

# Esercizi numeri complessi

Dal libro Marcellini

### ES 4.1:

a)  $(1+i) + (1-2i)$   
 b)  $(1-i) - (1+i)$

c)  $(1+i) \cdot (1-2i)$   
 d)  $(1+i) \cdot (1+i)$

a)  $1+i + 1-2i = 2-i$

b)  $1-i - 1-i = -2i$

c)  $(1+i) \cdot (1-2i) = 1-2i+i-2i^2 = 1-i+2 = 3-i$

d)  $(1+i)^2 = 1+i^2+2i = 1-1+2i = 2i$

### ES 4.2: Scrivere in forma algebrica

(a)  $\frac{i}{1-i}$

(b)  $\frac{1}{i}$

a)  $\frac{i}{1-i} \cdot \frac{1+i}{1+i} = \frac{i+i^2}{1+i-i^2} = \frac{i-1}{2} = \frac{i-1}{2} = \frac{1}{2} + \frac{1}{2}i$

(c)  $\frac{1-i}{1+i}$

(d)  $13 \frac{1+i}{2-3i}$

b)  $\frac{1}{i} \cdot \frac{i}{i} = \frac{1}{-1} = -i$

c)  $\frac{1-i}{1+i} \cdot \frac{1-i}{2-i} = \frac{(1-i)^2}{1-i+i-i^2} = \frac{(2-i)^2}{1+i} = \frac{1}{2} (2-i)^2 = \frac{1}{2} (4-4i-i^2) = \frac{1}{2}(4-4i+1) = \frac{1}{2} \cdot 5i = i$

d)  $13 \frac{1+i}{2-3i} \cdot \frac{2+3i}{2+3i} = 13 \frac{2+3i+2i+3i^2}{4+6i-6i-9i^2} = \cancel{13} \frac{2+5i-3}{\cancel{13}} = -1+5i$

### ES 4.3: Verifica l'identità

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$$(3+i) \cdot \left(1 + \frac{2i}{1-i}\right) \cdot \left(1 - \frac{2}{(1+i)^2}\right) = 10$$

$$= (3+i) \cdot \left(\frac{1-i+2i}{1-i}\right) \cdot \left(\frac{2i-2}{2i}\right) = 10 \Rightarrow (3+i) \left(\frac{1+i}{1-i}\right) \cdot \left(\frac{2i-2}{2i}\right) = 10$$

$$\Rightarrow a: \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = i \quad b: \frac{2i-2}{2i} = \frac{\cancel{2i}}{\cancel{2i}} - \frac{\cancel{2}}{\cancel{2i}} = \frac{1}{i} \cdot \frac{i}{i} = -i$$

$$\Rightarrow (3+i) \cdot i \cdot (-i) = 3i + i^2 \cdot (-i) = -3i^2 + i = \boxed{3+i = 10} \Rightarrow -7+i$$

Sappiamo che

$$\begin{aligned} \varphi &= \sqrt{a^2+b^2} \\ &= \sqrt{(-7)^2+1} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned}$$

$$\begin{cases} a = \cos \theta \\ b = \sin \theta \end{cases} \quad \begin{cases} \cos \theta = \\ \sin \theta = \end{cases}$$

?

### ES 4.4:

4.4 Calcolare il modulo e l'argomento dei numeri complessi

(a)  $1+i$       (b)  $2-2i$   
 (c)  $\sqrt{3}+i$       (d)  $-1+i\sqrt{3}$

a)  $1+i \quad \begin{cases} a = 1 \\ b = 1 \end{cases} \Rightarrow \varphi = \sqrt{1+1} = \sqrt{2} \quad \begin{cases} a = \cos \theta = \frac{a}{\rho} \\ b = \sin \theta = \frac{b}{\rho} \end{cases}$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2} \cos 45^\circ$$

$$\sin \theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2} \sin 45^\circ$$

