## Metodi di integrazione

- Integrazione per sostituzione:

$$\int f(x)\,dx = \int f[g(t)]\,g'(t)\,dt$$

- Integrazione per parti

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

## Formule di integrazione

- $\int x^b\,dx = \frac{x^{b+1}}{b+1} + c$
- $\int \frac{1}{x}\,dx = \ln|x| + c \qquad \int \frac{1}{f(x)} \cdot f'(x)\,dx = \ln|f(x)| + c$
- $\int \tan x\,dx = -\ln|\cos x| + c$
- $\int \frac{1}{\cos^2 x}\,dx = \tan x + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx \quad \arcsin x + c$
- $\int \frac{1}{1+x^2}\,dx \quad \arctan x + c$
- $\int a^x\,dx = \frac{a^x}{\ln a} + c$

- $\int x^3\,dx = \frac{x^4}{4} + c$
- $\int x^5\,dx = \frac{x^6}{6} + c$
- $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx \Rightarrow \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2\sqrt[2]{x^3}}{3} + c$
- $\int x^{1/2}\,dx = \frac{2}{3}\sqrt{x^3} + c = \frac{2}{3}\sqrt{x^2 \cdot x} + c = \frac{2}{3}\,x\sqrt{x} + c$
- $\int x^{\frac{2}{3}}\,dx = 2\,\frac{x^{\frac{5}{3}}}{5} + c = \frac{3\sqrt[3]{x^5}}{5} = \frac{3}{5}\,x\sqrt[3]{x} + c$
- $\int x^{-3}\,dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$
- $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}} + c = \frac{2}{\sqrt{x}} + c$
- $\int \frac{1}{\sqrt{x}} + 3\,dx = \int \frac{1}{\sqrt{x}} + 3\int dx = \int x^{-\frac{1}{2}}\,dx + 3\int dx = \frac{2}{\sqrt{x}} + 3x + c$
- $\int \frac{\sqrt{x}}{\sqrt[3]{x}}\,dx = \int x^{\frac{1}{2}} \cdot x^{-\frac{1}{3}}\,dx = \int x^{\frac{1}{2}-\frac{1}{3}}\,dx = \int x^{\frac{1}{6}}\,dx = 6\,\frac{x^{\frac{7}{6}}}{7} + c = \sqrt[6]{x^7} = x\sqrt[6]{x} \cdot \frac{6}{7} + c$
- $\int \frac{x^2 + x^3}{x^5}\,dx = \int \frac{x^2}{x^5}\,dx + \int \frac{x^3}{x^5}\,dx = \int x^{-3}\,dx + \int x^{-2}\,dx = \frac{x^{-2}}{2} + x^{-1} = \frac{1}{2x^2} + \frac{1}{x} + c$
- $\int \frac{2x+1}{x^3}\,dx = \int \frac{2x}{x^3} + \int x^{-3}\,dx = -\frac{2}{x} + -\frac{1}{2x^2} + c = \frac{-4x+1}{2x^2} = \frac{1-4x}{2x^2}$
- $\int \frac{1+x^3}{2x^2}\,dx = \frac{1}{2}\int x^{-2}\,dx + \frac{1}{2}\int \frac{x^3}{x^2}\,dx = \frac{1}{2}\,\frac{x^{-1}}{-1} + \frac{1}{2}\,\frac{x^2}{2} + c = \frac{1}{2}\left(-\frac{1}{x} + \frac{x^2}{2}\right) + c$
- $\int \frac{3+2x}{\sqrt{x}}\,dx = 3\int x^{-\frac{1}{2}}\,dx + 2\int x \cdot x^{-\frac{1}{2}}\,dx = 6\sqrt{x} + \cancel{6}^{3}\,\frac{x^{\frac{3}{2}}}{\cancel{2}} = 6\sqrt{x} + 3x\sqrt{x} + c$
- $\int 2x^2(1 - x + 2x^3)\,dx = 2\int x^2 - 2\int x^3 + 4\int x^5\,dx = 2\,\frac{x^3}{3} - \cancel{2}\,\frac{x^4}{\cancel{4}} + 4\,\frac{x^6}{6} + c$
- $\int x^2 + \frac{1}{x^2} - \frac{2}{x^3}\,dx = \int x^2 + \int x^{-2} - 2\int x^{-3}\,dx = \frac{x^3}{3} - \frac{1}{x} - \cancel{2}\,\frac{x^{-2}}{\cancel{-2}} = \frac{x^3}{3} - \frac{1}{x} + \frac{1}{x^2} + c$

- $\int 2(2x+1)^2\,dx = 2\int (2x+1)^2 dx = 2\int 4x^2+4x+1\,dx = 8\int x^2 dx + 8\int x\,dx + 2\int dx = 8\frac{x^3}{3} + 8\frac{x^2}{2} + 2x + c$

$= 2x\left(4\frac{x^2}{3} + 4\frac{x}{2} + 1\right) + c = 2x\left(\frac{8x^2+12x+1}{6}\right) = \frac{16x^3}{6} + \frac{24x^2}{6} + \frac{2x}{6} \;/2 \Rightarrow \frac{8x^3}{3} + \frac{12x^2}{3} + \frac{x}{3}$

Per sostituzione pongo $t = 2x+1$ $\quad dx = D(t)dt = 2\,dt$

$\Rightarrow \int 2(2x+1)^2 dx = 2\int t^2 \cdot 2\,dt = 4\frac{t^3}{3} + c$ Torno a $x$ $\quad \frac{4}{3}(2x+1)^3 + c$

- $\int \frac{1}{(x+1)^2} dx =$ Per sostituzione pongo $t = x+1 \Rightarrow dx = dt$

$\Rightarrow \int \frac{1}{t^2} dt = \int t^{-2} dt = \frac{t^{-1}}{-1} + c \Rightarrow -\frac{1}{t} + c \Rightarrow -\frac{1}{x+1} + c$ Ritorno a $t$

- $\int x^2\sqrt{x^3+1}\,dx$ pongo $t = x^3+1$ $\quad dx = \frac{1}{t'}dt = \frac{1}{3x^2}dt$

$\Rightarrow \int \cancel{x^2}\sqrt{x^3+1}\,\frac{1}{3\cancel{x^2}}dt = \frac{1}{3}\int\sqrt{x^3+1}\,dt$

$\Rightarrow \frac{1}{3}\int\sqrt{t}\,dt = \frac{1}{3}\int t^{\frac{1}{2}}dt \Rightarrow \frac{1}{3}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{9}t\sqrt{t}$ Torno ad $x$ $\quad \frac{2}{9}(x^3+1)\sqrt{x^3+1} + c$

- $\int x\sqrt[3]{1+x^2}\,dx$ pongo $t = 1+x^2 \Rightarrow dx = \frac{1}{t'}dt = \frac{1}{2x}dt$

$\Rightarrow \int \cancel{x}\sqrt{1+x^2}\cdot\frac{1}{2\cancel{x}}dt = \frac{1}{2}\int\sqrt{t}\,dt = \frac{1}{2}\int t^{\frac{1}{2}}dt = \frac{1}{\cancel{2}}\frac{t^{3/2}}{\frac{3}{\cancel{2}}} + c = \frac{t\sqrt{t}}{3} + c$

Torno a $x \Rightarrow \frac{1}{3}\,1+x^2\sqrt{1+x^2} + c$

- $\int x\sqrt[3]{1+x^2}\,dx$ pongo $t = 1+x^2 \rightarrow dx = \frac{1}{t'}dt = \frac{1}{2x}dt$

$\Rightarrow \int \cancel{x}\sqrt[3]{1+x^2}\,\frac{1}{2\cancel{x}}dt = \frac{1}{2}\int\sqrt[3]{t}\,dt = \frac{1}{2}\frac{t^{\frac{4}{3}}}{\frac{4}{3}} = \frac{3}{8}t\sqrt{t}$ Torno a $x \rightarrow \frac{3}{8}(1+x^2)\sqrt[3]{1+x^2}$

- $\int \sin(x+1)\cos x\,dx$

# Esercizi dal libro:

## Integrali immediati: Altre formule utili

- $\int [f(x)]^b f'(x)\,dx = \dfrac{[f(x)]^{b+1}}{b+1} + c$
- $\int \dfrac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$
- $\int e^{f(x)} \cdot f'(x)\,dx = e^{f(x)} + c$

4.4)

$$\int \operatorname{cotg} x\,dx = \int \frac{\cos x}{\sin x}\,dx = \text{uso } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c \Rightarrow \ln|\sin x| + c$$

$$\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{\overset{f'(x)}{-\sin x}}{\underset{f(x)}{\cos x}}\,dx = -\ln|\cos x| + c$$

- $\int \sin(ax)\,dx$ = pongo $t = ax \Rightarrow dx = \frac{1}{t'} = \frac{1}{a}dt \Rightarrow \int \sin(t)\frac{1}{a}\,dt \Rightarrow \frac{1}{a} - \cos(t) + c$
  Torno ad $x \to -\frac{1}{a}\cos(ax) + c$
- $\int \cos(ax)\,dx$ pongo $t = ax \Rightarrow dx = \frac{1}{t'}dt = \frac{1}{a}dt \Rightarrow \frac{1}{a}\int \cos(t)\,dt = \frac{1}{a}\sin(ax) + c$
- $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = 2\dfrac{x^{\frac{3}{2}}}{3} = \dfrac{2}{3}x\sqrt{x} + c$
- $\int \dfrac{1}{\sqrt{x+a}}\,dx = \int (x+a)^{-\frac{1}{2}}dx =$ pongo $t = x+a$ $\quad dx = 1 = dt \Rightarrow \int t^{-\frac{1}{2}}dx = 2t^{\frac{1}{2}} + c$

  Sostituire non era necessario

  $= 2\sqrt{t} + c$ Torno ad $x$ $= 2\sqrt{x+a} + c$
- $\int \dfrac{x}{\sqrt{a+x^2}}\,dx =$ Pongo $t = \sqrt{a+x^2} \Rightarrow dx = \dfrac{1}{t'} = \dfrac{1}{\frac{1}{2}(a+x^2)^{-\frac{1}{2}}}\,dt = \dfrac{1}{\frac{1}{2}\cdot\frac{1}{\sqrt{a+x^2}}\cdot 2x}\,dt = \dfrac{1}{\frac{1}{\cancel{2}\sqrt{a+x^2}}\cdot\cancel{2}x}\,dt$

  $\Rightarrow \int \dfrac{\cancel{x}}{\cancel{\sqrt{a+x^2}}} \cdot \dfrac{1}{\frac{\cancel{x}}{\cancel{\sqrt{a+x^2}}}}\,dt = \int dt = t + c$

  Torno ad $x = \sqrt{a+x^2} + c$
- $\int \dfrac{x}{\sqrt{a-x^2}}\,dx$ pongo $t = \sqrt{a-x^2} \Rightarrow dx = \dfrac{1}{\frac{1}{2}\frac{1}{\sqrt{a-x^2}}\cdot(-2x)}\,dt = \dfrac{1}{-\frac{\cancel{2}x}{\cancel{2}\sqrt{a-x^2}}}\,dt = \dfrac{1}{-\frac{x}{\sqrt{a-x^2}}} = -\dfrac{\sqrt{a-x^2}}{x}\,dt$

$\Rightarrow -\int \dfrac{\cancel{x}}{\cancel{\sqrt{a-x^2}}} \cdot \dfrac{\cancel{\sqrt{a-x^2}}}{\cancel{x}}\,dt = -\int dt = -t + c \to \sqrt{a-x^2} + c$

ES 4.5)

- $\int \dfrac{1}{\sqrt{a^2-x^2}}\,dx$ **Regola:** $\displaystyle\int \frac{f'(x)}{\sqrt{1-f^2(x)}}\,dx = \arcsin(f(x)) + c$

$$= \int \frac{\overset{\leftarrow f'(x)}{1}}{\sqrt{a^2\underset{\uparrow f^2(x)}{\left(1-\frac{x^2}{a^2}\right)}}}\,dx = \int \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{1}{a}\int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx = \frac{1}{a}\arcsin\left(\frac{x}{a}\right) + c$$

- $\int \dfrac{1}{a^2+x^2}\,dx = \int \dfrac{1}{a^2\left[1+\left(\frac{x^2}{a^2}\right)\right]}\,dx = \dfrac{1}{a^2}\int \dfrac{1}{1+\left(\frac{x}{a}\right)^2}\,dx = \operatorname{arctg}\left(\frac{x}{a}\right) + c$

• 4.6) $\int \frac{1}{(3+5x)^6}\,dx$ pongo $t = 3+5x \Rightarrow dx = \frac{1}{5}dt$

$\Rightarrow \frac{1}{5}\int \frac{1}{t^6}\,dt = \frac{1}{5}\,\frac{t^{-5}}{-5} + c = -\frac{t^{-5}}{25} + c = -\frac{(3+5x)^{-5}}{25} + c = -\frac{1}{25(3+5x)^5} + c$

4.7) $\int \sqrt{x+2}\,dx$ = pongo $t = x+2 \Rightarrow dx = 1\,dt \Rightarrow \int \sqrt{t}\,dt = \frac{2\,t\sqrt{t}}{3} + c \to \frac{2x+4\sqrt{2x+2}}{3}$

4.8) $\int \frac{1}{\sqrt[4]{2x+1}}\,dx = \frac{1}{2}\int (2x+1)^{-\frac{1}{4}}\,D(2x)\,dx = \frac{\cancel{4}^{\,2}}{\cancel{2}}\,\frac{(2x+1)^{3/4}}{3} = \frac{\cancel{2}\,4\sqrt[4]{(2x+1)^3}}{3}$

4.9) $\int x\sqrt{3-2x^2}\,dx = -\frac{1}{4}\int x^{(-4)}(3-2x^2)^{\frac{1}{2}}dx = -\frac{2}{12}\,\frac{(3-2x^2)^{\frac{3}{2}}}{} = -\frac{1}{6}(3-2x^2)^{\frac{3}{2}} = -\frac{1}{6}\sqrt{(3-2x^2)^3} + c$

4.10) $\int \frac{x}{\sqrt{2-3x^2}}\,dx = -\frac{1}{6}\int x^{(6)}\cdot(2-3x^2)^{-\frac{1}{2}}dx = -\frac{1}{\cancel{6}\,3}\cdot\frac{(2-3x^2)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} = -\frac{1}{3}\,\frac{1}{\sqrt{2-3x^2}} + c$

4.11) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{1}{x\sqrt{x^2\left(1-\left(\frac{1}{x}\right)^2\right)}}\,dx = \int \frac{1}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}}\,dx = \int x^{-2}\cdot\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\,dx$

BOH! $\quad$ $\arcsin\left(\frac{1}{x}\right)$

4.12) $\int \sin^5 x\cdot\cos x\,dx$ = pongo $t = \sin x \Rightarrow dx = \frac{1}{\cos x}dt$

$= \int \cos^5 t\cdot\cancel{\cos x}\cdot\frac{1}{\cancel{\cos x}}\,dt = \int t^5\,dt = \frac{1}{6}t^6 + c \longrightarrow \frac{1}{6}\sin^6(x) + c$

