Esercizi numeri complessi

Dal libro Marcellini

ES 4.1:

a) $(1+i)+(1-2i)$ c) $(1+i)\cdot(1-2i)$
b) $(1-i)-(1+i)$ d) $(1+i)\cdot(1+i)$

a) $1+i+1-2i = 2-i$
b) $1-i-1-i = -2i$
c) $(1+i)\cdot(1-2i) = 1-2i+i-2i^2 = 1-i+2 = 3-i$
d) $(1+i)^2 = 1+i^2+2i = 1-1+2i = 2i$

ES 4.2: Scrivere in forma algebrica

(a)	$\frac{i}{1-i}$	(b)	$\frac{1}{i}$
(c)	$\frac{1-i}{1+i}$	(d)	$13\,\frac{1+i}{2-3i}$

a) $\frac{i}{1-i}\cdot\frac{1+i}{1+i} = \frac{i+i^2}{1+\cancel{i}-\cancel{i}-\underbrace{i^2}_{1}} = \frac{i-1}{2} = \frac{i-1}{2} = -\frac{1}{2}+\frac{1}{2}i$

b) $\frac{1}{i}\cdot\frac{i}{i} = \frac{1}{-1} = -i$

C) $\frac{1-i}{1+i}\cdot\frac{1-i}{1-i} = \frac{(1-i)^2}{1-i+i-i^2} = \frac{(1-i)^2}{1+1} = \frac{1}{2}(1-i)^2 = \frac{1}{2}\,1-2i-i^2 = \frac{1}{2}(1-2i+1) = -\frac{1}{\cancel{2}}\cancel{2}i = -i$

d) $13\,\frac{1+i}{2-3i}\cdot\frac{2+3i}{2+3i} = 13\,\frac{2+3i+2i+3i^2}{4+\cancel{6i}-\cancel{6i}-9i^2} = \cancel{13}\,\frac{2+5i-3}{\cancel{13}} = -1+5i$

ES 4.3: Verifica l'identità

φ

$$(3+i)\cdot\left(1+\frac{2i}{1-i}\right)\cdot\left(1-\frac{2}{(1+i)^2}\right)=10$$

$$=(3+i)\cdot\Big(\underbrace{\frac{1-i+2i}{1-i}}_{a}\Big)\cdot\Big(\underbrace{\frac{2i-2}{2i}}_{b}\Big)=10 \Rightarrow (3+i)\Big(\underbrace{\frac{1+i}{1-i}}_{a}\Big)\cdot\Big(\underbrace{\frac{2i-2}{2i}}_{b}\Big)=10$$

$\Rightarrow$ a: $\frac{1+i}{1-i}\cdot\frac{1+i}{1+i} = i$ b) $\frac{2i-2}{2i} = \cancel{\frac{2i}{2i}} - \frac{\cancel{2}}{\cancel{2}i} = \frac{1}{i}\cdot\frac{i}{i} = -i$

$\Rightarrow (3+i)\cdot i\cdot(-i) = 3i+i^2\cdot(-i) = -3i^2+i = \boxed{3+i = 10} \Rightarrow -7+i$

Sappiamo che

$\varphi = \sqrt{a^2+b^2}$ e $\begin{cases} a=\cos\theta \\ b=\sin\theta \end{cases}$ $\begin{cases} \cos\theta = \\ \sin\theta = \end{cases}$

$= \sqrt{(-7)^2+1}$

$= \sqrt{50} = 5\sqrt{2}$

?

ES 4.4:

4.4 Calcolare il modulo e l'argomento dei numeri complessi

(a)	$1+i$	(b)	$2-2i$
(c)	$\sqrt{3}+i$	(d)	$-1+i\sqrt{3}$

a) $1+i$ $\quad a=1,\ b=1 \Rightarrow \varphi=\sqrt{1+1}=\sqrt{2}$ $\quad a=\cos\theta=\frac{a}{\varphi}$, $b=\sin\theta=\frac{b}{\varphi}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = $ \ $\quad 45^0$

$\sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = /$

$180 : \pi = 45 : x$

$\frac{\cancel{45}\pi}{\cancel{180}}\ \frac{9}{36}$ $\quad \frac{1}{4}\pi$

$\Rightarrow z=\varphi[\cos(\theta)+i\sin(\theta)] = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$

b) $z = \overset{a}{2} - \overset{b}{2i}$ $\qquad \varphi = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\theta /$ $a \quad \cos\theta = \frac{2}{2\sqrt{2}} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} =$

$b \quad \sin\theta = -\frac{2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2} =$

$45° = \frac{\pi}{4}$?