$180^\circ : \pi = 45^\circ : x$

$$\frac{45^\circ}{180^\circ} \cdot 360^\circ = \frac{1}{4} \cdot 360^\circ$$

$$\Rightarrow z = \varphi [\cos(\theta) + i \sin(\theta)] = \underline{\sqrt{2} \left[ \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]}$$

$$b) z = \frac{a}{2-2i} \quad \varphi = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\theta / \begin{aligned} a \cos \theta &= \frac{2}{2\sqrt{2}} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} \\ b \sin \theta &= -\frac{2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2} = ? \end{aligned} \quad \left. \begin{aligned} \theta &= 45^\circ = \frac{\pi}{4} \\ &? \end{aligned} \right\} z = 2\sqrt{2} \left[ \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$$

Trovo  $\theta$  con  $\tan$ :

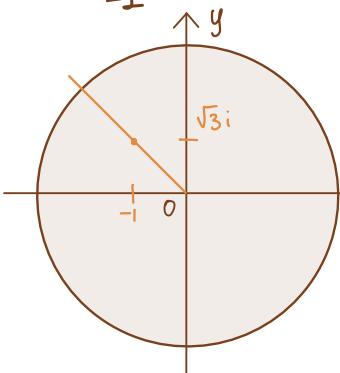
$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-2}{2} = -1 \Rightarrow \tan^{-1}(-1) = -45^\circ = \frac{-\pi}{4}$$

$$c) z = \sqrt{3} + i \quad \varphi = \sqrt{3+1} = \underline{2}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{b}{a} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = 30^\circ = \frac{\pi}{6} + 2k\pi \quad \left. \begin{aligned} z &= 2 \left[ \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right] \end{aligned} \right\}$$

$$d) z = -1 + i\sqrt{3} \quad \varphi = \sqrt{3+1} = \underline{2}$$

$$\tan \theta = \frac{\sqrt{3}}{-1} = -\sqrt{3} \Rightarrow \theta = \tan^{-1}(-\sqrt{3}) = -60^\circ = -\frac{\pi}{3} + 2k\pi \quad \left. \begin{aligned} z &= 2 \left[ \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right] \end{aligned} \right\}$$



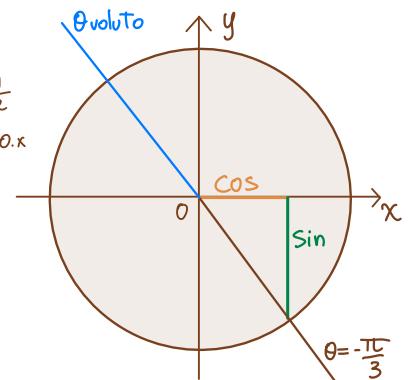
- Siccome  $z$  si trova nel III quadrante ma  $\theta = -\frac{\pi}{3}$  no, dobbiamo trovare un  $\theta$  avente lo stesso valore di  $\tan(-\frac{\pi}{3})$  ma giacente nel II quadrante

Quindi, guardando il disegno ②, dobbiamo ottenere un angolo opposto a quello che abbiamo; per farlo ci basta aggiungere "mezzo giro" ovvero  $\pi = 180^\circ$ :

$$\Rightarrow \theta = -\frac{\pi}{3} + \pi = -\frac{\pi + 3\pi}{3} = \underline{\frac{2\pi}{3}} \quad z = 2 \left[ \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right]$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\sin\left(-\frac{\pi}{3}\right) = -0.866$$



## Potenze e radici

$$z \cdot z' = [\varphi(\cos\theta + i\sin\theta)] \cdot [\varphi'(\cos\theta' + i\sin\theta')] = \varphi \cdot \varphi' [(\cos(\theta+\theta') + i\sin(\theta+\theta'))]$$

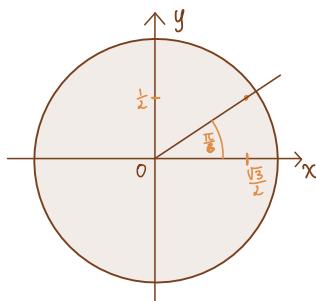
$$\text{Ponendo } z' = z \Rightarrow z^2 = \varphi^2 [\cos(2\theta) + i \sin(2\theta)] = z^n = \varphi^n [\cos(n\theta) + i \sin(n\theta)]$$

