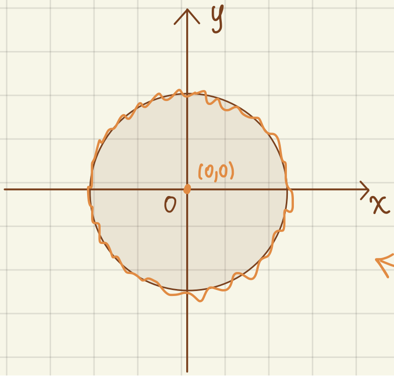
3B. Insiemi di definizione

(a) $z = \log\left(1 - x^2 - y^2\right)$

$\log(\text{arg}) \to \text{arg} > 0 \qquad 1-x^2-y^2>0 \to x^2+y^2<1$

Prendo $\overset{x\ y}{(0,0)} \to 1-0-0>0$? SI

⇒ Soluzione:

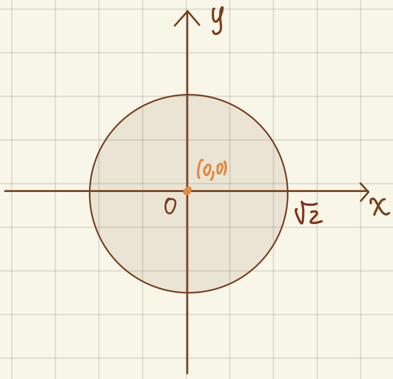


(b) $z = \sqrt{2 - x^2 - y^2}$

$2-x^2-y^2 \geq 0 \to x^2+y^2 = 2$

⇒ Scelgo $\overset{x\ y}{(0,0)} \to 2-0-0 \geq 0 \to 2 \geq 0$ SI

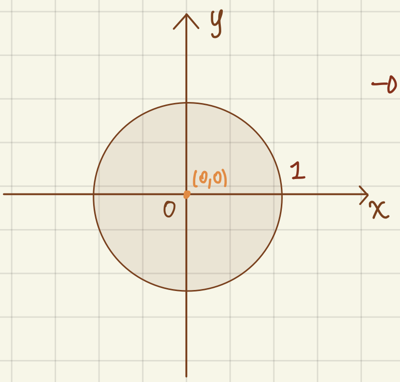
Cerchio compresa circonferenza



(c) $z = \log\left(x^2 + y^2 - 1\right)$

$x^2+y^2-1>0 \to x^2+y^2=1$

→ $0+0-1>0$ NO → f definita per valori esterni al cerchio, circ esclusa



(d) $z = \sqrt{-|x^2 + y^2 - 2|}$

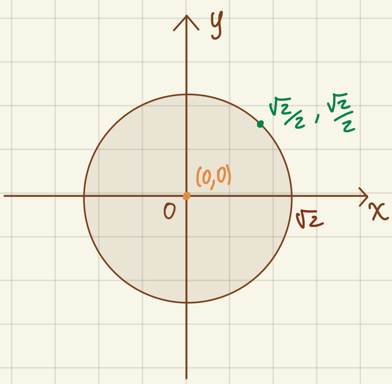
$x^2+y^2-2 \leq 0 \to x^2+y^2=2 \to$ Circ ⇒ $x^2+y^2=c^2$

⇒ $x^2+y^2=\sqrt{2}$

mai <0 ⇒ def per $x^2+y^2=2$

Prendo $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \to \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=2$ SI

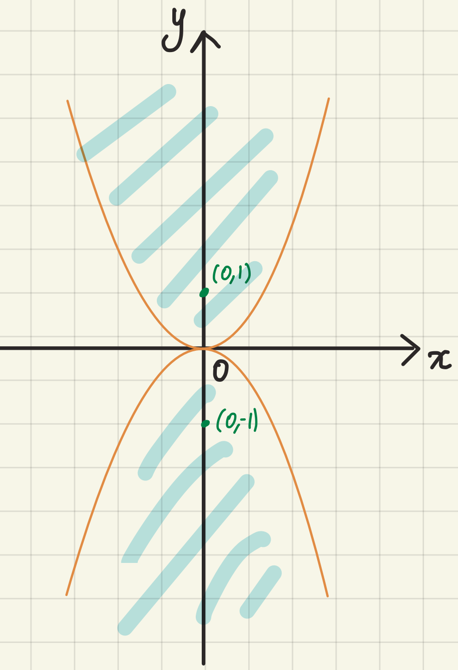
f definita sulla circonferenza di $c=(0,0)$ e $r=\sqrt{2}$



3.10 Rappresentare graficamente in un piano cartesiano x, y gli insiemi di definizione delle funzioni

(a) $f(x,y) = \sqrt{y^2 - x^4}$ (b) $f(x,y) = \sqrt{x^4 - y^2}$

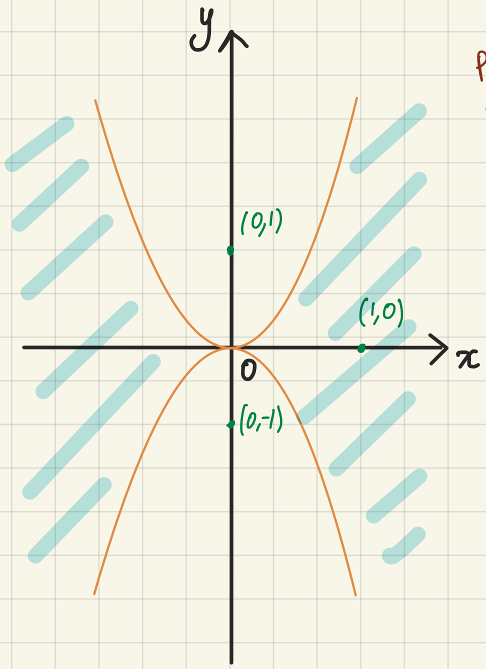
a) $y^2 - x^4 \geq 0 \rightarrow y = \pm x^2$



per $y = x^2$, prendo $(0,1) \rightarrow 1 \geq 0$? SI
Per $y = -x^2$, prendo $(0,-1) \rightarrow 1 \geq 0$? SI

