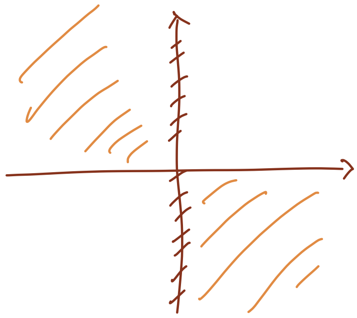
2. Studiare la seguente funzione e disegnarne il grafico: $y = \arctan\left(\frac{e^x}{e^x - 1}\right)$.



1) Dominio $\quad e^x - 1 \neq 0$ per $e^x \neq 1 \to x \neq \ln(1) \to x \neq 0$ ✓

2) Simm: $\arctan\left(\frac{e^{-x}}{e^{-x}-1}\right) = \arctan\left(\frac{e^{-x}}{\frac{1}{e^{x}}-1}\right) = \arctan\left(\frac{e^x}{1-e^{-x}}\right) \begin{cases} \neq f(x) \\ \neq -f(x) \end{cases}$

3) Segno $f(x)>0 \to \arctan\left(\frac{e^x}{e^x-1}\right)>0$ per $\frac{\cancel{e^x} \to >0\ \forall x\in\mathbb{R}}{e^x-1} \Rightarrow e^x > 1$ per $x>0$ ✓

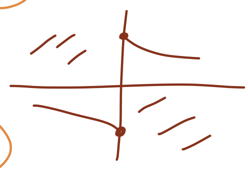
4) Intersez:

$\begin{cases} y = \arctan\left(\frac{e^x}{e^x-1}\right) \\ x=0 \quad \nexists x \in D \end{cases}$ $\qquad \begin{cases} y = \arctan\left(\frac{e^x}{e^x-1}\right) \\ y=0 \end{cases} \to \arctan\left(\frac{e^x}{e^x-1}\right)>0$ per $\frac{\cancel{e^x} \to \neq 0}{e^x-1} = 0 \quad \nexists x\in\mathbb{R} \Rightarrow$ NO INT

5) Asintoti

$\lim_{x\to 0^+} \arctan\left(\frac{e^x}{e^x-1}\right) = \arctan\left(\frac{1}{1-1}\right) = \arctan\left(\frac{1}{0^+}\right) = \arctan(+\infty) \to +\frac{\pi}{2} \Rightarrow$ No Asintoto

$\lim_{x\to 0^-} f(x) = \arctan\left(\frac{1}{1^- -1}\right) = \arctan\left(\frac{1}{0^-}\right) \to \arctan(-\infty) \to -\frac{\pi}{2}$



$\lim_{x\to +\infty} \arctan\left(\frac{e^x}{e^x-1}\right) = \arctan\left(\frac{\cancel{e^x}}{\cancel{e^x}(1-0)}\right) = \arctan(1) \to \frac{\pi}{4} \to y = \frac{\pi}{4}$ A.O.Dx

$\lim_{x\to -\infty} \arctan\left(\frac{e^x}{e^x-1}\right) \to e^{-\infty} = \frac{1}{e^{\infty}} \to 0 \Rightarrow \arctan\left(\frac{0}{0-1}\right) = \arctan(0) \to \emptyset$ A.O.Sx

6) Derivata

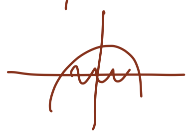
$$f'(x) = \frac{\frac{e^x(e^x-1)-e^{2x}}{(e^x-1)^2}}{1+\left(\frac{e^x}{e^x-1}\right)^2} = \frac{\frac{e^{2x}-e^x-e^{2x}}{e^{2x}-2e^x+1}}{1+\frac{e^{2x}}{e^{2x}-2e^x+1}} = \frac{-e^x}{\frac{e^{2x}-2e^x+1+e^{2x}}{e^{2x}-2e^x+1}} = \frac{\frac{-e^x}{2e^{2x}-2e^x+1}}{e^{2x}-2e^x+1} =$$

$$= \frac{-e^{3x}+2e^{2x}-e^x}{2e^{2x}-2e^x+1} > 0$$

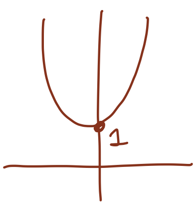
N: $-t^3+2t^2-t>0$ per $t(-t^2+2t-1)>0$

$\to t>0 \qquad t\neq 1$

$\Delta = 4-4\cdot(-1)(-1) = 0 \to t_{1/2} = \frac{-2}{-2} = 1 \qquad a<0,\ eq>0$ Val interni



