

Esercizi numeri complessi

Dal libro Marcellini

ES 4.1:

a) $(1+i) + (1-2i)$
 b) $(1-i) - (1+i)$

c) $(1+i) \cdot (1-2i)$
 d) $(1+i) \cdot (1+i)$

a) $1+i + 1-2i = 2-i$

b) $1-i - 1-i = -2i$

c) $(1+i) \cdot (1-2i) = 1-2i+i-2i^2 = 1-i+2 = 3-i$

d) $(1+i)^2 = 1+i^2+2i = 1-1+2i = 2i$

ES 4.2: Scrivere in forma algebrica

(a) $\frac{i}{1-i}$

(b) $\frac{1}{i}$

(c) $\frac{1-i}{1+i}$

(d) $13 \frac{1+i}{2-3i}$

a) $\frac{i}{1-i} \cdot \frac{1+i}{1+i} = \frac{i+i^2}{1-i-i^2} = \frac{i-1}{2} = \frac{i-1}{2} = \frac{1}{2} + \frac{1}{2}i$

b) $\frac{1}{i} \cdot \frac{1}{i} = \frac{1}{-1} = -i$

c) $\frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i-i+i^2}{1-i+i-i^2} = \frac{1-2i-1}{1} = -2i ?$

d) $13 \frac{1+i}{2-3i} \cdot \frac{2+3i}{2+3i} = 13 \frac{2+3i+2i+3i^2}{4+6i-6i-9i^2} = 13 \frac{2+5i-3}{13} = -1+5i$

ES 4.3: Verifica l'identità

✓

$$(3+i) \cdot \left(1 + \frac{2i}{1-i}\right) \cdot \left(1 - \frac{2}{(1+i)^2}\right) = 10$$

$$= (3+i) \cdot \left(\frac{1-i+2i}{1-i}\right) \cdot \left(\frac{2i-2}{2i}\right) = 10 \Rightarrow (3+i) \left(\frac{1+i}{1-i}\right) \cdot \left(\frac{2i-2}{2i}\right) = 10$$

$$\Rightarrow a: \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = i \quad b: \frac{2i-2}{2i} = \frac{2i}{2i} - \frac{2}{2i} = \frac{1}{i} \cdot \frac{i}{i} = -i$$

$$\Rightarrow (3+i) \cdot i \cdot (-i) = 3i + i^2 \cdot (-i) = -3i^2 + i = 3+i = 10 \Rightarrow -7+i$$

Sappiamo che

$$\begin{aligned} \varphi &= \sqrt{a^2+b^2} \\ &= \sqrt{(-7)^2+1} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned}$$

$$\begin{cases} a = \cos \theta \\ b = \sin \theta \end{cases} \quad \begin{cases} \cos \theta = \\ \sin \theta = \end{cases}$$

?

a) $1+i \quad a=1 \quad b=1 \Rightarrow \varphi = \sqrt{1+1} = \sqrt{2} \quad a = \cos \theta = \frac{a}{\varphi} \\ b = \sin \theta = \frac{b}{\varphi}$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2} \cos 45^\circ$$

$$\sin \theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2} \sin 45^\circ$$

$$180^\circ : \pi = 45^\circ : x$$

$$\frac{45\pi}{180} \cdot \frac{\pi}{4} = \frac{1}{4} \pi$$

$$\Rightarrow z = \varphi [\cos(\theta) + i \sin(\theta)] = \sqrt{2} \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$$

$$b) z = \frac{a}{2} - \frac{b}{2}i$$

$$\theta / \begin{aligned} a \cos \theta &= \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \\ b \sin \theta &= -\frac{2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{aligned}$$

$$\varphi = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\left. \begin{aligned} \frac{\sqrt{2}}{2} &= 45^\circ = \frac{\pi}{4} \\ ? & \end{aligned} \right\} z = 2\sqrt{2} \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$$

Trovo θ con \tan :

