Addendum to Dual-antennae GNSS-aided INS stationary alignment with SPE

Jay A. Farrell, Felipe O. Silva

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Abstract

This document works through the observability for Sections III.B and IV of the paper "Dual-antennae GNSS-aided INS stationary alignment with sensor parameter estimation", submitted for consideration of publication by the IEEE Transactions on Instrumentation and Measurement (TIM).

1 Edits to Section III.B

Paper equations (19-23) are repeated here with new equation numbers:

$$\eta_N = \frac{1}{r_x}\left[\frac{r_z}{g_D} b_{f_x} + b_{r_x}\right] + \omega_{\eta_N} \tag{1}$$

$$\eta_E = \frac{1}{r_x}\left[\frac{r_z}{g_D} b_{f_x} + b_{r_x} - \frac{r_x}{g_D}\left(b_{f_z} + b_{g_D}\right)\right] + \omega_{\eta_E} \tag{2}$$

$$\eta_D = \frac{-1}{g_D}\left[b_{f_z} + b_{g_D}\right] + \omega_{\eta_D} \tag{3}$$

$$o_E = \frac{1}{2\,r_x}\left[-\frac{r_x}{g_D} b_{f_x} + b_{r_z} + \frac{r_z}{g_D}\left(b_{f_z} + b_{g_D}\right)\right] + \omega_{o_E} \tag{4}$$

This Section analyzes the algebraic structure of the problem (i.e., observability); therefore, the noise terms (last term on the right-hand side of each equation) will be dropped in the following.

These four pseudo-measurement equations depend on five unknown systematic sensor parameters (b_{f_x}, b_{f_z}, b_{r_x}, b_{r_z} and b_{g_D}). However, the equations are not linearly independent; for example (dropping noise terms),

$$\eta_E - \eta_N = \eta_D.$$

Therefore, this system of equations cannot be solved for the five unknowns. The question that remains is: which unknowns (or linear combinations of unknowns) are observable?

Note that the variables b_{f_z} and b_{g_D} only appear in these equations as the sum

$$x_{S_1} = (b_{f_z} + b_{g_D}).$$

The orthogonal complement of x_{S_1} is

$$x_{D_1} = (b_{f_z} - b_{g_D}).$$

Using these variables, equation (1-4) can be written in the matrix form

$$\mathbf{Y_S} = \mathbf{H_S}\,\mathbf{x_S} + \eta_{\mathbf{S}} \tag{5}$$

where $\mathbf{x} = [x_{S_1},\ b_{f_x},\ b_{r_x},\ b_{r_z},\ x_{D_1}]$ and

$$\mathbf{H_S} = \begin{bmatrix} 0 & \frac{r_z}{r_x\,g_D} & \frac{1}{r_x} & 0 & 0 \\ \frac{-1}{g_D} & \frac{r_z}{r_x\,g_D} & \frac{1}{r_x} & 0 & 0 \\ \frac{-1}{g_D} & 0 & 0 & 0 & 0 \\ \frac{r_z}{2\,r_x\,g_D} & \frac{-1}{2\,g_D} & 0 & \frac{1}{2\,r_x} & 0 \end{bmatrix}. \tag{6}$$

Replacing row 2 with (row 2 minus row 1 minus row 3) yields

$$\mathbf{H_1} = \begin{bmatrix} 0 & \frac{r_z}{r_x\,g_D} & \frac{1}{r_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{g_D} & 0 & 0 & 0 & 0 \\ \frac{r_z}{2\,r_x\,g_D} & \frac{-1}{2\,g_D} & 0 & \frac{1}{2\,r_x} & 0 \end{bmatrix}. \tag{7}$$

Multiplying each row by a constant to clear its denominator and using row 3 to clear column 1 yields

$$\mathbf{H_2} = \begin{bmatrix} 0 & r_z & g_D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -r_x & 0 & g_D & 0 \end{bmatrix}. \tag{8}$$

The algebraic structure of $\mathbf{H_1}$ shows that the four nonzero equations, with the variables ordered as $\mathbf{x} = [x_{S_1},\ b_{f_x},\ b_{r_x},\ b_{r_z},\ x_{D_1}] \in \Re^5$, the observable space is spanned by the three vectors

$$\mathcal{O}_{\mathcal{S}} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ r_z \\ g_D \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -r_x \\ 0 \\ g_D \\ 0 \end{bmatrix} \right\}.$$

Any choice of three linearly independent variables within $\mathcal{O}_{\mathcal{S}}$ can be observed. One choice of observable variables is

$$x_{O_1} = x_{S_1}, \tag{9}$$
$$x_{O_2} = r_z\,b_{f_x}/g_D + b_{r_x}, \tag{10}$$
$$x_{O_3} = -r_x\,b_{f_x}/g_D + b_{r_z}. \tag{11}$$

The unobservable space, in turn, is spanned by the two vectors

$$\mathcal{U}_{\mathcal{S}} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ g_D \\ -r_z \\ r_x \\ 0 \end{bmatrix} \right\}.$$

Any choice of two linearly independent variables within $\mathcal{U}_{\mathcal{S}}$ is unobservable. One choice of unobservable variables is

$$x_{U_1} = x_{D_1} \tag{12}$$
$$x_{U_2} = g_D\, b_{f_x} - r_z\, b_{r_x} + r_x\, b_{r_z}. \tag{13}$$

In this choice of variables, with $\mathbf{z_S} = \begin{bmatrix} x_{O_1} & x_{O_2} & x_{O_3} & x_{U_1} & x_{U_2} \end{bmatrix}$, equation (5) can be rewritten as

$$\mathbf{Y_S} = \mathbf{G_S}\, \mathbf{z_S} + \eta_\mathbf{S} \tag{14}$$

with

$$\mathbf{G_S} = \begin{bmatrix} 0 & \frac{1}{r_x} & 0 & 0 & 0 \\ \frac{-1}{g_D} & \frac{1}{r_x} & 0 & 0 & 0 \\ \frac{-1}{g_D} & 0 & 0 & 0 & 0 \\ \frac{r_z}{2\, r_x\, g_D} & 0 & \frac{1}{2\, r_x} & 0 & 0 \end{bmatrix},$$

which corresponds to equation (35) of the original paper.