$z = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

Trovo θ con Tan:

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-2}{2} = -1 \Rightarrow \tan^{-1}(-1) = -45 = -\frac{\pi}{4}$

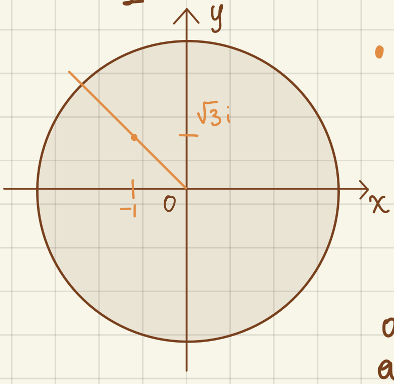
c) $z = \sqrt{3} + i$ $\qquad \varphi = \sqrt{3+1} = 2$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = 30° = \frac{\pi}{6} + 2k\pi$

$z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

d) $z = -1 + i\sqrt{3}$ $\qquad \varphi = \sqrt{3+1} = 2$

$\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3} \Rightarrow \theta = \tan^{-1}(-\sqrt{3}) = -60° = -\frac{\pi}{3} + 2k\pi$

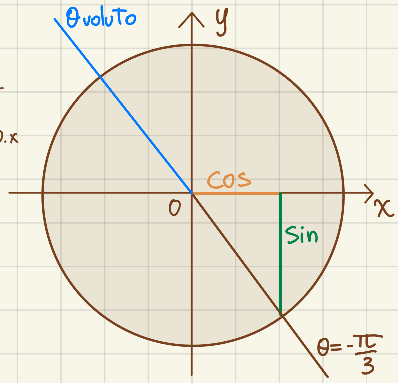


- Siccome z si trova nel II quadrante ma $\theta = -\frac{\pi}{3}$ no, dobbiamo trovare un θ avente lo stesso valore di $\tan\left(-\frac{\pi}{3}\right)$ ma giacente nel II quadrante

Quindi, guardando il disegno ②, dobbiamo ottenere un angolo opposto a quello che abbiamo; per farlo ci basta aggiungere "mezzo giro" ovvero $\pi = 180$:

$\cos -\frac{\pi}{3} = \frac{1}{2}$

$\sin -\frac{\pi}{3} = -0.x$



$\Rightarrow \theta = -\frac{\pi}{3} + \pi = \frac{-\pi + 3\pi}{3} = \frac{2}{3}\pi$ $\qquad z = 2\left[\cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right)\right]$

Potenze e radici

$$z \cdot z' = [\varphi(\cos\theta + i\sin\theta)] \cdot [\varphi'(\cos\theta' + i\sin\theta')] = \varphi \cdot \varphi'[(\cos(\theta+\theta') + i\sin(\theta+\theta')]$$

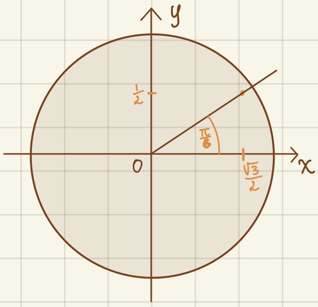
Ponendo $z' = z \Rightarrow \quad z^2 = \varphi^2[\cos(2\theta) + i\sin(2\theta)] = z^n = \varphi^n[\cos(n\theta) + i\sin(n\theta)]$

ES 4.6:

4.6 Verificare che la potenza sesta del numero complesso $z = (\sqrt{3} + i)/2$ vale -1.

$$\frac{\sqrt{3}+i}{2} = \frac{\sqrt{3}}{2} + \frac{1}{2}i \qquad \varphi = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

$$\theta: \quad \tan\theta = \frac{b}{a} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} = \frac{1}{\sqrt{3}} = 30^0 = \frac{\pi}{6} \checkmark$$



$$\Rightarrow z = \left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

$$z^6 = \varphi^6\left[\cos\left(6\frac{\pi}{6}\right) + i\sin(\pi)\right] = -1$$ Verificato

4.7 Determinare la forma algebrica dei numeri complessi

(a) $(1-i)^6$ (b) $(1+i)^8$ (c) $(1-i)^{12}$

a) $(1-i)^6 = (1-i)^3(1-i)^3 = (1-3i+3i^2-i^3)^2 = (1-3i-3+i)^2 = (-2-2i)^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\underbrace{i^2}_{-1} \cdot \underbrace{i}_{i}$

$= 4 + 8i + 4\cancel{i^2}^{-1} = 4 + 8i - 4 = 8i$

b) $(1+i)^8 = [(1+i)^2]^4 \Rightarrow [\cancel{1} + 2i + \cancel{i^2}]^4 = (2i)^4 = 16i^4 = 16 \cdot \cancel{i^2} \cdot \cancel{i^2} = 16$

$-1 \times -1 = 1$

c) $(1-i)^{12} = \underbrace{(1-i)^8}_{16} \cdot (1-i)^4 = 16(1-i)^4 = 16[(1-i)^2]^4 = 16[\cancel{1} - 2i - \cancel{i^2}]^2 = 16[-2i]^2 = 16 \cdot 4i^2$
$= 16 \cdot (-4) = -64$

4.8 Determinare la forma algebrica dei numeri complessi

(a) $(\sqrt{3}+i)^6$ (b) $\left(\frac{1}{i}\right)^4$ (c) $\left(\frac{1+i}{i-1}\right)^3$

$[(\sqrt{3}+i)^2]^3 = [3 + 2\sqrt{3}i + i^2]^3 = [2 + 2\sqrt{3}i]^3 = 2^3 - [3 \cdot 2^2 \cdot (2\sqrt{3}i)] + [3 \cdot 2 \cdot (2\sqrt{3}i)^2] + (2\sqrt{3}i)^3$
$= 8 - [12(2\sqrt{3}i)] + [6i^2 6] + 12i^2 = 8 - [24\sqrt{3}i] - 36 + 12 = \underbrace{-16}_{a}\ \underbrace{24\sqrt{3}i}_{b}$

a) $(\sqrt{3}+i)^6 = (\sqrt{3}+i)^3 \cdot (\sqrt{3}+i)^3 = [(\sqrt{3}+i)^3]^2 = [3\sqrt{3} - 9i + 3\sqrt{3}i^2 + i^3]^2 = [3\sqrt{3} + 9i - 3\sqrt{3} - i]^2$

$= [-8i]^2 = 64i^2 \Rightarrow z = 64$ F. Algebrica

b) $\left(\frac{1}{i}\right)^4 = \frac{1}{i^4} = 1 \Rightarrow z = 1$

i^4: $-1 \cdot -1$

c) $\left(\frac{1+i}{i-1}\right)^3 = \frac{1+i}{-1+i} \cdot \frac{-1-i}{-1+i} = \frac{\cancel{-1} - i - i - \cancel{i^2}}{\cancel{1} - i - i + \cancel{i^2}} = \frac{-2i}{-2i} = i^3 = -i$

RADICI

$$w_k = \underbrace{\rho^{1/n}}_{\sqrt[n]{\varphi}}\left[\cos\frac{\vartheta+2k\pi}{n} + i\,\text{sen}\,\frac{\vartheta+2k\pi}{n}\right],$$

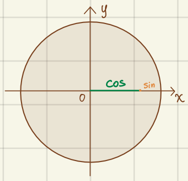
Una radice $\sqrt[n]{x}$ ha esattamente n radici distinte.
Inoltre, se z è una radice, anche $\bar{z}$ lo è.