4.13) $\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx$ = pongo $t = \arcsin^2 x \Rightarrow dx = \frac{1}{\frac{1}{\sqrt{1-x^2}}}\,dt = \sqrt{1-x^2}\,dt$

$\Rightarrow \int \frac{\arcsin^2 x}{\cancel{\sqrt{1-x^2}}}\cdot\cancel{\sqrt{1-x^2}}\,dt = \int t^2\,dt = \frac{1}{3}t^3 + c = \frac{1}{3}\arcsin^3 x + c$

4.14) $\int \frac{1}{\arcsin x\cdot\sqrt{1-x^2}}\,dx \quad \int \frac{\arcsin^{-1} x}{\sqrt{1-x^2}}\,dx$ pongo $t = \arcsin x \Rightarrow dx = \frac{1}{\frac{1}{\sqrt{1-x^2}}}\,dt = \sqrt{1-x^2}\,dt$

$\Rightarrow \int \frac{\arcsin^{-1} x}{\cancel{\sqrt{1-x^2}}}\cdot\cancel{\sqrt{1-x^2}}\,dt = \int t^{-1}\,dt = \int \frac{1}{t}\,dt = \ln|t| + c = \ln|\arcsin x| + c$

4.15) $\int \frac{1}{(1+x^2)\,\mathrm{arctg}\,x}\,dx = \int \frac{\mathrm{arctg}^{-1} x}{(1+x^2)}\,dx$ pongo $t = \mathrm{arctg}\,x \Rightarrow dx = \frac{1}{\frac{1}{1+x^2}}\,dt = 1+x^2\,dt$

$\Rightarrow \int \frac{\mathrm{arctg}^{-1} x}{\cancel{(1+x^2)}}\cdot\cancel{(1+x^2)}\,dt = \int \frac{1}{t}\,dt = \ln|\mathrm{arctg}\,x| + c$

4.16) $\int \frac{1}{x\ln x}\,dx \quad \int \frac{\ln x}{x}\,dx$ pongo $t = \ln x \Rightarrow dx = \frac{1}{\frac{1}{x}}\,dt = x\,dt$

$\Rightarrow \int \frac{\ln x^{-1}}{\cancel{x}}\cdot\cancel{x}\,dt = \int \frac{1}{t}\,dt = \ln|\ln(x)| + c$

4.17) $\int \frac{\ln(x)}{x}\,dx \Rightarrow$ pongo $t = \ln x \Rightarrow dx = x\,dx \Rightarrow \int t\,dt = \frac{1}{2}\ln^2(x) + c$

4.18) $\int \frac{\ln(x)^n}{x}\,dx$ = pongo $t = \ln(x) \Rightarrow dx = x\,dt \Rightarrow \int \frac{t^n}{\cancel{x}}\cdot\cancel{x}\,dt = \frac{t^{n+1}}{n+1} + c = \frac{\ln(x)^{n+1}}{n+1} + c$

4.19) $\int \frac{1}{x(\ln x)^n}\,dx = \int \frac{(\ln x)^{-n}}{x}\,dx$ pongo $t = \ln x \Rightarrow dx = x\,dt \Rightarrow \int t^{-n}\,dt = \frac{t^{-n+1}}{-n+1} + c = \frac{\ln^{1-n} x}{1-n} + c$

4.20) $\int 3x\,e^{x^2}\,dx = \frac{3}{2}\int 2x\,e^{x^2} = \frac{3}{2}\int e^{x^2}\cdot D(e^{x^2}) = \frac{3}{2}e^{x^2} + c$ ← usando $\int e^{f(x)}\cdot f'(x)\,dx = e^{f(x)} + c$

$D(e^{x^2}) = 2x\,e^{x}$

Verifico con sostituzione pongo $t = e^{x^2} \Rightarrow dx = \frac{1}{e^{x^2}\cdot 2x}\,dt \Rightarrow \frac{3}{2}\int \cancel{e^{x^2}\cdot 2x}\cdot\frac{1}{\cancel{e^{x^2}\cdot 2x}}\,dx = \frac{3}{2}\int dt =$

$= \frac{3}{2}t + c \to \frac{3}{2}e^{x^2} + c$

4.21) $\int \frac{1}{\sin(x+a)}\,dx =$ pongo $t = x+a \Rightarrow dx = dt \Rightarrow \int \frac{1}{\sin(t)} \cdot \frac{\sin(t)}{\sin(t)}\,dt = \int \frac{\sin(t)}{\sin^2(t)}\,dt$

$\Rightarrow \sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x \Rightarrow \int \frac{\sin(t)}{1-\cos^2 t}\,dt =$ pongo $u = \cos t \Rightarrow dt = \frac{1}{\sin t}\,du$

$\Rightarrow \int \frac{\cancel{\sin(t)}}{1-u^2} \cdot \frac{1}{\cancel{\sin(t)}}\,du = \int \frac{1}{1-u^2}\,du =$ BOH

**4.22** $\int \frac{1}{\cos x}\,dx \cdot \frac{\cos x}{\cos x} = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$ pongo $t = \sin x \Rightarrow dx = -\frac{1}{\cos x}\,dt$

$\Rightarrow \int \frac{\cancel{\cos x}}{1-t^2} \cdot \frac{1}{\cancel{\cos x}}\,dt = \int \frac{1}{1-t^2}\,dt = \int \frac{1}{(1-t)(1+t)}\,dt =$

**4.26** $\int \operatorname{sen}^n x \cos x\,dx$ pongo $t = \sin x \Rightarrow dx = \frac{1}{\cos x}\,dt \Rightarrow \int \sin^n x \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}}\,dt$

$= \int t^n\,dt = \frac{t^{n+1}}{n+1} + c = \frac{\sin^{n+1} x}{n+1} + c$

**4.27** $\int \cos^n x \operatorname{sen} x\,dx =$ pongo $t = \cos x \Rightarrow dx = -\frac{1}{\sin x}\,dt \Rightarrow -\int \cos^n x \frac{\cancel{\sin x}}{\cancel{\sin x}}\,dt = -\int t^n\,dt$

$= -\frac{\cos^{n+1} x}{n+1} + c$

**4.28** $\int \frac{1}{\operatorname{sen} x \cos x}\,dx = \int \frac{1}{\sin x} \cdot \frac{1}{\cos x}\,dx$ BOH

**4C. Integrazione per decomposizione in somma**

**4.34** $\int (hx+k)\,dx = h\int x + k\int dx = \frac{h}{2}x^2 + kx + c$

**4.36** $\int \frac{x}{1+x}\,dx = \int \frac{x+1-1}{1+x}\,dx = \int \frac{\cancel{x+1}}{\cancel{x+1}} - \int \frac{1}{x+1}\,dx = \int dx - \int \frac{1}{x+1}\,dx = x - \ln|x+1| + c$

**4.37** $\int \frac{3x+2}{4x+5}\,dx = \int \frac{3x}{4x+5}\,dx + \int \frac{2}{4x+5}\,dx = 3\int \frac{x}{4x+5}\,dx + 2\int \frac{1}{4x+5}\,dx$

$\rightarrow$ pongo $t = 4x+5 \Rightarrow dx = \frac{1}{4}\,dt \Rightarrow \frac{1}{4}\int \frac{t-5}{4t}\,dt = \frac{1}{4}\int \frac{t}{4t}\,dt - \frac{1}{4}\int \frac{5}{4t}\,dt = \frac{1}{8}\int dt - \frac{5}{8}\int \frac{1}{t}\,dt$

$\Rightarrow x = \frac{t-5}{4}$ $\Rightarrow \frac{1}{8}t - \frac{5}{8}\ln|t| + c$

$\Rightarrow \frac{3}{8}t - \frac{15}{8}\ln|t| + 2\ln|4x+5| + c \rightarrow$ Torno ad $x \rightarrow \frac{3}{8}\cdot(4x+5) - \frac{15}{8}\ln|4x+5| + 2\ln|4x+5| + c$

$= \frac{3}{2}x + \frac{15}{8} - \frac{15}{8}\ln|4x+5| + \ln|4x+5|^2 + c$

# Nuovi esercizi integrali

## IMMEDIATI

Verifica di alcune formule di integ.

1) $\int [f(x)]^b f'(x)\,dx = \frac{f(x)^{b+1}}{b+1}$ $\quad D\left[\frac{f(x)^{b+1}}{b+1}\right] = \cancel{b+1}\cdot f(x)^b \cdot f'(x)\cdot \cancel{(b+1)} - \underbrace{f(x)^{b+1}(0)}_{0} \cdot \frac{1}{\cancel{(b+1)^2}}$

$= f^b(x)\cdot f'(x)$ ✓

2) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c \quad \to D[\ln|f(x)|] \quad \frac{1}{f(x)}\cdot f'(x)$ ✓

**4.4** Calcolare i seguenti integrali:

$$\int \operatorname{cotg} x\,dx \qquad \int \operatorname{tg} x\,dx$$
$$\int \operatorname{sen} ax\,dx \qquad \int \cos ax\,dx$$
$$\int \sqrt{x}\,dx \qquad \int \frac{1}{\sqrt{x+a}}\,dx$$
$$\int \frac{x}{\sqrt{a+x^2}}\,dx \qquad \int \frac{x}{\sqrt{a-x^2}}\,dx \quad (a>0)$$

a) $\int \operatorname{cotg} x\,dx = \frac{\cos x}{\sin x} \qquad \int \sin x = -\cos x$

$\Rightarrow -\int \frac{-\cos x}{\sin x}\,dx = \ln|\sin x|+c$

b) $\int \operatorname{tg} x = \int \frac{\sin x}{\cos x}\,dx \qquad \int \frac{f'(x)}{f(x)} = \ln|f(x)|+c$

$\Rightarrow \int \cos x = \sin x \Rightarrow \int \frac{D(\cos x)}{\cos x} = \ln|\cos x|+c$

c) $\int \sin(\alpha x)\,dx \qquad \sin(f(x))\cdot f'(x) = -\cos(f(x))+c \qquad D[\alpha x] = \alpha \Rightarrow \frac{1}{\alpha}\int \sin(\alpha x)\,\alpha\,dx = \frac{1}{\alpha}-\cos(\alpha x)+c$

d) $\int \cos(\alpha x)\,dx \to$ stesse considerazioni di sopra $\to \frac{1}{\alpha}\int \cos(\alpha x)\cdot \alpha\,dx = \frac{1}{\alpha}\sin(\alpha x)+c$

e) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}} = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+2}+c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

f) $\int \frac{1}{\sqrt{x+a}}\,dx = \int (x+a)^{-\frac{1}{2}}\cdot \underbrace{D[x+a]}_{1}\,dx = \frac{(x+a)^{\frac{1}{2}}}{\frac{1}{2}}+c = 2\sqrt{x+a}+c$

g) $\int \frac{x}{\sqrt{a+x^2}}\,dx = \frac{1}{2}\int \underbrace{2x}_{f'(x)}\cdot (a+x^2)^{-\frac{1}{2}}\,dx = \frac{1}{\cancel{2}}\,\frac{(a+x^2)^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} = \sqrt{a+x^2}+c$

h) $\int \frac{x}{\sqrt{a-x^2}}\,dx \to -\frac{1}{2}\int -2x\cdot (a-x^2)^{-\frac{1}{2}}\,dx = -\frac{1}{\cancel{2}}\,\frac{\sqrt{a-x^2}}{\frac{1}{\cancel{2}}} = -\sqrt{a-x^2}+c$

**4.5** Calcolare i seguenti integrali:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx\,; \qquad \int \frac{1}{a^2+x^2}\,dx\,; \qquad \int \frac{1}{\sqrt{a^2+x^2}}\,dx\,;$$
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx\,; \qquad \int \frac{1}{a^2-x^2}\,dx$$

a) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx \quad$ da $\int \frac{1}{\sqrt{1-f^2(x)}}\,dx = \arcsin f(x)$

$\Rightarrow \int \frac{1}{\sqrt{\frac{a^2}{a^2}-\frac{x^2}{a^2}}}\,dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+c$

b) $\int \frac{1}{a^2+x^2}\,dx \to$ da $\int \frac{1}{1+f^2}\,dx = \operatorname{arctg} f(x)+c \Rightarrow \int \frac{1}{1+\left(\frac{x}{a}\right)^2} = \operatorname{arctg}\left(\frac{x}{a}\right)+c$

Calcolare i seguenti integrali e verificare che si ottiene il risultato indicato:

**4.6** $\displaystyle\int \frac{1}{(3+5x)^6}\,dx = \frac{1}{5}\int (3+5x)^{-6}D(5x)\,dx =$

$$= \frac{1}{5}\,\frac{(3+5x)^{-5}}{-5}+c = -\frac{1}{25(3+5x)^5}+c$$

$\int \frac{1}{(3+5x)^6}\,dx = \int (3+5x)^{-6}\,dx$

$\to D[3+5x] = 5 \Rightarrow \frac{1}{5}\int (3+5x)^{-6}\cdot 5\,dx$

$= \frac{(3+5x)^{-5}}{-5}\cdot\frac{1}{5} = \frac{1}{-25}\cdot (3+5x)^{-5}+c$

**4.7** $\displaystyle\int \sqrt{x+2}\,dx = \int (x+2)^{1/2}\,dx = \frac{(x+2)^{3/2}}{3/2}+c$

$\int \sqrt{x+2}\,dx = \int (x+2)^{\frac{1}{2}}\,dx \qquad D[x+2]=1 \to \int (x+2)^{\frac{1}{2}}\,dx = \frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2\sqrt{(x+2)^3}}{3}$

**4.8** $\displaystyle\int \frac{1}{\sqrt[4]{2x+1}}\,dx = \frac{1}{2}\int (2x+1)^{-1/4} D(2x)\,dx = \frac{1}{2}\frac{(2x+1)^{3/4}}{3/4} + c =$

$\displaystyle = \frac{2}{3}\sqrt[4]{(2x+1)^3} + c$

$$\int \frac{1}{\sqrt[4]{2x+1}}\,dx = \int (2x+1)^{-\frac{1}{4}}\,dx \qquad D[2x+1] = 2$$

$$\to \frac{1}{2}\int (2x+1)^{\frac{1}{4}} \cdot 2 = \frac{(2x+1)^{\frac{3}{4}}}{\frac{3}{4}} = \frac{4\sqrt[4]{(2x+1)^3}}{3}$$

**4.9** $\displaystyle\int x\sqrt{3-2x^2}\,dx = -\frac{1}{4}\int (3-2x^2)^{1/2}\cdot(-4x)\,dx =$

$\displaystyle = -\frac{1}{4}\int (3-2x^2)^{1/2} D(3-2x^2)\,dx = -\frac{1}{4}\frac{(3-2x^2)^{3/2}}{3/2} + c = -\frac{1}{6}\sqrt{(3-2x^2)^3} + c$

$$\int x\sqrt{3-2x^2}\,dx \qquad D[3-2x^2] = -4x \to -\frac{1}{4}\int x\,(3-2x^2)^{\frac{1}{2}} \cdot 4\,dx = -\frac{\cancel{2}(3-2x^2)^{\frac{3}{2}}}{6}$$