2 Edits to Section IV

Paper equations (40-45) are also repeated here with new equation numbers:

$$\eta_N = \frac{1}{r_N^2 + r_E^2}\left[\frac{r_N r_D}{g_D} b_{f_N} + \frac{r_E r_D}{g_D} b_{f_E} - \frac{r_E^2}{g_D}(b_{f_D} + b_{g_D}) + r_N b_{r_N} + r_E b_{r_E}\right] + w_{\eta_N} \tag{15}$$

$$\eta_E = \frac{1}{r_N^2 + r_E^2}\left[\frac{r_N r_D}{g_D} b_{f_N} + \frac{r_E r_D}{g_D} b_{f_E} - \frac{r_N^2}{g_D}(b_{f_D} + b_{g_D}) + r_N b_{r_N} + r_E b_{r_E}\right] + w_{\eta_E} \tag{16}$$

$$\eta_D = -\frac{1}{g_D}\left[b_{f_D} + b_{g_D}\right] + w_{\eta_D} \tag{17}$$

$$o_N = \frac{r_E}{2(r_N^2 + r_E^2)}\left[-\frac{r_N}{g_D} b_{f_N} - \frac{r_E}{g_D} b_{f_E} + \frac{r_D}{g_D}(b_{f_D} + b_{g_D}) + b_{r_D}\right] + w_{o_N} \tag{18}$$

$$o_E = \frac{r_N}{2(r_N^2 + r_E^2)}\left[-\frac{r_N}{g_D} b_{f_N} - \frac{r_E}{g_D} b_{f_E} + \frac{r_D}{g_D}(b_{f_D} + b_{g_D}) + b_{r_D}\right] + w_{o_E} \tag{19}$$

$$o_D = \frac{r_N r_E}{g_D(r_N^2 + r_E^2)}\left[b_{f_D} + b_{g_D}\right] + w_{o_D} \tag{20}$$

Because any row can be multiplied by any convenient constant, all the fractional multiplicative coefficients of each row are irrelevant to the algebraic structure and can be removed. The resulting problem

$$\mathbf{Y_G} = \mathbf{H_G}\,\mathbf{x_G} + \eta_{\mathbf{G}}$$

has matrix

$$\mathbf{H_G} = \begin{bmatrix} \frac{r_D\,r_N}{g_D} & \frac{r_D\,r_E}{g_D} & \frac{r_E^2}{g_D} & r_N & r_E & 0 \\ \frac{r_D\,r_N}{g_D} & \frac{r_D\,r_E}{g_D} & -\frac{r_N^2}{g_D} & r_N & r_E & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{r_N}{g_D} & -\frac{r_E}{g_D} & \frac{r_D}{g_D} & 0 & 0 & 1 \\ -\frac{r_N}{g_D} & -\frac{r_E}{g_D} & \frac{r_D}{g_D} & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

with

$$\mathbf{Y_G} = \begin{bmatrix} \eta_N & \eta_E & \eta_D & o_N & o_E & o_D \end{bmatrix}^\top$$

and

$$\mathbf{x_G} = \begin{bmatrix} b_{f_N} & b_{f_E} & (b_{f_D} + b_{g_D}) & b_{r_N} & b_{r_E} & b_{r_D} \end{bmatrix}.$$

Using the third row to clear the third column, subtracting the first row from the second row, then subtracting the fourth row from the fifth row, and multiplying each row by a factor to clear the denominators:

$$\mathbf{H_{G_1}} = \begin{bmatrix} r_D\,r_N & r_D\,r_E & 0 & r_N\,g_D & r_E\,g_D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -r_N & -r_E & 0 & 0 & 0 & g_D \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

The algebraic structure of $\mathbf{H_{G_1}}$ shows that the six equations, with the variables ordered as $\mathbf{x_G} \in \Re^6$, the observable space is spanned by the three linearly independent vectors

$$\mathcal{O}_{\mathcal{G}} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} r_N \\ r_E \\ 0 \\ 0 \\ 0 \\ -g_D \end{bmatrix}, \begin{bmatrix} r_N\,r_D \\ r_E\,r_D \\ 0 \\ g_D\,r_N \\ g_D\,r_E \\ 0 \end{bmatrix} \right\}.$$

Any choice of three linearly independent variables within $\mathcal{O}_{\mathcal{G}}$ can be observed.

The unobservable space is spanned by the three linearly independent vectors

$$\mathcal{U}_{\mathcal{G}} = \text{span}\left\{ \begin{bmatrix} r_E \\ -r_N \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ r_E \\ -r_N \\ 0 \end{bmatrix}, \begin{bmatrix} g_D \\ g_D \\ 0 \\ -r_D \\ -r_D \\ (r_N + r_E) \end{bmatrix} \right\}.$$

Any choice of three linearly independent variables within $\mathcal{U}_{\mathcal{G}}$ are unobservable.