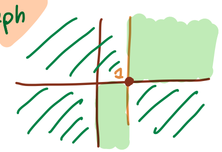
1. Studiare la seguente funzione e disegnarne il grafico: $y = \ln^3 x$.

Tempo: 18" No graph
20" graph.



1) Dominio: $\ln^3 x \to$ $x>0$

2) Segno: $f(x)>0 \to \ln^3 x > 0$ per $\ln x > 0 \to$ per $e^{\ln(x)} > e^0 \to$ $x>1$

3) Simm $\to$ $f(-x) = \ln^3(-x) \neq \begin{cases} f(x) \\ -f(x) \end{cases}$ No simm

4) Inters.

$\begin{cases} y = \ln^3 x \\ x = 0 \end{cases} \to \cancel{y = \ln^3 0}$ $\quad 0 \notin D$

$\begin{cases} y = \ln^3 x \\ y = 0 \end{cases} \to \ln^3 x = 0$ per $x = 1$

$\Rightarrow (1,0) \in f(x)$

5) Asintoti

$\lim_{x \to 0^+} \ln^3 x = \ln^3 0^+ \to -\infty \Rightarrow$ $y=0$ A.V.

$\lim_{x \to +\infty} \ln^3 x = \ln^3 \infty = +\infty \Rightarrow$ NO A.O. $\quad \lim_{x \to -\infty} \ln^3 \cancel{(-\infty)} =$

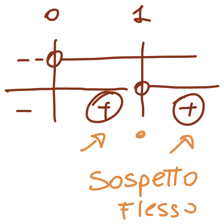
6) Max Min

$f'(x) = 3\ln^2(x) \cdot \dfrac{1}{x} \Rightarrow f'(x) > 0$ per $\quad$ N: $3\ln^2(x) > 0$ $\quad$ D: $x>0$

$\ln(x) > 0$ per $x > 1$

$3\ln^2(x) > 0 \ \forall x \in \mathbb{R} - \{1\}$

$\Rightarrow f'(x) > 0$




$0 < x < 1 \cup x > 1$

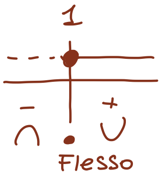
oppure $f'(x) > 0$ per $\{x \,/\, x > 0 - \{x = 1\}\}$

7) Flessi e concavita'

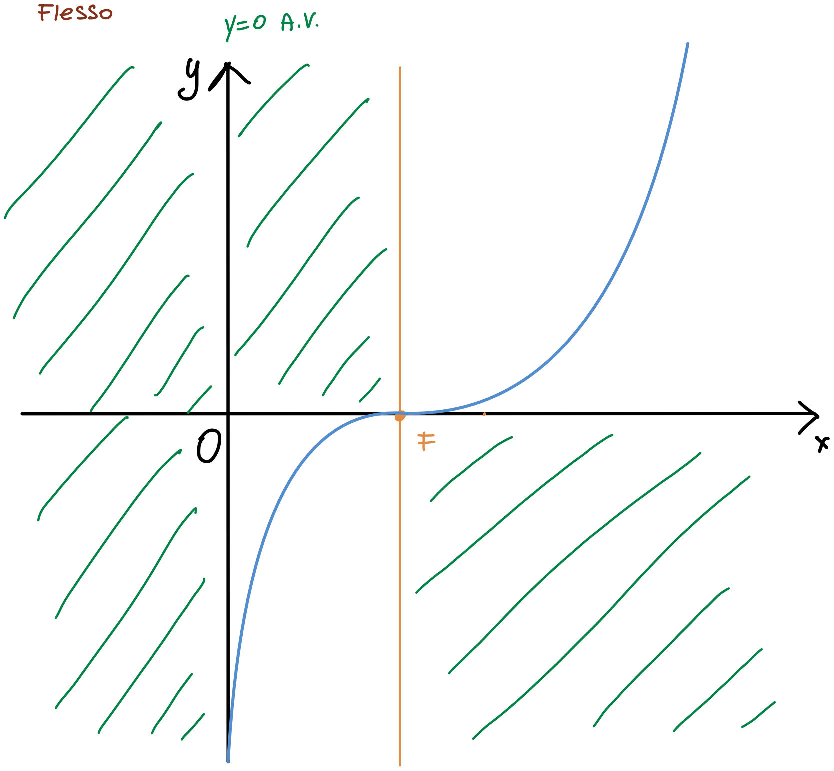
$f''(x) = D\left[\dfrac{3\ln^2(x)}{x}\right] \to \dfrac{6\ln(x) - 3\ln(x)}{x^2} = \dfrac{3\ln(x)}{x^2} > 0$

