# 4A. Equazioni differenziali lineari del primo ordine

Un'equazione differenziale lineare di I ordine è un' eq del tipo:

$y' = b(x)$ ed ha soluzione $y = B(x) + c$, infatti ci basta integrare per avere:

$\int y' = \int b(x)\, dx \Rightarrow y = B(x)$

Primitiva

Un altro esempio è l'eq:

$y' = y$ → Soluzioni → $ce^x$, con $c \in \mathbb{R}$, infatti:

$\int y' = \int y\, dx \Rightarrow y = ce^x$

Se invece $y(x)$ è una soluzione di $y' = y(x)$ otteniamo:

$y' - y(x) = 0$ Se moltiplichiamo per $e^{-x}$ otteniamo:

$e^{-x} y'(x) - e^{-x} y(x) = 0$, ovvero $\frac{d}{dx}[e^{-x} y(x)] = 0$

Infatti: $D(e^{-x} y(x)) = -e^{-x} y(x) + e^{-x} \cdot y'(x) = e^{-x} y'(x) - e^{-x} y(x) = 0$

Ne consegue che $e^{-x} y(x) = c \Rightarrow y(x) = ce^{x}$

Formula generale dell'eq lineare $y' = a(x)y + b(x)$:

$y(x) = e^{A(x)} \cdot \left[ \int e^{-A(x)}\, b(x)\, dx \right]$

**4.4** Risolvere l'equazione differenziale lineare omogenea $y' = 8xy$.

$y' - 8xy = 0$ ; $a(x)$

1) Omogenea associata: $y' - 8xy = 0$

2) Var sep: $\frac{y'}{y} - 8x = 0$ ; $\frac{y'}{y} = 8x$

3) Integro: $\int \frac{y'}{y} = \int 8x \, dx = \ln|y| = 4x^2 + c_1 \Rightarrow e^{\ln|y|} = e^{4x^2} \cdot e^{c_1} \Rightarrow y(x) = ce^{4x^2}$

**4.5** Risolvere l'equazione differenziale lineare omogenea

$$y' = \frac{x}{x^2+1}y$$

$\frac{y'}{y} = \frac{x}{x^2+1} \Rightarrow \int \frac{y'}{y} = \int \frac{x}{x^2+1} dx \Rightarrow \ln|y| = \int \frac{x}{x^2+1} dx$

Pongo $t = x^2+1 \Rightarrow dx = \frac{1}{2x} dt \Rightarrow$ $\ln|y| = \int \frac{\cancel{x}}{x^2+1} \cdot \frac{1}{2\cancel{x}} dt = \ln|y| = \frac{1}{2}\int \frac{1}{t} dt \Rightarrow \ln|y| = \frac{1}{2}\ln|t| + c_1$

$\Rightarrow \ln|y| = \frac{1}{2}\ln|x^2+1| + c_1 \Rightarrow e^{\frac{1}{2}\ln|x^2+1|} \cdot e^{c_1} \Rightarrow y(x) = c\sqrt{x^2+1}$

**4.7** Determinare l'integrale generale delle seguenti equazioni differenziali lineari *omogenee*

$y' = 3y \Rightarrow \frac{y'}{y} = 3 \Rightarrow \int \frac{y'}{y} dx = 3\int dx = \ln|y| = 3\frac{x^2}{2} + c_1 \Rightarrow e^{\ln|y|} = e^{3x+c_1}$

$\Rightarrow y(x) = ce^{3x}$

$y' = 2xy \Rightarrow \frac{y'}{y} = 2x \Rightarrow \int \frac{y'}{y} dx = \int 2x \, dx \Rightarrow \ln|y| = 2\frac{x^2}{2} \Rightarrow y(x) = e^{x^2+c_1} \Rightarrow y(x) = ce^{x^2}$

$y' = (x-1)y\frac{1}{x} \Rightarrow \frac{y'}{y} = \frac{(x-1)}{x} \Rightarrow \frac{y'}{y} = 1 - \frac{1}{x} \Rightarrow \int \frac{y'}{y} dx = \int 1 - \frac{1}{x} dx \Rightarrow \ln|y| = \int dx - \int \frac{1}{x} dx$