$\cup$

b) $x^4 - y^2 \geq 0$ per $-y^2 = -x^4 \rightarrow y^2 = x^4 \rightarrow y = \pm x^2$



Per $y = x^2$ prendo $(0,1) \rightarrow -1 \geq 0$? NO
prendo $(1,0) \rightarrow 1 \geq 0$? SI

Per $y = -x^2$ prendo $(1,0) \rightarrow 1 \geq 0$? SI

3.11 Rappresentare graficamente l'insieme di definizione delle funzioni

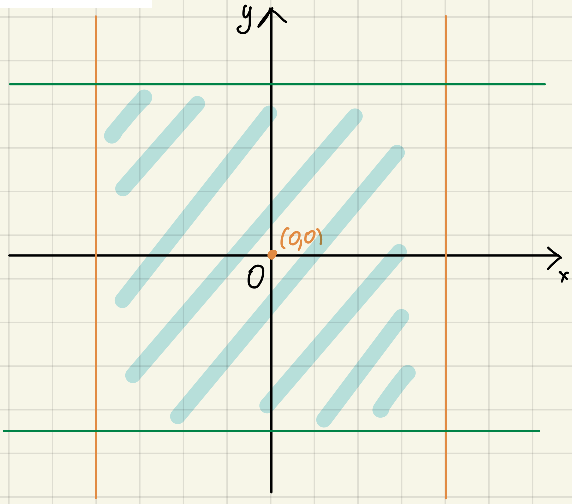
(a) $z = \log(1 - x^2) + \log(1 - y^2)$ (b) $z = \log \dfrac{1 - x^2}{1 - y^2}$

(c) $z = \log(x^2 - 1) + \log(1 - y^2)$ (d) $z = \log \dfrac{x^2 - 1}{1 - y^2}$

a) $\underset{a}{1 - x^2 > 0} \cup \underset{b}{1 - y^2 > 0}$

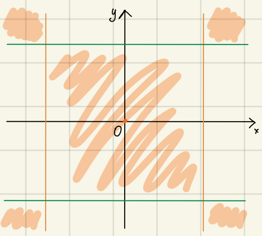
a) $x^2 = 1$ $\hookrightarrow x = \pm 1$ b) $y^2 = 1$ $\hookrightarrow y = \pm 1$

a) $1 - 0 > 0$? Si
b) $1 - y > 0$? Si

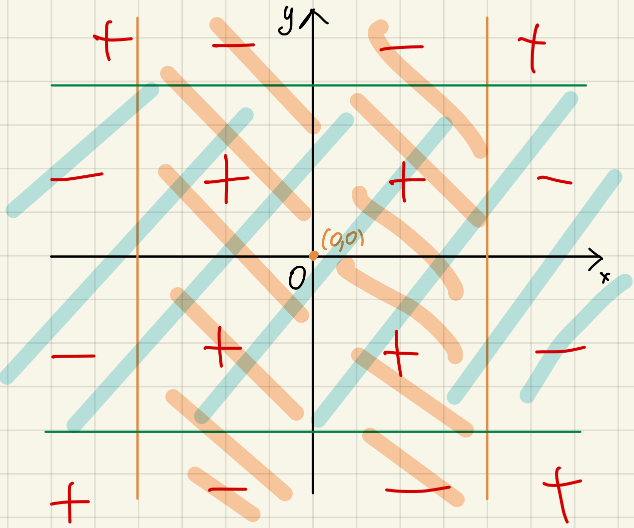


B) $\begin{cases} x^2 - 1 > 0 \to x = \pm 1 \\ 1 - y^2 > 0 \to y = \pm 1 \end{cases}$ stesso disegno ↑

Prendo $O(0,0) \to$ $\begin{cases} 0 - 1 > 0 ? \text{ NO} \\ 1 - 0 > 0 ? \text{ Si} \end{cases}$

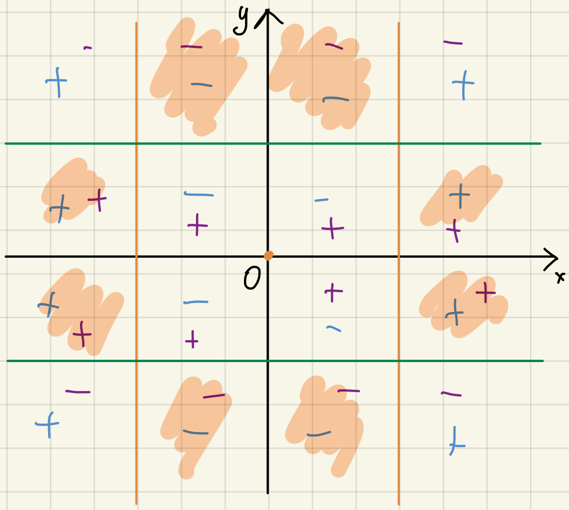


Soluz



C) $\begin{cases} x^2 - 1 > 0 \to x = \pm 1 \\ 1 - y^2 > 0 \to y = \pm 1 \end{cases}$

Prendo $(0,0) \to$ $\begin{cases} 0 - 1 > 0 ? \text{ NO} \\ 1 - 0 > 0 ? \text{ Si} \end{cases}$