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-2}{2} = -1 \Rightarrow \tan^{-1}(-1) = -45 = \frac{-\pi}{4}$$

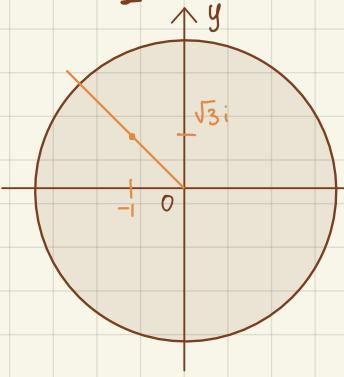
$$c) z = \sqrt{3} + i \quad \varphi = \sqrt{3+1} = 2$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{b}{a} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = 30^\circ = \frac{\pi}{6} + 2k\pi$$

$$\left. \right\} z = 2 \left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right]$$

$$d) z = -1 + i\sqrt{3} \quad \varphi = \sqrt{3+1} = 2$$

$$\tan \theta = \frac{\sqrt{3}}{-1} = -\sqrt{3} \Rightarrow \theta = \tan^{-1}(-\sqrt{3}) = -60^\circ = -\frac{\pi}{3} + 2k\pi$$



- Siccome z si trova nel III quadrante ma $\theta = -\frac{\pi}{3}$ no, dobbiamo trovare un θ avente lo stesso valore di $\tan(-\frac{\pi}{3})$ ma giacente nel II quadrante

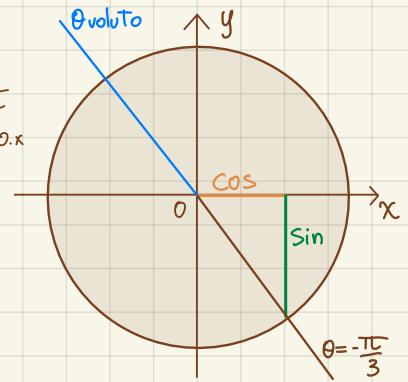
Quindi, guardando il disegno ②, dobbiamo ottenere un angolo opposto a quello che abbiamo; per farlo ci basta aggiungere "mezzo giro" ovvero $\pi = 180^\circ$:

$$\Rightarrow \theta = -\frac{\pi}{3} + \pi = -\frac{\pi + 3\pi}{3} = \frac{2\pi}{3}$$

$$z = 2 \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right]$$

$$\cos -\frac{\pi}{3} = \frac{1}{2}$$

$$\sin -\frac{\pi}{3} = -0,866$$



Potenze e radici

$$z \cdot z' = [\varphi (\cos \theta + i \sin \theta)] \cdot [\varphi' (\cos \theta' + i \sin \theta')] = \varphi \cdot \varphi' [(\cos(\theta+\theta') + i \sin(\theta+\theta'))]$$

Ponendo $z' = z \Rightarrow z^2 = \varphi^2 [\cos(2\theta) + i \sin(2\theta)] = z^n = \varphi^n [\cos(n\theta) + i \sin(n\theta)]$

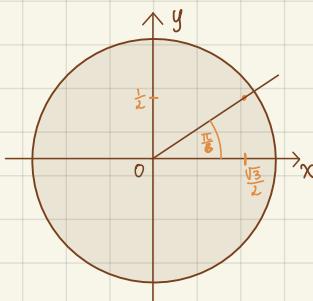
ES 4.6:

4.6 Verificare che la potenza sesta del numero complesso $z = (\sqrt{3} + i)/2$ vale -1.

$$\frac{\sqrt{3}+i}{2} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$\begin{aligned}\varphi &= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{3}{4} + \frac{1}{4}} \\ &= \sqrt{1} = 1\end{aligned}$$

$$\theta: \tan \theta = \frac{b}{a} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = 30^\circ = \frac{\pi}{6} \checkmark$$



$$\Rightarrow z = [\cos(\frac{\pi}{6}) + i \sin(\frac{\pi}{6})]$$

$$z^6 = \varphi^6 [\cos(6 \cdot \frac{\pi}{6}) + i \sin(6 \cdot \frac{\pi}{6})] = -1 \quad \text{Verificato}$$