$\Rightarrow \ln|y| = x - \ln|x| + c \Rightarrow y(x) = e^x \cdot \frac{1}{e^{\ln x}} \cdot e^{c_1} \Rightarrow y(x) = \frac{1}{x} \cdot e^x \cdot c = \frac{ce^x}{x}$

$y' = (\cos x)y \Rightarrow \frac{y'}{y} = \cos x \Rightarrow \int \frac{y'}{y} dx = \int \cos x \Rightarrow \ln|y| = \sin x + c \Rightarrow y(x) = ce^{\sin x}$

$y' = -e^x y \Rightarrow \frac{y'}{y} = -e^x \Rightarrow y(x) = -\int e^x dx \Rightarrow y(x) = ce^{-e^x}$

$y' = 2xe^{x^2}y \Rightarrow \frac{y'}{y} = 2xe^{x^2} \Rightarrow \ln|y| = 2\int xe^{x^2}$ pongo $t = e^{x^2} \Rightarrow dx = \frac{1}{e^{x^2}2x} dt$

$\Rightarrow \ln|y| = \cancel{2}\int \cancel{x}\cancel{e^{x^2}} \cdot \frac{1}{\cancel{e^{x^2}} \cdot \cancel{2x}} dt \Rightarrow \ln|y| = \int dt \Rightarrow y(x) = ce^t \Rightarrow y(x) = ce^{e^{x^2}}$

$y' = (\mathrm{tg}\,x)y \Rightarrow \frac{y'}{y} = \mathrm{tg}\,x \Rightarrow \int \frac{y'}{y} = \int \mathrm{tg}\,x \, dx \Rightarrow \ln|y| = \int \frac{\sin x}{\cos x} dx$ pongo $t = \cos x \Rightarrow dx = -\frac{1}{\sin x} dt$

$\Rightarrow \ln|y| = -\int \frac{\cancel{\sin x}}{\cos x} \cdot \frac{1}{\cancel{\sin x}} dt \Rightarrow \ln|y| = -\int \frac{1}{t} dt \Rightarrow \ln|y| = c_1 - \ln|\cos x| \Rightarrow y(x) = \frac{c}{\cos x}$

$y' = -y/2x \Rightarrow \frac{y'}{y} = -\frac{1}{2x} \Rightarrow \ln|y| = -\frac{1}{2}\int \frac{1}{x} dx \Rightarrow \ln|y| = \ln|x|^{-\frac{1}{2}} + c = \frac{c}{\sqrt{x}}$

$y' = \frac{2y}{x} \Rightarrow \frac{y'}{y} = \frac{2}{x} \Rightarrow y(x) = cx^2$

$y' = -\mathrm{cotg}\,x \Rightarrow$ Come faccio senza $y$?

$y' = \sqrt{x}\,y \Rightarrow \frac{y'}{y} = \sqrt{x} \Rightarrow \ln|y| = \int x^{\frac{1}{2}} dx \Rightarrow y(x) = e^{\frac{2}{3}x\sqrt{x}}$

$y' = y \cdot \frac{1}{\sqrt{x+5}} = \int \frac{y'}{y} = \int \frac{1}{\sqrt{x+5}} dx \to \int (x+5)^{-\frac{1}{2}} dx \Rightarrow 2(x+5)^{\frac{1}{2}} + c = 2\sqrt{x+5} \Rightarrow y(x) = ce^{2\sqrt{x+5}}$

$y' = y \log x \cdot \frac{1}{x} = \ln|y| = \int \frac{\log x}{x} dx$ Pongo $\log x = t \Rightarrow dx \frac{1}{\frac{1}{x}} dt \Rightarrow dx = x\,dt$

$\Rightarrow \ln|y| = \int \frac{\log x}{\cancel{x}} \cdot \cancel{x} dt = \int t \, dt = \frac{t^2}{2} + c \Rightarrow y(x) = ce^{\frac{1}{2}\log x^2}$

$$y' = xy \cdot \frac{1}{x^2-1} \Rightarrow \ln|y| = \int \frac{x}{x^2-1}\,dx \quad \text{pongo } t = x^2-1 \to dx = \frac{1}{2x}\,dt$$

$$\Rightarrow \ln|y| = \frac{1}{2}\int \frac{\cancel{x}}{x^2-1}\cdot\frac{1}{\cancel{x}}\,dt = \frac{1}{2}\int\frac{1}{t}\,dt \Rightarrow \ln|y| = \ln(x^2-1)^{\frac{1}{2}} + c \Rightarrow y(x) = c\sqrt{x^2-1}$$