3.16 Rappresentare graficamente in un piano cartesiano x, y l'insieme di definizione della funzione

$$f(x,y)=\sqrt{\frac{2x-(x^2+y^2)}{x^2+y^2-x}}$$

$$\begin{cases} x^2+y^2-x>0 \\ 2x-(x^2+y^2)\geq 0 \end{cases} \to \quad y^2=x-x^2 \to \quad y=\sqrt{x-x^2}$$

$y=\sqrt{x-x^2}$ $\quad D = x-x^2 \geq 0$ per $x(1-x)\geq 0$

$\hookrightarrow x\geq 0$, $x\leq 1$ $\}$ $0\leq x\leq 1$

Radici: $\begin{cases} x=0 \\ y=0 \end{cases}$ $\begin{cases} y=0 \\ x(1-x)=0 \end{cases}$ $\hookrightarrow x=0$, $\hookrightarrow x=1$

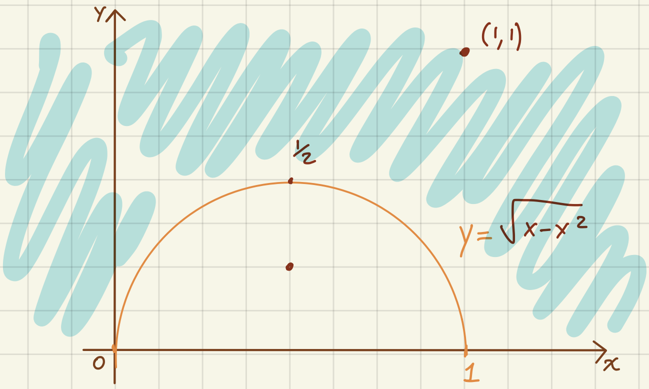
Deriv. $f'(x)=(x-x^2)^{\frac{1}{2}} = \frac{1}{2}(x-x^2)^{-\frac{1}{2}}\cdot 1-2x = \frac{1-2x}{2\sqrt{x-x^2}}$

$f'(x)\geq 0$ per $1-2x>0 \to x<\frac{1}{2}$

$\frac{1}{2}$ $\nearrow$ | $\searrow$ Max

$D''= 2\left(2\sqrt{x-x^2}\right)-(1-2x)\left(\frac{1}{\sqrt{x-x^2}}\right)\cdot 1-2x$

$= 4\sqrt{x-x^2}-\left[\frac{(1-2x)^2}{\sqrt{x-x^2}}\right]$



Prendo $\left(\frac{1}{3},\frac{1}{3}\right) \to \left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2-\frac{1}{3}>0$? $\to \frac{1+1-2}{9}>0$? NO

Prendo $(1,1)\to 1+1-1>0$? SI

prendo $\left(\frac{1}{2},\frac{1}{2}\right)\to\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2-\frac{1}{2}>0$? $\frac{1+2-2}{4}=0$

$\hookrightarrow$ Circonf. escluso

$2x-(x^2+y^2)\geq 0$ per $2x-x^2-y^2\geq 0$; $y^2<-x^2+2x \to y<\sqrt{2x-x^2}$

$\to$ riscrivo $y^2+x^2=2x$ cerchio di c = 1, e r = 2

Equazioni Circonferenza

F. ESPLICITA

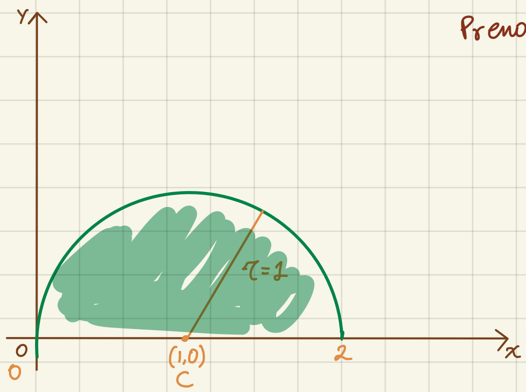
Eq = $y^2+x^2=r^2 \to (x-x_c)^2+(y-y_c)^2=r^2 \to x^2+y^2+ax+by+c=0$

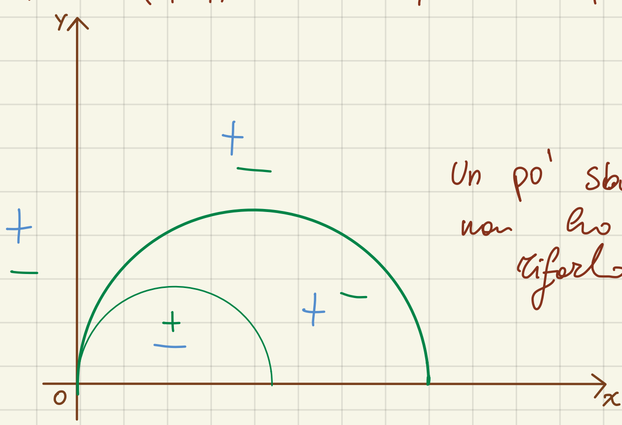
Raggio: $\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2-c}$ $\qquad$ Centro $C=\left(-\frac{a}{2},-\frac{b}{2}\right)$

ES: $y^2+x^2=2x \to y^2+x^2+0y-2x+0=0 \Rightarrow r=\sqrt{\left(-\frac{2}{2}\right)^2-c}=1$

$C=\left(\frac{-2}{2},0\right)\to C=(1,0)$, Siccome $2x-(x^2+y^2)\geq 0$ scriviamo:

Prendo $\left(\frac{1}{2},\frac{1}{2}\right)\to 1-\left(\frac{1}{4}+\frac{1}{4}\right)>0$? $1-\frac{2}{4}>0$? $\frac{4-2}{4}>0 \Rightarrow$ SI



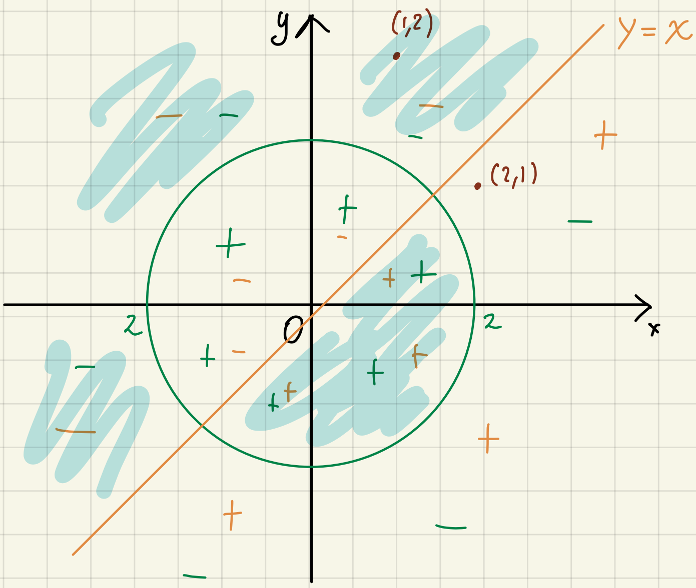


Un po' sbagliato non ho voglia di rifarlo.

3.17 Determinare l'insieme di definizione delle funzioni di due variabili

(a) $f(x,y) = \sqrt{\dfrac{4-x^2-y^2}{x-y}}$

$$\begin{cases} x - y > 0 & \rightarrow \quad y = x \\ 4 - x^2 - y^2 \geq 0 & \rightarrow \quad y^2 - x^2 = 4 \quad r = 2 \quad c = (0,0) \end{cases}$$

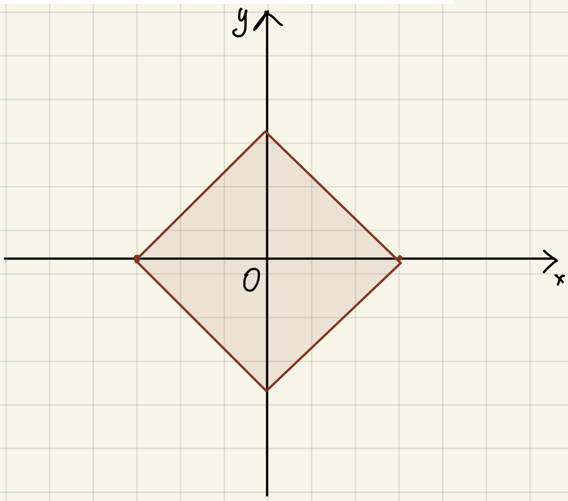


a) $(1,2) \rightarrow 1-2>0$? NO
$(2,1) \rightarrow 2-1>0$? Si circ esclusa

b) $(0,0) \rightarrow 4 \geq 0$? Si circ inclusa

(b) $f(x,y) = \sqrt{\dfrac{(|x|-1)(|y|-1)}{|x|+|y|-1}}$

$$\begin{cases} (|x|-1)(|y|-1) \geq 0 \\ |x| + |y| - 1 \quad \rightarrow \quad |x| + |y| = 1 \end{cases}$$



Derivate parziali

3.41 Calcolare, nel punto $(x,y)=(4,7)$, le derivate parziali della funzione
$$f(x,y)=x^3+y^2-xy$$

$$f_x = 3x^2 - 1y \quad , \quad f_y = 2y - x$$

nei punti $(4,7) \to f_x(4,7) = 3\cdot 16 - 7 = 41 \qquad f_y(4,7) = 2\cdot 7 - 4 = 10$

3.42 Calcolare le derivate parziali f_x, f_y delle seguenti funzioni nei punti interni ai rispettivi insiemi di definizione.

$f = x^2 + 2y$ $\qquad [f_x = 2x; \quad f_y = 2]$

$$f_x = 2x \qquad f_y = 2$$

$f = xy$

$f = (x+y)(x-y)$

$f = \dfrac{x}{y}$

$f = \dfrac{x-y}{x+y}$

$f = \dfrac{x+y}{1-xy}$

$f = \dfrac{x}{x^2+y^2}$

$$f_x = y \qquad f_y = x$$

$$f_x = [1][x-y] + [x+y][1] = x - y + x + y = 2x$$

$$f_y = [1][x-y] + [x+y][-1] = -2y$$

$$f = x \cdot y^{-1} \to f_x = \frac{1}{y} \qquad f_y = -\frac{1}{y^2}$$

$$f_x = \frac{(\cancel{x}+y) - (\cancel{x}-y)}{(x+y)^2} = \frac{2y}{(x+y)^2} \qquad f_y = \frac{-(x+\cancel{y}) - (x-\cancel{y})}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$f_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{\cancel{x^2} - \cancel{2}x^2 + y^2}{//} = -\frac{x^2+y^2}{(x^2+y^2)^2}$$

$$f_y = \frac{-x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$f = \dfrac{y}{\operatorname{sen} x}$

$$f_x = \frac{-y(\cos x)}{\sin^2 x} = \frac{-y\cos x}{\sin^2 x} \qquad f_y = \frac{\sin x}{\sin^2 x} = \frac{1}{\sin x}$$

$f = \dfrac{\operatorname{tg} x}{\operatorname{tg} y}$

$$f_x = \frac{1}{\cos^2 x}\cdot(\cancel{\operatorname{tg} y})\cdot\frac{1}{\operatorname{tg}^{\cancel{2}} y} = \frac{1}{\cos^2 x \operatorname{tg} y} \qquad f_y = -\operatorname{tg} x\left(\frac{1}{\cos^2 y}\right)\cdot\frac{1}{\operatorname{tg}^2 y} = \frac{\operatorname{tg} x}{\operatorname{tg}^2 y}\cdot\frac{1}{\cos^2 y}$$

Massimi e minimi

1) Calcoliamo le Derivate II parziali

2) Calcoliamo l'Hessiano

3) Se Hf è $>0 \to$ probabile Max/min

Se Hf è $<0 \to$ No min/Max.

