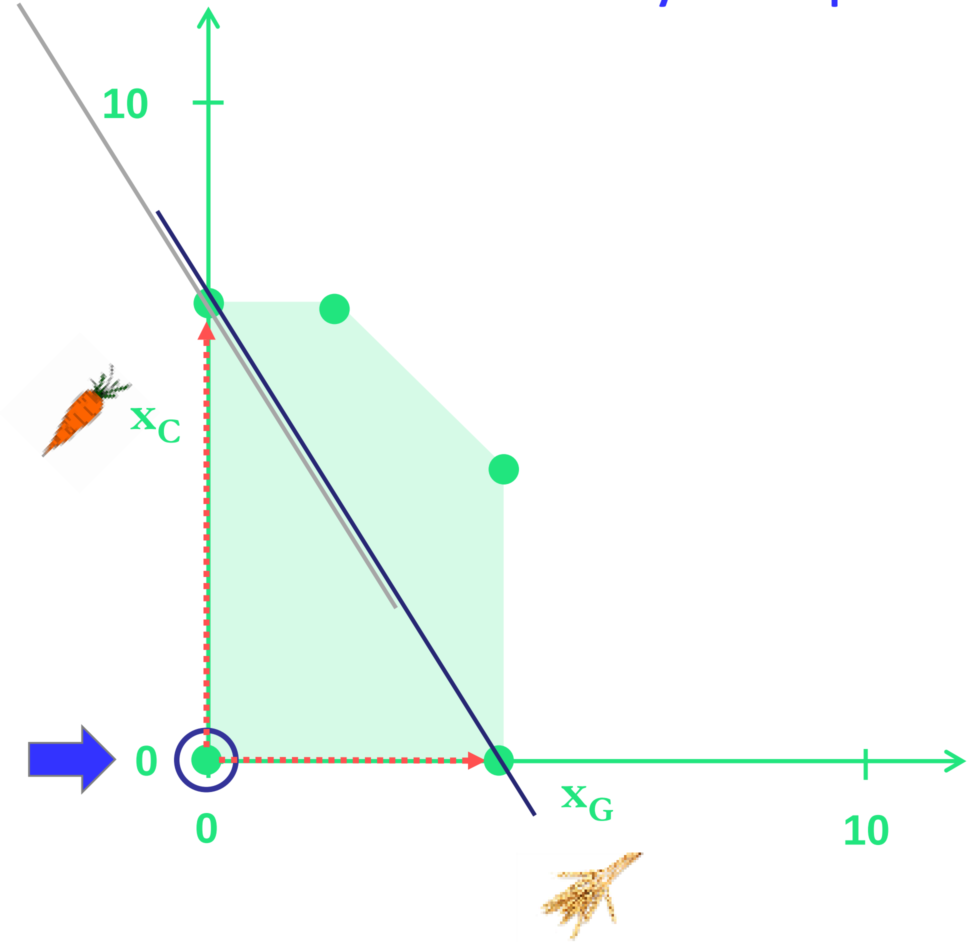
direction of gain?*

optimal solution

select direction of highest gain

Select node

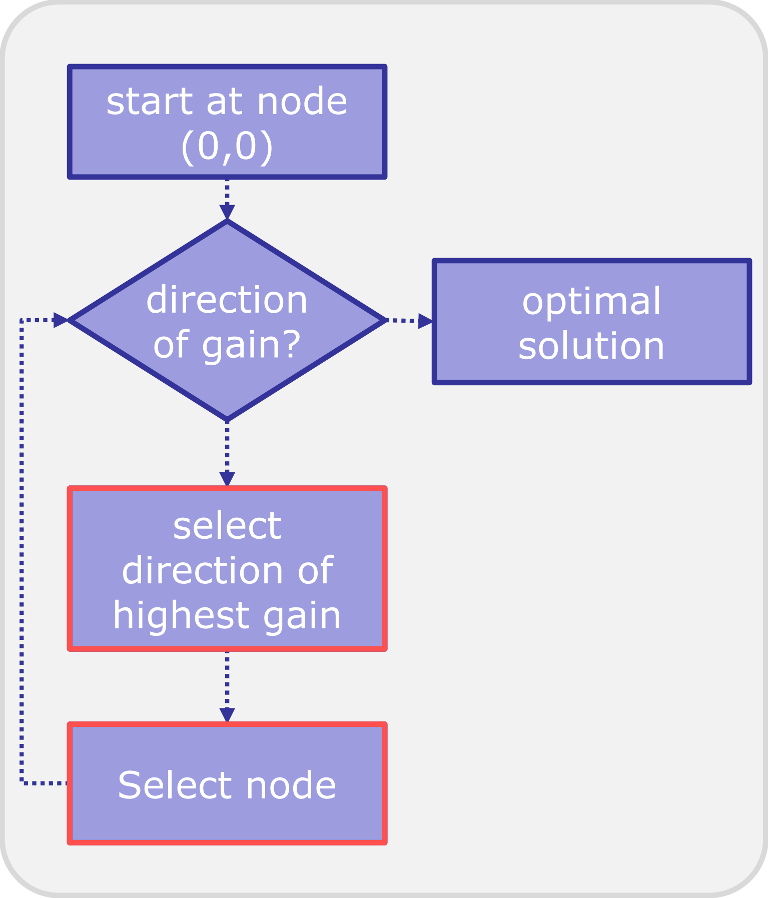
Maximize for y-intercept



* A move to a neighboring node which results into a better objective function value

Simplex Algorithm

[G. Dantzig]



start at node (0,0)

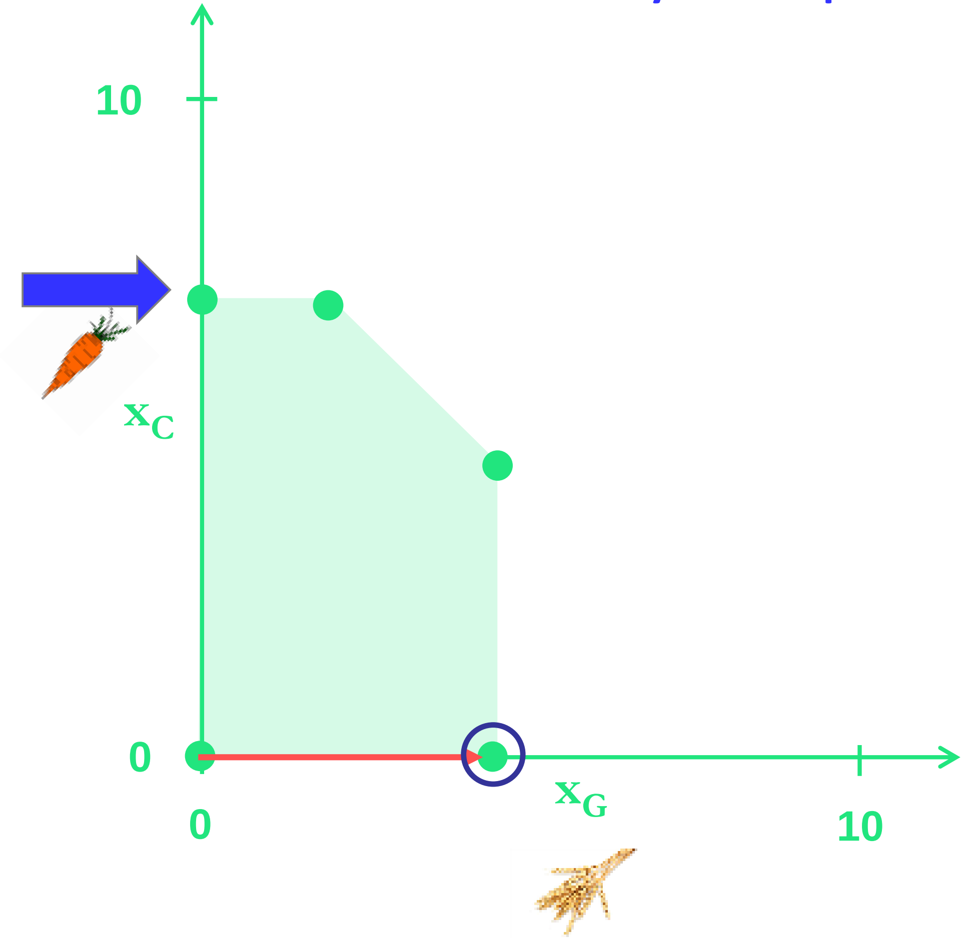
direction of gain?