**4.10** $\displaystyle\int \frac{x}{\sqrt{2-3x^2}}\,dx = -\frac{1}{6}\int (2-3x^2)^{-1/2}\cdot(-6x)\,dx =$

$\displaystyle = -\frac{1}{6}\int (2-3x^2)^{-1/2} D(2-3x^2)\,dx = -\frac{1}{6}\frac{(2-3x^2)^{1/2}}{1/2} + c = -\frac{1}{3}\sqrt{2-3x^2} + c$

$$\int \frac{x}{\sqrt{2-3x^2}}\,dx = \int x\,(2-3x^2)^{-\frac{1}{2}}\,dx \qquad D[2-3x^2] = -6x \to -\frac{1}{6}\int x\,(2-3x^2)^{-\frac{1}{2}} \cdot 6\,dx = -\frac{(2-3x^2)^{\frac{1}{2}}}{3}$$

**4.11** $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{1}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}}\,dx =$ $\qquad (x > 0)$

$\displaystyle = -\int \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot D\left(\frac{1}{x}\right)\,dx = -\operatorname{arcsen}\frac{1}{x} + c$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{f'}{\sqrt{1-f^2}}\,dx = \arcsin(f(x)) + c \to \int \frac{1}{x\sqrt{x^2\left(1-\frac{1}{x^2}\right)}}\,dx = \int \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}\,dx = \int \frac{1}{\underbrace{x^2}_{x^{-2}}\sqrt{1-\left(\frac{1}{x}\right)^2}}$$

$$D\left[\frac{1}{x}\right] = D(x^{-1}) = -x^{-2} \Rightarrow -\int \frac{-x^{-2}}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\,dx = \arcsin\left(\frac{1}{x}\right) + c$$

**4.12** $\displaystyle\int \operatorname{sen}^5 x\cos x\,dx = \int \operatorname{sen}^5 x \cdot D\operatorname{sen} x\,dx = \frac{1}{6}\operatorname{sen}^6 x + c$

$$\int \sin^5 x \cdot \cos x\,dx = -\int \sin^5 x \cdot D[\sin x]\,dx = \frac{\sin^{5+1}(x)}{5+1}\,dx$$

Con sostituz. pongo $t = \sin^5 x \Rightarrow dx = \frac{1}{\cos x}\,dt \to \int \sin^5 x \cdot \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}}\,dt = \int t^5\,dt$

$$\to \frac{t^6}{6} + c \to \frac{\sin^6 x}{6} + c$$

**4.13** $\displaystyle\int \frac{\operatorname{arcsen}^2 x}{\sqrt{1-x^2}}\,dx = \int \operatorname{arcsen}^2 x \cdot D(\operatorname{arcsen} x)\,dx = \frac{1}{3}\operatorname{arcsen}^3 x + c$

$$\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c \to \int \arcsin^2 x \cdot \left[\frac{1}{\sqrt{1-x^2}}\right]dx = \frac{\arcsin^3 x}{3} + c$$

Sostituzione: $t = \arcsin x \to dx = \frac{1}{\frac{1}{\sqrt{1+x^2}}}\,dt \to dx = \sqrt{1+x^2}\,dt \Rightarrow \int \frac{\arcsin^2(x)}{\cancel{\sqrt{1+x^2}}} \cdot \cancel{\sqrt{1+x^2}}\,dx$

$$\to \int t^2\,dt = \frac{t^3}{3} \to \frac{\arcsin^3 x}{3} + c$$

**4.14** $\displaystyle\int \frac{1}{(\operatorname{arcsen} x)\sqrt{1-x^2}}\,dx = \int \frac{D\operatorname{arcsen} x}{\operatorname{arcsen} x}\,dx = \log\,|\operatorname{arcsen} x| + c$

$\displaystyle\int \frac{1}{(\arcsin x)\sqrt{1-x^2}}\,dx \to \int \frac{\arcsin^{-1} x}{\sqrt{1-x^2}}\,dx$ pongo $t = \arcsin x \to dx = \sqrt{1+x^2}\,dt \to \int \arcsin^{-1} x\,dt$

$\to \int t^{-1}\,dt = \ln|t| + c = \ln|\arcsin x| + c$ Piu' semplice con SOST

**4.15** $\displaystyle\int \frac{1}{(1+x^2)\operatorname{arctg} x}\,dx = \int \frac{D\operatorname{arctg} x}{\operatorname{arctg} x}\,dx = \log\,|\operatorname{arctg} x| + c$

$\displaystyle\int \frac{1}{(1+x^2)\operatorname{arctg} x} = \int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x \to \int \frac{1}{1+x^2}\cdot \operatorname{arctg}^{-1}(x)$ $\quad \int f^b(x)\cdot f'(x) \to \frac{f^{b+1}(x)}{b+1} + c$ (m) ma $b \neq -1$

$\to$ Quando abbiamo lo schema (m) ma con $b = -1$, guardiamo il probl da un'altra prospettiva:

$\displaystyle\int \underbrace{\frac{1}{1+x^2}}_{f'(x)}\ \underbrace{\frac{1}{\operatorname{arctg} x}}_{f(x)} \to \int \frac{D(\operatorname{arctg} x)}{\operatorname{arctg}}\,dx = \ln|\operatorname{arctg} x| + c$

**4.16** $\displaystyle\int \frac{1}{x\log x}\,dx = \int \frac{D\log x}{\log x}\,dx = \log\,|\log x| + c$

$\displaystyle\int \frac{1}{x\ln x}\,dx = \int \underbrace{\frac{1}{x}}_{D(\ln x)}\cdot \frac{1}{\ln x} = \ln|\ln x| + c$

**4.17** $\displaystyle\int \frac{\log x}{x}\,dx = \int \log x \cdot D\log x\,dx = \frac{1}{2}\log^2 x + c$

$\displaystyle\int \frac{\ln(x)}{x}\,dx = \int \underbrace{\frac{1}{x}}_{D(\ln x)}\cdot \ln x$ $\quad \int f^b\cdot f'(x) = \frac{f^{b+1}}{b+1} \to \frac{\ln^2 x}{2} + c$

**4.18** $\displaystyle\int \frac{(\log x)^n}{x}\,dx = \int (\log x)^n \cdot D\log x\,dx = \frac{(\log x)^{n+1}}{n+1} + c$

$\displaystyle\int \frac{\ln^n(x)}{x}\,dx = \int \ln^n(x)\cdot \frac{1}{x}$ $\quad D[\ln x] = \frac{1}{x} \to \int f^b_x \cdot f'(x) \to \frac{\ln^{n+1}(x)}{n+1} + c$

**4.19** $\displaystyle\int \frac{1}{x(\log x)^n}\,dx = \int (\log x)^{-n} D\log x\,dx = \frac{(\log x)^{1-n}}{1-n} + c \qquad (n \neq 1)$

$n \neq -1$

$\displaystyle\int \frac{1}{x(\ln x)^n}\,dx = \int \frac{1}{x}\cdot \frac{1}{(\ln x)^n} = \int \ln x^{-n}\cdot \underbrace{\frac{1}{x}}_{D(\ln x = \frac{1}{x})} \to \frac{\ln^{-n+1} x}{-n+1} + c$

**4.20** $\displaystyle\int 3xe^{x^2}\,dx = \frac{3}{2}\int e^{x^2} Dx^2\,dx = \frac{3}{2}e^{x^2} + c$

Sost $\displaystyle\int 3x\,e^{x^2}\,dx =$ pongo $e^{x^2} = t \to dx\,\frac{1}{2xe^x}\,dt \to 3\int x\,\frac{\cancel{e^{x^2}}}{2\cancel{x}\cancel{e^x}}\,dt = \frac{3}{2}\int 1\,dt = \frac{3}{2}t \to \frac{3}{2}e^{x^2} + c$

$\displaystyle\int e^{f(x)}\cdot f'(x)\,dx = e^{f(x)} + c \to \frac{3}{2}e^{x^2} + c$

**4.21** $\displaystyle\int \frac{1}{\operatorname{sen}(x+a)}\,dx = \int \frac{1}{2\operatorname{sen}\frac{x+a}{2}\cos\frac{x+a}{2}}\,dx = \int \frac{1}{\operatorname{tg}\frac{x+a}{2}\cdot 2\cos^2\frac{x+a}{2}}\,dx =$

$\displaystyle = \int \frac{D\operatorname{tg}\frac{x+a}{2}}{\operatorname{tg}\frac{x+a}{2}}\,dx = \log\left|\operatorname{tg}\frac{x+a}{2}\right| + c$

$\displaystyle\int \frac{1}{\sin(x+a)}\,dx = \quad \sin(2x) = 2\sin(x)\cdot\cos(x) \Rightarrow \int \frac{1}{2\sin\left(\frac{x+a}{2}\right)\cos\left(\frac{x+a}{2}\right)}\,dx =$ Non l'ho capito molto ?

**4.22** $\displaystyle\int \frac{1}{\cos x}\,dx = \int \frac{1}{\operatorname{sen}(x+\frac{\pi}{2})}\,dx = \log\left|\operatorname{tg}(\frac{\pi}{2}+\frac{\pi}{4})\right| + c$

$\displaystyle\int \frac{1}{\cos x}\,dx\cdot\frac{\cos x}{\cos x} = \int \frac{\cos x}{\cos^2 x}\,dx \to \int \frac{\cos x}{1-\sin^2 x}\,dx$ pongo $t = \sin x \to dx = \frac{1}{\cos x}\,dt$

$\displaystyle\Rightarrow \int \frac{\cancel{\cos x}}{1-\sin^2 x}\cdot\frac{1}{\cancel{\cos^2 x}}\,dt = \int \frac{1}{1-t^2}\,dt$ siccome $\displaystyle\int \frac{1}{1+f(x)^2}\,dx = \operatorname{arctg} x + c$

$\Rightarrow$ pongo $z = 1-t^2 \to dt = \frac{1}{-2t}\,dz = -\int \frac{1}{1-t^2}\cdot t^2\,dz = -\int \frac{1}{\cancel{t^2}(\frac{1}{t^2}-1)}\,\cancel{t^2}\,dz \to \int \frac{1}{\frac{1}{z}-1} = \int \frac{1}{\frac{1-z}{z}}\,dz$

$\displaystyle = -\int \frac{z}{1-z}\,dz$ BOH

Verificare che ($n \in \mathbb{N}$):

**4.26** $\displaystyle\int \operatorname{sen}^n x\cos x\,dx = \int \operatorname{sen}^n x\cdot D\operatorname{sen} x\,dx = \frac{\operatorname{sen}^{n+1} x}{n+1} + c$

$\displaystyle\int \sin^n x\cdot\cos x\,dx \qquad D[\sin x] = \cos x \to \int \sin^n x\cdot D[\sin x] = \frac{\sin^{n+1} x}{n+1} + c$

**4.27** $\displaystyle\int \cos^n x\operatorname{sen} x\,dx = -\int \cos^n x\cdot D\cos x\,dx = -\frac{\cos^{n+1} x}{n+1} + c$

$\displaystyle\int \cos^n x\,\sin x\,dx \qquad D[\cos n] = -\sin x \to -\int \cos^n x\cdot D[\cos x](-1)\,dx = -\frac{\cos^{n+1}(x)}{n+1} + c$

**4.28** $\displaystyle\int \frac{1}{\operatorname{sen} x\cos x}\,dx = \int \frac{1}{\operatorname{tg} x\cos^2 x}\,dx = \int \frac{D\operatorname{tg} x}{\operatorname{tg} x}\,dx = \log|\operatorname{tg} x| + c$

$\displaystyle\int \frac{1}{\sin x\cos x}\,dx = \quad \frac{\cancel{2}\sin x\cos x}{\cancel{2}} = \frac{\sin(2x)}{2} \quad \int \frac{2}{\sin(2x)} \qquad \int \frac{f'(x)}{f(x)}$ pongo $2x = t \to dx = \frac{1}{2}\,dt$

obbiettivo

$\displaystyle\to \int \frac{\cancel{2}}{\sin(t)}\cdot\frac{1}{\cancel{2}}\,dt = \int \frac{1}{\sin(t)}\cdot\frac{\sin t}{\sin t}\,dt \to \int \frac{\sin t}{\sin^2 t}\,dt \to \int \frac{\sin t}{1-\cos^2 t}\,dt$ Pongo $z = \cos^2 t \to dt = \frac{1}{-2\sin(t)}$

$\displaystyle -\frac{1}{2}\int \frac{\cancel{\sin t}}{1-z^2}\cdot\frac{1}{\cancel{\sin t}}\,dz = -\frac{1}{2}\int \frac{1}{1-z^2}$

## 4C. Integrazione per decomposizione in somma

Verificare che:

**4.34** $\displaystyle\int (hx+k)\,dx = h\int x\,dx + k\int 1\,dx = h\frac{x^2}{2} + kx + c$

$$\int (hx+k)\,dx = \int hx + \int k\,dx = h\int x + k\int dx = h\frac{x^2}{2} + kx + c = x\left(\frac{h}{2}x + k\right) + c$$

**4.35** $\displaystyle\int (a_0+a_1x+\ldots+a_nx^n)\,dx = a_0\int dx + a_1\int x\,dx + \ldots + a_n\int x^n\,dx =$

$$= a_0x + \frac{a_1}{2}x^2 + \ldots + \frac{a_n}{n+1}x^{n+1} + c$$

$$\int a_0 + a_1x + \ldots + a_nx^n\,dx = a_0\int dx + a_1\int x\,dx + a_2\int x^2\,dx + \ldots\ a_n\int x^n\,dx = a_0x + a_1x^2 + \ldots + a_n\frac{x^{n+1}}{n+1} + c$$

**4.36** $\displaystyle\int \frac{x}{1+x}\,dx = \int \frac{1+x-1}{1+x}\,dx = \int 1\,dx - \int \frac{1}{1+x}\,dx =$

$$= x - \log|1+x| + c$$

$$\int \frac{x}{1+x}\,dx = \int \frac{1+x-1}{1+x}\,dx = \int \frac{1+x}{1+x}\,dx - \int \frac{1}{1+x}\,dx = x - \ln|1+x| + c$$

$$\int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$$

**4.37** $\displaystyle\int \frac{3x+2}{4x+5}\,dx = \frac{1}{4}\int \frac{12x+8}{4x+5}\,dx = \frac{1}{4}\int \frac{12x+15-7}{4x+5}\,dx =$

$$= \frac{3}{4}\int 1\,dx - \frac{7}{4}\int \frac{1}{4x+5}\,dx = \frac{3}{4}\int 1\,dx - \frac{7}{16}\int \frac{4}{4x+5}\,dx =$$

$$= \frac{3}{4}x - \frac{7}{16}\log|4x+5| + c$$

$$\int \frac{3x+2}{4x+5}\,dx = \quad D[4x+5] = 4 \quad \int \frac{3x+x-x+2+3-3}{4x+5}\,dx = \int \frac{4x+5}{4x+5}\,dx + \int \frac{-x-3}{4x+5}$$

