1. Calcolare le seguenti potenze di i:

$a)$ i^5 [i]

$b)$ $\frac{1}{i^3}$ [i]

$c)$ i^{63} [-i]

$d)$ i^{-9} [-i]

$a)$ i^5 $b)$ $\frac{1}{i^3}$ $c)$ i^{63} $d)$ i^{-9}

[RISPOSTE: i , i , $-i$, $-i$]

a) $i^5 \Rightarrow i^2 = -1 \Rightarrow -i\cdot(-i)\cdot i = i$

b) $\frac{1}{i^3} = \frac{1}{-i}\cdot\frac{i}{i} = \cancel{+}\frac{i}{i^2} \Rightarrow \cancel{-1} = i$

c) $i^{63} = i^{60+3} = \underbrace{i^{60}}_{4\cdot 15 \,=\, i}\cdot i^3 = i\cdot(-i\cdot -i)\cdot i = i$

d) $i^{-9} = \frac{1}{i^9} = \frac{1}{i^8\cdot i} = \frac{1}{i}\cdot\frac{i}{i} = \frac{i}{\underbrace{i^2}_{-1}} = -i$

2. Semplificare le seguenti espressioni:

$a)$ $(2-3i)(-2+i)$ $[-1+8i]$

$b)$ $\frac{1+2i}{3-i}+\frac{2-i}{5i}$ $[-\frac{1}{10}+\frac{3}{10}i]$

$c)$ $(1-i)^4$ $[-4]$

$d)$ $\frac{(1+i)^2}{3-4i}$ $[-\frac{8}{25}+\frac{6}{25}i]$

a) $(2-3i)(-2+i) = -4+2i+6i-\underbrace{3i^2}_{3} = 8i-1$ ✓

b) $\frac{1+2i}{3-i}+\frac{2-i}{5i} = \frac{(1+2i)5i+(2-i)(3-i)}{(3-i)(5i)} = \frac{//}{15i-5i^2} = \frac{5i+10i^2+6-2i-3i+i^2}{15i-5i^2} = \frac{\cancel{5i}-10+6-\cancel{5i}-1}{15i+5}$

c) $(1-i)^4 = \left[(1-i)^2\right]^2 = \left[1-2i+i^2\right]^2 = \left[\cancel{1}-2i-\cancel{1}\right]^2 = 4i^2 = -4$ ✓

d) $\frac{(1+i)^2}{3-4i} = \frac{1+2i+i^2}{3-4i} = \frac{2i}{3-4i}\cdot\frac{3+4i}{3+4i} = \frac{6i+8i^2}{9-\cancel{12i}+\cancel{12i}-16i^2} = \frac{6i-8}{9+16} = \frac{6i-8}{25}$

$= \frac{6i}{25}-\frac{8}{25}$ ✓

3. Verificare che $z = i\frac{1\pm\sqrt{5}}{2}$ soddisfa l'equazione $z^2-iz+1=0$.

$\Delta = (-i)^2-4\cdot1\cdot1 = i^2-4 = -1-4 = -5$

$z_{1,2} = \frac{i\pm\sqrt{-5}}{2} \Rightarrow \sqrt{-5} = \sqrt{i^2 5} = i\sqrt{5} \Rightarrow z = \frac{i\pm\sqrt{5}\,i}{2} = i\,\frac{1\pm\sqrt{5}}{2}$ ✓

4. Calcolare il modulo dei seguenti numeri complessi :

$a)$ $\frac{1}{1-i}+\frac{2i}{i-1}$ $[\sqrt{\frac{5}{2}}]$

$b)$ $\frac{3-i}{(1+i)^2}-\frac{1}{1-i}$ $[\sqrt{5}]$

$c)$ $\left(\frac{1-3i}{1+i}-i\right)^3$ $[10\sqrt{10}]$

Trovare il modulo

Per Trovare il modulo dobbiamo ricondurci alla forma $z = a+ib$, ovvero pte imm e reale.