optimal solution

select direction of highest gain

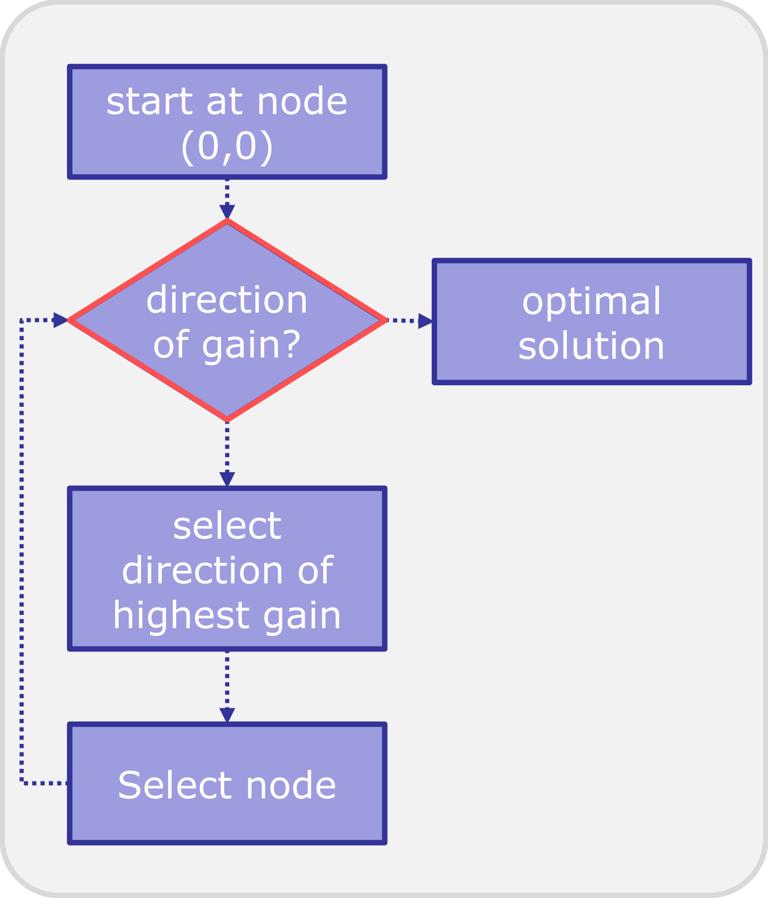
Select node

Maximize for y-intercept



Simplex Algorithm

[G. Dantzig]



start at node (0,0)

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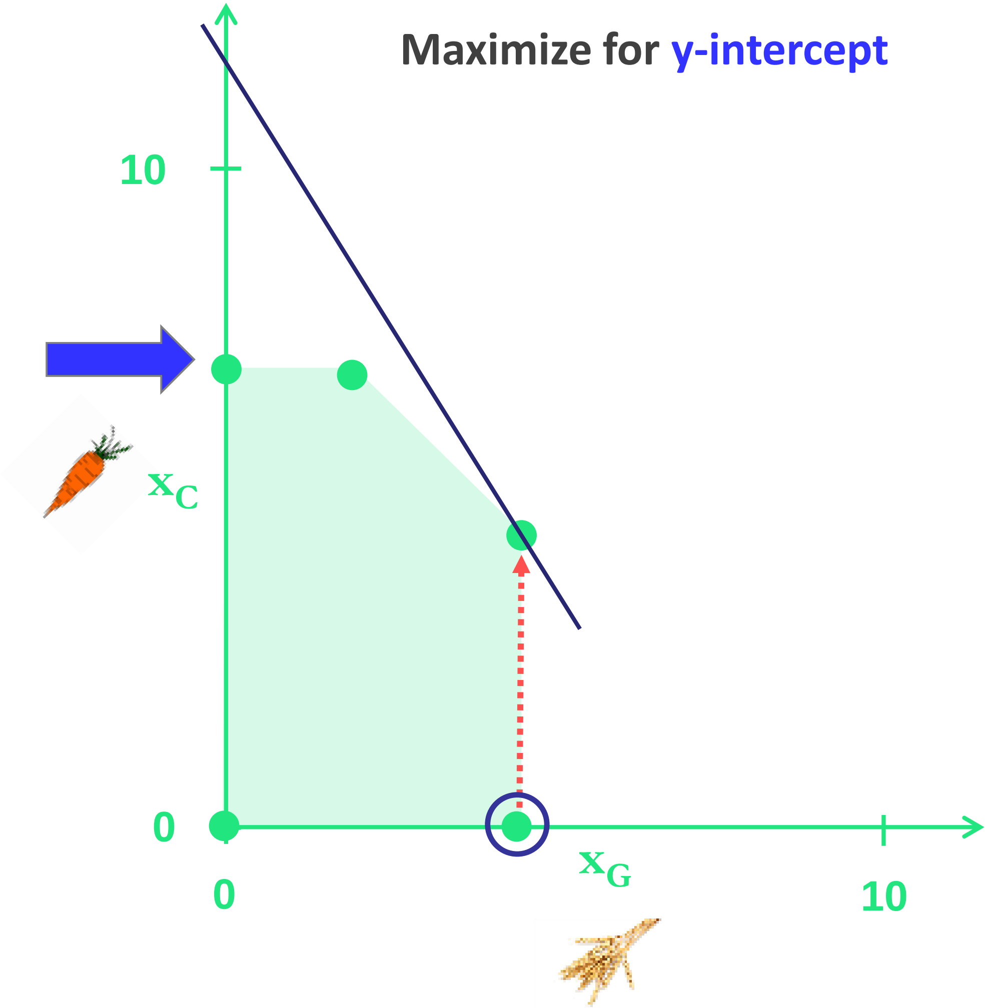
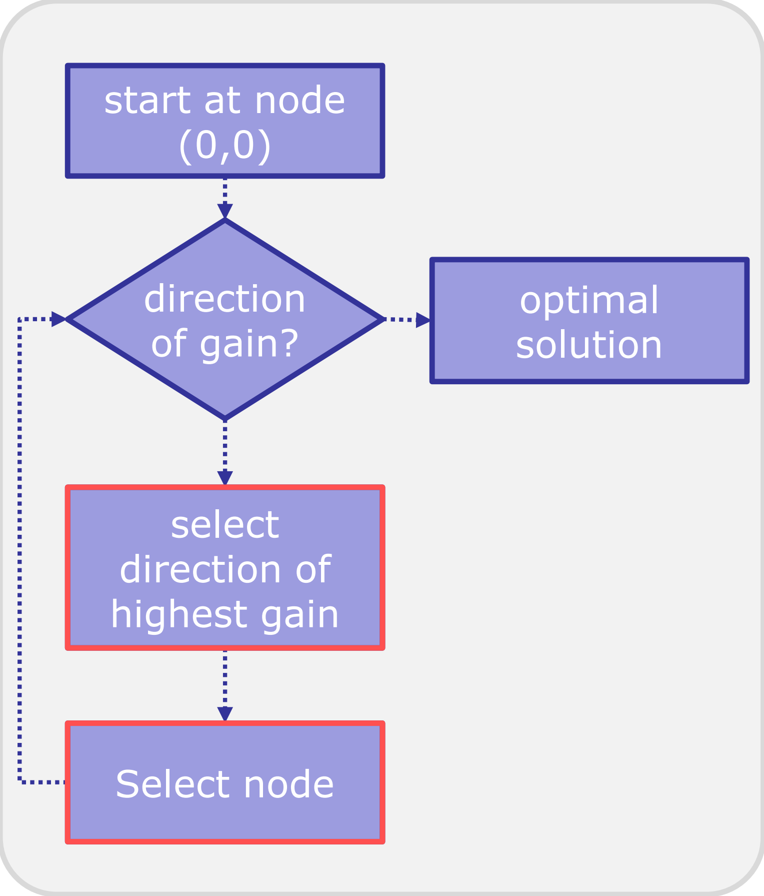
Select node

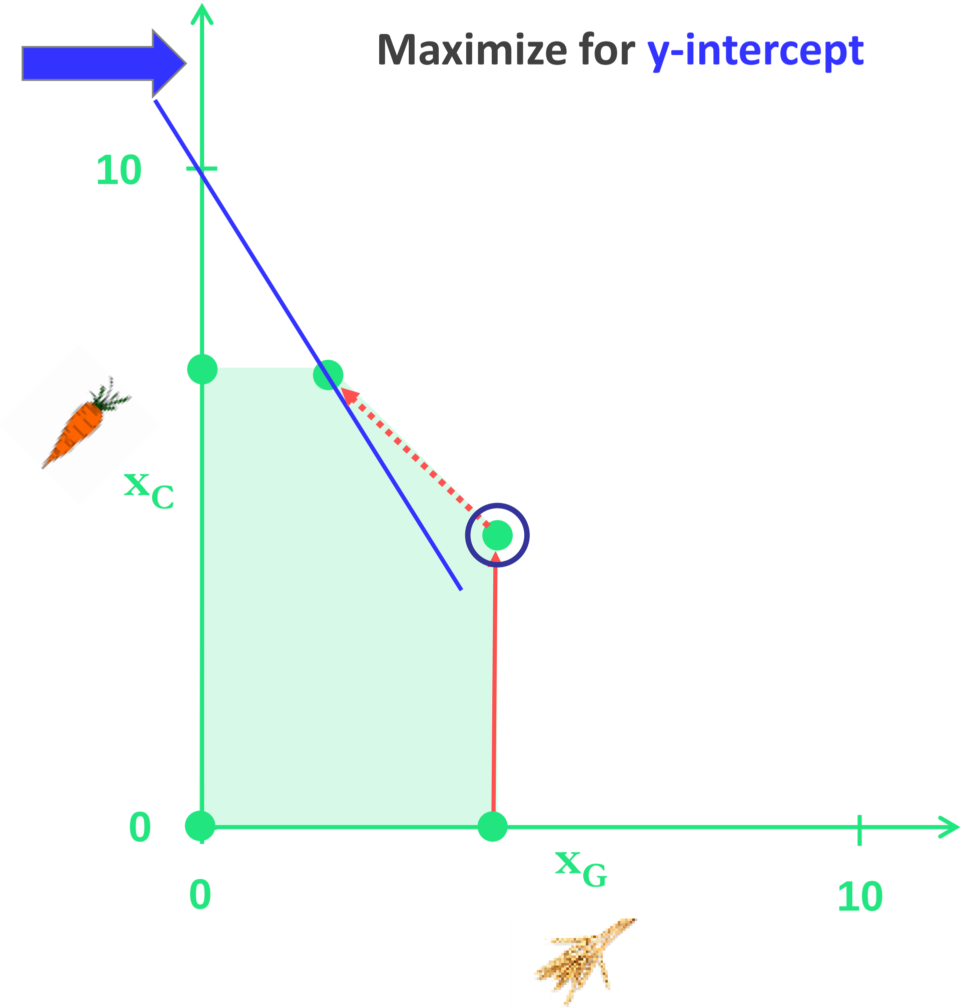


Simplex Algorithm

[G. Dantzig]