4.7 Determinare la forma algebrica dei numeri complessi

- (a) $(1-i)^6$ (b) $(1+i)^8$ (c) $(1-i)^{12}$

$$\begin{aligned}a) \quad (1-i)^6 &= (1-i)^3 (1-i)^3 = (1-3i+3i^2-i^3)^2 = (1-3i-3+i)^2 = (-2-2i)^2 \\ &= (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad \begin{matrix} \uparrow \\ i^3 \\ \downarrow \\ -1 \end{matrix} \quad \begin{matrix} \uparrow \\ i^2 \\ \downarrow \\ -1 \end{matrix} \quad \begin{matrix} \uparrow \\ i \\ \downarrow \\ 1 \end{matrix}\end{aligned}$$

$$= 4 + 8i + 4i^2 = 4 + 8i - 4 = 8i$$

$$b) \quad (1+i)^8 = [(1+i)^2]^4 \Rightarrow [1+2i+i^2]^4 = (2i)^4 = 16i^4 = 16 \cdot i^2 \cdot i^2 = 16$$

$$c) \quad (1-i)^{12} = (1-i)^8 \cdot (1-i)^4 = 16(1-i)^4 = 16[(1-i)^2]^4 = 16[1-2i-i^2]^2 = 16[-2i]^2 = 16 \cdot 4i^2 \\ = 16 \cdot (-4) = -64$$

4.8 Determinare la forma algebrica dei numeri complessi

- (a) $(\sqrt{3}+i)^6$ (b) $\left(\frac{1}{i}\right)^4$ (c) $\left(\frac{1+i}{i-1}\right)^3$

$$\begin{aligned} &[(1-i)^2]^3 + [2+2i+1]^3 = [4+2i]^3 = 4^3 \cdot 5^3 \cdot i^3 + \\ &= 4 \cdot [1+2i+1] \cdot [4+1] = 16^3 = 4^3 \cdot [4+1] = 56 \cdot 16 = -16 \cdot 2i \cdot 56 = -9216i \end{aligned}$$

$$a) \quad (\sqrt{3}+i)^6 = (\sqrt{3}+i)^3 \cdot (\sqrt{3}+i)^3 = [(\sqrt{3}+i)^3]^2 = [3\sqrt{3}-9i+3\sqrt{3}i^2+i^3]^2 = [3\sqrt{3}+9i-3\sqrt{3}-i]^2$$

$$= [-8i]^2 = 64i^2 \Rightarrow z = 64 \quad \text{f. Algebrica}$$

$$b) \quad \left(\frac{1}{i}\right)^4 = \frac{1}{i^4} = 1 \Rightarrow z = 1$$

$$\begin{matrix} \downarrow \\ -1 \\ \downarrow \\ -1 \end{matrix}$$

$$c) \quad \left(\frac{1+i}{i-1}\right)^3 = \frac{1+i}{-1+i} \cdot \frac{-1-i}{-1+i} = \frac{i-i-i-i^2}{i-i-i+i^2} = \frac{-2i}{-2i} = i^3 = -i$$

RADICI

$$w_k = \rho^{1/n} \left[\cos \frac{\vartheta + 2k\pi}{n} + i \sin \frac{\vartheta + 2k\pi}{n} \right],$$