Quindi $|z| = \sqrt{a^2+b^2}$

a) $\frac{1}{1-i}+\frac{2i}{i-1} = \frac{(i-1)+(2i)\cdot(1-i)}{(1-i)(i-1)} = \frac{i-1+2i-2i^2}{i-1-i^2+i} = \frac{i-1+2i+2}{2i}$ $\frac{3i+1}{2i}\cdot\frac{2i}{2i} = \frac{6i^2+2i}{4i^2}$

$= \frac{-6+2i}{-4} = \frac{-6}{-4}+\frac{2i}{-4} = -\frac{1}{2}i+\frac{3}{2} \Rightarrow |z| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{9}{4}} = \sqrt{\frac{1+9}{4}} = \sqrt{\frac{10}{4}}$

$= \sqrt{\frac{5}{2}}$ ✓

b) $\frac{3-i}{(1+i)^2} - \frac{1}{1-i} = \frac{3-i}{2i} - \frac{1}{1-i} = \frac{(3-i)(1-i)\cdot 2i}{2i+2} = \frac{3-3i-i-1}{2i+2} = \frac{2-4i}{2i+2} \cdot \frac{2i-2}{2i-2} =$

$= \frac{4i-4-8i^2+8i}{4i^2-4i+4i-4} = \frac{12i-4+8}{-4-4} = \frac{12i+4}{-8} = -\frac{3}{2}i - \frac{1}{2}$

$|z| = \sqrt{\frac{9}{4}+\frac{1}{4}} = \sqrt{\frac{9+1}{4}} = \sqrt{\frac{5}{2}}$

c) $\left(\frac{1-3i}{1+i} - i\right)^3$ Calcolo senza potenza $\rightarrow \frac{1-3i \quad -i(1+i)}{1+i} = \frac{1-3i-i-i^2}{1+i} = \frac{1-4i+1}{1+i} = \frac{2-4i}{1+i} \cdot \frac{1-i}{1-i}$

$= \frac{2-2i-4i+4i^2}{1\cancel{-i}\cancel{+i}-i^2} = \frac{2-6i-4}{1+1} = \frac{-2-6i}{2} = -1-3i$ Ripristino la potenza

$\rightarrow (-1-3i)^3 = (-1-3i)^2 \cdot (-1-3i) = 1+6i+9i^2 \cdot (-1-3i) = 1+6i-9(-1-3i) = -1-6i+9-3i-18i^2+27i$

$= -1-6i+9-3i+18+27i = 26-18i \Rightarrow \sqrt{26^2+18^2} = \sqrt{676+324} = \sqrt{10^3} = 10\sqrt{10}$

5. Mettere in forma trigonometrica e in forma esponenziale i seguenti numeri complessi

a) $z = -1$ $[z = \cos\pi + i\sin\pi \ ; \ z = e^{i\pi}]$

b) $z = i(1+i)$ $[z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \ ; \ z = \sqrt{2}e^{i\frac{3\pi}{4}}]$

c) $z = \frac{1+i}{1-i}$ $[z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \ ; \ z = e^{i\frac{\pi}{2}}]$

d) $z = \frac{i(i-1)}{(i+1)^2}$ $[z = \frac{\sqrt{2}}{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \ ; \ z = \frac{\sqrt{2}}{2}e^{i\frac{3\pi}{4}}]$

e) $z = \frac{3}{\left(-1+\frac{i}{\sqrt{3}}\right)^4}$ $[z = \frac{27}{16}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \ ; \ z = \frac{27}{16}e^{i\frac{2\pi}{3}}]$

Forma Polare / Trigonometrica
Tramite modulo ed argomento θ e φ

$z = r[\cos(\theta) + i\sin(\theta)]$

Forma cartesiana $z = a+ib$
Forma esponenziale
$z = r[\cos\theta + i\sin\theta] = \varphi e^{i\theta}$
$|z| = \varphi$