4.9 Determinare le tre radici cubiche del numero $z = 1$.

$z = 1$; $\sqrt[3]{1} = \begin{matrix} z_1 \\ z_2 \\ z_3 \end{matrix}$

$\varphi_z = 1$
$\theta = 0$

$\tan\frac{b}{a} = \frac{0}{1} \Rightarrow \tan^{-1} 0 = 0$
$\Rightarrow \cos 0 = 90 = \frac{\pi}{2}$
$\sin 0 = 0$

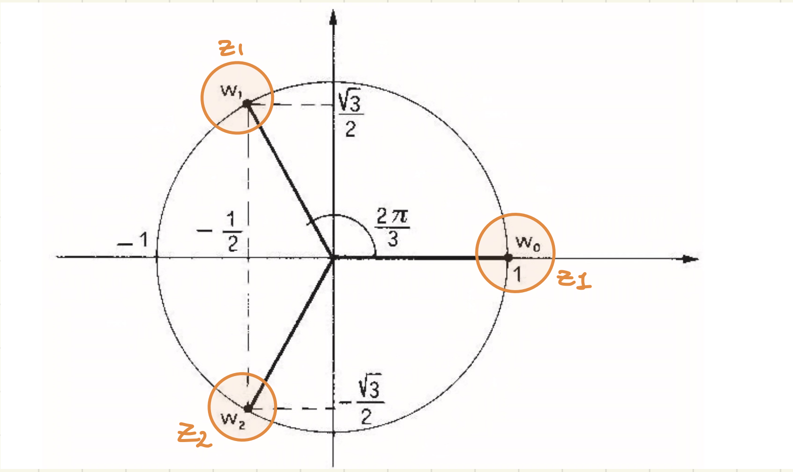


$\Rightarrow z = 1[\cos 0 + i \sin 0]$

$z_0 = \sqrt[3]{1}\left[\cos\left(\frac{0+20\pi}{3}\right) + i \sin(0)\right] = \overset{0}{\cos} 0 + i\,\overset{1}{\sin} 0 = 1$

$z_1 = \sqrt[3]{1}\left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2}{3}\pi\right)\right] = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$



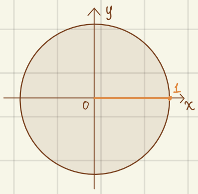
4.10 Determinare le radici quadrate dei numeri complessi

(a) 1 (b) −1 (c) i (d) $-i$

$z = 1 \Rightarrow \varphi = 1 \quad \theta = 0 \Rightarrow 1[\cos(0) + i \sin(0)] = 1$

$z_0 = \sqrt{1}[\cos(0) + i \sin(0)] = 1$

$z_1 = \sqrt{1}[\cos(2\pi) + i \sin(2\pi)] = 1$



Ad occhio... $z = 1 \Rightarrow z = i^2 \Rightarrow \sqrt{i^2} = \pm 1$

b) $z = -1 \Rightarrow \sqrt{-1} = i$ per definizione

c) $z = i \Rightarrow \sqrt{i} \Rightarrow$ per definizione $\quad z = bi,\ b>1 \quad \begin{cases} \varphi = |b| = 1 \\ \theta = \frac{\pi}{2} \end{cases}$

$\Rightarrow z = 1\left[\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right] = i$

$z_0 = \sqrt{1}[\cos(0) + i \sin(0)] = 1$
$z_1 = \sqrt{i}\left[\cos\left(\frac{2\pi}{2}\right) + i \sin\left(\frac{2\pi}{2}\right)\right] = 1$

Equazioni con numeri complessi

4.16 Risolvere le equazioni algebriche di primo grado

$(a)\quad iz+1=0 \qquad (b)\quad (2+i)z-4+3i=0$

a) $iz+1=0 \Rightarrow z=-\frac{1}{i}$; $z=\frac{-1}{i}\cdot\frac{i}{i}=\frac{-i}{-1}=\underline{i}$

b) $(2+i)z-4+3i=0$; $\cancel{(2+i)}z=\frac{4-3i}{2+i}$; $z=\frac{4-3i}{2+i}\cdot\frac{2-i}{2-i}=\frac{8-4i-6i+3i^2}{4-\cancel{2i}+\cancel{2i}-i^2}=\frac{8-10i-3}{5}$