## ES 4.6:

**4.6** Verificare che la potenza sesta del numero complesso  $z = (\sqrt{3} + i)/2$  vale  $-1$ .

$$\begin{aligned} \frac{\sqrt{3}+i}{2} &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ \theta^o &= \frac{\pi}{6} \quad \checkmark \\ \varphi &= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{3}{4} + \frac{1}{4}} \\ &= \sqrt{1} = \underline{\underline{1}} \end{aligned}$$

$$\theta: \quad \tan \theta = \frac{b}{a} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = 30^\circ = \frac{\pi}{6}$$



$$= D \ z = \left[ \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right]$$

$$z^6 = \varphi^6 \left[ \cos\left(6 \frac{\pi}{6}\right) + i \sin\left(\pi\right) \right] = -1 \quad \text{Verificato}$$

#### 4.7 Determinare la forma algebrica dei numeri complessi

- $$(a) \quad (1-i)^6 \quad (b) \quad (1+i)^8 \quad (c) \quad (1-i)^{12}$$

$$a) (1-i)^6 = \underbrace{(1-i)^3}_{(a+b)^3} \underbrace{(1-i)^3}_{a^3+3a^2b+3ab^2+b^3} = \left(1-3i+3i^2-i^3\right)^2 = \left(1-3i-3+i\right)^2 = (-2-2i)^2$$

$$= 4 + 8i + 4i^2 = 4 + 8i - 4 = 8i$$

$$b) (1+i)^8 = \left[ (1+i)^2 \right]^4 \Rightarrow \left[ 1+2i+i^2 \right]^4 = (2i)^4 = 16i^4 = 16 \cdot i^2 \cdot i^2 = 16$$

$$c) (1-i)^{12} = \cancel{(1-i)^8} \cdot (1-i)^4 = 16(1-i)^4 = 16[(1-i)^2]^4 = 16[(-2i)^2]^2 = 16[-2i]^2 = 16 \cdot 4i^2 \\ = 16 \cdot (-4) = \cancel{-64}$$

#### 4.8 Determinare la forma algebrica dei numeri complessi

- $$(a) \quad (\sqrt{3} + i)^6 \quad (b) \quad \left(\frac{1}{i}\right)^4 \quad (c) \quad \left(\frac{1+i}{i-1}\right)^3$$

$$\text{a) } (\sqrt{3}+i)^6 = (\sqrt{3}+i)^3 \cdot (\sqrt{3}+i)^3 = \left[ (\sqrt{3}+i)^3 \right]^2 = \left[ 3\sqrt{3} - 9i + 3\sqrt{3}i^2 + i^3 \right]^2 = \left[ 3\sqrt{3} + 9i - 3\sqrt{3} - i \right]^2 \\ = [-8i]^2 = 64i^2 \Rightarrow \boxed{z = 64} \text{ F. Algebraica}$$

$$b) \left(\frac{1}{1}\right)^4 = \frac{1}{1^4} = 1 \Rightarrow z=1$$

$$c) \left( \frac{1+i}{i-1} \right)^3 = \frac{1+i}{-1+i} \cdot \frac{-1-i}{-1+i} = \frac{\cancel{i}-i-\cancel{i}-\cancel{i^2}}{\cancel{i}-\cancel{i}-\cancel{i}+\cancel{i^2}} = \frac{-2i}{-2i} = i^3 = \textcolor{brown}{(-i)}$$

# RADICI

$$w_k = \rho^{1/n} \left[ \cos \frac{\vartheta + 2k\pi}{n} + i \sin \frac{\vartheta + 2k\pi}{n} \right],$$