La somma non funz.

$$\int \frac{3x+2}{4x+5}\,dx \cdot 4 \to \frac{1}{4}\int \frac{12x+15-7}{4x+5}\,dx \div 3 \to \frac{3}{4}\int \frac{4x+5}{4x+5} - \frac{3}{4}\int \frac{7}{4x+5}\,dx = \frac{3}{4}x - \frac{7}{4\cdot 4}\int \frac{1\cdot 4}{4x+5} = \frac{3}{4}x + \frac{7}{16}\ln|4x+5| + c$$

$$\frac{3}{4}x$$

**4.42** $\displaystyle\int \frac{1-x^4+x}{1+x^2}\,dx = \int \frac{1-x^4}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx =$

$$\int (1-x^2)\,dx + \frac{1}{2}\int \frac{2x}{1+x^2}\,dx =$$

$$= x - \frac{x^3}{3} + \frac{1}{2}\log(1+x^2) + c$$

$$\int \frac{1-x^4+x}{1+x^2}\,dx = \underbrace{\int \frac{1}{1+x^2}}_{\arctan(x)} + \frac{1}{2}\underbrace{\int \frac{2x}{1+x^2}}_{\frac{1}{2}\ln|1+x^2|} - \int \frac{x^4}{1+x^2}\,dx$$

a b c

$$c/a) \quad \int \frac{1-x^4}{1+x^2}\,dx = \int \frac{\cancel{(1+x^2)}(1-x^2)}{\cancel{1+x^2}} = \int 1-x^2\,dx$$

$$= \int dx - \int x^2\,dx = x - \frac{x^3}{3} + \frac{1}{2}\ln|1+x^2| + c$$

**4.43** $\displaystyle\int \frac{1}{x(1+x)}\,dx = \int \frac{1+x-x}{x(1+x)}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{1+x}\,dx =$

$$\int \frac{1}{x(1+x)}\,dx = \int \frac{1+x-x}{x(1+x)}\,dx = \int \frac{\cancel{1+x}}{x\cancel{(1+x)}}\,dx - \int \frac{x}{x(1+x)}\,dx$$

$$\to \int \frac{1}{x}\,dx - \int \frac{1}{1+x}\,dx \quad D[1+x] = 1 = \ln|x| - \ln|1+x| + c$$

**4.44** $\displaystyle\int \frac{1}{x(1+x^2)}\,dx = \int \frac{1+x^2-x^2}{x(1+x^2)}\,dx = \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx =$

$$= \log|x| - \frac{1}{2}\log(1+x^2) + c = \log\frac{|x|}{\sqrt{1+x^2}} + c$$

$$\int \frac{1}{x(1+x^2)}\,dx = \int \frac{1+x^2-x^2}{x(1+x^2)}\,dx$$

$$= \int \frac{\cancel{1+x^2}}{x\,\cancel{(1+x^2)}}\,dx - \int \frac{x^{\cancel{2}}}{\cancel{x}(1+x^2)}\,dx = \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|1+x| + c$$

**4.45** Calcolare gli integrali indefiniti

(a) $\displaystyle\int \frac{x^2+x}{x^2-x}\,dx$

(b) $\displaystyle\int \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}\,dx$

[(a) Vale la scomposizione

a) $\displaystyle\int \frac{x^2+x}{x^2-x}\,dx = \int \frac{x^2-x+2x}{x^2-x}\,dx$

$$= \int \frac{\cancel{x}(x-1+2)}{\cancel{x}(x-1)}\,dx = \int \frac{x-1}{x-1}\,dx + 2\int \frac{1}{x-1}\,dx$$

$$= x\,dx + 2\ln|x-1| + c$$

Inoltre $\displaystyle\frac{x^2+x}{x^2-x} = \frac{x^2-x+2x}{x^2-x} = \frac{\cancel{x^2-x}}{\cancel{x^2-x}} + \frac{2x}{x^2-x} = 1 + \frac{2x}{x^2-x} = 1 + \int \frac{2x-1}{x^2-x} + \frac{1}{x^2-x}$

$\displaystyle\int \frac{f'}{f} = \ln|f| + c$

$\to \dfrac{1}{x^2-x}$ ? $\to \dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1} = \dfrac{1}{x(x-1)} \to x(x-1)\left[\dfrac{A}{x} + \dfrac{B}{x-1}\right] = \dfrac{1}{x(x-1)} \cdot x(x-1)$

$\Rightarrow A(x-1) + Bx = 1 \to Ax + Bx - A = 1$ affinché l'eq $= 1$, il sistema deve essere rispettato:

$$\begin{cases} Ax + Bx = 0 \\ A = -1 \end{cases} \Rightarrow -x + Bx = 0 \Rightarrow x(B-1) = 0 \Rightarrow B = 1$$

Riscrivendo $\displaystyle\int \frac{A}{x} + \int \frac{Bx}{x-1}\,dx = -\int \frac{1}{x}\,dx + \int \frac{x}{x-1}\,dx$ pongo $t = x-1 \to x = t+1$

$$\Rightarrow -\ln|x| + \int \frac{t+1}{t}\,dt = \int dt + \int \frac{1}{t}\,dt = -\ln|x| + x + 1 + \ln|x|$$

**4.47** $\displaystyle\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx = \operatorname{arctg}(x+1) + c$

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1} = \int \frac{1}{1+f^2} = \operatorname{arctg} f$$

$$= \int \frac{1}{1+(x+1)^2}\,dx = \operatorname{arctg}(x+1)^2 + c$$

**4.48** $\displaystyle\int \frac{1}{x^2+x+2}\,dx = \frac{2}{\sqrt{7}}\operatorname{arctg}\frac{2x+1}{\sqrt{7}} + c$

$$\int \frac{1}{x^2+x+2}\,dx = \int \text{———}$$

Verificare che:

**4.51** $\displaystyle\int \frac{3x+2}{x^2+x+1}\,dx = \frac{3}{2}\int \frac{2x+4/3}{x^2+x+1}\,dx = \frac{3}{2}\int \frac{2x+1+1/3}{x^2+x+1}\,dx =$

$$= \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx =$$

$$= \frac{3}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + c$$

$$\int \frac{3x+2}{x^2+x+1}\,dx = D[x^2+x+1] = 2x+1$$

$$\Rightarrow \int \frac{2x+1 + x+1}{x^2+x+1} = \int \frac{2x+1}{x^2+x+1}\,dx + \overset{a)}{\int \frac{x+1}{x^2+x+1}\,dx}$$

a) $\displaystyle\int \frac{x}{x^2+x+1}\,dx + \int \frac{1}{x^2+x+1}\,dx$

$$\Rightarrow \frac{1}{2}\int \frac{2x+1}{x^2+x+1} - \cancel{\int \frac{1}{x^2+x+1}\,dx} + \cancel{\int \frac{1}{x^2+x+1}\,dx} \to \ln|x^2+x+1| + \frac{1}{2}\ln|x^2+x+1| + c$$

Sbagliato

**4.68** Utilizzando il metodo di integrazione per parti, calcolare i seguenti integrali

(a) $\int x \operatorname{sen} x dx$ (b) $\int x \cos x dx$

Formula $\int \underset{\text{Derivare}}{f(x)}\, \underset{\text{Integrare}}{g(x)'} = f(x)\cdot g(x) - \int f'(x)\, g(x)\, dx$

a) $\int \underset{f}{x} \underset{g}{\sin x}\, dx$ $\quad D[x] = 1 \quad ; \int \sin x\, dx = -\cos x$

$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + c$

1) mi conviene scegliere come $f(x)$ una $f$ che poi si annulla nell'integrale DERIVANDO

b) $\int \overset{f}{x} \overset{g}{\cos x}$ $\quad D[x] = 1 \quad , \int \cos x = \sin x$

$\Rightarrow \int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + c$

c) $\int \underset{f}{x}\, \underset{g}{e^x}\, dx \to x e^x - \int e^x dx = x e^x - e^x = e^x(x-1) + c$

(c) $\int x\, e^x\, dx$ (d) $\int x \log |x|\, dx$

d) $\int \underset{g}{x} \underset{f}{\ln x}\, dx = \int f \cdot g' = f \cdot g - \int f' \cdot g\, dx \Rightarrow \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \ln x \cdot \frac{x^2}{2} - \frac{1}{2}\frac{x^2}{2} + c = \frac{x^2}{2}\left(\ln - \frac{1}{2}\right) + c$

**4.69** Calcolare i seguenti integrali

(a) $\int \operatorname{arcsen} x\, dx$ (b) $\int \arccos x\, dx$

(c) $\int \operatorname{arctg} x\, dx$ (d) $\int \log |x|\, dx$

a) $\int \arcsin x\, dx = \int \arcsin(x) \cdot \underset{D[x]}{1} = \arcsin x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x$

pongo $t = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$ $\quad D = \frac{1}{2}\frac{1}{\sqrt{1-x^2}} \cdot (-2x) \to dx = \frac{1}{\frac{1}{2\sqrt{1-x^2}} \cdot (-2x)} dt = -\frac{\cancel{2}\sqrt{1-x^2}}{\cancel{2}x} dt$

$\Rightarrow x \arcsin x + \int \frac{\cancel{x}}{\cancel{\sqrt{1-x^2}}} \cdot \frac{\cancel{\sqrt{1-x^2}}}{\cancel{x}} dt = x \arcsin x + x + c \to x(\arcsin x + 1) + c$

c) $\int \operatorname{arctg} x$ $\quad \int \frac{1}{1+x^2} dx = \operatorname{arctg} x \Rightarrow \int \overset{f}{\operatorname{arctg}(x)} \cdot \overset{g'}{D[x]}\, dx = \operatorname{arctg}(x) \cdot x - \int \frac{1}{1+x^2} \cdot x\, dx$

pongo $t = 1 + x^2 \to dx = \frac{1}{2x} dt \to x \operatorname{arctg}(x) - \frac{1}{2}\int \frac{\cancel{x}}{1+x^2} \cdot \frac{1}{\cancel{x}} dx = x \operatorname{arctg} x - \frac{1}{2}\int \frac{1}{1+x^2} dt$

$x^2 = t - 1$

$\Rightarrow x \operatorname{arctg} x - \frac{1}{2}\int \frac{1}{1+t-1} dt = x \operatorname{arctg} x - \frac{1}{2} \ln|1+x^2| + c$

d) $\int \ln|x| + c = \int \overset{f}{\ln|x|} \cdot \overset{g'}{D[x]}\, dx$ $\quad \int \frac{1}{x} dx = \ln|x| \Rightarrow \ln|x| \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} dx = x \ln|x| - x + c$

② $f(x)$ è quella di cui NON voglio fare l'integrale!

③ $f(x)$ resta uguale nella prima parte, $g$ si integra (sia nella prima che seconda)

**4.70** Calcolare i seguenti integrali

(a) $\int x^2 \operatorname{sen} x dx$ (b) $\int x^2 \cos x dx$

(c) $\int x^2 e^x \, dx$ (d) $\int x^2 \log x \, dx$

a) $\int \underset{f}{x^2} \underset{g'}{\sin x} \, dx = -x^2 \cos x + \int \underset{f}{2x} \underset{g}{\cos x} \, dx$

Di nuovo: $-x^2 \cos x + 2\left[x \sin x - \int \sin x \, dx\right]$

$= -x^2 \cos x + 2x \sin x + 2\cos x + c$

④ Quando ho un membro del tipo $x^n$ posso applicare n volte la formula ponendo $x^n$ sempre come $f(x)$ in modo che derivando ad ogni iterazione arrivo alla n-esima iterazione che $f(x) = 1$ [eliminando un termine].

$$\boxed{\int f(x) \cdot g(x) = f(x) \cdot \left[\int g(x)\right] - \int D[f(x)] \cdot \left[\int g(x)\right]}$$

FORMULA PIU' FACILE DA RICORDARE!

b) $\int \underset{f}{x^2} \underset{g}{\cos x} \, dx \to x^2 \cdot \sin x - \int 2x \cdot \sin x \, dx$

$= x^2 \sin x - 2\left[-x \cos x + \int \cos x \, dx\right] = x^2 \sin x + 2x \cos x - 2\sin x + c$

$= \sin x (x^2 - 2) + 2x \cos x + c$

c) $\int \underset{f}{x^2} \underset{g}{e^x} \, dx = x^2 e^x - \int 2x e^x \, dx$

$\to x^2 e^x - 2\left[x e^x - \int e^x dx\right] = x^2 e^x - 2x e^x + 2e^x + c = e^x (x^2 - 2x + 2) + c$

d) $\int x^2 \ln x = x^2 \cdot \int \ln x$

$\to \int \ln x \cdot D[x] \, dx = \ln x \cdot x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x = x(\ln x - 1) + c$

$\Rightarrow \int \underset{f}{x^2} \underset{g}{\ln x} = x^2 \cdot x(\ln(x) - 1) - \int x \ln(x) - \int dx = x^2 \cdot x(\ln x - 1) - x - \left[x \cdot x(\ln(x) - 1) - \int \ln(x) \, dx\right]$

$= x^3 \ln(x) - 1 - x - x^2 \ln(x) - 1 + x \ln(x) - 1 + c$

**4.71** $\int \log^2 x \, dx = \int \log^2 x \cdot Dx \, dx = x \log^2 x - 2 \int \log x \, dx =$

$= x \log^2 x - 2x \log x + 2x + c$

$\int \ln^2 x \, dx \cdot D[x] = x \ln^2 x - \int \cancel{x} \, 2 \frac{\ln x}{\cancel{x}} \, dx$

$= x \ln^2 x - 2 \int \ln x \, dx \cdot D[x] =$

$= x \ln^2 x - 2\left[x \ln x - \int \cancel{x} \frac{1}{\cancel{x}} \, dx\right] = x \ln^2 x - 2x \ln x - x + c = x(\ln^2 x - 2\ln x - 1) + c$

**4.72** $\int \operatorname{sen}^2 x \, dx = \int \operatorname{sen} x \cdot D(-\cos x) \, dx =$

$= -\operatorname{sen} x \cos x + \int \cos^2 x \, dx = -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x) \, dx =$

$= -\operatorname{sen} x \cos x + \int dx - \int \operatorname{sen}^2 x \, dx$

da cui

$\int \underset{f}{\sin^2 x} \, dx = \int \underset{g}{\sin x} \cdot D[\underset{f}{\overset{\sin x}{-\cos x}}] \, dx$

$\sin^2 x + \cos^2 x = 1 \to \sin^2 x = 1 - \cos^2 x$

non mi trovo

**4.73** $\int x \operatorname{sen}^2 x \, dx = \int x \operatorname{sen} x \cdot D(-\cos x) \, dx =$

$= -x \operatorname{sen} x \cos x + \int (\operatorname{sen} x + x \cos x) \cos x \, dx =$

$= -x \operatorname{sen} x \cos x + \int \operatorname{sen} x \cos x \, dx + \int x(1 - \operatorname{sen}^2 x) \, dx$

da cui

$\int x \operatorname{sen}^2 x \, dx = -\frac{1}{2} x \operatorname{sen} x \cos x + \frac{1}{4}(x^2 + \operatorname{sen}^2 x) + c$