$\sqrt[n]{\rho}$

4.9 Determinare le tre radici cubiche del numero $z = 1$.

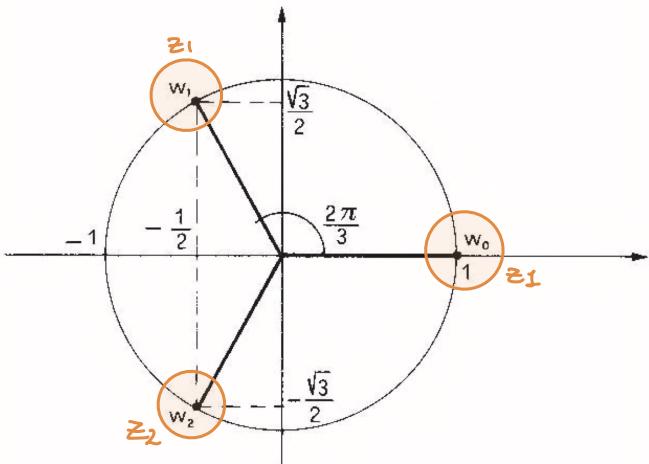
$$z = 1 ; \quad \sqrt[3]{1} = \frac{z_1}{z_2} = z_3$$

$$\Rightarrow z = 1 [\cos \theta + i \sin \theta]$$

$$z_0 = \sqrt[3]{1} \left[\cos \left(\frac{0+20\pi}{3} \right) + i \sin \left(0 \right) \right] = \cos^0 + i \sin^0 = 1$$

$$z_1 = \sqrt[3]{1} \left[\cos \left(\frac{2\pi}{3} \right) + i \sin \left(\frac{2}{3}\pi \right) \right] = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$



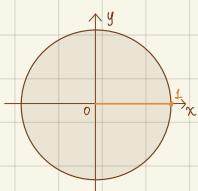
4.10 Determinare le radici quadrate dei numeri complessi

- (a) 1 (b) -1 (c) i (d) -i

$$z = 1 \Rightarrow \varphi = 1 \quad \theta = 0 \Rightarrow 1 [\cos(\theta) + i \sin(\theta)] = 1$$

$$z_0 = \sqrt{1} \left[\cos(0) + i \sin(0) \right] = 1$$

$$z_1 = \sqrt{1} \left[\cos(2\pi) + i \sin(2\pi) \right] = 1$$



$$\text{Ad occhio... } z = 1 \Rightarrow z = i^2 \Rightarrow \sqrt{i^2} = \pm 1$$

$$\text{b) } z = -1 \Rightarrow \sqrt{-1} = i \text{ per definizione}$$

$$\text{c) } z = i \Rightarrow \sqrt{i} \text{ per definizione}$$

$$\Rightarrow z = 1 \left[\cos \left(\frac{\pi}{2} \right) + i \sin \left(\frac{\pi}{2} \right) \right] = i$$

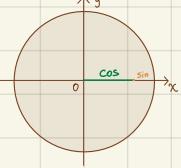
$$z_0 = \sqrt{1} \left[\cos(0) + i \sin(0) \right] = 1$$

$$z_1 = \sqrt{1} \left[\cos \left(\frac{2\pi}{2} \right) + i \sin \left(\frac{2\pi}{2} \right) \right] = 1$$

Una radice $\sqrt[n]{x}$ ha esattamente n radici distinte
Inoltre, se z è una radice, anche \bar{z} lo è.

$$\varphi_z = 1 \quad \theta = 0$$

$$\tan \frac{b}{a} = \frac{0}{1} \Rightarrow \tan^{-1} 0 = 0 \\ \Rightarrow \cos 0 = \cos 0 = \frac{\pi}{2} \\ \sin 0 = 0$$



4.10 Determinare le radici quadrate dei numeri complessi



$$z = bi, \quad b > 1 \quad \begin{cases} \varphi = \frac{|b|}{b} = 1 \\ \theta = \frac{\pi}{2} \end{cases}$$