1.3 Determinare i punti di massimo o di minimo relativo delle seguenti funzioni

(a) $f(x,y) = x^3+y^3+xy$

1) $f_x = 3x^2+y$ $\quad f_y = 3y^2+x$ $\quad f_{xx} = 6x$ $\quad f_{yy} = 6y$ $\quad f_{xy} = f_{yx} = 1$

2) Calcolo Hessiano $\begin{vmatrix} 6x & 1 \\ 1 & 6y \end{vmatrix} = 36xy - 1$

3) Ricerca punti Stazionari Sistema 2 eq 2 inc.

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ $\begin{cases} 3x^2+y=0 \to y = -3x^2 \; (1) \\ 3y^2+x=0 \to 3\cdot(-3x^2)^2+x=0 \; (2) \end{cases}$

$\to 3\cdot 9x^4 + x = 0 \to 27x^4+x=0$

per $x(27x^3+1)=0$

$\to x = 0$

$\to 27x^3 = -1$ per $x = \sqrt[3]{-\frac{1}{27}}$

Sostituiamo nella (1)

Se $x=0 \to y = -3\cdot 0^2 = 0 \Rightarrow (0,0)$ Pto. ST.

Se $x = \sqrt[3]{-\frac{1}{27}} = -\frac{1}{3} \to y = -3\cdot\left(-\frac{1}{3}\right)^2 = -3\cdot\frac{1}{9} = -\frac{1}{3}$

$\Rightarrow \left(-\frac{1}{3}, -\frac{1}{3}\right)$ Pto. ST

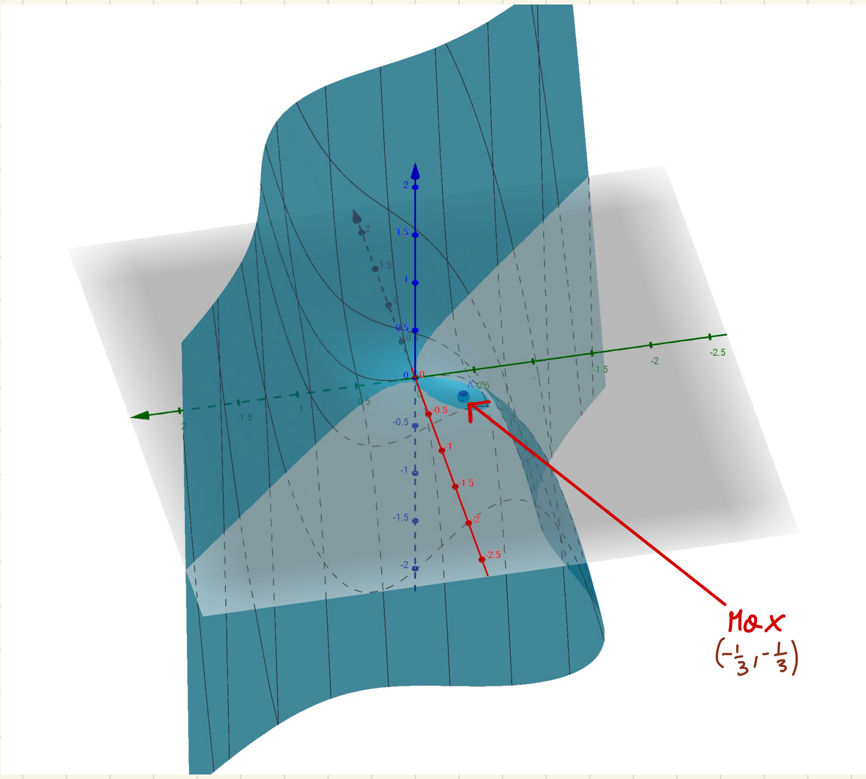
4) Sostituisco a Hf e valuto

$Hf(0,0) = 36(0\cdot 0) - 1 = -1 < 0$ Ne Max Ne min

$Hf\left(-\frac{1}{3}, -\frac{1}{3}\right) = 36\cdot\left(-\frac{1}{3}\cdot -\frac{1}{3}\right) - 1 = \frac{36}{9} - 1 = 3 > 0$ Sicuro Max/min

Per capire se $\left(-\frac{1}{3}, -\frac{1}{3}\right)$ è max o min sostituiamo a f_{xx} o f_{yy}

$f_{xx} = 6x \to f_{xx}\left(-\frac{1}{3}\right) = 6\cdot\left(-\frac{1}{3}\right) = -\frac{6}{3} = -2 < 0 \to$ Massimo



(b) $f(x,y) = x^3-y^3+xy$

$f_x = 3x^2+y \quad f_y = -3y^2+x \quad f_{xx} = 6x \quad f_{yy} = -6y \quad f_{xy} = f_{yx} = 1$

$Hf = 6x\cdot(-6y)-1 = -36xy-1$

Cerco pti ST:

$\begin{cases} 3x^2+y=0 \to 27y^4+y=0 \to y(27y^3+1)=0 \\ -3y^2+x=0 \to x=3y^2 \end{cases}$

$\hookrightarrow y=0$

$\hookrightarrow y=\sqrt[3]{-\frac{1}{27}} = -\frac{1}{3}$

$\Rightarrow x=3y^2 \to x\big|_0 = 0 \qquad x\big|_{-\frac{1}{3}} = 3\frac{1}{9} = \frac{1}{3}$

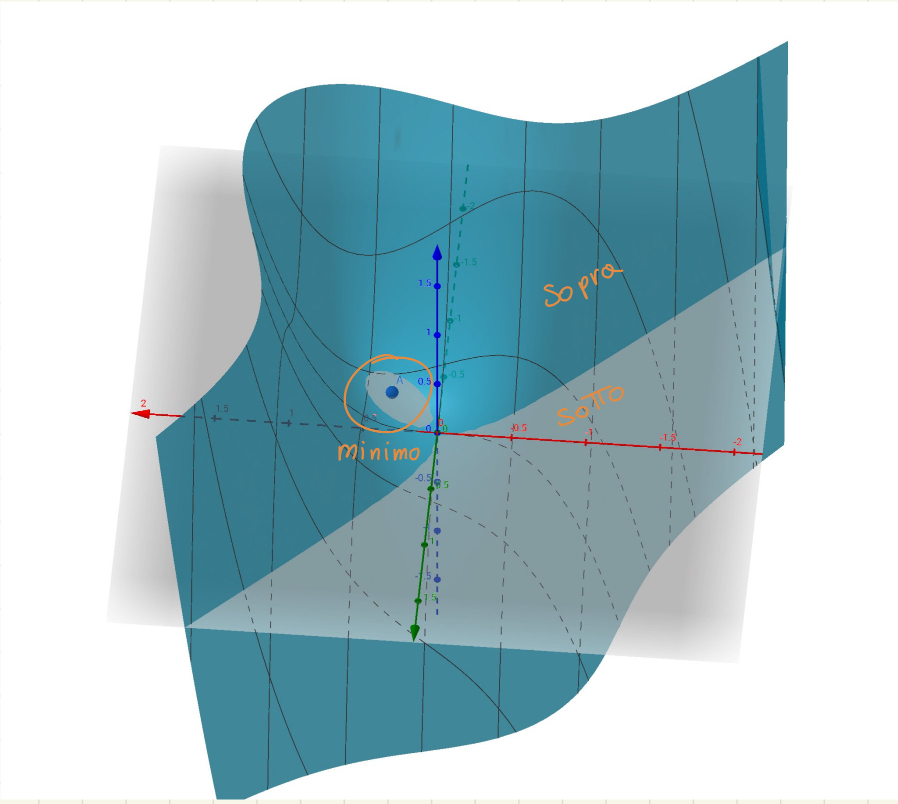
$\left.\begin{matrix}(0,0)\\ (\frac{1}{3},-\frac{1}{3})\end{matrix}\right\}$ pti stazionari

Sostituisco a Hessiano

$Hf(0,0) = -1 < 0 \to$ No max/min

$Hf(\frac{1}{3},-\frac{1}{3}) = -36\cdot(-\frac{1}{3}\cdot\frac{1}{3})-1 = \frac{36}{9}-1 = 3 > 0$ Sicuro max/min

$f_{xx} = 6x\big|_{\frac{1}{3}} = \frac{6}{3} = 2 > 0$ $\quad(\frac{1}{3},-\frac{1}{3})$ punto di minimo



1.1 Determinare i punti di massimo o di minimo relativo della funzione

$$f(x,y) = 4y^4 - 16x^2y + x$$

1) $f_x = -32xy + 1$ $\quad f_y = 16y^3 - 16x^2$

$f_{xx} = -32y$ $\quad f_{yy} = 48y^2$ $\quad f_{xy} = f_{yx} = -32x$

2) Punti Staz

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \begin{cases} -32xy + 1 = 0 \\ 16y^3 - 16x^2 = 0 \end{cases} \quad x = \frac{1}{32y} \to 16y^3 - 16\left(\frac{1}{32y}\right)^2 = 0 \to 16y^3 - \frac{16}{1024y^2} = 0 \to 16y^3 - \frac{1}{64y} = 0$$

$$\to 16y^3 = \frac{1}{64y}\ ; \quad y^4 = \frac{1}{64 \cdot 16} \Rightarrow y = \sqrt[4]{\frac{1}{1024}} = \frac{1}{\sqrt[4]{1024}} = \frac{1}{\sqrt[4]{16 \cdot 64}} = \frac{1}{\sqrt[4]{2^2 \cdot 2^8}} = \frac{1}{2^2\sqrt[4]{2^2}} = \frac{1}{4\sqrt{2}}$$

$$\to y = \pm\frac{\sqrt{2}}{8}$$

$$\Rightarrow x = \left.\frac{1}{32y}\right|_{+\frac{\sqrt{2}}{8}} = \frac{1}{\cancel{32} \cdot \frac{\sqrt{2}}{\cancel{8}}} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8} \to \left(\frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{8}\right) \ 1^\circ \text{ pto staz.}$$

$$x = \left.\frac{1}{32y}\right|_{-\frac{\sqrt{2}}{8}} = \frac{1}{32 \cdot \left(-\frac{\sqrt{2}}{8}\right)} = -\frac{1}{4\sqrt{2}} \to \left(-\frac{\sqrt{2}}{8}, -\frac{\sqrt{2}}{8}\right) \ 2^\circ \text{ pto staz}$$