$\int \underset{f}{x} \underset{g}{\sin^2 x} \, dx = x \int \sin^2 x - \int \int \sin^2 x \, dx$

$\int \sin^2 x = \int \underset{f}{\sin x} \cdot \underset{g}{\sin x} = -\sin x \cos x - \int \cos x \cdot (-\cos x) dx$

$\Rightarrow -\cos x \sin x + \int \cos^2 x \, dx = -\cos x \sin x + \int 1 - \sin^2 x \, dx$

$\Rightarrow -\cos x \sin x + \int dx - \int \sin^2 x \, dx = \int \sin^2 x \, dx$ Pongo $I = \int \sin^2 x \, dx \Rightarrow -\cos x \sin x + x - I = I$

$\to 2I = -\cos x \sin x + x$ ; $I = \int \sin^2 x \, dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} x + c = \frac{1}{2}(-\cos x \sin x + x) + c$

$\Rightarrow x \int \sin^2 x - \int \int \sin^2 x = -\frac{x}{2} \cos x \sin x + \frac{1}{2} x^2 - \left[-\int \frac{1}{2} \cos x \sin x + \frac{1}{2} \int x \, dx\right] = -\frac{x}{2} \cos x \sin x + \frac{1}{2} x^2$

Troppo lungo

**4.74** $\displaystyle\int x\cos^2 x\,dx = \frac{1}{2}x\,\mathrm{sen}\,x\cos x + \frac{1}{4}(x^2+\cos^2 x) + c$ $\qquad \int \underset{f}{x}\,\underset{g}{\cos^2 x}\,dx = \int f\cdot g = f\cdot\int g - \int Df\cdot\int g$

$\int \cos^2 x\,dx = \int \overset{f}{\cos x}\cdot\overset{g}{\cos x} = \cos x\sin x - \int -\sin x\cdot\sin x\,dx \Rightarrow \cos x\sin x + \int \sin^2 x\,dx \Rightarrow \cos x\sin x + \int dx - \int \cos^2 x\,dx$

$\Rightarrow \cos x\sin x + x - \int\cos^2 x\,dx = \int\cos^2 x\,dx$ $\quad$ pongo $I = \int\cos^2 x\,dx \Rightarrow \cos x\sin x + x - I = I$

$\Rightarrow I = \int\cos^2 x\,dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}x + c$

$\Rightarrow \int x\cos^2 x\,dx = x\cdot\left[\frac{1}{2}\cos x\sin x + \frac{1}{2}x\right] - \frac{1}{2}\int\cos x\sin x + \frac{1}{2}\int x\,dx = \frac{1}{2}x\cos x\sin x + \frac{1}{2}x^2 - \frac{1}{4}x^2 - \frac{1}{2}\int\cos x\sin x\,dx$

$\int \underset{f}{\cos x}\,\underset{g}{\sin x}\,dx = -\cos x\cos x - \int -\sin x\cdot -\cos x = -\cos^2 x - \int\sin x\cos x\,dx = \int\sin x\cos x$ $\quad$ pongo $I = \int\sin x\cos x$

$\Rightarrow -\cos^2 x - I = I \Rightarrow I = -\frac{\cos^2 x}{2}$

$\Rightarrow \frac{1}{2}x\cos x\sin x + \frac{1}{4}x^2 + \frac{1}{4}\cos^2 x$

Integrazione $\int x^2\,dx$ per parti $\quad \int \underset{f}{x}\cdot\underset{g}{x}\,dx = x\cdot\frac{x^2}{2} - \frac{1}{2}\int x^2 \Rightarrow \frac{x^3}{2} - \frac{1}{2}\int x^2 = \int x^2\,dx$

pongo $t = \int x^2\,dx \Rightarrow \frac{x^3}{2} - \frac{1}{2}t = t$

$\Rightarrow \frac{x^3}{2} = \frac{2t+t}{2} \Rightarrow \frac{\cancel{2}x^3}{3\cancel{2}} = \frac{\cancel{3}}{\cancel{2}}t \Rightarrow t = \int x^2\,dx = \frac{x^3}{3}$

**4.75** $\displaystyle\int x\,e^{2x}\,dx = \frac{1}{2}\int xDe^{2x}\,dx = \frac{1}{2}x\,e^{2x} - \frac{1}{4}e^{2x} + c$ $\qquad \int \underset{f}{x}\,\underset{g}{e^{2x}}\,dx\ ;\quad \int e^{2x}\,dx = \int a^{f(x)}\cdot f'(x)$

$\Rightarrow \frac{1}{2}\int e^{2x}\cdot 2\,dx = \frac{1}{2}e^{2x} \Rightarrow \int xe^{2x} = \frac{1}{2}x\,e^{2x} - \frac{1}{2}\int e^{2x}\,dx = \frac{1}{2}x\,e^{2x} - \frac{1}{4}e^{2x} + c = \frac{1}{2}e^{2x}\left(x - \frac{1}{4}\right) + c$

**4.76** $\displaystyle\int \frac{x}{\mathrm{sen}^2 x}\,dx = \int x\,D(-\mathrm{cotg}\,x)\,dx = -x\,\mathrm{cotg}\,x + \log|\mathrm{sen}\,x| + c$

$\int \frac{x}{\sin^2 x}\,dx = \int \underset{f}{x}\cdot\underset{g}{\frac{1}{\sin^2 x}}\,dx = x\cdot\underbrace{\int\frac{1}{\sin^2 x}\,dx}_{\text{integrale notevole } = -\cot(x)} - \underbrace{\int\int\frac{1}{\sin^2 x}\,dx}\,dx \Rightarrow -x\cot(x) + \int\cot(x)\,dx$

$= -x\cot(x) + \int\frac{\cos x}{\sin x}\,dx$ $\quad$ pongo $t = \sin x \Rightarrow dx = \frac{1}{\cos x}\,dt \Rightarrow -x\cot(x) + \int\frac{\cancel{\cos x}}{\sin x}\cdot\frac{1}{\cancel{\cos x}}\,dt$

$= -x\cot(x) + \int\frac{1}{t}\,dt = -x\cot(x) + \ln|\sin x| + c$

**4.77** $\displaystyle\int \frac{x}{\cos^2 x}\,dx = \int x\,D\mathrm{tg}\,x\,dx = x\,\mathrm{tg}\,x + \log|\cos x| + c$

$\int\frac{x}{\cos^2 x}\,dx = \int \underset{f}{x}\cdot\underset{g}{\frac{1}{\cos^2 x}}\,dx = x\cdot\int\frac{1}{\cos^2 x}\,dx - \int\int\frac{1}{\cos^2 x}\,dx\,dx = x\,tg(x) - \int tg(x)\,dx = x\,tg(x) - \int\frac{\sin x}{\cos x}\,dx$

pongo $t = \cos x \Rightarrow dx = \frac{1}{\sin x}\,dt \Rightarrow x\,tg\,x + \int\frac{\cancel{\sin x}}{\cos x}\cdot\frac{1}{\cancel{\sin x}}\,dt = x\,tg\,x + \ln|\cos x| + c$

**4.78** $\displaystyle\int x\,\mathrm{tg}^2 x\,dx = \int x\,\frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\frac{x}{\cos^2 x}\,dx - \int x\,dx$ $\qquad \int\frac{\sin x}{\cos x}\,dx = \ln|\cos x| \qquad \int\frac{1}{\cos^2 x} = tg\,x$

$\int \underset{f}{x}\,\underset{g}{tg^2 x}\,dx$ $\quad$ risolvo prima $\quad \int tg^2 x\,dx = \int \underset{f}{tg\,x}\cdot\underset{g}{tg(x)} = tg\,x\cdot\int tg\,x - \int\frac{1}{\cos^2 x}\cdot\int tg\,x\,dx\,dx$

$\int tg\,x = \int\frac{\sin x}{\cos x}\,dx$ $\quad$ pongo $\cos x = t \Rightarrow dx = -\frac{1}{\sin x}\,dt \Rightarrow -\int\frac{1}{t}\,dt = -\ln|\cos x| + c$

$\Rightarrow \int tg^2 x\,dx = -tg\cdot\ln|\cos x| + \underbrace{\int\frac{\ln|\cos x|}{\cos^2 x}\,dx}_{Q}$

$$\int \underset{f}{\ln|\cos x|} \cdot \underset{g}{\frac{1}{\cos^2 x}}\, dx \qquad \int \frac{1}{\cos^2 x}\, dx = tg\, x \Rightarrow \int \ln|\cos x| \cdot \frac{1}{\cos^2 x} = \underbrace{\ln|\cos x| \cdot tg\, x + \int tg^2 x\, dx}_{a}$$

$$\Rightarrow \int tg^2 x\, dx = -tg \ln|\cos x| + \ln|\cos x|\, tg\, x + \int tg^2 x\, dx$$

Calcoli fatti 2 volte NON mi TROVO

Posto $n \in \mathbb{R}$, $\alpha \in \mathbb{R} - \{0\}$

$$I_n = \int x^n \operatorname{sen} \alpha x\, dx \qquad J_n = x^n \cos \alpha x\, dx\,,$$

verificare, mediante l'integrazione per parti, che per $n > 1$, si ha:

$$I_n = -\frac{1}{\alpha} x^n \cos \alpha x + \frac{n}{\alpha} J_{n-1} \qquad J_n = \frac{1}{\alpha} x^n \operatorname{sen} \alpha x - \frac{n}{\alpha} I_{n-1}\,.$$

$$\int \underset{f}{x^n} \underset{g}{\sin(\alpha x)}\, dx \to x^n \cdot \int \sin(\alpha x)\, dx - \frac{1}{n} \int x^{n-1} \cdot \int \sin(\alpha x)$$

$$\frac{1}{\alpha} \int \sin(\alpha x) \cdot \alpha = -\frac{1}{\alpha} \cos(\alpha x) + c$$

$$\Rightarrow \int x^n \sin(\alpha x)\, dx = -\frac{1}{\alpha} x^n \cos(\alpha x) + \frac{n}{\alpha} \int x^{n-1} \cdot \cos(\alpha x) + c = -\frac{1}{\alpha} x^n \cos(\alpha x) + \frac{n}{\alpha} \int J_{n-1} = I_n$$

## Integrazione func. razionali

**4.97** Calcolare gli integrali

(a) $\displaystyle\int \frac{x+3}{x^2-6x}\, dx$ (b) $\displaystyle\int \frac{x+3}{x^2-6x+5}\, dx$

(c) $\displaystyle\int \frac{3x-1}{x^2-x-6}\, dx$ (d) $\displaystyle\int \frac{x^4-10x^3+20x^2+11x-20}{x^2-10x+21}\, dx$

(e) $\displaystyle\int \frac{x}{(x-3)^2}\, dx$ (f) $\displaystyle\int \frac{x^4-4x^3+3x^2+5x-4}{x^2-4x+4}\, dx$

a) $\displaystyle\int \frac{x+3}{x^2-6x}\, dx = \int \frac{P(x)}{Q(x)}$ con $P(x) < Q(x)$

$\Rightarrow \Delta(Q(x)) = 36 > 0 \Rightarrow x^2 - 6x = x(x-6)$

$$\Rightarrow \frac{x+3}{x^2-6x} = \frac{A}{x} + \frac{B}{x-6} = \frac{Ax-6A+Bx}{x(x-6)} = \frac{x+3}{x(x-6)}$$

$$\Rightarrow \frac{x(A+B)-6A}{x(x-6)} \Rightarrow \begin{cases} A+B=1 \Rightarrow B = 1+\frac{1}{2} = \frac{3}{2} \\ -6A = 3 \to A = -\frac{1}{2} \end{cases}$$

$$\Rightarrow \frac{x+3}{x(x-6)} = \frac{-\frac{1}{2}}{x} + \frac{\frac{3}{2}}{x-6} = -\frac{1}{2} \cdot \frac{1}{x} + \frac{3}{2} \cdot \frac{1}{x-6} \Rightarrow -\frac{1}{2} \int \frac{1}{x} + \frac{3}{2} \int \frac{1}{x-6}\, dx = -\frac{1}{2} \ln|x| + \frac{3}{2} \ln|x-6| + c$$

b) $\displaystyle\int \frac{x+3}{x^2-6x+5}$ $\begin{matrix} Px \\ Qx \end{matrix}$ $Px < Qx \Rightarrow \Delta(Qx) = 36 - 4 \cdot 5 = 16 > 0 \Rightarrow$ $\alpha_{1,2} = \frac{6 \pm 4}{2}$ $\begin{matrix} 5 \\ 1 \end{matrix}$

$$\Rightarrow x^2 + px + q = (x-\alpha_1)(x-\alpha_2) \Rightarrow (x^2-6x+5) = (x-5)(x-1) = \underline{x^2 - x - 5x + 5}$$

$$\Rightarrow \frac{A}{x-5} + \frac{Bx+C}{x-1} = \frac{A(x-1) + Bx(x-5) + C(x-5)}{(x-5)(x-1)} = \frac{Ax - A + Bx^2 - 5Bx + Cx - 5C}{(x-5)(x-1)} = \frac{x+3}{//}$$

$$\Rightarrow x^2 B + x(A - 5B + C) - A - 5C = x + 3$$

$$\begin{cases} B = 0 \\ A - 5B + C = 1 \Rightarrow A + C = 1 \Rightarrow A = 1 - C \Rightarrow A = 2 \\ -A - 5C = 3 \Rightarrow -(1-C) - 5C = 3\,;\ -1 + C - 5C = 3\,; \quad C = -1 \end{cases} \Rightarrow \frac{x+3}{x^2-6x+5} = \frac{2}{x-5} - \frac{1}{x-1}$$

$$\Rightarrow \int \frac{x+3}{x^2-6x+5}\, dx = 2\int \frac{1}{x-5} - \int \frac{1}{x-1}\, dx = 2 \ln|x-5| - \ln|x-1| + c$$

il libro dice $4\ln|x-5| \ldots$ ma è corretto $2\ln \ldots$

c) $\int \frac{3x-1}{x^2-x-6}\,dx$ $\quad \frac{P(x)}{Q(x)}$, $P<Q \to \Delta(Q_x) = 1+4\cdot 6 = 25 > 0 \Rightarrow x_{1/2} = \frac{1\pm 5}{2} \begin{cases} 3 \\ 2 \end{cases} \Rightarrow$