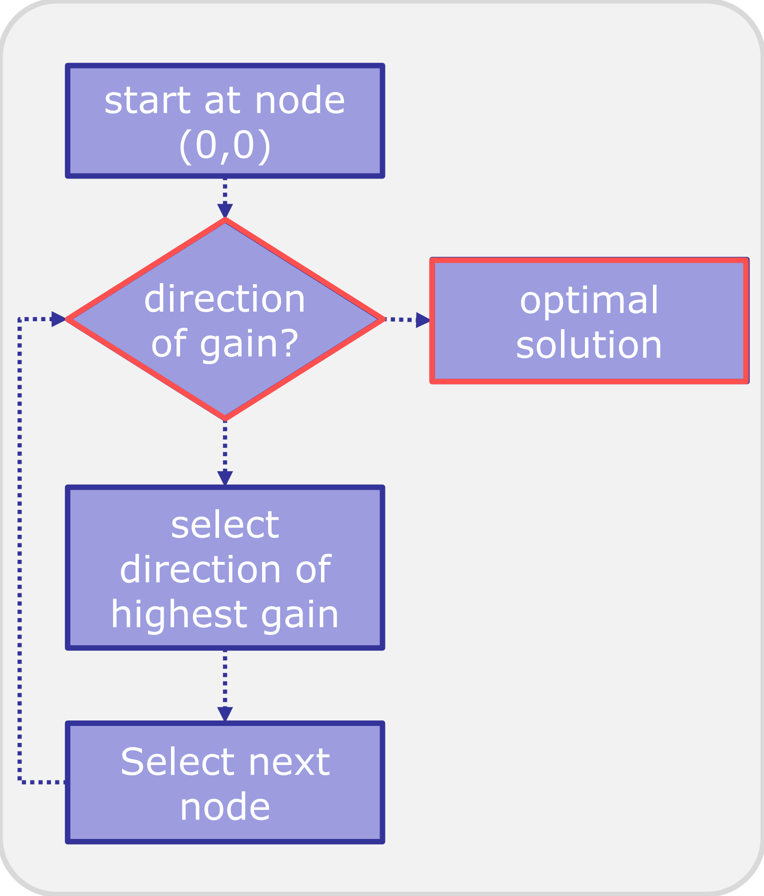


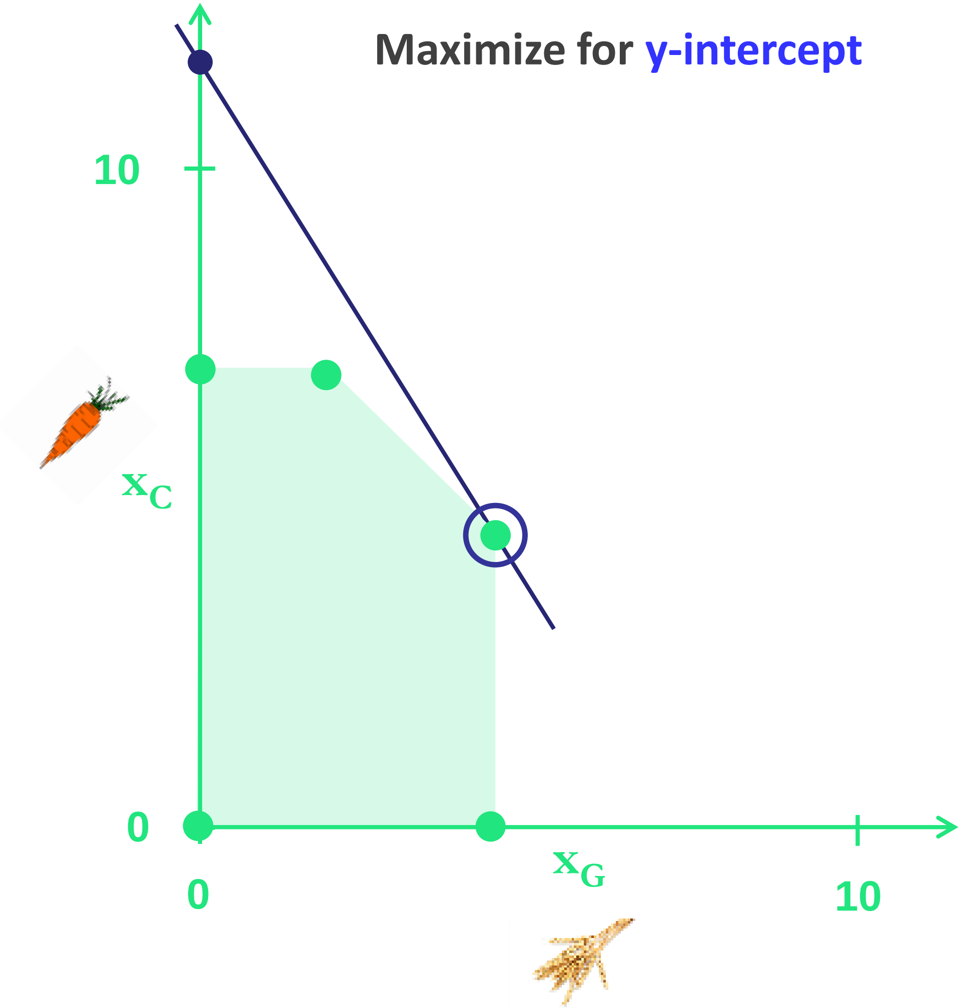
Maximize for y-intercept



Simplex Algorithm

[G. Dantzig]





Notation of an optimization model

Linear program
for the
Farmer's Choice Problem

$$\begin{array}{llcccl}
\textbf{MAX} & 5000 \cdot x_C + 8000 \cdot x_G & & & & \\
\text{s.t.} & & x_C & + & x_G & \leq 9 \\
& & & & x_G & \leq 4.5 \\
& & x_C & & & \leq 7 \\
& & & & x_C, x_G \in R_0^+ &
\end{array}$$

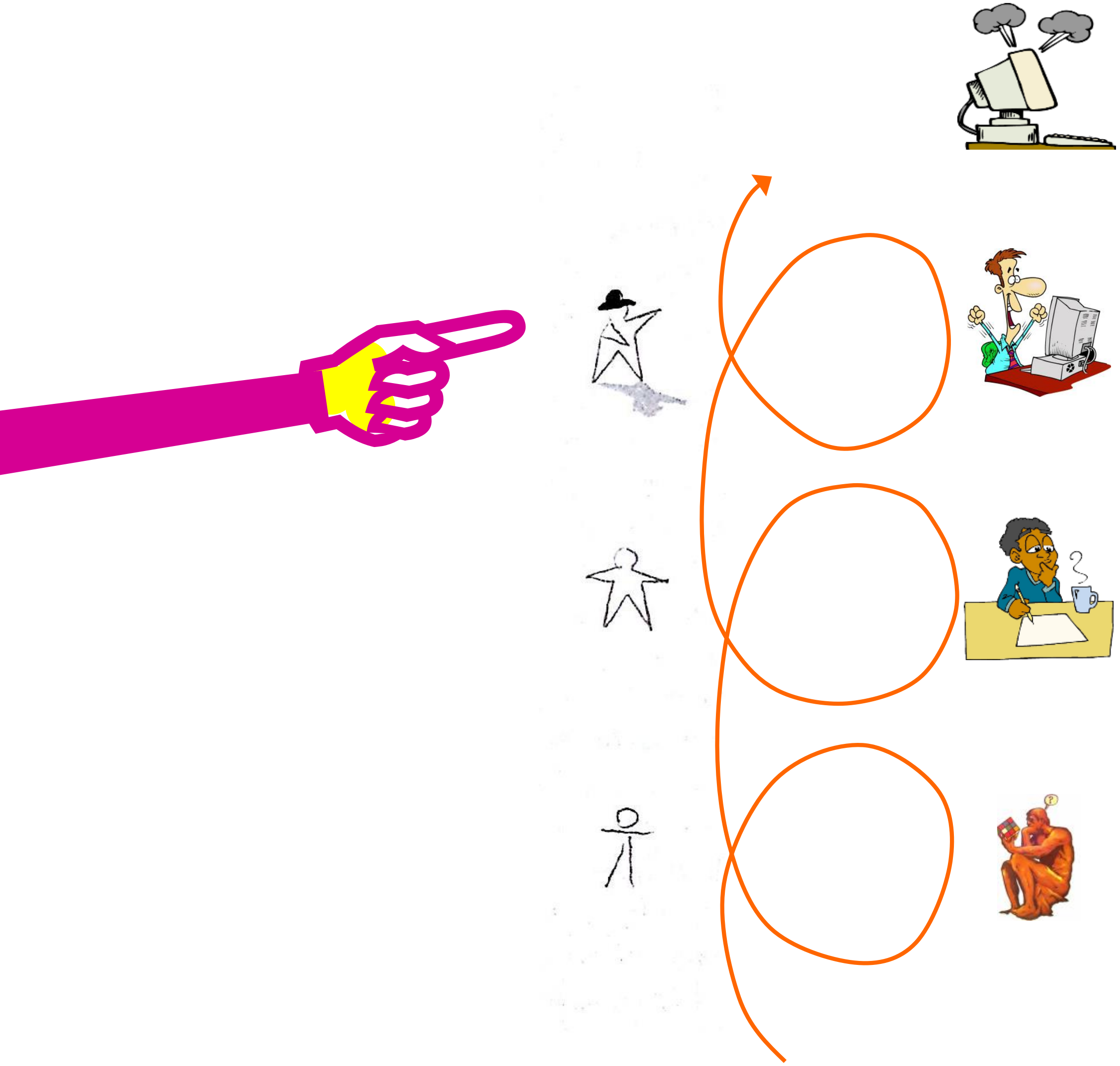
s.t. : subject to

add «0»

$$\begin{array}{llcccl}
\textbf{MAX} & \mathbf{5000} \cdot x_C & + & \mathbf{8000} \cdot x_G & & \\
\text{s.t.} & \mathbf{1} \cdot x_C & + & \mathbf{1} \cdot x_G & \leq & \mathbf{9} \\
& \mathbf{0} \cdot x_C & + & \mathbf{1} \cdot x_G & \leq & \mathbf{4.5} \\
& \mathbf{1} \cdot x_C & + & \mathbf{0} \cdot x_G & \leq & \mathbf{7} \\
& & & x_C, x_G \in R_0^+ & &
\end{array}$$

conceptualize & represent

implement



solve

implement

conceptualize &
represent

Matrix notation of an optimisation model

MAX $5000 \cdot x_C + 8000 \cdot x_G$

s.t.