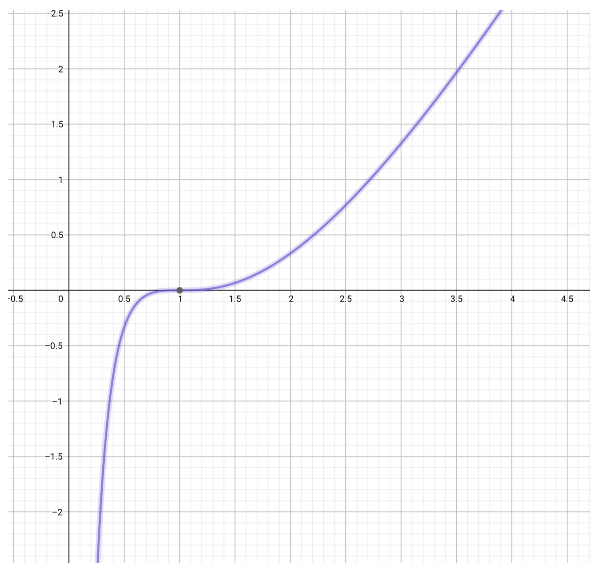
$3\ln(x) \to > 0$ per $x > 1$

$x^2 \to$ sempre pos




$f(1) = 0 \Rightarrow (1,0)$ FLESSO



2. Calcolare il seguente integrale: $\displaystyle\int_1^{+\infty} \frac{1}{e^{2x}+1}dx.$

$$\int_1^{+\infty} \frac{1}{e^{2x}+1}\,dx \qquad deg(N) < deg(D) \quad \checkmark$$

Pongo $t = e^{2x}+1 \Rightarrow dx = \frac{1}{2}\frac{1}{e^{2x}}dt \Rightarrow \frac{1}{2}\int_2^{+\infty} \frac{1}{t}\cdot\frac{1}{e^{2x}}\,dt \underset{e^{2x}=t-1}{=} \frac{1}{2}\int_1^{+\infty} \frac{1}{t}\cdot\frac{1}{t-1}\,dt$

Tempo 57'

$$= \frac{1}{2}\int_1^{+\infty} \frac{1}{t(t-1)}\,dt$$

$$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1} = \frac{At-A+Bt}{t(t-1)} = 1 \Rightarrow \begin{cases} A+B=0 \Rightarrow -1+B=0 \Rightarrow B=1 \\ -A = 1 \Rightarrow A=-1 \end{cases}$$

$$\Rightarrow -\frac{1}{t} + \frac{1}{t-1} \Rightarrow -\frac{1}{2}\int_1^{+\infty}\frac{1}{t}\,dt + \frac{1}{2}\int_1^{+\infty}\frac{1}{t-1}\,dt = -\frac{1}{2}\left[\ln(t)\right]_1^{+\infty} + \frac{1}{2}\left[\ln(t-1)\right]_1^{+\infty} \qquad \underline{t = e^{2x}+1}$$

$$\Rightarrow -\frac{1}{2}\left[\ln(e^{2x}+1)\right]_1^{+\infty} + \frac{1}{2}\left[\ln(e^{2x})\right]_1^{+\infty} = -\frac{1}{2}\left[\ln(e^{2x}+1)\right]_1^{+\infty} + \frac{1}{\cancel{2}}\cancel{2}x = \left[x\right]_1^{+\infty} - \frac{1}{2}\left[\ln(e^{2x}+1)\right]_1^{+\infty}$$