$\Rightarrow (x^2-x-6) = (x-3)(x+2) = x^2+2x-3x-6$

$$\Rightarrow \frac{3x-1}{x^2-x-6} = \frac{3x-1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{Bx+C}{x+2} = \frac{Ax+2A+Bx^2-3Bx+Cx-3C}{(x-3)(x+2)} = \frac{x(A+C-3B)+Bx^2+2A-3C}{//} = 3x-1$$

$$\left.\begin{cases} A+C-3B=1 \Rightarrow A+C=1 \to A=1-C \Rightarrow A=1+\frac{1}{5}=\frac{6}{5} \\ B=0 \\ 2A-3C=-1 \Rightarrow 2(1-C)-3C=-1 \to 2-2C-3C=-1\ ; \quad C=-\frac{1}{5} \end{cases}\right\} \Rightarrow \frac{3x+1}{x^2-x-6} = \frac{6/5}{x-3} - \frac{1/5}{x+2}$$

$$\Rightarrow \int \frac{3x-1}{x^2-x-6}\,dx = \frac{6}{5}\int\frac{1}{x-3} - \frac{1}{5}\int\frac{1}{x+2}\,dx = \frac{6}{5}\ln|x-3| - \frac{1}{5}\ln|x+2| + c$$

sbagliati

Processo alternativo

$$\frac{3x-1}{\cancel{(x+2)(x-3)}} = \frac{A}{x+2} + \frac{B}{x+3} \cdot (x+2)(x-3) \Rightarrow 3x-1 = \frac{Ax-3A+Bx+2B}{\cancel{(x+2)(x-3)}} \cdot \cancel{(x+2)(x-3)} \to 3x-1 = x(A+B)-3A+2B$$

soddisfo:
$$\begin{cases} A+B=3 \to A=3-B \to A=3-\frac{8}{5}=\frac{15-8}{5}=\frac{7}{5} \\ -3A+2B=-1 \to -3(3-B)+2B=-1 = -9+3B+2B=-1 \to B=\frac{8}{5} \end{cases}$$

$$\Rightarrow \frac{7/5}{x+2} + \frac{8/5}{x-3} = \frac{3x-1}{x^2-x-6} \Rightarrow \frac{7}{5}\int\frac{1}{x+2}\,dx + \frac{8}{5}\int\frac{1}{x-3}\,dx = \frac{7}{5}\ln|x+2| + \frac{8}{5}\ln|x-3| + c$$ ← giusto

d) $\int \frac{x^4-10x^3+20x^2+11x-20}{x^2-10x+21}\,dx$ $\quad \frac{P_x}{Q_x}$, $P_x > Q_x \Rightarrow$ Divisione

$$\begin{array}{l|l} x^4-10x^3+20x^2+11x-20 & x^2-10x+21 \\ \cline{2-2} x^4-10x^3+21x^2 & x^2-1 \\ \hline // \quad // \quad -x^2+11x-20 & \\ \quad -x^2+10x-21 & \\ \hline \quad // \quad x+1 & \end{array}$$

Possiamo riscrivere come $(x^2-1) + \frac{x+1}{x^2-10x+21}$

$$\Rightarrow \int x^2-1\,dx + \int\frac{x+1}{x^2-10x+21}$$

$\frac{P_x}{Q_x}$, $P_x<Q_x \to \Delta(Q_x) = 100-4\cdot 21 = 16 > 0$

$\Rightarrow x_{1/2} = \frac{10\pm 4}{2} \begin{cases} 7 \\ 3 \end{cases} \to x^2-10x+21 = (x-3)(x-7)$

$$\frac{x+1}{\cancel{(x-3)(x-7)}} = \frac{A}{x-3} + \frac{B}{x-7} \cdot (x-3)(x-7) \Rightarrow x+1 = \frac{Ax-7A+Bx-3B}{\cancel{(x-3)(x-7)}} \cdot \cancel{(x-3)(x-7)} \to x+1 = x(A+B)-7A-3B$$

$$\begin{cases} A+B=1 \to A=1-B \to A=1-2=-1 \\ -7A-3B=1 \to -7(1-B)-3B=1 \to -7+7B-3B=1 \to B=2 \end{cases} \Rightarrow \frac{x+1}{(x-3)(x-7)} = -\frac{1}{x-3} + 2\frac{1}{x-7}$$

$$\Rightarrow \int x^2\,dx - \int dx - \int\frac{1}{x-3}\,dx + 2\int\frac{1}{x-7}\,dx \to \frac{x^3}{3} - x - \ln|x-3| + 2\ln|x-7| + c$$

e) $\int \frac{x}{(x-3)^2}\,dx$ $\quad$ Pongo $x-3=t \to x=t+3 \to \int\frac{t+3}{t^2}\,dt = \frac{1}{2}\int\frac{2t}{t^2}\,dt + 3\int\frac{1}{t^2}\,dt$

$$= \frac{1}{2}\ln|t^2| + 3t \to \frac{1}{2}\ln|(x-3)^2| + 3\int\frac{1}{t^2}\,dt \to \int t^{-2}\,dt = \frac{1}{2}\ln|(x-3)^2| - \frac{3}{x-3} + c$$

f) $\int \frac{x^4-4x^3+3x^2+5x-4}{x^2-4x+4}$ $\quad \frac{P_x}{Q_x}$, $P_x > Q_x \to$

$$\begin{array}{l|l} x^4-4x^3+3x^2+5x-4 & x^2-4x+4 \\ \hline x^4-4x^3+4x^2 & x^2-1 \\ // \quad // \quad -x^2+5x-4 & \\ \quad -x^2+4x-4 & \\ \quad // \quad x & \end{array}$$

$\Rightarrow \frac{P_x}{Q_x} = (x^2-1) + \frac{x}{x^2-4x+4} \to \frac{x}{(x-2)^2} \quad \frac{P_x}{Q_x}$

TEST: $\frac{x^4-4x^3+4x^2-x^2+4x-4+x}{x^2-4x+4}$ ✓

Separo $\frac{x}{x^2-4x+4} \Rightarrow \Delta = 16-4\cdot 4 = 0 \to x^2-4x+4 = (x-2)^2$

$\Rightarrow \frac{A}{x-2} + \frac{B}{(x-2)^2} \to \quad Ax-2A+B = x \quad \begin{cases} A=1 \\ -2A+B=0 \to -2+B=0 \to B=2 \end{cases}$

$\Rightarrow \int x^2 - \int dx + \int \frac{1}{x-2}dx + 2\int \frac{}{(x-2)^2}dx =$ pongo $t = x-2 \Rightarrow \int \frac{1}{t^2}dt = -\frac{1}{t} = -\frac{1}{x-2}$

$\Rightarrow \frac{x^3}{3} - x + \ln|x-2| + \frac{2}{x-2} + C$

**4.98** Calcolare l'integrale $\int \frac{x^2}{x^2-3x-4}\,dx$ $\quad \int \frac{x^2}{x^2-3x-4} \quad \frac{P_x}{Q_x}$, $P_x = Q_x$

Divido:

$$\begin{array}{l|l} x^2+0x+0 & x^2-3x-4 \\ \hline x^2-3x-4 & 1 \\ // \quad 3x+4 & \end{array}$$

$\Rightarrow 1 + \frac{3x+4}{x^2-3x-4} \quad \frac{P_x}{Q_x} \quad P_x < Q_x$ posso procedere

$x^2-3x-4 = (x-4)(x+1) \Rightarrow \frac{3x+4}{x^2-3x-4} = \frac{A}{x-4} + \frac{B}{x+1} = Ax+A+Bx-4B = 3x+4$

$\Rightarrow \begin{cases} A+B=3 \Rightarrow A = 3-B \Rightarrow A = 3+\frac{1}{5} = \frac{16}{5} \\ A-4B=4 \Rightarrow \quad B = -\frac{1}{5} \end{cases} \Rightarrow \frac{3x+4}{x^2-3x-4} = \frac{\frac{16}{5}}{x-4} - \frac{1/5}{x+1}$

$\Rightarrow \int dx + \frac{16}{5}\int \frac{1}{x-4}dx - \frac{1}{5}\int \frac{1}{x+1}dx = x + \frac{16}{5}\ln|x-4| - \frac{1}{5}\ln|x+1| + c$

**4.99** Calcolare l'integrale $\int \frac{x+2}{x^2+2}\,dx$ $\quad \int \frac{x+2}{x^2+2}dx \quad \frac{P_x}{Q_x} \quad P_x < Q$

NON riesco a scomporlo

**4.100** Calcolare l'integrale $\displaystyle\int \frac{3x+2}{x(x^2+1)}\,dx$

$$\int \frac{3x+2}{x(x^2+1)}\,dx \qquad \frac{P_x}{Q_x} \qquad P_x < Q_x$$

$$\Rightarrow \frac{3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = Ax^2 + A + Bx^2 + Cx \rightarrow \begin{cases} A+B=0 \Rightarrow B=-2 \\ C=3 \\ A=2 \end{cases}$$

$$\Rightarrow \frac{3x+2}{x(x^2+1)} = \frac{2}{x} + \frac{-2x+3}{x^2+1} \Rightarrow 2\int \frac{1}{x}\,dx - 2\int \frac{x}{x^2+1} + 3\int \frac{1}{x^2+1}\,dx = 2\ln|x| - \ln|x^2+1| + 3\,\mathrm{arctg}\,x + c$$

**4.101** Calcolare l'integrale $\displaystyle\int \frac{x^2-7x+12}{(x-2)^3}\,dx$

$$\int \frac{x^2-7x+12}{(x-2)^3}\,dx \qquad \frac{P_x}{Q_x}, \; P_x < Q_x$$

$$(x-2)^3 = (x-2)(x-2)^2$$

$$\frac{x^2-7x+12}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} = \frac{A(x-2)^2 + B(x-2) + C}{(x-2)^3} = Ax^2 - 2Ax + 4A + Bx - 2B + C = x^2 - 7x + 12$$

$$\begin{cases} A = 1 \\ -2A + B = -7 \rightarrow B = -5 \\ 4A + C = 12 \rightarrow C = 8 \end{cases} \Rightarrow \int \frac{1}{x-2}\,dx - 5\int \frac{1}{(x-2)^2}\,dx + 8\int \frac{1}{(x-2)^3}\,dx = \ln|x-2| + 5\,\frac{1}{x-2} - 4\,\frac{1}{(x-2)^2} + c$$

**4.102** Calcolare l'integrale $\displaystyle\int \frac{1}{x^4-3^4}\,dx$

$$\int \frac{1}{x^4-3^4}\,dx = \frac{1}{x^4-81} \qquad \Delta = -4\cdot(-81) = 324 > 0$$

$$x_{1,2} = \pm\frac{18}{2} \begin{cases} +9 \\ -9 \end{cases}$$

$$\Rightarrow \frac{1}{x^4-3^4} = \frac{1}{(x^2+9)(x^2-9)} = \frac{A}{(x^2+9)} + \frac{B}{(x^2-9)} =$$

$$= Ax^2 - 9A + Bx^2 + 9B = 1 \rightarrow \begin{cases} A+B=0 \rightarrow A=-B \rightarrow A=-\frac{1}{18} \\ -9A+9B=1 \rightarrow 9B+9B=1 \rightarrow B=\frac{1}{18} \end{cases}$$

$$\Rightarrow \frac{-\frac{1}{18}}{(x^2+9)} + \frac{\frac{1}{18}}{(x^2-9)} = -\frac{1}{18}\int \frac{1}{x^2+9}\,dx + \frac{1}{18}\int \frac{1}{x^2-9}\,dx = -\frac{1}{54}\,\mathrm{arctg}\left(\frac{x}{3}\right)$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\left(\frac{x}{a}\right)$$

# Integrazione per sostituzione

**4.104** Calcoliamo l'integrale

$$\int \sqrt{a^2 - x^2}\, dx \qquad (a > 0)\,.$$

$\int \sqrt{a^2-x^2}\, dx = \int (a^2-x^2)^{\frac{1}{2}}$

pongo $x = a\sin t \to dx = a\cos t\, dt \Rightarrow \int \sqrt{a^2-(a\sin t)^2} \cdot a\cos t\, dt = \int \sqrt{a^2-a^2\sin^2 t} \cdot a\cos t\, dt$

$= \int \sqrt{a^2(1-\sin^2 t)} \cdot a\cos t\, dt = \int a^2\sqrt{1-\sin^2 t} \cdot \cos t\, dt = a^2\int\sqrt{\cos^2 t} \cdot \cos t\, dt = a^2\int \cos^2 t\, dt$

$= \frac{a^2}{2}t + \frac{1}{4}\sin(2t)$

**4.105** Calcoliamo l'integrale $\displaystyle\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\, dx$ .

$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$ pongo $x = t^2$ $dx = 2t\,dt$

$t = \sqrt{x}$

$\Rightarrow \int \frac{\sin t}{t} \cdot 2t\, dt = 2\int \sin t\, dt = -2\cos t + c \to -2\cos\sqrt{x} + c$

**4.106** Calcoliamo l'integrale $\displaystyle\int x^5 e^{x^2}\, dx$ .

$\int \underset{f}{x^5}\, \underset{g}{e^{x^2}}\, dx$ Pongo $t = x^2 \to x = \sqrt{t}$

$\Rightarrow dx = D[\sqrt{t}]\, dt = \frac{1}{2\sqrt{t}}\, dt$

$\Rightarrow -\int x^5 \cdot e^{x^2} \cdot \frac{1}{2\sqrt{t}}\, dt = -\int x^2 \cdot x^2 \cdot x \cdot e^{x^2} \cdot \frac{1}{2\sqrt{t}}\, dt = -\int t^2 \cdot \cancel{\sqrt{t}} \cdot e^t \cdot \frac{1}{2\cancel{\sqrt{t}}}\, dt = \frac{1}{2}\int \underset{f}{t^2}\, \underset{g}{e^t}\, dt$ per parti

$\frac{1}{2}\left[t^2 e^t - 2\int t e^t dt\right] = \frac{1}{2}\left\{\left[t^2 e^t - 2te^t - \int e^t dt\right\}\right] = \frac{1}{2}t^2 e^t - te^t - e^t + c = e^t\left(\frac{1}{2}t^2 - t - 1\right) + c$

siccome $t = x^2 \to e^{x^2}\left(\frac{1}{2}x^4 - x^2 - 1\right) + c$

**4.107** Calcoliamo l'integrale $\displaystyle\int \sqrt{3 - x^2}\, dx$ .

$\int \sqrt{3-x^2}\, dx = \int (3-x^2)^{\frac{1}{2}}$ ci serve $[-2x]$

$\Rightarrow$ pongo $x = \sqrt{3}\sin t \to dx = \sqrt{3}\cos t\, dt \Rightarrow \int \sqrt{3 - 3\sin^2 t} \cdot \sqrt{3}\cos t\, dt$

$\to \int \sqrt{3(1-\sin^2 t)} \cdot \sqrt{3}\cos t\, dt = 3\int \sqrt{\cos^2 t} \cdot \cos t\, dt = 3\int \cos^2 t\, dt$

$\int \cos t \cdot \cos t = \cos t \cdot \sin t + \int \sin t \cdot \sin t\, dt \to \cos t \sin t + \int dx - \int \cos^2 dt = \int \cos^2 dt$

$\Rightarrow \cos t \sin t + x = 2I \Rightarrow \int \cos^2 t = \frac{1}{2}(\cos t \sin t + x)$

$\Rightarrow x = \sqrt{3}\sin t \Rightarrow \sin t = \frac{x}{\sqrt{3}} \Rightarrow t = \arcsin\left(\frac{x}{\sqrt{3}}\right) \Rightarrow \frac{1}{2}\left(\cos\left(\arcsin\left(\frac{x}{\sqrt{3}}\right)\right) \cdot \sin\left(\arcsin\left(\frac{x}{\sqrt{3}}\right)\right) + \arcsin\left(\frac{x}{\sqrt{3}}\right)\right)$