Integrazione per parti

$$y' = (1+\ln x)y \Rightarrow y(x) = \int dx + \int \ln x\,dx = \quad \Rightarrow \int f(x)\cdot g'(x)\,dx = f(x)g(x) - \int f'(x)\cdot g(x)$$

$$\Rightarrow \int \overset{g'}{1}\cdot \overset{f(x)}{\ln(x)}\,dx \Rightarrow \ln(x)x - \int\frac{1}{\cancel{x}}\,\cancel{x}\,dx = x\ln x - x + c$$

$$\Rightarrow \ln|y| = x + x\ln x - x \Rightarrow y(x) = e^{c\cdot\cancel{x}}\cdot e^{x\ln x}\cdot\frac{1}{\cancel{e^x}} = c\cancel{e}^{\ln x^x} \Rightarrow y(x) = cx^x$$

$$y' = y\cdot\frac{1}{\sin(x+1)} \Rightarrow \ln y = \int\frac{1}{\sin(x+1)}\,dx \quad \text{pongo } t = x+1$$

$$\Rightarrow \ln|y| = \int\frac{1}{\sin t}\cdot\frac{\sin t}{\sin t}\,dt = \int\frac{\sin t}{\sin^2 t}\,dt \qquad \cos^2+\sin^2 = 1 \Rightarrow \sin^2 = 1-\cos^2$$

$$\Rightarrow \int\frac{\sin t}{1-\cos^2 t}\,dt = \text{Pongo } \gamma = \cos x \Rightarrow dx = \frac{1}{-\sin t}\,d\gamma = -\int\frac{\cancel{\sin t}}{1-\cos^2 t}\cdot\frac{1}{\cancel{\sin t}}\,d\gamma = -\int\frac{1}{1-\gamma^2}\,d\gamma$$

$$= \int\frac{1}{\gamma^2-1^2}\,dx \Rightarrow \int\frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\cdot\ln\left|\frac{x-a}{x+a}\right| = \frac{1}{-2}\ln\left|\frac{\gamma+1}{\gamma-1}\right| + c$$

$$\Rightarrow \ln|y| = -\frac{1}{2}\ln\left|\frac{\gamma+1}{\gamma-1}\right| + c_1 \Rightarrow y(x) = ce^{\ln\left|\frac{\gamma+1}{\gamma-1}\right|^{-\frac{1}{2}}} = \frac{c}{\sqrt{\frac{\gamma-1}{\gamma+1}}}$$

$$y' = xy\sin x \Rightarrow \ln|y| = \int \overset{f}{x}\,\overset{g'}{\sin x} = f\cdot g - \int f'\cdot g \Rightarrow -x\cos x + \int\cos x\,dx = -x\cos x + \sin x + c$$

$$\Rightarrow y(x) = e^{-x\cos x}\cdot e^{\sin x}\,c = \frac{ce^{\sin x}}{e^{x\cos x}}$$

$$y' = -\sin 2x\;y \Rightarrow \ln|y| = -\int\sin(2x)\,dx \Rightarrow \text{pongo } t = 2x \Rightarrow dx = \frac{1}{2}\,dt \Rightarrow \ln|y| = \frac{1}{2}\cos(2x)$$

$$\Rightarrow y(x) = ce^{\frac{1}{2}\cos(2x)}$$

$$y' = \mathrm{arcTg}\,x\;y \Rightarrow \mathrm{arcTg}\,x = \mathrm{Tan}^{-1}x = \left(\frac{\sin x}{\cos x}\right)^{-1} = \frac{\cos x}{\sin x} \Rightarrow \ln|y| = \int\frac{\cos x}{\sin x}\,dx \quad \text{pongo } t = \sin x$$

$$\Rightarrow dx = \frac{1}{\cos x}\,dt \Rightarrow \int\frac{\cancel{\cos x}}{\sin x}\cdot\frac{1}{\cancel{\cos x}}\,dt = \int\frac{1}{t}\,dt = \sin x + c \Rightarrow \ln|y| = ce^{\sin x}$$

Sbaglio qualcosa