Pongo $+\infty = c$

$$\Rightarrow \underbrace{c - 1}_{\text{cost}} - \frac{1}{2}\ln(e^{2c}+1) + \underbrace{\frac{1}{2}\ln(e^{2}+1)}_{\text{cost}} = \underset{\downarrow\ +\infty}{c} - 1 + \underset{\downarrow\ -\infty}{\frac{1}{2}\ln}\left(\frac{e^{2}+1}{e^{2c}+1}\right)_{\downarrow\ 0}$$

$$\Rightarrow \left\{\lim_{c\to+\infty} \underbrace{\left[c - \frac{1}{2}\ln(e^{2c}+1)\right]}\right\} + \frac{1}{2}\ln(e^2+1) - 1 = \left\{\frac{1}{2}\lim_{c\to+\infty} 2c - \ln(e^{2c}+1)\right\} + \frac{1}{2}\ln(e^2+1) - 1$$

$$\Rightarrow \frac{1}{2}\left(2c - \ln(e^{2c}+1)\right)$$

Pongo $t = 2c$

$$\overset{t=2c}{\Rightarrow} \left\{\frac{1}{2}\lim \underset{\ln(e^t)}{t} - \ln(e^t+1)\right\} + \frac{1}{2}\ln(e^2+1) - 1 = \left\{\frac{1}{2}\lim \ln(e^t) - \ln(e^t+1)\right\} + \dots$$

$$= \left\{\frac{1}{2}\lim \ln\left(\frac{e^t}{e^t+1}\right)\right\} + \dots = \frac{1}{2}\lim_{c\to+\infty} \ln\left(\frac{\cancel{e^t}}{\cancel{e^t}(1+0)}\right)\Bigg\} + \frac{1}{2}\ln(e^2+1) - 1$$

$\ln(1) = 0$

$$\Rightarrow \int_1^{+\infty}\frac{1}{e^{2x}+1}\,dx = \frac{1}{2}\ln(e^2+1) - 1$$

4. Determinare gli estremi relativi della seguente funzione: $f(x,y) = x^2 - xy^2 - y^3$.

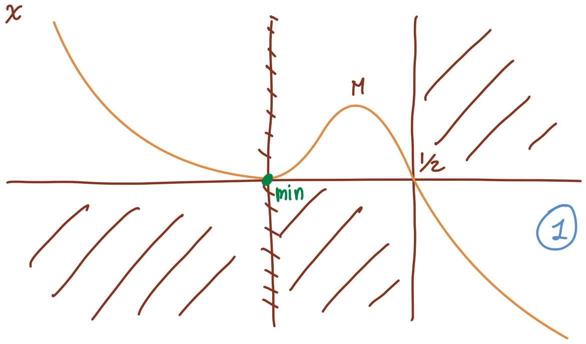
Si trovano due punti dove si annulla il gradiente: $P_0(0,0)$, $H=0$, $P_1(\frac{9}{2},-3)$, $H<0 \to$ Sella

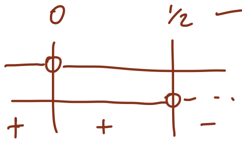
Studio la funzione in $I(0,0)$

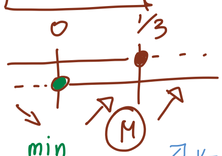
$\begin{cases} x^2-xy^2-y^3 \\ y=x \end{cases} \to x^2-x^3-x^3 \to x^2-2x^3$

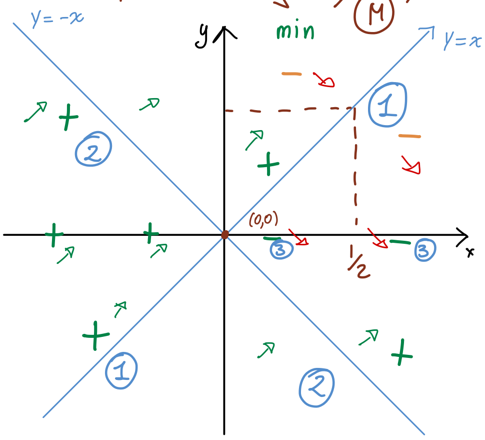
$f>0$ per $x^2(1-2x)>0$
$\to \forall x - \{x=0\}$
$\to x<\frac{1}{2}$