$= \frac{x}{\sqrt{3}} + \frac{1}{2}\arcsin\frac{x}{\sqrt{3}}$

**4.108** Calcolare l'integrale $\displaystyle\int \frac{\sqrt{x}}{2+\sqrt{x}}\,dx$ .

$$\int \frac{\sqrt{x}}{2+\sqrt{x}}\,dx \quad \text{pongo } x=t^2 \Rightarrow dx = 2t\,dt$$

$$\int \frac{t}{2+t}\,dt \quad \text{pongo } 2+t=z, \quad t=z-2$$

$$= \int dz - 2\int \frac{1}{z}\,dz = z - 2\ln|z| + c \underset{z=2+t}{=} 2+t-2\ln|2+t|$$

$$= 2\int \underbrace{\frac{t}{2+t}}_{g}\cdot \underbrace{t}_{f}\,dt = 2\left[t\cdot\int\frac{t}{2+t}\,dt - \int\int\frac{t}{2+t}\,dt\,dt\right]$$

$$= 2\left[t\cdot(2+t-2\ln|2+t|) - 2\int dt + \int t\,dt - 2\int \ln|2+t|\,dt\right]$$

$$= 2t(2+t-2\ln|2+t|) - 4t + t^2 - 4\int \ln|2+t|\,dt$$

$$= 4t + 2t^2 - 4\ln|2+t| - 4t + t^2 - 4\left[(2+t)\ln|2+t| - (2+t)\right] + c$$

Boh

$$\int \ln|2+t|\,dt \quad \text{pongo } 2+t=z$$

$$\int \ln|z|\cdot D[z]\,dz = \ln|z|\cdot z - \int \frac{1}{z}\cdot z\,dx = z\ln|z| - z + c \underset{z=2+t}{=} (2+t)\ln|2+t| - (2+t) + c$$

# Integrazione di funzioni Irrazionali

(1) $\displaystyle\int \frac{1}{x\sqrt{x+4}}\,dx$ (2) $\displaystyle\int x\sqrt{\frac{1-x}{1+x}}\,dx$

(3) $\displaystyle\int \frac{dx}{x+\sqrt{1+x^2}}\,dx$ .

a) è del tipo $x\sqrt{ax+b}$ = usiamo la sostituzione $t=\sqrt{x+4}$ , $t^2 = x+4 \to x = t^2-4$

se $x=t^2-4$ , $dx = 2t\,dt$

$$\Rightarrow \int \frac{1}{(t^2-4)\cdot \cancel{t}} \cdot 2\cancel{t}\,dt = 2\int \frac{1}{t^2-4}\,dt$$

Scomponiamo $t^2-4 = (t-2)(t+2) = t^2+2t-2t-4$

$$\frac{1}{t^2-4} = \frac{A}{t-2} + \frac{B}{t+2} = \frac{At+2A+Bt-2B}{(t+2)(t-2)} = 1 \to \begin{cases} A+B=0 \Rightarrow A=-B \Rightarrow A=\frac{1}{4} \\ 2A-2B=1 \Rightarrow -2B-2B=1 \end{cases}$$

$\to -4B=1 \Rightarrow B=-\frac{1}{4}$

$$\Rightarrow \frac{1}{2}\int \frac{1}{t-2} - \frac{1}{2}\int \frac{1}{t+2}\,dt = \frac{1}{2}\ln|t-2| - \frac{1}{2}\ln|t+2| + c$$

b) $\displaystyle\int x\sqrt{\frac{1-x}{1+x}}\,dx \to$ è del tipo $x\sqrt{\frac{ax+b}{cx+d}}$ , quindi poniamo $t=\sqrt{\frac{1-x}{1+x}}$

ho dimenticato dt → sbagliato

$$\Rightarrow t^2 = \frac{1-x}{1+x} \quad 1+x+t^2 = 1-x \to x = -\frac{t^2}{2} \quad \Rightarrow -\frac{1}{2}\int t^2\cdot t\,dt \Rightarrow -\frac{1}{6}t^3 + c \to -\frac{1}{6}\sqrt{\left(\frac{1-x}{1+x}\right)^3} + c$$

c) $\displaystyle\int \frac{1}{x+\sqrt{1+x^2}}$ è del tipo $(x, \sqrt{ax^2+bx+c})$ quindi poniamo $t = x+\sqrt{ax^2+bx+c}$

$$\Rightarrow t = x+\sqrt{1+x^2} \to t-x = \sqrt{1+x^2} = t^2-2tx+\cancel{x^2} = 1+\cancel{x^2} \to \frac{t^2-1}{2t} = x$$

$$dt = \frac{2t(2t)-(t^2-1)\cdot 2}{4t^2}\,dt = \frac{4t-2t^2+2}{4t^2} = \frac{1}{t} - \frac{1}{2} + \frac{1}{2t^2} = \frac{1}{2t^2} - \frac{1}{2}$$

$$\Rightarrow \int \frac{1}{t}\cdot\frac{1}{2t^2}\,dt - \frac{1}{2}\int dt = \frac{1}{2}\int \frac{1}{t^3}\,dt - \frac{1}{2}\int dt = \frac{1}{4t^2} - \frac{1}{2}t$$

**4.118** Calcolare l'integrale $\displaystyle\int x\sqrt{\frac{1-x}{1+x}}\,dx$ . $\displaystyle\int x\sqrt{\frac{1-x}{1+x}}\,dx$ pongo $t=\sqrt{\frac{1-x}{1+x}}$

$$\to t^2 = \frac{1-x}{1+x} \to t^2+xt^2 = 1-x$$

$$t^2+xt^2+x = 1 \; ; \quad xt^2+x = 1-t^2 \; ; \quad x = \frac{1-t^2}{t^2+1}$$

$$\Rightarrow dx = \frac{-2t(t^2+1)-(1-t^2)2t}{(t^2+1)^2} = \frac{\cancel{-2t^3}-2t-2t\cancel{+2t^3}}{(t^2+1)^2} = -\frac{4t}{(t^2+1)^2}\,dt$$

$$\Rightarrow -\int \frac{1-t^2}{t^2+1}\cdot\frac{4t^2}{(t^2+1)^2} = 4\int \underbrace{\frac{t^2-t^4}{(t^2+1)^3}}_{\text{VI grado}}\,dt =$$

# Integrazione di funzioni irrazionali

**4.122** Calcolare gli integrali

(a) $\displaystyle\int \frac{1}{x + x\log^2 x}\,dx$ (b) $\displaystyle\int \frac{\log 3x}{x\log 9x}\,dx$

$$\int \frac{1}{x + x\ln^2 x}\,dx \qquad \text{pongo } t = \ln x \to x = e^t \to dx = e^t dt$$

$$\Rightarrow \int \frac{1}{e^t + e^t \cdot t^2} \cdot e^t dt \to \int \frac{1}{\cancel{e^t}(1+t^2)} \cancel{e^t}\, dt = \int \frac{1}{1+t^2}\,dt = \operatorname{arctg}(t) + c = \operatorname{arctg}(\ln x) + c$$

b) $$\int \frac{\ln(3x)}{x\ln(9x)}\,dx \qquad \text{pongo } t = \ln x \to x = e^t \to dx = e^t dt \Rightarrow \int \frac{\ln x + \ln 3}{x\ln 9 + \ln(x)}\,dx = \int \frac{t + \ln(3)}{x\ln 9 + t} \cdot e^t dt$$

$$= \int \frac{e^t t + \ln(3)}{x\, 2\ln 9 + t}$$

# Nuovi esercizi - Integrali immediati

**6.1.** $\int\left(5x^2-\sqrt{x}+\frac{1}{x}-\frac{2}{\sqrt[3]{x}}+4\right)dx$ $\to 5\int x^2 - \int x^{\frac{1}{2}} + \int\frac{1}{x} - 2\int x^{-\frac{1}{3}} + 4\int dx = \frac{5}{3}x^3 - \frac{1}{2\sqrt{x}} - \ln|x| + \frac{4}{3}x^{\frac{2}{3}} + 4x + c$

$\int \ln x\, dx = f\cdot G - \int f'\cdot G \to \int \ln x \cdot \overset{1}{D[x]}$

$\to x\ln x - \int \frac{1}{x}\cdot x\, dx = x\ln x - x + c$

**6.2.** $\int (x^2-1)^2 dx$ $= \int x^4 - 2x^2 + 1 = \int x^4 - 2\int x^2 + \int dx$

$= \frac{x^5}{5} - \frac{2}{3}x^3 + x + c$

**6.3.** $\int\left(\frac{1}{1+x}+\frac{2}{1+x^2}-\frac{2}{\sqrt{1-x^2}}\right)dx$ $\to \int\frac{1}{1+x} + 2\int\frac{1}{1+x^2} - 2\int\frac{1}{\sqrt{1-x^2}}\,dx$

$\downarrow \ln|1+x|$ $\quad \downarrow 2\arctan$ $\quad \downarrow 2\arcsin x$

**6.4.** $\int (2x+1)^3 dx$ $= \int 4x^3 + 8x + 2x + 4x^2 + 4x + 1\, dx$

$= 4\int x^3 + 12\int x\, dx + 4\int x^2 + \int dx = x^4 + 6x^2 + \frac{4}{3}x^3 + x + c$

$(2x+1)(4x^2+3x+1) = 8x^3 + 8x^2 + 2x + 4x^2 + 4x + 1 = \int(8x^3 + 12x^2 + 6x + 1)$

$= 2x^4 + 4x^3 + 3x^2 + x = x(2x^3 + 4x^2 + 3x + 1)$

**6.5.** $\int\left(\frac{3}{1-5x}+2\sqrt{3x+1}\right)dx$ $= 3\int\overset{a}{\frac{1}{1-5x}} + 2\int\overset{b}{(3x+1)^{\frac{1}{2}}}\,dx$

a) $-\frac{3}{5}\int\frac{-5}{1-5x}dx = -\frac{3}{5}\ln|1-5x|$ + b) $t = 3x+1 \Rightarrow dx = \frac{1}{3}dt \to \frac{2}{3}\int t^{\frac{1}{2}}dt = \frac{2}{3}\,\frac{2}{3}t^{\frac{3}{2}}$

$= \frac{4}{9}\sqrt{(3x+1)^3} = \frac{4}{9}(3x+1)\sqrt{3x+1}$

$\to -\frac{3}{5}\ln|1-5x| + \frac{4}{9}\sqrt{(3x+1)^3} + c$

**6.6.** $\int(3^x - 2^{3x+1} + 5e^{-7x})dx$ $= \int 3^x - \int 2^{3x+1} + 5\int e^{-7x}dx$

$a^{f(x)}\cdot f'(x) = \frac{a^{f(x)}}{\ln(a)}$ $\Rightarrow \int 3^x\cdot 1 = \frac{3^x}{\ln 3}$ $\quad \frac{1}{3}\int 2^{3x+1}\cdot 3 = \frac{1}{3}\frac{2^{3x+1}}{\ln(2)}$

$\frac{1}{7}\int e^{-7x}\cdot(-7) = -7e^{-7x}$ $\Rightarrow \int // dx = \frac{3^x}{\ln 3} - \frac{1}{3}\frac{2^{3x+1}}{\ln(2)} - \frac{5}{7}e^{-7x}$

**6.8.** $\int(\tan x + 3\cot x)dx$ $\to \int\overset{a}{\tan x} + 3\int\cot x\, dx$

a) $\tan x = \frac{\sin x}{\cos x} \Rightarrow \int\frac{\sin x}{\cos x}dx$ $\quad D[\cos x] = -\sin x \Rightarrow \int\tan x\, dx = -\ln|\cos x|$

b) $\tan x = \frac{\cos x}{\sin x} \Rightarrow \int\frac{\cos x}{\sin x}dx \to D[\sin x] = \cos x \Rightarrow \int\cot x\, dx = 3\ln|\sin x|$

$\Rightarrow \int // dx = \ln\left|\frac{3\sin x}{\cos x}\right| = \ln|3\tan x| + c$

$$\bullet \int \ln^2(x)\cdot D[x]\,dx = x\ln^2(x) - 2\int \cancel{x}\cdot\frac{1}{\cancel{x}}\ln x\,dx = x\ln^2(x) - 2\int \ln x\,dx = x\ln^2(x) - 2x\ln x + 2x$$

Parti: $x\ln x - x$

$$D[\ln^2 x] = 2\ln x\cdot\frac{1}{x}$$

**Esempio 6.1.** Calcolare l'integrale definito:

$$\int_2^3 \frac{x^3}{x^2+x-2}\,dx.$$

$$\int \frac{x^3}{x^2+x-2}\,dx \quad \begin{matrix} N \\ D \end{matrix} \quad \text{con } d(N) > d(D)$$

Dividiamo

Divisore

$$\begin{array}{l|l} \cancel{x^3} & x^2+x-2 \\ \hline \cancel{x^3}+\cancel{x^2}-2x & x-1 \\ \cancel{x^2}-x-2 & \\ 3x-2 & \end{array}$$

Quoz.