$$1 \cdot x_C + 1 \cdot x_G \leq 9$$

$$0 \cdot x_C + 1 \cdot x_G \leq 4.5$$

$$1 \cdot x_C + 0 \cdot x_G \leq 7$$

$$x_C\ , x_G \in R_0^+$$

$$f = \begin{pmatrix} 5000 \\ 8000 \end{pmatrix} \qquad x = \begin{pmatrix} x_C \\ x_G \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \leq b = \begin{pmatrix} 9 \\ 4.5 \\ 7 \end{pmatrix}$$

Computer can read this format!

$$\max_{x} \ f^T \cdot x$$

s.t. $A \cdot x \leq b$

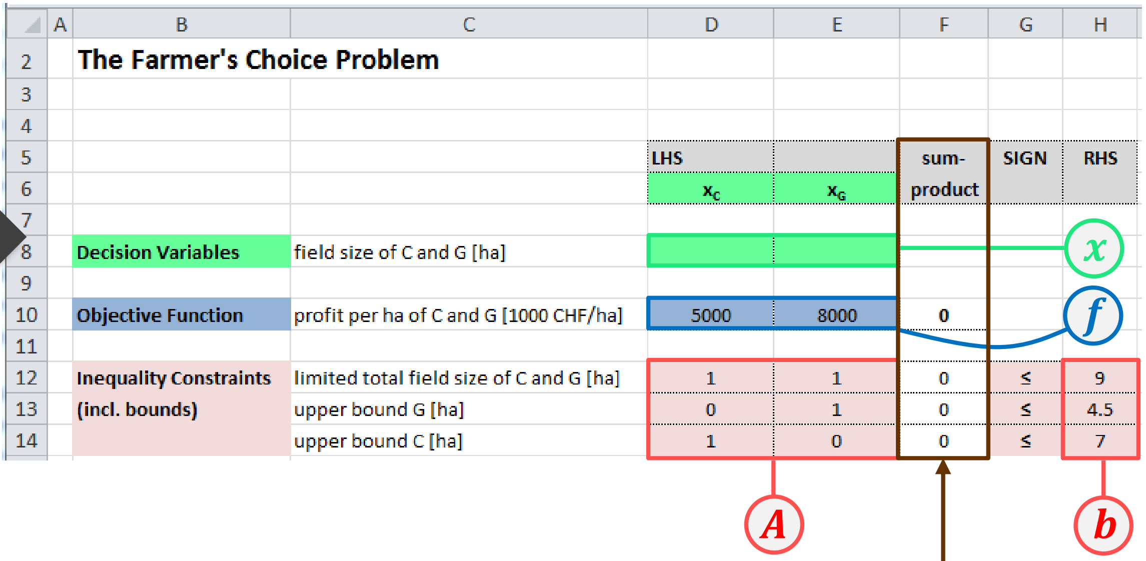
MS EXCEL – Set up of matrix notation model I

Specify *x*, *f*, *A* and *b*

$$f = \begin{pmatrix} 5000 \\ 8000 \end{pmatrix} \qquad x = \begin{pmatrix} [\,] \\ [\,] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \leq b = \begin{pmatrix} 9 \\ 4.5 \\ 7 \end{pmatrix}$$

Matrix notation model



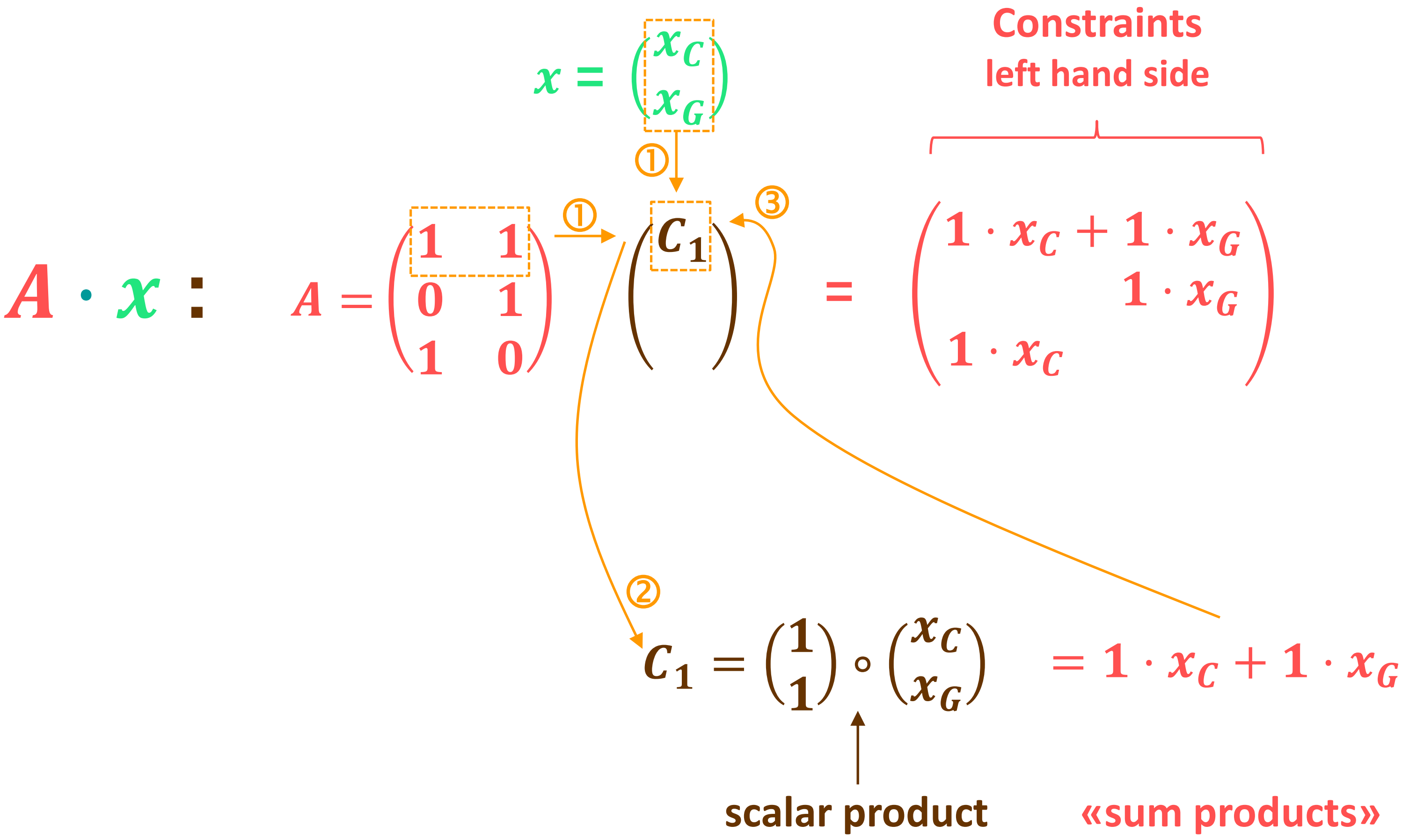
	A	B	C	D	E	F	G	H
2		The Farmer's Choice Problem						
3								
4								
5				LHS		sum-product	SIGN	RHS
6				x_C	x_G			
7								
8		Decision Variables	field size of C and G [ha]					
9								
10		Objective Function	profit per ha of C and G [1000 CHF/ha]	5000	8000	0		
11								
12		Inequality Constraints (incl. bounds)	limited total field size of C and G [ha]	1	1	0	≤	9
13			upper bound G [ha]	0	1	0	≤	4.5
14			upper bound C [ha]	1	0	0	≤	7

