5.6 Calcolare, in base alla definizione di integrale definito, l'integrale
$$\int_0^1 x\,dx\,.$$

$$\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \left[\frac{1^2}{2}\right] - \left[\frac{0}{2}\right] = \frac{1}{2}$$

5B. Calcolo di integrali definiti

5.12 Calcolare l'integrale
$$\int_0^{2\pi} \operatorname{sen}^2 x\,dx$$

$$\int_0^{2\pi} \sin^2 x\,dx = \int \underset{f}{\sin x}\ \underset{o}{\sin x}\,dx$$
$$= -\sin x\cos x + \int \cos^2 x\,dx = -\sin x\cos x + \int 1-\sin^2 x$$
$$= -\sin x\cos x + \int dx - \int \sin^2 x\,dx = \int \sin^2 x\,dx$$

$$I = \sin^2 x \Rightarrow \frac{-\sin x\cos x + x}{2} = I \Rightarrow \int \sin^2 dx = \frac{1}{2}x - \frac{\sin 2x}{4}$$
$$\Rightarrow \left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_0^{2\pi} = [\pi - 0] - \left[-\frac{1}{4}\right] = \pi + \frac{1}{4} =$$

5.13 Traendo spunto dall'esercizio precedente, utilizzare il significato geometrico di integrale definito per verificare che
$$\int_0^{2\pi} \operatorname{sen}^2 x\,dx = \int_0^{2\pi} \cos^2 x\,dx = \pi$$

1) $$\int_0^{2\pi} \sin^2 x\,dx \to \int \sin x\cdot\sin x\,dx$$
$$= -\sin x\cos x + \int \cos^2 x\,dx$$
$$= -\sin x\cos x + \int dx - \int \sin^2 x\,dx = \int \sin^2 x$$ pongo $I = \int \sin^2 x \to -\sin x\cos x + x - I = I$
$$\Rightarrow \int \sin^2 x\,dx = \frac{-\sin x\cos x + x}{2} \Rightarrow \left[\frac{1}{2}x - \frac{1}{2}\sin x\cos x\right]_0^{2\pi} = \pi$$

2) $$\int_0^{2\pi} \cos^2 x\,dx = \int \cos x\cos x\,dx = \cos x\sin x + \int \sin^2 x\,dx = \cos x\sin x + \int dx - \int \cos^2 x\,dx$$
$$= \frac{\cos x\sin x + x}{2} = I \Rightarrow \left[\frac{1}{2}\cos x\sin x + \frac{1}{2}x\right]_0^{2\pi} = \pi - [0] = \pi$$

5.14 Utilizzare il significato geometrico di integrale definito per verificare che
$$\int_0^{\pi} \operatorname{sen}^2 x\,dx = \int_0^{\pi} \cos^2 x\,dx = \frac{\pi}{2}\,,$$
$$\int_0^{\pi/2} \operatorname{sen}^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}\,.$$

a) $$\int_0^{\pi} \sin^2 x\,dx = \frac{-\sin x\cos x + x}{2}$$
$$\int_0^{\pi} \cos^2 x\,dx = \frac{\sin x\cos x + x}{2}$$

$$\Rightarrow \left[-\frac{1}{2}\sin x\cos x + \frac{1}{2}x\right]_0^{\pi} = \left[\frac{1}{2}\sin x\cos x + \frac{1}{2}x\right]_0^{\pi} = \frac{\pi}{2} = \frac{\pi}{2}$$

b) $$\int_0^{\pi/2} \sin^2 x = \int_0^{\pi/2} \cos^2 x\,dx = \left[-\frac{1}{2}\sin x\cos x + \frac{1}{2}x\right]_0^{\pi/2} = \left[\frac{1}{2}\sin x\cos x + \frac{1}{2}x\right]_0^{\pi/2} = -\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}\pi = \left[\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}\pi\right]$$

$$\to \frac{1}{4}\pi = \frac{1}{4}\pi$$

5.16 Calcolare l'integrale

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$$

$\int \frac{x}{\sqrt{x+1}}\,dx$ Pongo $t=\sqrt{x+1} \rightarrow t^2 = x+1 \rightarrow x = t^2-1$
$dx = 2t\,dt$

$$\rightarrow \int \frac{t^2-1}{t}\cdot 2t\,dt = 2\int t^2\,dt - 2\int dt$$

$$= \frac{2t^3}{3} - 2t + c$$

$t=\sqrt{x+1}$

$$\Rightarrow \frac{2}{3}\cdot\sqrt{x+1}\,(x+1) - 2\sqrt{x+1} = 2\sqrt{x+1}\left(\frac{1}{3}(x+1) - 1\right)$$

$$\Rightarrow \left[\frac{2}{3}\sqrt{(x+1)^3} - 2\sqrt{x+1}\right]_0^3 = \frac{2}{3}\sqrt{4^3} - 4 - \left[\frac{2}{3} - 2\right] = \frac{16}{3} - 4 - \frac{2}{3} + 2 = \frac{16-12-2+6}{3} = \frac{8}{3}$$

5.17 Calcolare l'integrale

$$\int_0^3 |x-1|\,dx$$

$\int_0^3 |x-1|\,dx = \quad x-1 \neq 0$ per $x \neq 1$

$$\Rightarrow \int_0^1 -(x-1) + \int_1^3 x-1 = \int_0^1 1-x\,dx + \int_1^3 x-1\,dx$$

$$\rightarrow \int_0^1 dx - \int_0^1 x\,dx + \int_1^3 x\,dx - \int_1^3 dx = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^3 = \left(1-\frac{1}{2}\right) + \left[\left(\frac{9}{2}-3\right) - \left(\frac{1}{2}-1\right)\right]$$

$$= \frac{1}{2} + \frac{3}{2} + \frac{1}{2} = \frac{5}{2}$$

5.19 Posto $f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, verificare che

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{120}(b-a)^4\,.$$

$$\int x^4 + a_3x^3 + a_2x^2 + a_1x + a_0\,dx = \int x^4 + a_3\int x^3 + a_2\int x^2 + a_1\int x + a_0\int dx$$

$$= \frac{x^5}{5} + a_3\frac{x^4}{4} + a_2\frac{x^3}{3} + a_1\frac{x^2}{2} + a_0x + c =$$