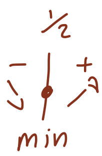
D: $2t^2-2t+1>0 \quad \Delta = 4-4\cdot 2\cdot 1 = <0$



$D_y = \forall x\in\mathbb{R}$ $\quad$ int $y \to y=1 \quad (0,1)\in f_1$

int $x \to y=0 \to 2t^2-2t+1=0$?

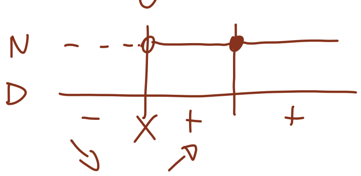
$f_1'(x) = 4t-2>0$ per $t>\frac{1}{2}$



$f\left(\frac{1}{2}\right) = \cancel{2}\cdot\frac{1}{\cancel{4}} - 2\frac{1}{2}+1 = \frac{1}{2}$

$2t^2-2t+1>0 \quad \forall x\in\mathbb{R}$

1- dovevano cambiare $t=e^x$ ma è uguale



$\Rightarrow$ No Max Min

$$f''(x) = \frac{(-3e^{3x}+4e^{2x}-e^x)(2e^{2x}-2e^x+1)-(-e^{3x}+2e^{2x}-e^x)(4e^{2x}-2e^x)}{\cancel{(2e^{2x}-2e^x+1)^2} > 0\ \forall x\in\mathbb{R}} > 0$$

$-6e^{5x}+6e^{4x}-3e^{3x}+8e^{4x}-8e^{3x}+4e^{2x}-2e^{3x}+2e^{2x}-e^x \,[+4e^{5x} - 2e^{4x} - 8e^{4x}+4e^{3x}+4e^{3x}-2e^{2x}] > 0$

$-10e^{4x} \qquad 8e^{3x}$

$-2e^{5x}+4e^{4x}-5e^{3x}+4e^{2x}-e^x>0$

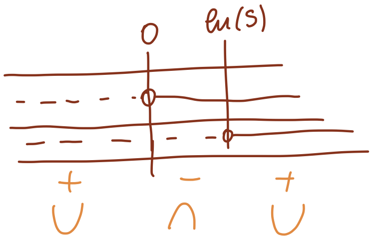
$e^x(-2e^{4x}+4e^{3x}-5e^{2x}+4e^x-1)>0$ ↳ $e^x>0\ \forall x\in\mathbb{R}$