Matrix multiplication

$$\min_{x} \; f^{T} \cdot x$$

$$\text{s.t.} \; A \cdot x \leq b$$

Matrix multiplication I



$$x = \begin{pmatrix} \boxed{x_C} \\ \boxed{x_G} \end{pmatrix}$$

① ① ③

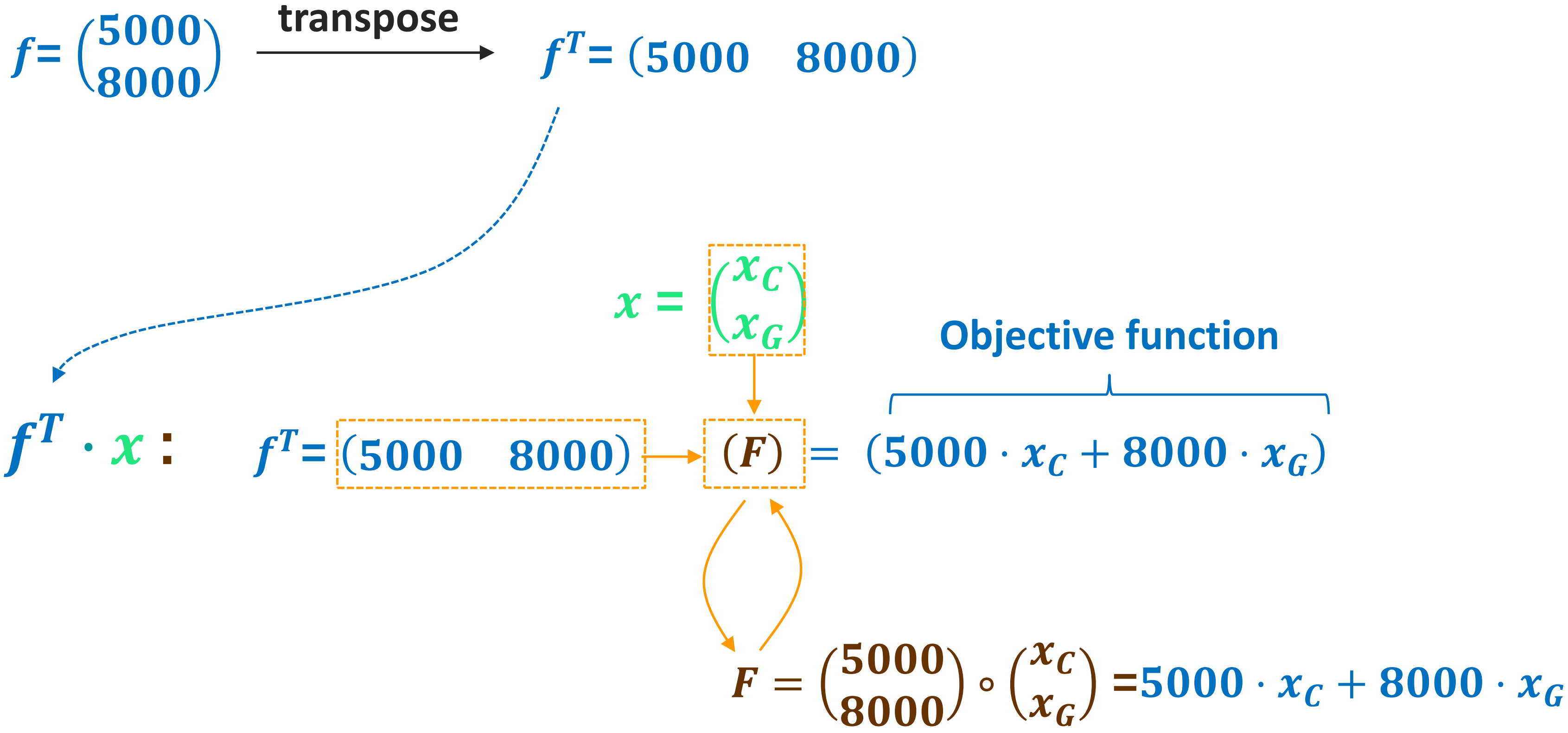
$$A \cdot x : \quad A = \begin{pmatrix} \boxed{1 \quad 1} \\ 0 \quad 1 \\ 1 \quad 0 \end{pmatrix} \begin{pmatrix} \boxed{c_1} \\ \\ \\ \end{pmatrix} = \overbrace{\begin{pmatrix} 1 \cdot x_C + 1 \cdot x_G \\ 1 \cdot x_G \\ 1 \cdot x_C \end{pmatrix}}^{\text{Constraints left hand side}}$$

②

$$c_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} x_C \\ x_G \end{pmatrix} \quad = 1 \cdot x_C + 1 \cdot x_G$$

scalar product «sum products»

Matrix multiplication II



$$f = \begin{pmatrix}5000\\8000\end{pmatrix} \xrightarrow{\text{transpose}} f^T = \begin{pmatrix}5000 & 8000\end{pmatrix}$$

$$x = \boxed{\begin{pmatrix}x_C\\x_G\end{pmatrix}}$$

$$f^T \cdot x: \quad f^T = \boxed{\begin{pmatrix}5000 & 8000\end{pmatrix}} \rightarrow \boxed{(F)} = \overbrace{\left(5000 \cdot x_C + 8000 \cdot x_G\right)}^{\text{Objective function}}$$

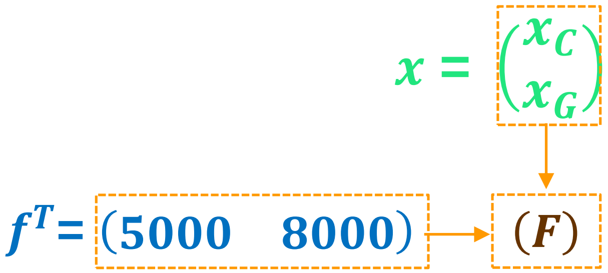
$$F = \begin{pmatrix}5000\\8000\end{pmatrix} \circ \begin{pmatrix}x_C\\x_G\end{pmatrix} = 5000 \cdot x_C + 8000 \cdot x_G$$

MS EXCEL – Set up of matrix notation model II

Implement the «scalar product»

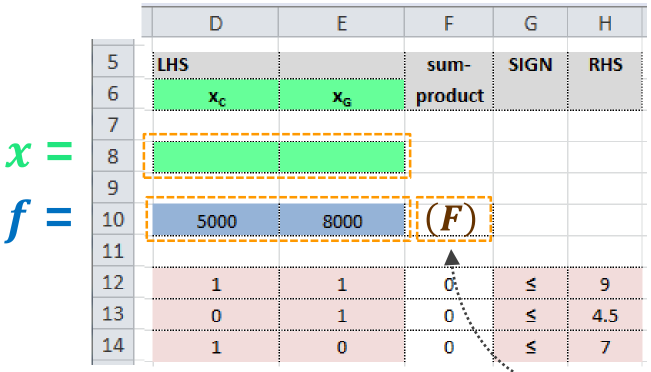
Linear algebra convention

Excel convention



$$x = \boxed{\begin{pmatrix} x_C \\ x_G \end{pmatrix}}$$

$$f^T = \boxed{(5000 \quad 8000)} \rightarrow \boxed{(F)}$$

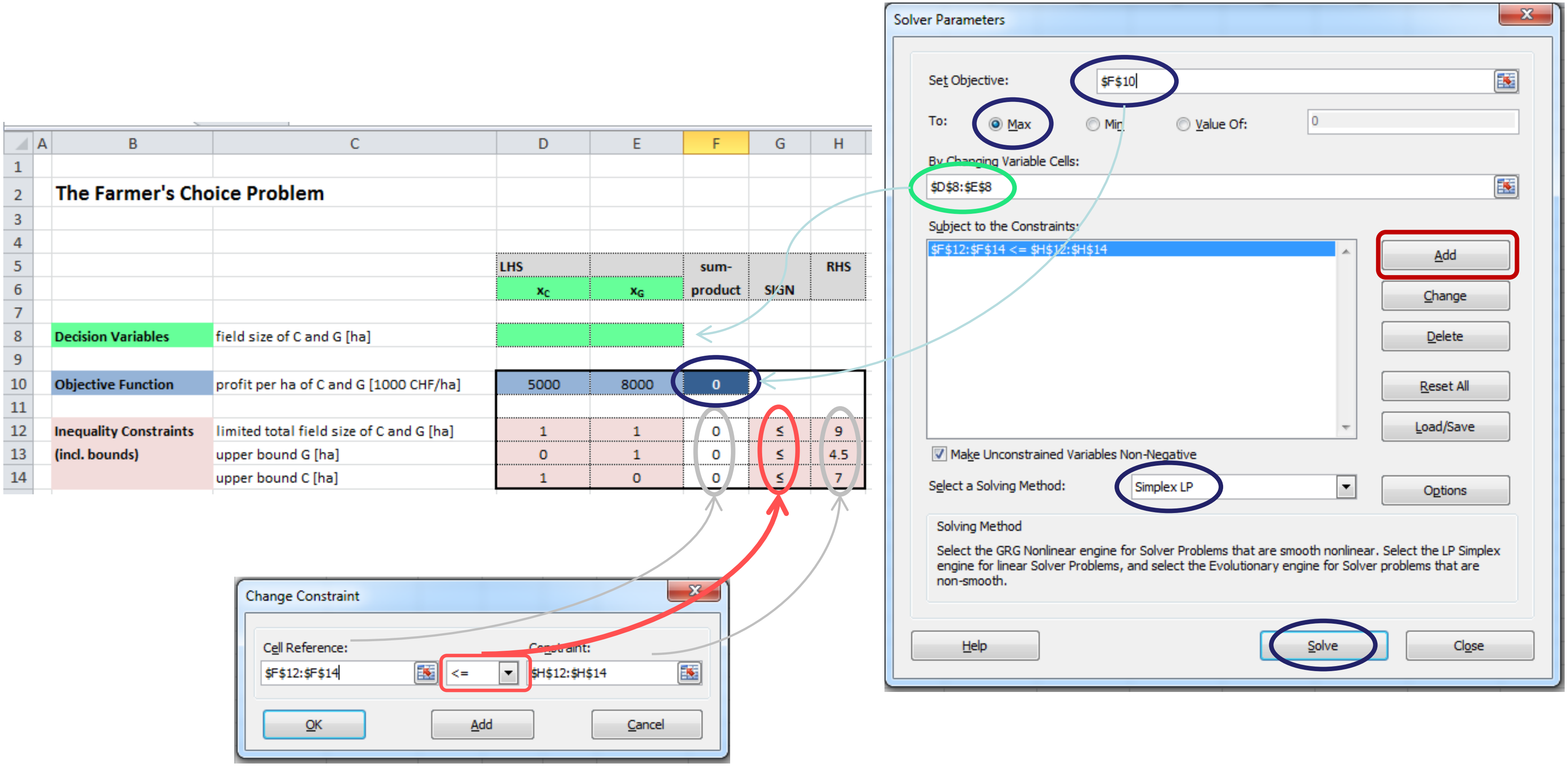


	D	E	F	G	H
5	LHS		sum-	SIGN	RHS
6	x_C	x_G	product		
7					
8					
9					
10	5000	8000	(F)		
11					
12	1	1	0	≤	9
13	0	1	0	≤	4.5
14	1	0	0	≤	7