$\sqrt[n]{\rho}$

4.9 Determinare le tre radici cubiche del numero  $z = 1$ .

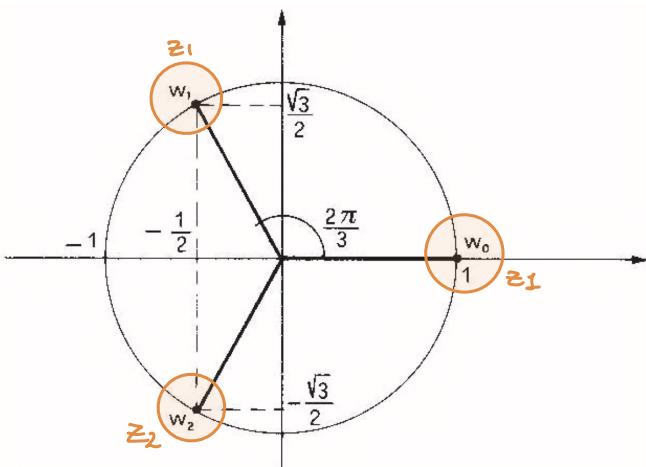
$$z = 1 ; \quad \sqrt[3]{1} = \frac{z_1}{z_2} = \frac{z_1}{z_3}$$

$$\Rightarrow z = 1 [\cos \theta + i \sin \theta]$$

$$z_0 = \sqrt[3]{1} \left[ \cos \left( \frac{0+2\cdot0\pi}{3} \right) + i \sin \left( 0 \right) \right] = \cos^0 + i \sin^0 = 1$$

$$z_1 = \sqrt[3]{1} \left[ \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right] = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$



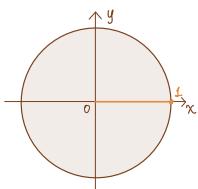
4.10 Determinare le radici quadrate dei numeri complessi

- (a) 1    (b) -1    (c) i    (d) -i

$$z = 1 \Rightarrow \varphi = 1 \quad \theta = 0 \Rightarrow 1 [\cos(\theta) + i \sin(\theta)] = 1$$

$$z_0 = \sqrt{1} \left[ \cos(0) + i \sin(0) \right] = 1$$

$$z_1 = \sqrt{1} \left[ \cos(2\pi) + i \sin(2\pi) \right] = 1$$



$$\text{Ad occhio... } z = 1 \Rightarrow z = i^2 \Rightarrow \sqrt{i^2} = \pm 1$$

$$\text{b) } z = -1 \Rightarrow \sqrt{-1} = i \text{ per definizione}$$

$$\text{c) } z = i \Rightarrow \sqrt{i} \text{ per definizione}$$

$$\Rightarrow z = 1 \left[ \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right] = i$$

$$z_0 = \sqrt{1} \left[ \cos(0) + i \sin(0) \right] = 1$$

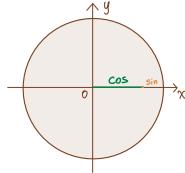
$$z_1 = \sqrt{1} \left[ \cos \left( \frac{2\pi}{2} \right) + i \sin \left( \frac{2\pi}{2} \right) \right] = 1$$

Una radice  $\sqrt[n]{x}$  ha esattamente  $n$  radici distinte

Inoltre, se  $z$  è una radice, anche  $\bar{z}$  lo è.

$$\varphi_z = 1 \quad \theta = 0$$

$$\tan \frac{b}{a} = \frac{0}{1} \Rightarrow \tan^{-1} 0 = 0 \\ \Rightarrow \cos 0 = \cos 0 = \frac{\sqrt{2}}{2} \\ \sin 0 = 0$$



$$z = bi, \quad b > 1 \quad \begin{cases} \varphi = \underline{|b|} = 1 \\ \theta = \frac{\pi}{2} \end{cases}$$

# Equazioni con numeri complessi

4.16 Risolvere le equazioni algebriche di primo grado

$$(a) iz + 1 = 0 \quad (b) (2+i)z - 4 + 3i = 0$$

$$a) iz + 1 = 0 \Rightarrow z = -\frac{1}{i} ; z = -\frac{1}{i} \cdot \frac{i}{i} = \frac{-i}{-1} = i$$

$$b) (2+i)z - 4 + 3i = 0 ; z = \frac{4-3i}{2+i} \cdot \frac{2-i}{2-i} = \frac{8-4i-6i+3i^2}{4-2i+2i-i^2} = \frac{8-10i-3}{5}$$