Resto

$$\Rightarrow \frac{x^3}{x^2+x-2} = Q + \frac{R}{D} = (x-1) + \frac{3x-2}{x^2-x-2}$$

$$\Rightarrow \int x - \int 1 + \int \frac{3x-2}{x^2-x-2}$$

$\frac{x^2}{2}$ $\quad -x$

Dobbiamo scomporre il denominatore

$$\Rightarrow x^2-x-2 = \quad \Delta = 1-4\cdot(-2) = 9$$

$$\Rightarrow x_{1/2} = \frac{1\pm 3}{2} \begin{cases} 2 \\ -1 \end{cases}$$

$$\Rightarrow x^2-x-2 = (x+x_0)(x+x_1) = (x+2)(x-1)$$

$$\Rightarrow \frac{3x-2}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)} = \frac{Ax-A+Bx+2B}{(x+2)(x-1)} = \frac{3x-2}{//}$$

$$\Rightarrow \begin{cases} A+B=3 \Rightarrow A=3-B \\ 2B-A=-2 \Rightarrow 2B-3+B=-2 \end{cases}$$

$$\Rightarrow 3B = -2+3 = \frac{1}{3}$$

B

$$\Rightarrow A = 3-\frac{1}{3} = \frac{9-1}{3} = \frac{8}{3}$$

A

$$\Rightarrow \frac{3x-2}{(x+2)(x-1)} = \frac{\frac{8}{3}}{(x+2)} + \frac{\frac{1}{3}}{(x-1)} = \frac{1}{3}\left(\frac{8}{x+2} + \frac{1}{x-1}\right)$$

$$\Rightarrow \frac{x^3}{x^2+x-2} = x-1+\frac{1}{3}\left(\frac{8}{x+2}+\frac{1}{x-1}\right) \Rightarrow \int // \,dx = \frac{x^2}{2} - x + \frac{8}{3}\int\frac{1}{x+2}\,dx + \frac{1}{3}\int\frac{1}{x-1}\,dx$$

$$= \frac{x^2}{2} - x + \frac{8}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + c$$

Integrale indefinito

$$\int_2^3 \frac{x^3}{x^2+x-2}\,dx = \left[\frac{x^2}{2} - x + \frac{8}{3}\ln|x+2| + \frac{1}{3}\ln|x-1|\right]_2^3 = \left[\frac{9}{2} - 3 + \frac{8}{3}\ln|5| + \frac{1}{3}\ln|2|\right] - \left[\frac{8}{3}\ln|4|\right] =$$

$3/2$

**Esempio 6.2.** Calcolare l'integrale indefinito:

$$\int \frac{x+5}{x^2+3x+4}\,dx.$$

$$\int \frac{x+5}{x^2+3x+4}\,dx \qquad N<D$$

$$x^2+3x+4 = \quad \Delta = 9-4\cdot 4 = -7<0$$

Irriducibile

$\Rightarrow$ Lo scopo è far uscire fuori la derivata del denominatore

$$D[x^2+3x+4] = 2x+3 \quad \Rightarrow \quad \frac{x+5}{x^2+3x+4} = \frac{1}{2}\,\frac{2x+3}{x^2+3x+4} + \frac{1}{2}\,\frac{(5\cdot 2)-3}{x^2+3x+4} = \underbrace{\frac{1}{2}\,\frac{2x+3}{x^2+3x+4}}_{\ln|x^2+3x+4|} + \overbrace{\frac{7}{2}\,\frac{1}{x^2+3x+4}}^{b}$$

$\frac{16}{4}=4$

b) $\frac{7}{2}\int \frac{1}{x^2+3x+4}\,dx$ Somma di quadrati

$$x^2+3x+4 = x^2+3x+\frac{9}{4}+\frac{7}{4}$$

$$\Rightarrow \left(x+\frac{3}{2}\right)^2+\frac{7}{4}$$

$$\Rightarrow \frac{7}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2+\frac{7}{4}}$$

Pongo $t = x+\frac{3}{2} \to dx = 1\,dt \Rightarrow \frac{7}{2}\int \frac{1}{\underbrace{t^2}_{x^2}+\underbrace{\frac{7}{4}}_{a^2 \Rightarrow a=\frac{\sqrt{7}}{2}}}\,dx = \frac{7}{2}\cdot\frac{1}{\frac{\sqrt{7}}{2}}\cdot \operatorname{arctg}\left(\frac{t}{\frac{\sqrt{7}}{2}}\right)$

$$\int \frac{1}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right)$$

$\frac{a}{b}\cdot\frac{d}{c}$

$\frac{c}{d}$

$$= \frac{7}{2}\cdot\frac{2}{\sqrt{7}}\cdot \operatorname{arctg}\left(\frac{t}{\frac{2}{2}\sqrt{7}}\right) = \frac{7}{\sqrt{7}}\operatorname{arctg}\left(\frac{\frac{2x+3}{2}}{\frac{\sqrt{7}}{2}}\right) = \frac{7}{\sqrt{7}}\left(\frac{2x+3}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{7}}\right)$$

$\frac{2x+3}{2}$

$$= \frac{\sqrt{7}}{\sqrt{7}}\,\frac{7}{\sqrt{7}}\operatorname{arctg}\left(\frac{2x+3}{\sqrt{7}}\right) = \sqrt{7}\operatorname{arctg}\left(\frac{2x+3}{\sqrt{7}}\right)$$

$$= \frac{\sqrt{7}\,\cancel{7}}{\cancel{7}} = \sqrt{7}$$

$$\Rightarrow \text{tutto} = \ln|x^2+3x+4| + \sqrt{7}\operatorname{arctg}\left(\frac{2x+3}{\sqrt{7}}\right)+c$$

**Esempio 6.3.** Calcolare l'integrale indefinito:

$$\int \frac{2x+1}{x^2+6x+9}dx.$$

$$= \int \frac{2x+1}{x^2+6x+9} \quad \begin{matrix} N \\ D \end{matrix} \qquad N<D \Rightarrow \text{NO DIVISIONE}$$

$$x^2+6x+9 = (x+3)^2 \Rightarrow \int \frac{2x+1}{(x+3)^2}dx \qquad \text{pongo } t = x+3 \Rightarrow dx = dt$$

$$\Rightarrow \int \frac{2x+1}{t^2}dt \Rightarrow 2x+1 = 2x+6-5 = 2(\underbrace{x+3}_{t})-5 \Rightarrow \int \frac{2t-5}{t^2}dt =$$

$$\frac{2t-5}{t^2} = \frac{2}{t} - \frac{5}{t^2} \Rightarrow \underbrace{\int \frac{2}{t}}_{2\ln|t|} - 5\int \frac{1}{t^2} = -5\int t^{-2}dt = +5\frac{1}{t} + c$$

$$\Rightarrow 2\ln|t| + \frac{5}{t} + c \underset{t=x+3}{=} 2\ln|x+3| + \frac{5}{x+3} + c$$

**Esempio 6.4.** Calcolare l'integrale indefinito:

$$\int \frac{1+2x^2}{x^4-1}dx.$$

$$\int \frac{1+2x^2}{x^4-1} \quad \begin{matrix} N \\ D \end{matrix} \qquad N<D \Rightarrow \underline{\text{No Div}}$$

POLINOMI IRRIDUCIBILI

$$x^4-1 = (x^2+1)\underbrace{(x^2-1)}_{\Rightarrow (x^2-1)=(x+1)(x-1)} = \Rightarrow (x^2+1)(x+1)(x-1)$$

$$\Rightarrow \frac{1+2x^2}{x^4-1} = \overbrace{\underbrace{\frac{A}{(x+1)} + \frac{Bx}{(x-1)}}_{(x^2-1)}}^{1^\circ} + \overbrace{\frac{Cx+D}{(x^2+1)}}^{1^\circ} = \frac{A\overbrace{[(x-1)(x^2+1)]}^{x^3-x^2+x-1} + Bx\overbrace{[(x+1)(x-1)]}^{(x^2-1)} + (Cx+D)[x^2-1]}{(x^2+1)(x^2-1)}$$

Pongo $B=-A$, $C=0$ $\Rightarrow$ $\frac{A}{(x+1)} - \frac{A}{(x-1)} + \frac{D}{(x^2+1)} = \frac{\cancel{Ax}+A-\cancel{Ax}+A}{(x^2-1)} + \frac{D}{(x^2+1)} = \frac{2A}{(x^2-1)} + \frac{D}{(x^2+1)}$

$$\Rightarrow \frac{2Ax^2+2A+Dx^2+D}{(x^2-1)(x^2+1)} = 1+2x^2 \Rightarrow \begin{cases} 2A+D=2 \Rightarrow D=2-2A \Rightarrow D=\frac{1}{2} \\ 2A-D=1 \Rightarrow 2A-2+2A=1 \Rightarrow A=\frac{3}{4} \Rightarrow B=-\frac{3}{4} \end{cases}$$

$$\Rightarrow \frac{3/2}{(x^2-1)} + \frac{1/2}{(x^2+1)} \Rightarrow \frac{3}{2}\int \underbrace{\frac{1}{(x^2-1)}}_{(x+1)(x-1)}dx + \frac{1}{2}\int \underbrace{\frac{1}{(x^2+1)}}_{\operatorname{atan}(x)}dx = \frac{3}{2}\int \frac{1}{(x+1)} - \int \frac{1}{(x-1)}dx = \frac{3}{4}\ln\left|\frac{x+1}{x-1}\right|$$

$$\Rightarrow \frac{3}{4}\ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2}\operatorname{atan}(x)$$

**6.11.★** $\displaystyle\int \frac{1+x}{x^2+3}dx$

$$\int \frac{1+x}{x^2+3}dx \quad \begin{matrix} N \\ D \end{matrix} \Rightarrow N<D = \int \frac{1}{x^2+3}dx + \frac{1}{2}\int \frac{2x}{x^2+3}dx = \underline{\frac{1}{2}\ln|x^2+3|}$$

$$+$$

$$a) \int \frac{1}{x^2+3}dx \Rightarrow \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\operatorname{atan}\left(\frac{x}{a}\right) \Rightarrow \underline{\frac{1}{\sqrt{3}}\operatorname{atan}\left(\frac{x}{\sqrt{3}}\right)}$$

**6.12.★** $\displaystyle\int_{-1}^{1} \frac{3x}{x^2+2x+5}dx$

$$\int \frac{3x}{x^2+2x+5}dx \qquad x^2+2x+5 \quad \text{Irriducibile}$$

$$\Rightarrow D = 2x+2$$

$$\Rightarrow \frac{3}{\cancel{3}}\cdot\frac{1}{2}\int \frac{\cancel{2}\cancel{3}x+2-2}{x^2+2x+5} = \frac{3}{2}\ln|x^2+2x+5| - \frac{2}{3}\int \frac{1}{x^2+2x+5} \Rightarrow \frac{3}{2}\ln|\ldots| - \frac{3}{\cancel{2}}\int \frac{\cancel{2}}{(x+1)^2+4}dx$$

$$x^2+2x+1 = (x+1)^2+4$$

$$= \frac{3}{2}\ln|x^2+2x+5| - 3\int \frac{1}{\underbrace{(x+1)^2}_{f^2}+\underbrace{4}_{a^2}}\,dx = \frac{3}{2}\ln|x^2+2x+5| - 3\cdot\frac{1}{2}\,\mathrm{atan}\left(\frac{x+1}{2}\right)$$

$$= \left[\frac{3}{2}\ln|x^2+2x+5| - \frac{3}{2}\,\mathrm{atan}\left(\frac{x+1}{2}\right)\right]_{-1}^{1} = \left[\frac{3}{2}\ln|8| - \frac{3}{2}\,\frac{\pi}{4}\right] - \left[\frac{3}{2}\ln|4|\right] = \frac{3}{2}\ln|8| - \frac{6-\pi}{4} - \frac{3}{2}\ln 4$$

$$= \frac{3}{2}\ln|2| - \frac{3}{8}\pi \quad \checkmark$$

**6.13.★** $\displaystyle\int \frac{x^2}{x^2+3x+2}\,dx$

$$\int \frac{x^2}{x^2+3x+2}\,dx \qquad \Delta = 9-4\cdot 2 = 1 \to \qquad x_{1,2} = \frac{-3\pm 1}{2} \begin{cases} -1 \\ -2 \end{cases}$$

$$x^2+3x+2 = (x+1)(x+2)$$

$$\begin{array}{l|l} \cancel{x^2} & x^2+3x+2 \\ \cancel{x^2}+3x+2 & 1 \\ -3x-2 & \end{array}$$

$$\frac{x^2}{x^2+3x+2} = 1 - \frac{3x-2}{x^2+3x+2} = -\int dx - \int \frac{3x-2}{(x+1)(x+2)}\,dx$$

$$-x$$

$$\to \frac{3x-2}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{Ax+2A+Bx+B}{//} = 3x-2 \to \begin{cases} A+B=3 \to A = 3-B \\ 2A+B=-2 \to 6-2B+B=-2 \\ \qquad \to B = 8 \end{cases}$$

$$A = -5$$

$$\to -x + 5\int \frac{1}{x+1} - 8\int \frac{1}{x+2}\,dx = -x + 5\ln|x+1| - 8\ln|x+2|$$

**6.14.★** $\displaystyle\int \frac{x^2-3}{x^2+3x+3}\,dx$

$$\int \frac{x^2-3}{x^2+3x+3}\,dx \qquad \begin{matrix} N \\ D \end{matrix} \qquad N = D \to \text{Div}$$

$$\to \begin{array}{l|l} \cancel{x^2} - 3 & x^2+3x+3 \\ \cancel{x^2}+3x+3 & 1 \\ -3x-6 & \end{array} \qquad 1 - \frac{3x-6}{x^2+3x+3} = 1 - 3\,\frac{x+2}{x^2+3x+3} = \int dx - 3\int \frac{x+2}{x^2+3x+3}\,dx$$

$$D[x^2+3x+3] = 2x+3 \to \int dx - \frac{3}{2}\int \frac{\overbrace{2x+2+3}^{1}-3}{x^2+3x+3}\,dx = \underbrace{\int dx}_{x} - \frac{3}{2}\underbrace{\int \frac{2x+3}{x^2+3x+3}\,dx}_{\frac{3}{2}\ln|x^2+3x+3|} - \frac{3}{2}\int \frac{\overset{a}{1}}{x^2+3x+3}\,dx$$

a) $\displaystyle -\frac{3}{2}\int \frac{1}{x^2+3x+3}\,dx \qquad \int \frac{1}{ax^2+b} = \frac{\mathrm{atan}\left(\sqrt{\frac{a}{b}}\,x\right)}{\sqrt{ab}}$ BOH

Integ. Per Parti

**Esempio 6.5.** Calcolare l'integrale definito:

$$\int_0^{\frac{\pi}{3}} x^2 \sin x \, dx.$$

$$\int \overset{f}{x^2} \overset{g}{\sin x} = -x^2 \cos x + 2\int x \cos x \, dx$$

$$= -x^2 \cos x + 2x \sin x - 2\int \sin x \, dx$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x$$

$$= \left[\cos x (2 - x^2) + 2x \sin x\right]_0^{\frac{\pi}{3}} = \left[\underbrace{\cos\left(\frac{\pi}{3}\right)}_{1/2}\left(2 - \frac{\pi^2}{9}\right) + 2\frac{\pi}{3}\sin\left(\frac{\pi}{3}\right)\right] - \left[\underbrace{\cos(0)}_{1}(2) + 0\right] = -1 - \frac{\pi^2}{18} + \frac{\pi}{\sqrt{3}}$$

**Esempio 6.6.** Calcolare l'integrale indefinito:

$$\int e^{-x} \sin x \cos x \, dx.$$

$$\int e^{-x} \sin x \cos x \, dx$$

Siccome $2 \sin x \cos x = 2 \sin(x)$

$$\Rightarrow \sin x \cos x = \frac{1}{2}\sin(2x)$$

$$\Rightarrow \frac{1}{2}\int e^{-x} \sin(2x) = \frac{1}{2}\left\{-\frac{1}{2}e^{-x}\cos(2x) - \frac{1}{2}\int e^{-x}\cos(2x)\,dx\right\} = \frac{1}{2}\left\{-\frac{1}{2}e^{-x}\cos(2x) - \frac{1}{2}\left[\frac{1}{2}e^{-x}\sin(2x) + \frac{1}{2}\int e^{-x}\sin(2x)\,dx\right]\right\}$$

$$= \frac{1}{2}\left\{-\frac{1}{2}e^{-x}\cos(2x) - \frac{1}{4}e^{-x}\sin(2x) - \frac{1}{4}\int e^{-x}\sin(2x)\,dx\right\} = \int e^{-x}\sin(2x)\,dx$$

$$= -\frac{1}{4}e^{-x}\cos(2x) - \frac{1}{8}e^{-x}\sin(2x) - \frac{1}{8}\int e^{-x}\sin(2x)\,dx = \int e^{-x}\sin(2x)\,dx$$

$\rightarrow$