=sumproduct(D8:E8;D10:E10)

MS Excel – Solve the model

DATA tab ➲ SOLVER



	A	B	C	D	E	F	G	H
1								
2		**The Farmer's Choice Problem**						
3								
4								
5				LHS		sum-		RHS
6				x_C	x_G	product	SIGN	
7								
8		**Decision Variables**	field size of C and G [ha]					
9								
10		**Objective Function**	profit per ha of C and G [1000 CHF/ha]	5000	8000	0		
11								
12		**Inequality Constraints**	limited total field size of C and G [ha]	1	1	0	≤	9
13		**(incl. bounds)**	upper bound G [ha]	0	1	0	≤	4.5
14			upper bound C [ha]	1	0	0	≤	7

Solver Parameters

Set Objective: $F$10

To: ● Max ○ Min ○ Value Of: 0

By Changing Variable Cells: $D$8:$E$8

Subject to the Constraints:

$F$12:$F$14 <= $H$12:$H$14

☑ Make Unconstrained Variables Non-Negative

Select a Solving Method: Simplex LP

Solving Method

Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.

Change Constraint

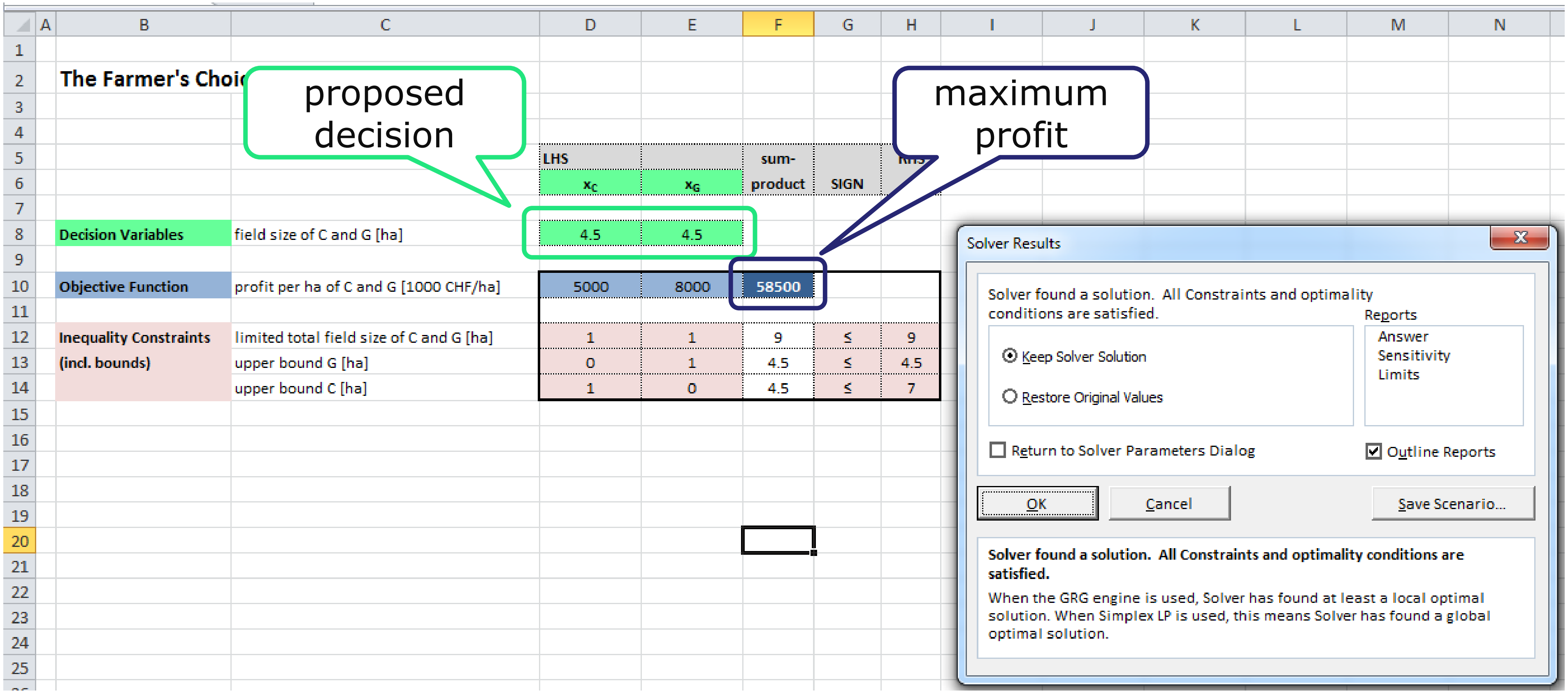
Cell Reference: $F$12:$F$14

<=

Constraint: $H$12:$H$14

Define the «sign»

MS Excel – Optimization result



proposed decision

maximum profit

	A	B	C	D	E	F	G	H
1								
2		**The Farmer's Choi**						
5				LHS		sum-		RHS
6				x_C	x_G	product	SIGN	
8		**Decision Variables**	field size of C and G [ha]	4.5	4.5			
10		**Objective Function**	profit per ha of C and G [1000 CHF/ha]	5000	8000	58500		
12		**Inequality Constraints**	limited total field size of C and G [ha]	1	1	9	≤	9
13		**(incl. bounds)**	upper bound G [ha]	0	1	4.5	≤	4.5
14			upper bound C [ha]	1	0	4.5	≤	7

Solver Results

Solver found a solution. All Constraints and optimality conditions are satisfied.

Reports
- Answer
- Sensitivity
- Limits

● Keep Solver Solution

○ Restore Original Values

☐ Return to Solver Parameters Dialog

☑ Outline Reports

Solver found a solution. All Constraints and optimality conditions are satisfied.

When the GRG engine is used, Solver has found at least a local optimal solution. When Simplex LP is used, this means Solver has found a global optimal solution.

MATLAB - Set up of matrix notation model I

Provide the information that Matlab needs to understand the optimization model!

Matrix notation model

$$f = \begin{pmatrix} 5000 \\ 8000 \end{pmatrix} \qquad x = \begin{pmatrix} x_C \\ x_G \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \leq b = \begin{pmatrix} 9 \\ 4.5 \\ 7 \end{pmatrix}$$

MATLAB syntax

$$f = \begin{pmatrix} 5000 \\ 8000 \end{pmatrix} \qquad xType = \begin{pmatrix} 'C' \\ 'C' \end{pmatrix}$$

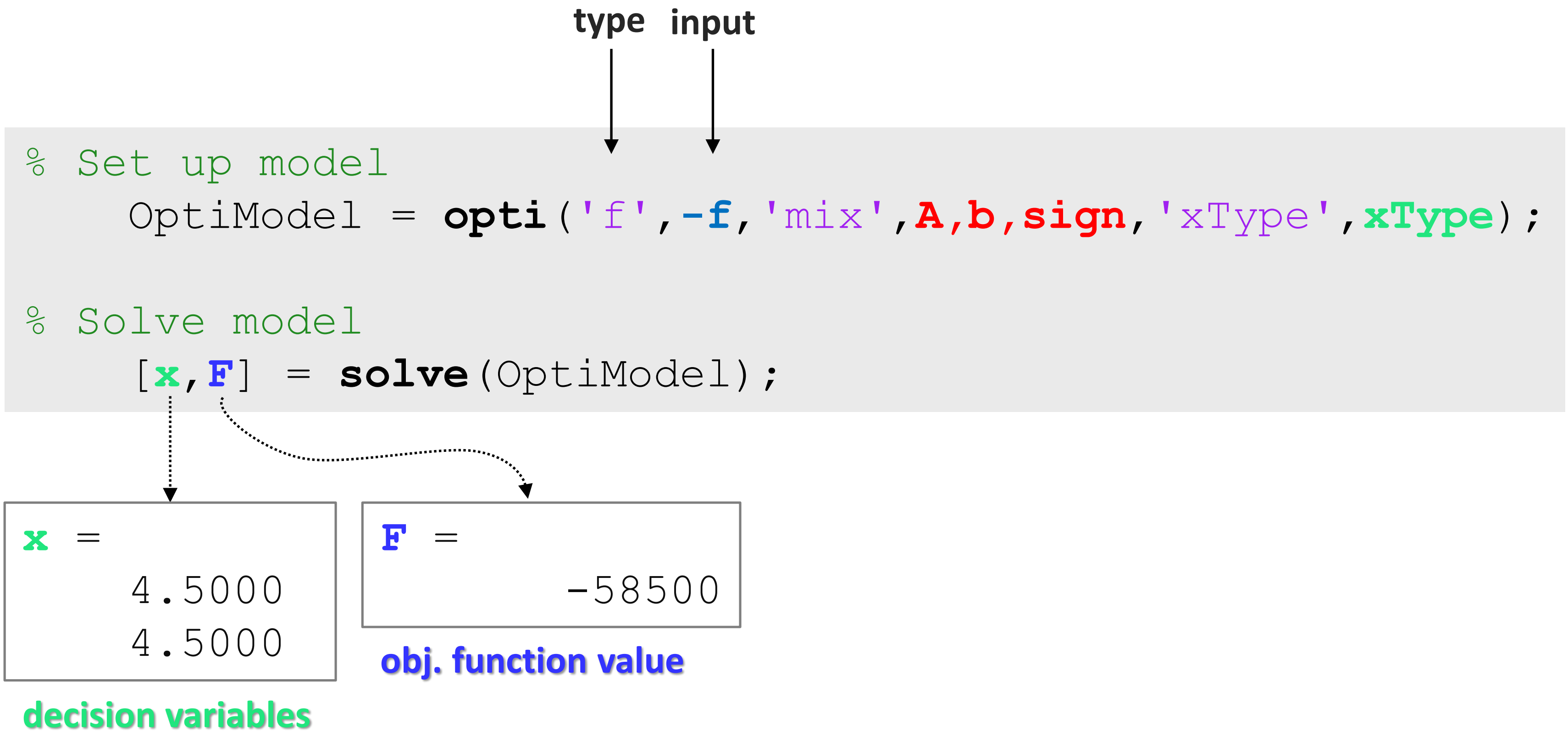
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad sign = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad b = \begin{pmatrix} 9 \\ 4.5 \\ 7 \end{pmatrix}$$