$e^x(-2e^{3x}+4e^{2x}-5e^x+4)>1$ ↳ $e^x>1$ per $x>0$

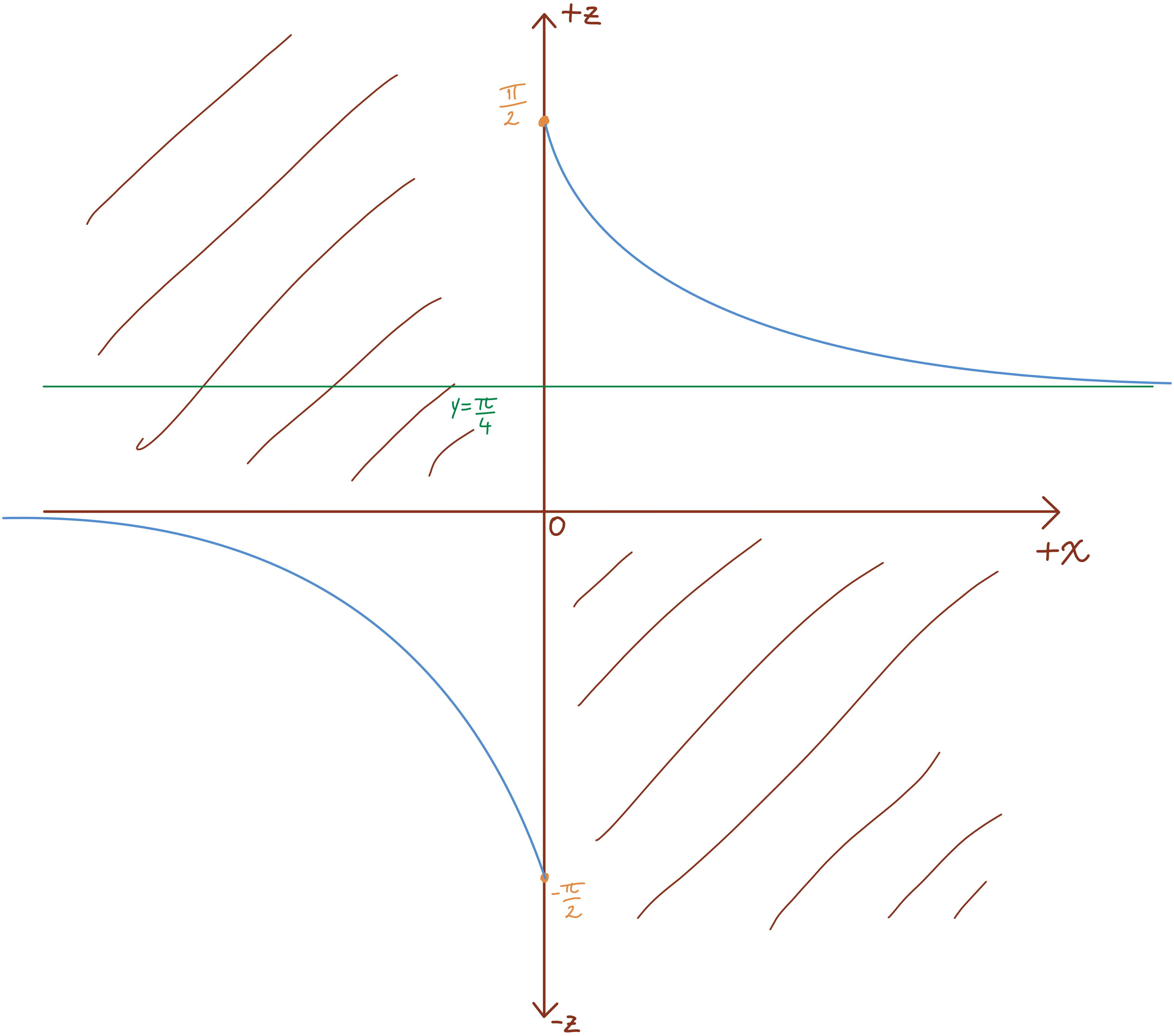
$e^x(-2e^{2x}+4e^x-5)>-4$ ↳ $e^x>-4\ \forall x\in\mathbb{R}$

$e^x(-2e^x+4)>5$ ↳ $e^x>5$ per $x>\ln(5)$

$e^x<2\ \nexists x\in\mathbb{R}$



Ho sbagliato qualcosa



5. Calcolare l'integrale del seguente problema di Cauchy:

$$\begin{cases} y'' + y' - 2y = \cos x \\ y(0) = y'(0) = 0. \end{cases}$$

$\lambda^2 + \lambda - 2 = 0 \to \Delta = 1 - 4(-2) = 9 > 0$

$\lambda_{1,2} = \frac{-1 \pm 3}{2} \begin{matrix} \nearrow 1 \\ \searrow -2 \end{matrix} \Rightarrow y_0(x) = c_1 e^{x} + c_2 e^{-2x}$

$f(x) = \cos x \to \gamma = 0$ NO RADICE

$\to y_p(x) = e^{\gamma x}\left[A\cos(wx) + B\sin(wx)\right] \to y_p(x) = A\cos(x) + B\sin(x)$

$\to y_p'(x) = -A\sin(x) + B\cos(x)$, $y_p''(x) = -A\cos(x) - B\sin(x)$

$\Rightarrow -A\cos(x) - B\sin(x) - A\sin(x) + B\cos(x) - 2A\cos(x) - 2B\sin(x) = \cos(x)$