Equazioni con numeri complessi

4.16 Risolvere le equazioni algebriche di primo grado

$$(a) iz + 1 = 0 \quad (b) (2+i)z - 4 + 3i = 0$$

$$a) iz + 1 = 0 \Rightarrow z = -\frac{1}{i}; z = -\frac{1}{i} \cdot \frac{i}{i} = \frac{-i}{-1} = i$$

$$b) (2+i)z - 4 + 3i = 0; (2+i)z = \frac{4-3i}{2+i}; z = \frac{4-3i}{2+i} \cdot \frac{2-i}{2-i} = \frac{8-4i-6i+3i^2}{4-2i+2i-i^2} = \frac{8-10i-3}{5}$$

$$z = \frac{5-10i}{5}; z = \frac{5}{5} - \frac{10}{5}i; z = 1 - 2i$$

4.17 Risolvere le equazioni di primo grado

$$(a) (1-i)z - 2 = 0 \quad (b) iz + 1 - i = 0$$

$$a) (1-i)z - 2 = 0; z = \frac{2}{1-i} \cdot \frac{1+i}{1+i} = \frac{2+2i}{1+i-i^2}; z = \frac{2+2i}{2} \Rightarrow z = 1+i$$

$$b) iz + 1 - i = 0; z = -\frac{1+i}{i} \cdot \frac{1}{1} \Rightarrow z = -\frac{i+i^2}{i^2}; z = \frac{-i-1}{-1} \Rightarrow z = 1+i$$

4.18 Risolvere l'equazione di secondo grado $z^2 + z + 1 = 0$.

$$z^2 + z + 1 = 0 \quad z_{1,2} = \frac{-1 \pm \sqrt{1-4 \cdot 1 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 + \sqrt{-1} \cdot \sqrt{3}}{2} = \frac{-1 + i\sqrt{3}}{2}$$

$$\quad \quad \quad z_2 = \frac{-1 - \sqrt{-3}}{2} = \frac{-1 - i\sqrt{3}}{2}$$

4.19 Risolvere nel campo complesso le equazioni di secondo grado

$$(a) z^2 + 4 = 0 \quad (b) z^2 - 4 = 0$$

$$a) z^2 + 4 = 0 \Rightarrow z^2 = -4 \Rightarrow z = \sqrt{-4} \Rightarrow z = \pm 2i$$

$$b) z^2 - 4 = 0 \Rightarrow z = \pm 2$$

4.20 Risolvere nel campo complesso le equazioni

$$(a) 5z^2 - 4z + 1 = 0 \quad (b) z^2 + 4z + 5 = 0$$

$$a) 5z^2 - 4z + 1 = 0 \quad z_{1,2} = -\left(\frac{b}{2}\right) \pm \sqrt{\left(\frac{b}{2}\right)^2 - ac} = \frac{2 \pm \sqrt{4-5}}{5} = \frac{2+i}{5} \quad z_1 = \frac{2+i}{5}$$

$$z_2 = \frac{2-i}{5}$$

$$b) z^2 + 4z + 5 = 0 \quad z_1 = -2 + \sqrt{-16} = -2 + 4i \quad ? \quad \text{Il risultato dovrebbe essere } 2+4i \\ z_{1,2} = -2 \pm \sqrt{4-4 \cdot 5} = z_2 = -2 - \sqrt{-16} = -2 - 4i \quad ?$$

4.21 Risolvere l'equazione $4z^2 - 4z + 3 - 2\sqrt{3}i = 0$.

$$4z^2 - 4z + 3 - 3\sqrt{3}i = 0; 4(z^2 - z) + 3(1 - \sqrt{3}i) = 0$$

$$\frac{4 \pm \sqrt{16-4 \cdot 4 \cdot (3-3\sqrt{3}i)}}{8} \quad z_1 = \frac{4 + \sqrt{16-16(3-3\sqrt{3}i)}}{8} = \frac{4 + \sqrt{16-48+48\sqrt{3}i}}{8}$$

$$z_2 = \frac{4 - \sqrt{16-16(3-3\sqrt{3}i)}}{8} = \frac{4 - \sqrt{16-48+48\sqrt{3}i}}{8}$$