Number type	*xType*
Real number	‘C’
Integer	‘I’
Binary (0;1)	‘B’

Signum	*sign*
≤	-1
=	0
≥	+1

MATLAB – Set up and solve the model with two lines of code

solves MINIMIZATION problems only ➲ set **–f** for MAXIMIZATION



```
% Set up model
    OptiModel = opti('f',-f,'mix',A,b,sign,'xType',xType);

% Solve model
    [x,F] = solve(OptiModel);
```

```
x =
    4.5000
    4.5000
```

decision variables

```
F =
    -58500
```

obj. function value

detailed description >> see assignment document!

Literature

Lecture notes *System Modeling and Optimization* by Prof. H.-J. Lüthi, IFOR, ETH Zürich (2011)
- *3 Modeling Linear Problems* [definitions, representation]
- *4 Solving Linear Programs* [Excel-Solver]

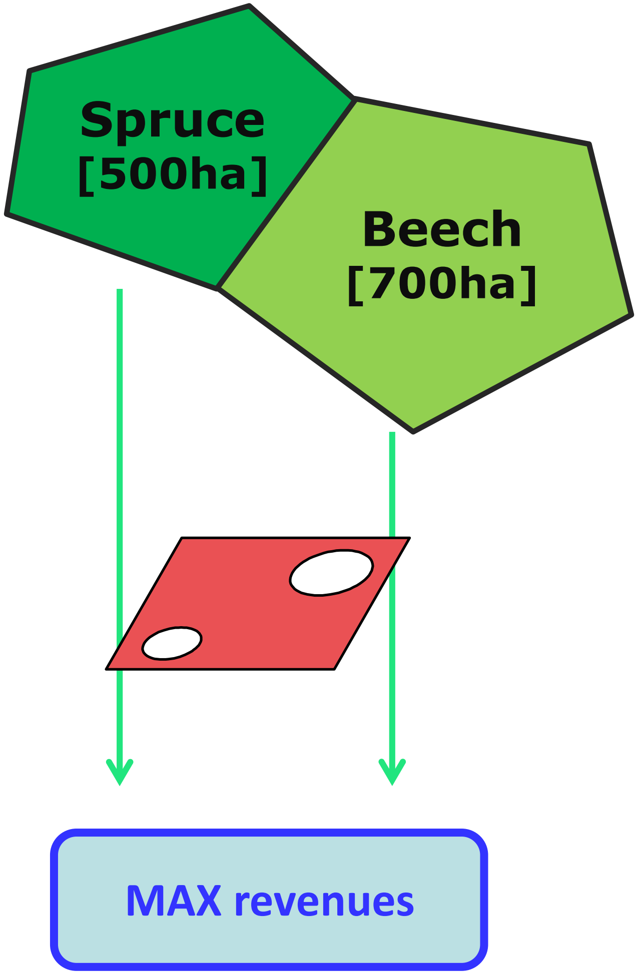
OPTI TOOLBOX website: http://www.i2c2.aut.ac.nz/Wiki/OPTI/

- C -

Exercise 5

Product Portfolio Problem

Problem

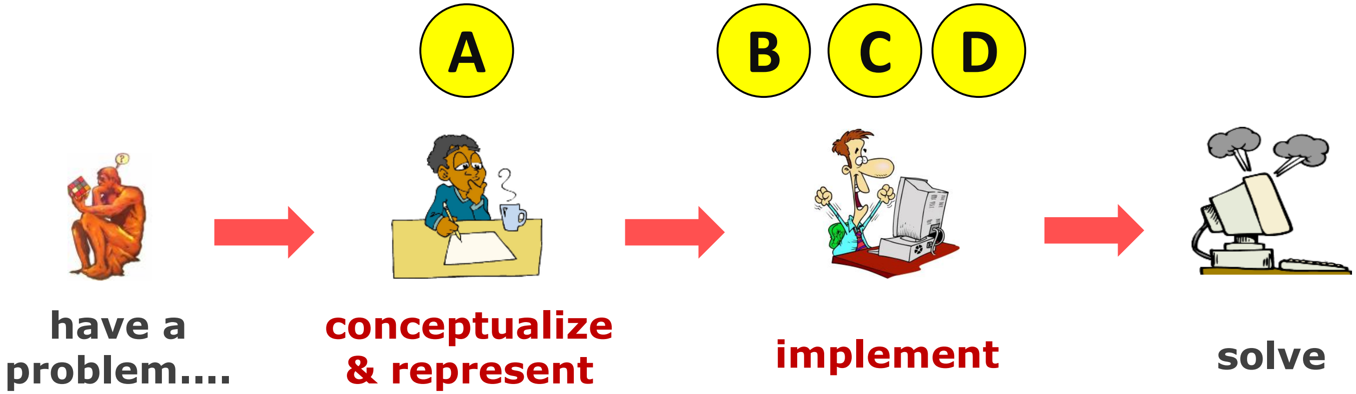


A forest company would like to plan the harvest for their forest stands of two qualities, spruce and beech trees, for the upcoming year in order to maximize the revenues from timber sale.

Constraints:

1) Maximum harvested volume is limited to sustainable yield
2) labor force of the company for a year is limited to 1’800h.

Tasks



A

B C D

have a problem....

conceptualize & represent

implement

solve

- **A** Formulate a Linear Program (LP)
- **B** Solve the LP graphically
- **C** Implement the LP in Excel and solve it using the Solver Add-in
- **D** Implement the LP in MATLAB using the OPTI toolbox

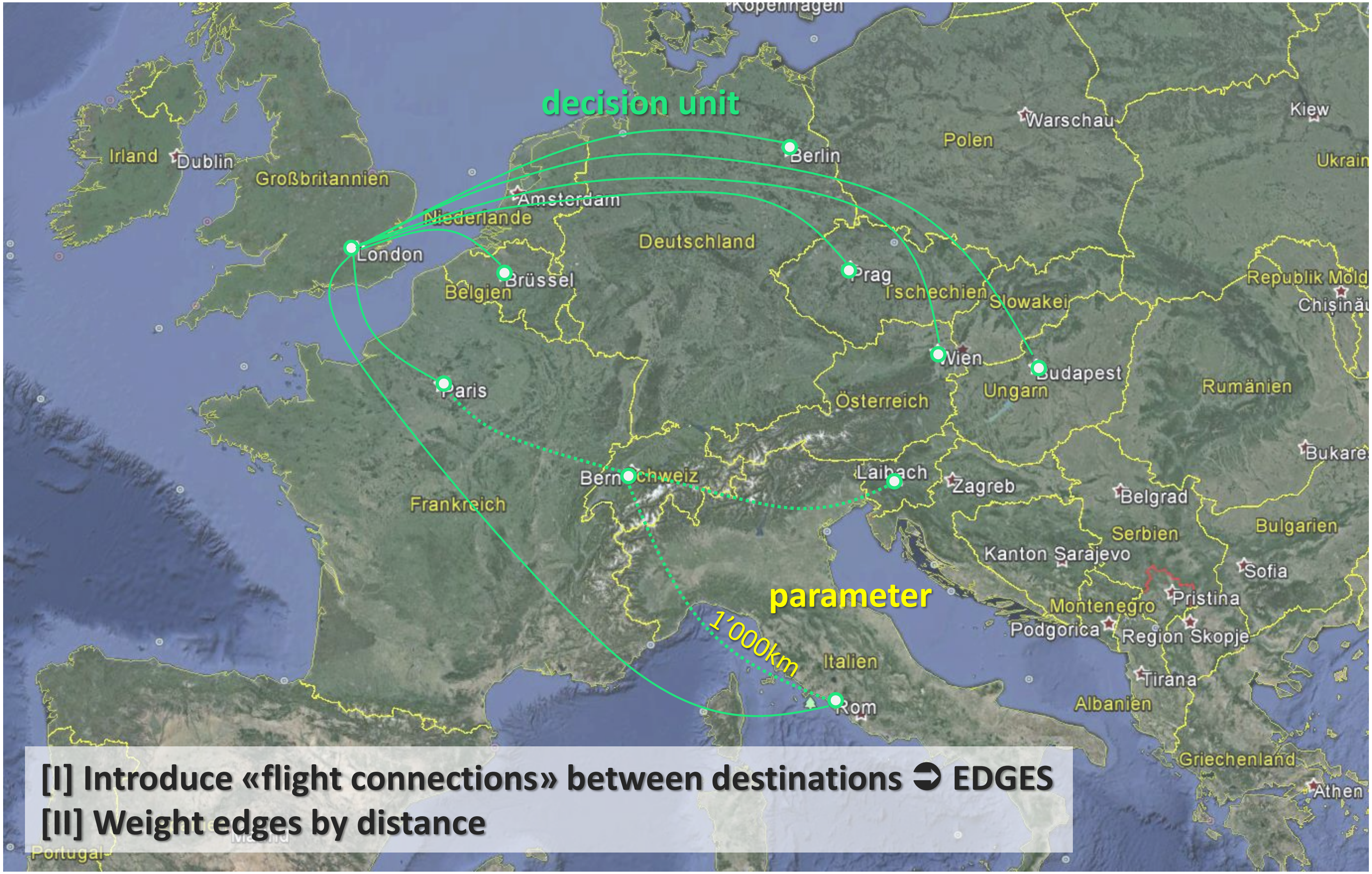
A warm-up for REPRESENTATION

Plan a round-trip for:

London, Rome, Berne Berlin, Prague, Vienna, Bruxelles, Budapest, Laibach and Paris

OBJECTIVE **Minimize distance**

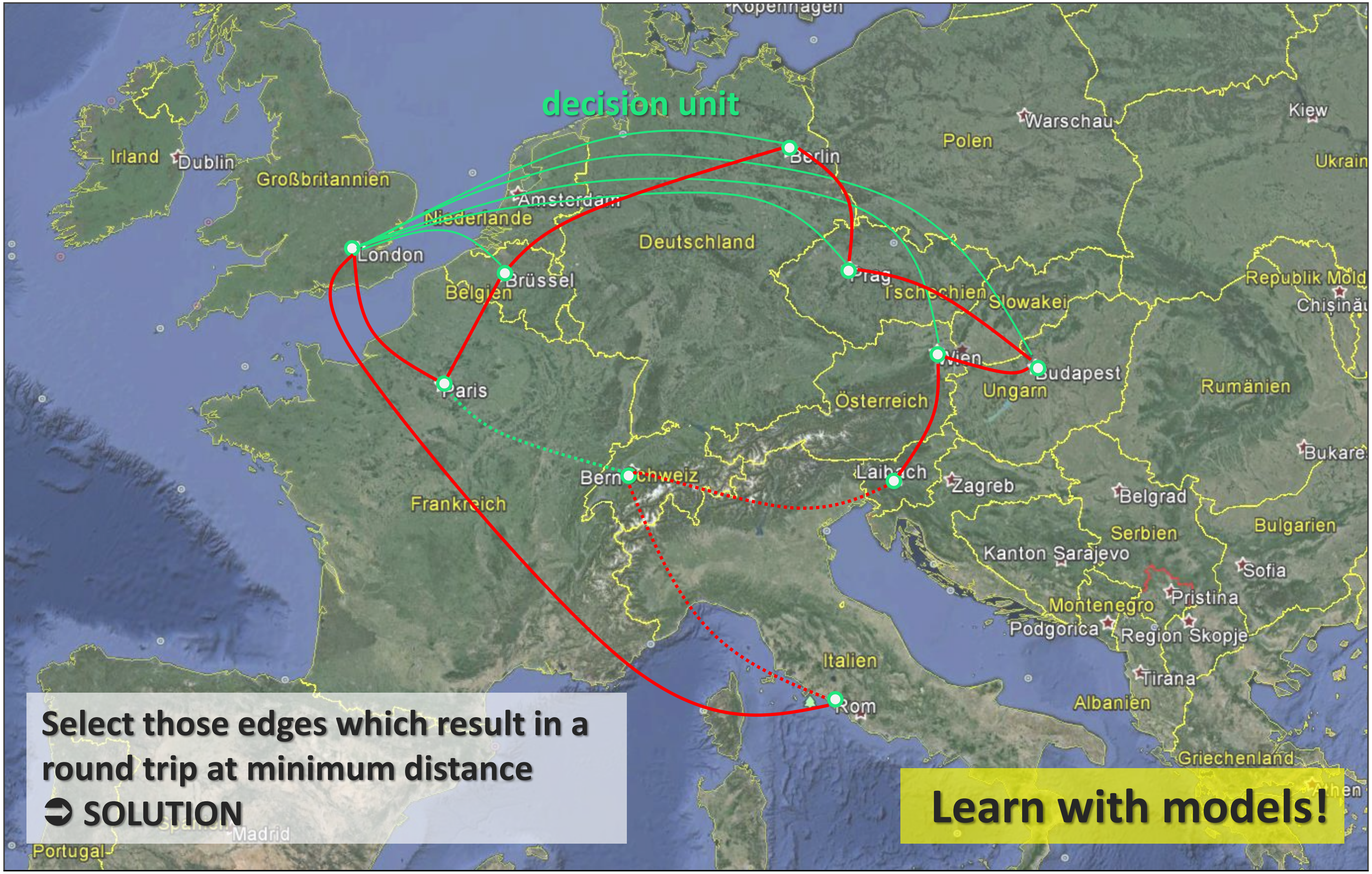
Problem Representation



[I] Introduce «flight connections» between destinations ➲ EDGES

[II] Weight edges by distance

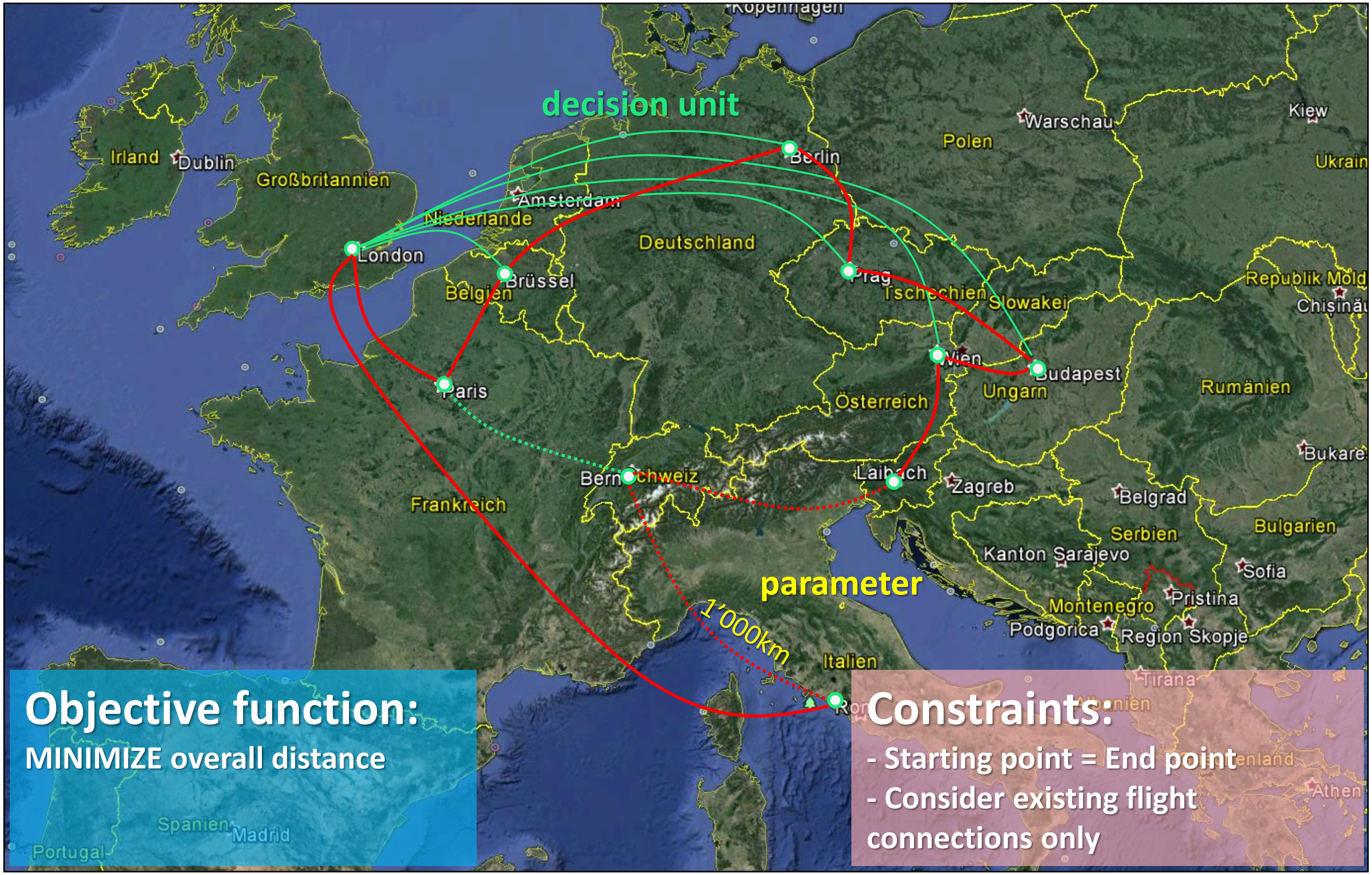
Solution



Select those edges which result in a round trip at minimum distance
➲ SOLUTION

Learn with models!

FS - Application to the «round trip problem»



Objective function:

MINIMIZE overall distance

Constraints:

- Starting point = End point
- Consider existing flight connections only