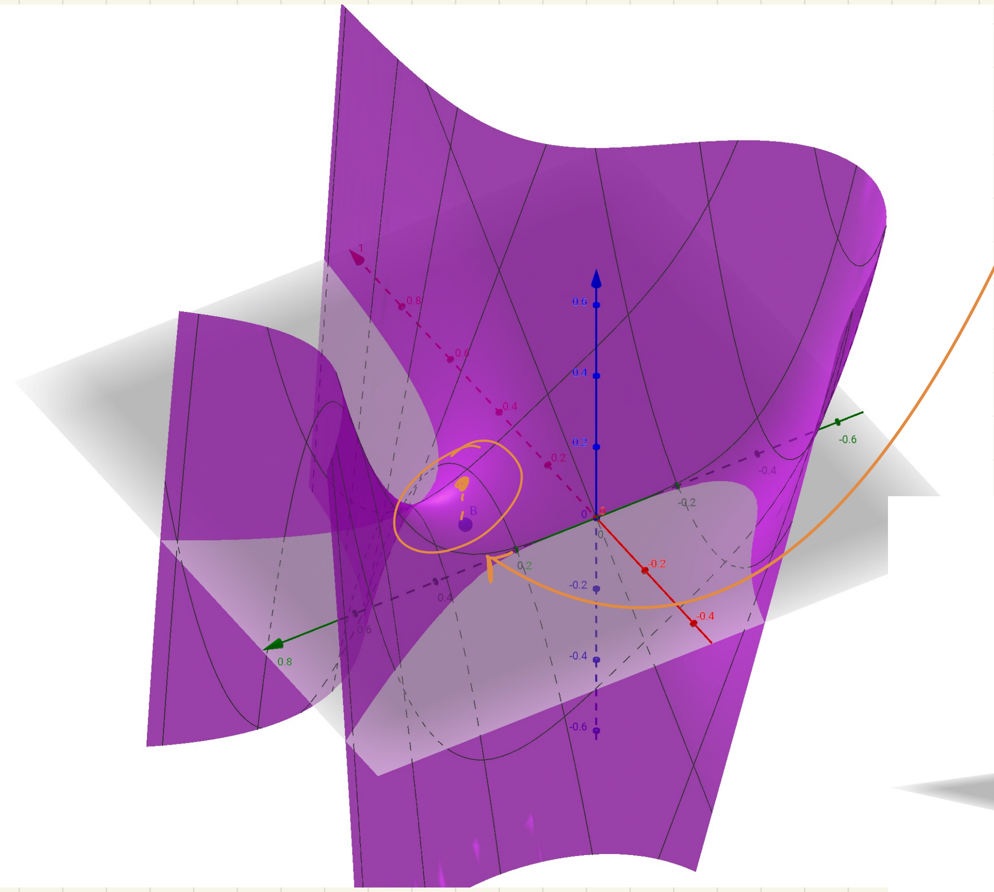
3) Sostituisco ad $Hf = (-32y)\cdot(48y^2) - (-32x)^2 = -1536y^3 - 1024x^2$

$$Hf\left(\frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{8}\right) = -1536 \cdot \left(\frac{\sqrt{2}}{8}\right)^3 - 1024\left(\frac{\sqrt{2}}{8}\right)^2 = -1536\left(\frac{4}{512}\right) - 1024\left(\frac{2}{64}\right) = n < 0$$

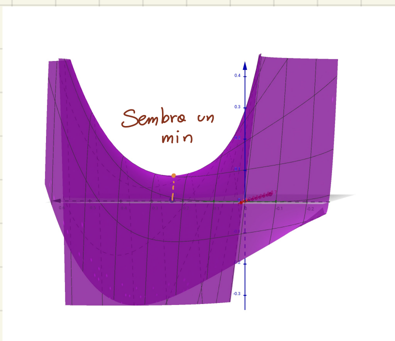
Ne Max ne min

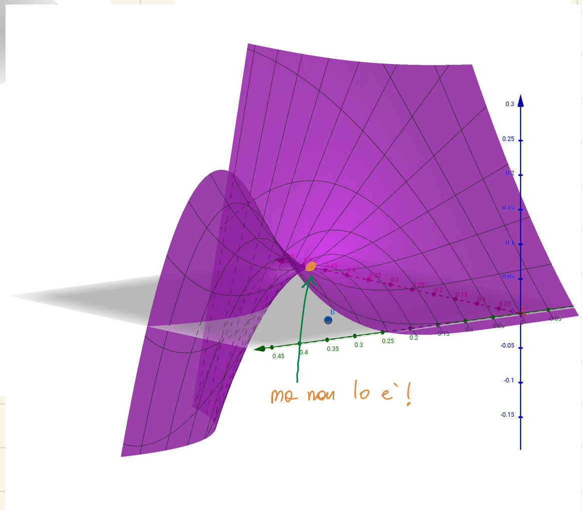
$$Hf\left(-\frac{\sqrt{2}}{8}, -\frac{\sqrt{2}}{8}\right) = -1536\left(-\frac{\sqrt{2}}{8}\right)^3 - 1024\left(-\frac{\sqrt{2}}{8}\right)^2 < 0 \quad \text{Ne max ne min}$$

* i calcoli potrebbero essere sbagliati ma Hf_1 e $Hf_2 < 0$



Punto sella





(a) $f(x,y) = 2\ (x^2+y^2+1) - (x^4+y^4)$

$2x^2+2y^2+2-x^4-y^4$

$\Rightarrow f_x = 4x-4x^3 \quad f_y = 4y-4y^3$

$f_{xx} = 4-12x^2 \quad f_{yy} = 4+12y^2 \quad f_{xy} = f_{yx} = \emptyset$

$\Rightarrow Hf = (4-12x^2)(4-12y^2) - 0 = 16 - 48y^2 - 48x^2 + 144x^2y^2$

2) Cerco pti St.