$z=\frac{5-10i}{5}$; $\frac{5}{5}-\frac{10}{5}i$; $\underline{z=1-2i}$

4.17 Risolvere le equazioni di primo grado

$(a)\quad (1-i)z-2=0 \qquad (b)\quad iz+1-i=0$

a) $(1-i)z-2=0$; $z=\frac{2}{1-i}\cdot\frac{1+i}{1+i}$; $z=\frac{2+2i}{1+\cancel{i}-\cancel{i}-i^2}$; $z=\frac{2+2i}{2}\Rightarrow \underline{z=1+i}$

b) $iz+1-i=0$; $z=-\frac{1+i}{i}\cdot\frac{i}{i}\Rightarrow z=\frac{-i+i^2}{i^2}$; $\frac{-i-1}{-1}\Rightarrow \underline{z=1+i}$

4.18 Risolvere l'equazione di secondo grado $z^2+z+1=0$.

$z^2+z+1=0 \qquad z_{1,2}=\frac{-1\pm\sqrt{1-4\cdot1\cdot1}}{2}=$

$z_1=\frac{-1+\sqrt{-3}}{2}=\frac{-1+\sqrt{-1}\cdot\sqrt{3}}{2}=\underline{\frac{-1+i\sqrt{3}}{2}}$

$z_2=\frac{-1-\sqrt{-3}}{2}=\underline{\frac{-1-i\sqrt{3}}{2}}$

4.19 Risolvere nel campo complesso le equazioni di secondo grado

$(a)\quad z^2+4=0 \qquad (b)\quad z^2-4=0$

a) $z^2+4=0 \Rightarrow z^2=-4 \Rightarrow z=\sqrt{-4}\Rightarrow \underline{z=\pm i2}$

b) $z^2-4=0\Rightarrow \underline{z=\pm2}$

4.20 Risolvere nel campo complesso le equazioni

$(a)\quad 5z^2-4z+1=0 \qquad (b)\quad z^2+4z+5=0$

a) $5z^2-4z+1=0 \qquad z_{1,2}=\frac{-\left(\frac{b}{2}\right)\pm\sqrt{\left(\frac{b}{2}\right)^2-ac}}{a}=\frac{2\pm\sqrt{4-5}}{5}=$ $\underline{z_1=\frac{2+i}{5}}$, $\underline{z_2=\frac{2-i}{5}}$

b) $z^2+4z+5=0$

$z_{1,2}=-2\pm\sqrt{4-4\cdot5}=$ $z_1=-2+\sqrt{-16}=-2+i4$, $z_2=-2-\sqrt{-16}=-2-i4$?

Il risultato dovrebbe essere $2\pm i$

4.21 Risolvere l'equazione $4z^2-4z+3-2\sqrt{3}\,i=0$.

$4z^2-4z+3-3\sqrt{3}i=0$; $4(z^2-z)+3(1-\sqrt{3}i)=0$

$\frac{4\pm\sqrt{16-4\cdot4\cdot(3-3\sqrt{3}i)}}{8}$

$z_1=\frac{4+\sqrt{16-16(3-3\sqrt{3}i)}}{8}=\frac{4+\sqrt{16-48+48\sqrt{3}i}}{8}$

$z_2=\frac{4-\sqrt{16-16(3-3\sqrt{3}i)}}{8}=\frac{4-\sqrt{16-48+48\sqrt{3}i}}{8}$

ES: Risolvere l'eq: $\frac{2z}{1+i} = -1$

$\frac{\cancel{2}z}{\cancel{1+i}} \cdot \frac{\cancel{1+i}}{\cancel{2}} = -\frac{1+i}{2}$; $\Rightarrow$ $z = -\frac{1-i}{2}$ $\Rightarrow$ $z = -\frac{1}{2} - \frac{1}{2}i$

ES: $z^3 = 1$ $\Rightarrow$ $z = \sqrt[3]{1}$