Passare da cart. a polari
1. Trovo il modulo: $|z| = \varphi = \sqrt{a^2+b^2}$ dove a, b vengono presi da $z = \underline{a}+i\underline{b}$
2. Trovo l'argomento θ usando le formule:

$\begin{cases} a = \varphi\cos\theta \\ b = \varphi\sin\theta \end{cases} \Rightarrow \begin{cases} \cos\theta = \frac{a}{\varphi} \text{ reale} \\ \sin\theta = \frac{b}{\varphi} \text{ complx} \end{cases}$

ES. $z = 1+i \Rightarrow$ (1) $\varphi = \sqrt{1+1} = \sqrt{2}$
(2) $\begin{cases} \cos\theta = \frac{1}{\sqrt{2}} \\ \sin\theta = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \theta = 45 \Rightarrow z = 1+i = \sqrt{2}[\cos\underbrace{45}_{\pi/4} + i\sin 45]$

a) $z = -1$ (1) $\varphi = \sqrt{1} = 1$
(2) $\begin{cases} \cos\theta = -\frac{1}{1} \\ \sin\theta = 0 \end{cases} \Rightarrow \theta = \pi \Rightarrow z = [\cos\pi + i\sin\pi] = e^{i\pi}$

b) $z = i(1+i) = i^2+i = i-1 \Rightarrow$ (1) $\varphi = \sqrt{2}$
(2) $\begin{cases} \cos\theta = \frac{1}{\sqrt{2}} \\ \sin\theta = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow z = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}e^{i\frac{\pi}{4}}$

c) $z = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{\cancel{1}+i+i+\cancel{i^2}}{1+\cancel{i}-\cancel{i}-i^2} = \frac{\cancel{2}i}{\cancel{2}} = i \Rightarrow \varphi = 1$

② $\begin{cases} \cos\theta = 0 \\ \sin\theta = 1 \end{cases} \Rightarrow \theta = 90 = \frac{\pi}{2} \Rightarrow z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

d) $z = \frac{i(i-1)}{(i+1)^2} = \frac{i^2 - i}{\cancel{i^2} + 2i + \cancel{1}} = \frac{-1-i}{2i} \cdot \frac{2i}{2i} = \frac{-2i-2i^2}{4i^2} = \frac{2-2i}{-4} = \overset{a}{-\frac{1}{2}} + \overset{b}{\frac{1}{2}i}$

$\Rightarrow \varphi = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt2}\cdot\frac{\sqrt2}{\sqrt2} = \frac{\sqrt2}{2}$

② $\begin{cases} \cos\theta = -\frac{1}{2} \Big/ \frac{\sqrt2}{2} \Rightarrow \frac{-\frac{1}{2}}{\frac{\sqrt2}{2}} = -\frac{1}{\sqrt2}\cdot\frac{\sqrt2}{\sqrt2} = -\frac{\sqrt2}{2} \\ \sin\theta = \frac{1}{2} \Big/ \frac{\sqrt2}{2} = \frac{\frac{1}{2}}{\frac{\sqrt2}{2}} = \frac{1}{\sqrt2} = \frac{\sqrt2}{2} \end{cases}$

7. Trovare le radici dei seguenti numeri complessi e disegnarle sul piano di Gauss.

a) $(2i)^{\frac{1}{2}}$ $[1+i\ ;\ -1-i]$

b) $\sqrt[3]{\sqrt5}$ $\left[\pm\sqrt[6]{5}\ ;\ \pm\frac{\sqrt[6]{5}}{2}(1+i\sqrt3)\ ;\ \pm\frac{\sqrt[6]{5}}{2}(-1+i\sqrt3)\right]$

c) $\sqrt[4]{-1+\sqrt3 i}$ $\left[\pm\frac{\sqrt[4]{2}}{2}(\sqrt3+i)\ ;\ \pm\frac{\sqrt[4]{2}}{2}(-1+i\sqrt3)\right]$