$\to \cos(x)\left[-A + B - 2A\right] + \sin(x)\left[-B - A - 2B\right] = \cos(x)$

$\Rightarrow \begin{cases} B - 3A = 1 \\ -3B - A = 0 \end{cases} \begin{cases} B = 1 + 3A \to B = 1 - \frac{9}{10} = \frac{10-9}{10} = \frac{1}{10} \; B \\ -3(1+3A) - A = 0 \to -3 - 9A - A = 0 \to -10A = 3 \to A = -\frac{3}{10} \end{cases}$

$\Rightarrow \boxed{y_p(x) = c_1 e^{x} + c_2 e^{-2x} - \frac{3}{10}\cos(x) + \frac{1}{10}\sin(x)}$

$y_p'(x) = c_1 e^{x} - 2c_2 e^{-2x} + \frac{3}{10}\sin x + \frac{1}{10}\cos(x)$

$y(0) = c_1 + c_2 - \frac{3}{10} = 0 \to c_1 = -c_2 + \frac{3}{10} \Rightarrow c_1 = -\frac{2}{15} + \frac{3}{10} = \frac{-20+45}{150} = \frac{25}{150} \to c_1 = \frac{1}{6}$

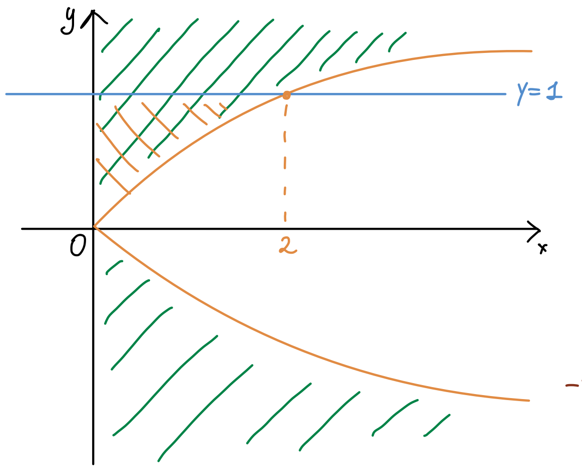
$y'(0) = c_1 - 2c_2 + \frac{1}{10} = 0 \to \frac{3}{10} - c_2 - 2c_2 + \frac{1}{10} = 0 \to -3c_2 = -\frac{4}{10} \to c_2 = \frac{4}{30} \to c_2 = \frac{2}{15}$

$c_1 = -c_2 + \frac{3}{10}$

$\Rightarrow$ Soluzione $\Rightarrow \frac{1}{6}e^{x} + \frac{2}{15e^{2x}} - \frac{3}{10}\cos(x) + \frac{1}{10}\sin x$

6. Calcolare il seguente integrale doppio $\iint_D y e^{y^2+x}\,dx\,dy$, dove

$$D=\left\{(x,y)\in\mathbb{R}^2 : 0\le x\le 2y^2,\ 0\le y\le 1\right\}.$$



$x=2y^2 \Rightarrow y=\pm\sqrt{\frac{1}{2}x}$

$D_x: \{(x,y) / 0<x<2, \sqrt{\frac{1}{2}x}<y<1\}$

$$\Rightarrow \int_0^2\int_{\sqrt{\frac{1}{2}x}}^1 y e^{y^2+x}\,dy\,dx = \int_0^2 e^x \underbrace{\int_{\sqrt{\frac{1}{2}x}}^1 y e^{y^2}\,dy}_{a}\,dx$$