5.20 Calcolare i seguenti integrali

(a) $\displaystyle\int_0^2 \frac{x^3+3x^2}{x^2+4x+4}\,dx$ (b) $\displaystyle\int_0^1 \frac{x^3-13x}{x^2+5x+4}\,dx$

(c) $\displaystyle\int_0^1 \frac{16x^4-3}{4x^2+1}\,dx$ (d) $\displaystyle\int_0^2 \frac{x}{(x^2+2)^3}\,dx$

a) $\displaystyle\int_0^2 \frac{x^3+3x^2}{x^2+4x+4}\,dx = \int \frac{x^3}{x^2+4x+4} + 3\int \frac{x^2}{x^2+4x+4}$

$$\begin{array}{l|l} x^3+3x^2 & x^2+4x+4 \\ \hline x^3+4x^2+4x & x-1 \\ // \; -x^2-4x & \\ -x^2-4x-4 & \\ // \; // \; 4 & \end{array}$$

$$\Rightarrow (x-1) + \frac{4}{x^2+4x+4} \rightarrow x^2+4x+4 = (x+2)^2$$

$$\Rightarrow (x-1) + \frac{4}{(x+2)^2}$$

$$\Rightarrow \int x\,dx - \int dx + 4\int \frac{1}{(x+2)^2}\,dx$$

pongo $t = x+2$, $x = t-2$ $\rightarrow dx = dt$ $\rightarrow \int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = -\frac{1}{t}$

$$= \frac{x^2}{2} - x - \frac{4}{x+2} = \left[\frac{x^2}{2} - x - \frac{4}{x+2}\right]_0^2 = (2-2-1) - (0-0-2) = -1+2 = 1$$

b) $\displaystyle\int_0^1 \frac{x^3-13x}{x^2+5x+4}\,dx$

$$\begin{array}{l|l} x^3-13x & x^2+5x+4 \\ \hline x^3+5x^2+4x & x-5 \\ // \; -5x^2-9x & \\ -5x^2-25x-20 & \\ \hline // \; 16+20=36 & \end{array}$$

$$\rightarrow (x-5) + \frac{36}{x^2+5x+4} = (x+1)(x+4) = x^2+4x+x+4$$

$$\frac{36}{x^2+5x+4} = \frac{36}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4} = Ax+4A+Bx+B = 36 \Rightarrow \begin{cases} A+B=0 \rightarrow A=-B \Rightarrow A=12 \\ 4A+B=36 \rightarrow -4B+B=36 \\ \rightarrow B=-12 \end{cases}$$

$$\Rightarrow \frac{12}{x+1} - \frac{12}{x+4} \Rightarrow \int x\,dx - 5\int dx + 12\int \frac{1}{x+1} - 12\int \frac{1}{x+4}\,dx = \left[\frac{x^2}{2} - 5x + 12\ln|x+1| - 12\ln|x+4|\,dx\right]_0^1$$

$$= \left[\frac{1}{2} - 5 + 12\ln|2| - 12\ln|5|\right] - \left[12\ln|1| - 12\ln|4|\right]$$

c) $$\int_0^1 \frac{16x^4-3}{4x^2+1}\,dx \quad \frac{Px}{Qx} \quad Px>Qx \rightarrow$$

$$\begin{array}{r|l} 16x^4-3 & 4x^2+1 \\ \hline 16x^4+4x^2 & 4x^2-1 \\ \hline -4x^2-3 & \\ -4x^2-1 & \\ \hline -2 & \end{array} \rightarrow 4x^2-1 - \frac{2}{4x^2-1}$$

$$\Rightarrow 4x^2-1 = (2x-1)(2x+1) \Rightarrow \frac{2}{4x^2-1} = \frac{A}{2x-1} + \frac{B}{2x+1} = 2Ax+A+2Bx-B \Rightarrow \begin{cases} 2A+2B=0 \\ A-B=2 \end{cases}$$

$$\Rightarrow 2(A+B)=0\ ;\ A=-B \quad \Rightarrow -B-B=2\ ;\ B=-1\ ;\ A=1$$

$$\Rightarrow \frac{2}{4x^2-1} = \frac{1}{2x-1} - \frac{1}{2x+1} \Rightarrow 4\int x^2 - \int dx - \int \frac{1}{2x-1}\,dx - \int \frac{1}{2x+1}\,dx = 4\frac{x^3}{3} - x - \ln|2x-1| - \ln|2x+1|$$

$$\Rightarrow \left[\frac{4}{3}x^3 - x - \ln|2x-1| - \ln|2x+1|\right]_0^1 = \left(\frac{4}{3} - 1 - \ln|1| - \ln|3|\right) - \left(-\ln|-1| - \ln|1|\right) =$$

5.21 Verificare che, se m ed n sono due interi non negativi, si ha

$$\int_0^\pi \operatorname{sen} mx \operatorname{sen} nx\,dx = \begin{cases} 0 & \text{se} \quad m \neq n \\ \pi/2 & \text{se} \quad m = n > 0 \end{cases}$$

$$\int \underbrace{\sin(mx)}_{f}\ \underbrace{\sin(nx)}_{g}\,dx = -\sin(mx)\cos(nx) + \int \cos(nx)\cos(mx)\,dx$$