$$z = \frac{5-10i}{5} ; z = \frac{5}{5} - \frac{10}{5}i ; z = 1 - 2i$$

4.17 Risolvere le equazioni di primo grado

$$(a) (1-i)z - 2 = 0 \quad (b) iz + 1 - i = 0$$

$$a) (1-i)z - 2 = 0 ; z = \frac{2}{1-i} \cdot \frac{1+i}{1+i} ; z = \frac{2+2i}{1+i-i^2} ; z = \frac{2+2i}{2} \Rightarrow z = 1+i$$

$$b) iz + 1 - i = 0 ; z = -\frac{1+i}{i} \cdot \frac{1}{1} \Rightarrow z = -\frac{i+i^2}{i^2} ; z = \frac{-i-1}{-1} \Rightarrow z = 1+i$$

4.18 Risolvere l'equazione di secondo grado  $z^2 + z + 1 = 0$ .

$$z^2 + z + 1 = 0 \quad z_{1,2} = \frac{-1 \pm \sqrt{1-4 \cdot 1 \cdot 1}}{2} = \frac{z_1 = -1 + \sqrt{-3}}{2} = \frac{-1 + \sqrt{-1} \cdot \sqrt{3}}{2} = \frac{-1 + i\sqrt{3}}{2}$$

$$z_2 = \frac{-1 - \sqrt{-3}}{2} = \frac{-1 - i\sqrt{3}}{2}$$

4.19 Risolvere nel campo complesso le equazioni di secondo grado

$$(a) z^2 + 4 = 0 \quad (b) z^2 - 4 = 0$$

$$a) z^2 + 4 = 0 \Rightarrow z^2 = -4 \Rightarrow z = \sqrt{-4} \Rightarrow z = \pm 2i$$

$$b) z^2 - 4 = 0 \Rightarrow z = \pm 2$$

4.20 Risolvere nel campo complesso le equazioni

$$(a) 5z^2 - 4z + 1 = 0 \quad (b) z^2 + 4z + 5 = 0$$

$$a) 5z^2 - 4z + 1 = 0 \quad z_{1,2} = \frac{-\left(\frac{b}{2}\right) \pm \sqrt{\left(\frac{b}{2}\right)^2 - ac}}{a} = \frac{2 \pm \sqrt{4-5}}{5} = \frac{z_1 = \frac{2+i}{5}}{z_2 = \frac{2-i}{5}}$$

$$b) z^2 + 4z + 5 = 0 \quad z_{1,2} = \frac{-2 \pm \sqrt{4-4 \cdot 5}}{2} = \frac{z_1 = -2 + \sqrt{-16}}{2} = \frac{z_2 = -2 - \sqrt{-16}}{2} = \frac{-2 + 4i}{2} = -2 + 4i \quad ? \quad \text{Il risultato dovrebbe essere} \\ 2+4i$$

4.21 Risolvere l'equazione  $4z^2 - 4z + 3 - 2\sqrt{3}i = 0$ .

$$4z^2 - 4z + 3 - 3\sqrt{3}i = 0 ; 4(z^2 - z) + 3(1 - \sqrt{3}i) = 0$$

$$\frac{4 \pm \sqrt{16 - 4 \cdot 4 \cdot (3 - 3\sqrt{3}i)}}{8} \quad z_1 = \frac{4 + \sqrt{16 - 16(3 - 3\sqrt{3}i)}}{8} = \frac{4 + \sqrt{16 - 48 + 48\sqrt{3}i}}{8}$$

$$z_2 = \frac{4 - \sqrt{16 - 16(3 - 3\sqrt{3}i)}}{8} = \frac{4 - \sqrt{16 - 48 + 48\sqrt{3}i}}{8}$$

ES: Risolvere l'eq:  $\frac{2z}{1+i} = -1$

$$\frac{2z}{1+i} \cdot \frac{1-i}{2} = -\frac{1+i}{2}; \Rightarrow z = -\frac{1-i}{2} \Rightarrow z = -\frac{1}{2} - \frac{1}{2}i$$

ES:  $z^3 = 1 \Rightarrow z = \sqrt[3]{1}$