$\Rightarrow$ pongo $t=y^2 \Rightarrow dy=\frac{1}{2y}dt$

$$\Rightarrow \int_{\sqrt{\frac{1}{2}x}}^1 \cancel{y} e^{y^2}\cdot\frac{1}{2\cancel{y}}\,dt = \frac{1}{2}\int e^t\,dt = \frac{1}{2}\left[e^{y^2}\right]_{\sqrt{\frac{1}{2}x}}^1$$

$$= \frac{1}{2}\left[e - e^{\sqrt{\frac{1}{2}x}}\right] \Rightarrow \frac{1}{2}\int_0^2 e^x\cdot\left[e-e^{\sqrt{\frac{1}{2}x}}\right] = \underbrace{\frac{1}{2}\int_0^2 e^{2x}\,dx}_{a)} - \frac{1}{2}\int_0^2 e^{\sqrt{\frac{1}{2}x}\,x}\,dx$$

a) $\frac{1}{4}\left[e^{2x}\right]_0^2 = \frac{1}{4}e^4 - \frac{1}{4}e \Rightarrow \frac{1}{4}e\left[e^3-1\right] \sim 12.9$

b) $-\frac{1}{2}\int e^{x\sqrt{\frac{1}{2}x}}\,dx \quad t=\frac{1}{2}x \Rightarrow dx=2dt \Rightarrow -\int e^{2t\sqrt{t}}\,dt \overset{\text{Parti}}{=} -te^{2t\sqrt{t}} - \int 3\sqrt{t}\,e^{2t\sqrt{t}}\,dt$

$\Rightarrow x=2t$

$$e^{2t\sqrt{t}} = \left(2\sqrt{t}+2t\frac{1}{2\sqrt{t}}\right)\cdot e^{2t\sqrt{t}} = \frac{4t+2t}{2\sqrt{t}} = \frac{3t}{\sqrt{t}}e^{2t\sqrt{t}}$$

$$= \frac{3\cancel{t}\sqrt{t}}{\cancel{t}} = 3\sqrt{t}\,e^{2t\sqrt{t}}$$

$$\overset{\text{Parti}}{=} -te^{2t\sqrt{t}} - \left[\frac{1}{2}t\sqrt{t}\,e^{2t\sqrt{t}} - \frac{3}{2}\int t^2 e^{2t\sqrt{t}}\,dt\right]$$

PARTI

4. Dopo averne studiato il carattere, calcolare l'integrale: $\int_{2/\pi}^{+\infty} \frac{1}{x^2} \cos^3 \frac{1}{x} dx.$

$$\int \frac{1}{x^2} \cos^3\left(\frac{1}{x}\right) dx$$

$$= \text{pongo } t = \frac{1}{x} \Rightarrow dx = \frac{1}{-\frac{1}{x^2}} = -x^2 dt \Rightarrow -\int \frac{1}{\cancel{x^2}} \cos^3\left(\frac{1}{x}\right) \cancel{x^2} dt = -\int \cos^3(t)\, dt$$

$$= -\int \underbrace{\cos^2(t)} \cdot \cos(t)\, dt$$

Tempo ~ 15'

$$\Rightarrow \int \cos^2 t \cdot \cos t\, dt = \int (1 - \sin^2 t) \cos t\, dt = \int \cos t\, dt - \int \sin^2 t \cos t\, dt$$

$$\underset{\text{Parti}}{=} \sin^3 t - 2\int \sin^2 t \cos t\, dt = \text{Pongo } I = \int \sin^2 t \cos t\, dt$$

$$\Rightarrow \sin^3 t - 2I = I \Rightarrow I = \frac{1}{3}\sin^3 t + c$$

$$\Rightarrow \sin t - \frac{1}{3}\sin^3 t = \sin t\left(1 - \frac{1}{3}\sin^2 t\right) + c \quad \Rightarrow t = 1/x \Rightarrow \sin\left(\frac{1}{x}\right) - \frac{1}{3}\sin^3\left(\frac{1}{x}\right) + c$$

$$\Rightarrow \left[\sin\left(\frac{1}{x}\right) - \frac{1}{3}\sin^3\left(\frac{1}{x}\right)\right]_{\frac{2}{\pi}}^{\infty} = \lim_{m \to +\infty} \underbrace{\sin}_{0} \underbrace{\frac{1}{m}}_{0}\left(\underbrace{1}_{1} - \cancel{\frac{1}{3}\sin^2\left(\frac{1}{m}\right)}_{0}\right) - \left[\sin\left(\frac{\pi}{2}\right) - \frac{1}{3}\sin^3\left(\frac{\pi}{2}\right)\right] =$$

$$= -\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin^3\left(\frac{\pi}{2}\right)$$