$\begin{cases} 4x-4x^3=0 \\ 4y+4y^3=0 \end{cases} \quad \begin{cases} 4x = 4x^3 \\ \hookrightarrow \dfrac{x}{x^3} = 1 \end{cases} \rightarrow \dfrac{1}{x^2} = 1 \rightarrow x = \pm 1$

$x = 0$

* sistema sbagliato ma $x <^{\pm 1}_{0} \quad y <^{\pm 1}_{0}$

$\begin{cases} 4y = -4y^3 \\ \hookrightarrow y = \emptyset \end{cases}$

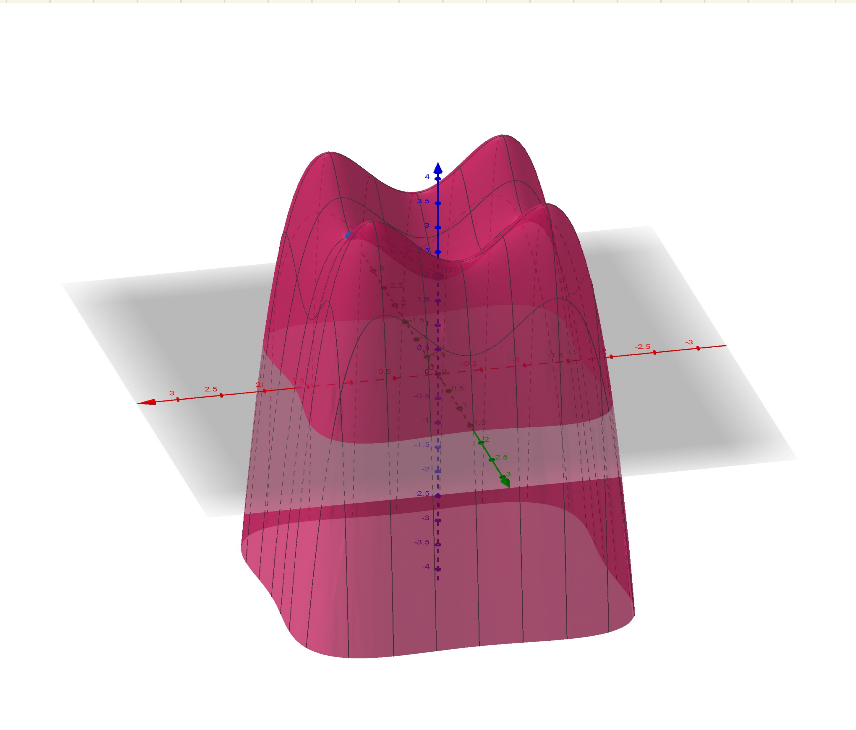
$4y = (1+y^2) = 0$
$\hookrightarrow y = 0$
$\hookrightarrow y = \pm 1$

testo Hf

$Hf|_{\substack{x=0\\y=0}} = 16 > 0$ max/min $\rightarrow f_{xx}|_{\substack{x=0\\y=0}} = 4 > 0 \Rightarrow$ minimo $\quad (0,0)$ minimo

$Hf|_{\substack{x=\pm1\\y=\pm1}} = 16-48-48+144 = 64 > 0$ max/min $\Rightarrow f_{xx}|_{\substack{x=\pm1\\y=\pm1}} = 4-12 < 0 \quad \Rightarrow \quad (\pm1, \pm1)$ massimi

$Hf|_{\substack{x=\pm1\\y=0}} = 16-48 < 0$ NO max/min $\quad Hf|_{\substack{x=0\\y=\pm1}} = 16-48 < 0$ NO max/min



(b) $f(x,y) = 2(x^4+y^4+1) - (x+y)^2$ $\quad 2x^4+2y^4+2-x^2-2xy-y^2$

$f_x = 8x^3-2x-2y \qquad f_y = 8y^3-2x-2y \qquad f_{xx} = 24x^2-2 \qquad f_{yy} = 24y-2 \qquad f_{xy} = f_{yx} = -2$

2) Trovo pti Staz.

$$\begin{cases} 8x^3-2x-2y=0 \\ 8y^3-2x-2y=0 \end{cases} \quad \begin{matrix} \to y = 2x-8x^3 \\ \Rightarrow 8(2x-8x^3)^3 - 2x - 2(2x-8x^3) = 0 \end{matrix} ; \quad 8[(4x^2-32x^4+64x^9)(2x-8x^3)] - 2x - 4x + 16x^3)$$

$$= 8[8x^3-32x^5-64x^5+256x^7+128x^{10}-512x^{12}] - 6x+16x^3 = 0$$

$$= 8[-512x^{12}+128x^{10}+256x^7-96x^5+8x^3]-6x+16x^3=0$$

$$\to -512x^{12}+128x^{10}+256x^7-96x^5+10x^3-\frac{3}{4}x=0$$

$$x\left(-512x^{11}+128x^9+256x^6-96x^4+10x^2-\frac{3}{4}\right)=0 \qquad \to x_1=0$$

$$x^2(-512x^9+128x^7+256x^4-96x^2+10)+\frac{3}{4}=0 \quad \to \quad x^2=-\frac{3}{4} \quad \to \quad x_{2,3}=\pm\sqrt{-\frac{3}{4}}$$

$$x^2(-512x^7+128x^5+256x^2-96)+10=0 \quad \to \quad x^2=-10 \quad \Rightarrow \quad x=\cancel{\sqrt{-10}}$$