Trovare le radici n-esime

1) Calcolare la forma trigonometrica

2) le radici cubiche di 1 sono date da: $z_K = \sqrt[3]{\varphi}\cdot\left[\cos\left(\frac{0+2K\pi}{3}\right) + i\sin\left(\frac{0+2K\pi}{3}\right)\right]$ con $K = 0, 1, 2$

a) $(2i)^{\frac{1}{2}} \to z = \overset{b}{2}i$ $\varphi = 2 \Rightarrow \begin{cases}\cos\theta = 0 \\ \sin\theta = 1\end{cases} \Rightarrow \theta = \frac{\pi}{4}$

$\Rightarrow z_1 = \sqrt2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt2 e^{i\frac{\pi}{4}} = -\cancel{\sqrt2}\cdot\frac{1}{\cancel{\sqrt2}} - i\cancel{\sqrt2}\frac{1}{\cancel{\sqrt2}} = -1+i$

$z_2 = \sqrt2\left[\cos\left(\frac{\pi}{4}+\frac{2\pi}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{2\pi}{4}\right)\right] = \sqrt2 e^{i\frac{3}{4}\pi} = -\cancel{\sqrt2}\frac{1}{\cancel{\sqrt2}} + \cancel{\sqrt2}\frac{1}{\cancel{\sqrt2}} = -1+i$

Non mi trovo col segno

b) $\sqrt[3]{\sqrt5} \to z = \sqrt5$ $\varphi = \sqrt5 \Rightarrow \begin{cases}\cos\theta = 1 \\ \sin\theta = 0\end{cases} \Rightarrow \theta = 0 \Rightarrow z_1 = \sqrt[3]{\sqrt5}[\cos 0 + i\sin 0]$

$z_2 = \sqrt[3]{\sqrt5}\left[\cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right)\right]$ $z_3 = \sqrt[3]{\sqrt5}\left[\cos\left(\frac{4}{3}\pi\right) + i\sin\left(\frac{4}{3}\pi\right)\right]$

8. Risolvere e rappresentare sul piano di Gauss le soluzioni delle seguenti equazioni:

a) $z^2 + i\sqrt3 z + 6 = 0$ $[i\sqrt3\ ;\ -2i\sqrt3]$

$\Delta = -3 - 4\cdot1\cdot6 = -27$ $z_{1,2} = \frac{-i\sqrt3 \pm \sqrt{-27}}{2} = \frac{-i\sqrt3 \pm i\sqrt{27}}{2} = i\,\frac{-\sqrt3\pm\sqrt{27}}{2}$

$z_1 = -i\,\frac{\sqrt3+\sqrt{27}}{2}$

$z_2 = i\,\frac{\sqrt3-\sqrt{27}}{2}$

$z_1 = \frac{-i\sqrt3 + i\sqrt{3^2\cdot3}}{2} = \frac{-i\sqrt3 + i3\sqrt3}{2} = \frac{\cancel{2}\sqrt3 i}{\cancel{2}} = \sqrt3 i$ z_1

$z_2 = \frac{-\sqrt3 i - 3\sqrt3 i}{2} = -\frac{4\sqrt3 i}{2} = -2\sqrt3 i$ z_2

b) $(z+i)^2 = (\sqrt3+i)^3$

$z^2 + 2iz + i^2 = (\sqrt3+i)^2(\sqrt3+i) \to z^2 + 2iz - 1 = (3 + 2\sqrt3 i + 1)(\sqrt3+i)$

$\to z^2 + 2zi - 1 = (3 + 2\sqrt3 i + i^2)(\sqrt3+i) \to z^2 + 2zi - 1 = 3\sqrt3 + 3i + 6i + 2\sqrt3 i^2 + \sqrt3 i^2 + i^3$

$\to z^2 + 2zi - 1 = 3\sqrt3 + 9i + 3\sqrt3 i^2 + i^3 \to$ Troppo lungo non ho voglia.