$f' = 2x-6x^2>0$
per $x(2-6x)>0$
$\to x>0$
$x<\frac{1}{3}$



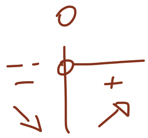



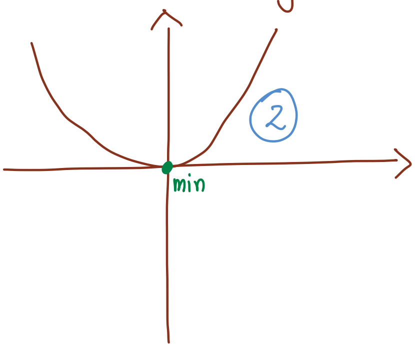




Lungo $y=-x$

$\begin{cases} x^2-xy^2-y^3 \\ y=-x \end{cases} \to x^2-\cancel{x^3}+\cancel{x^3} \to f=x^2 \Rightarrow f>0 \;\forall x\in\mathbb{R}$

$f' = 2x>0$ per $x>0$



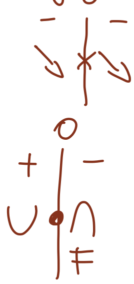



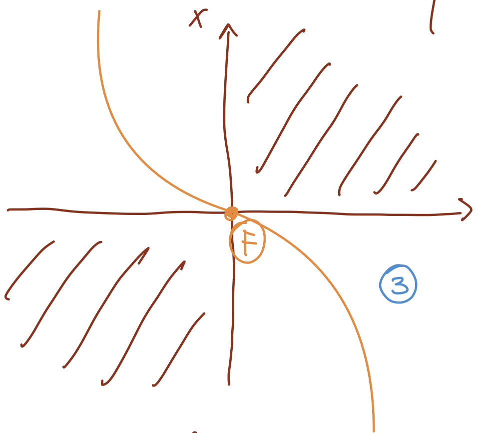

Lungo $x=0$

$\begin{cases} x^2-xy^2-y^3 \\ x=0 \end{cases} \to -y^3>0$ per $y<0$

$f' = -3y^2>0 \;\forall x\in\mathbb{R}$

$f'' = -6y>0$
per $y<0$




$\Rightarrow$ Lungo le rette $y=x$ e $y=-x$ (bisettrici) la situazione sembra indicare la presenza di un minimo, ma controllando anche lungo la retta $x=0$ (asse $y$) si nota che il punto è un punto sella.

5. Calcolare l'integrale del seguente problema di Cauchy: $\begin{cases} y'' + y' = x^2 - 1, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$

$$y'' + y' = x^2 - 1$$

$$\lambda^2 + \lambda = 0 \Rightarrow \lambda(\lambda + 1) = 0 \quad \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = -1 \end{matrix} \Rightarrow y_0(x) = c_1 + c_2 e^{-x}$$

$$\Rightarrow f(x) = x^2 - 1 \Rightarrow \gamma = 0 \quad \text{sol di } \underset{h=1}{\text{mul } 1} \Rightarrow x^h \cdot e^{\gamma x} \cdot (P^n(x)) \Rightarrow y_0(x) = x \cdot (Ax^2 + Bx + C)$$
$$= Ax^3 + Bx^2 + Cx$$

$$\Rightarrow y_0'(x) = 3Ax^2 + 2Bx + C \qquad y_0''(x) = 6Ax + 2B$$

$$\Rightarrow 6Ax + 2B + 3Ax^2 + 2Bx + C = x^2 - 1 \Rightarrow x^2(3A) + x(6A + 2B) + 2B + C = x^2 - 1$$

$$\Rightarrow \begin{cases} 3A = 1 \Rightarrow A = 1/3 \\ 6A + 2B = 0 \quad 2 + 2B = 0 \Rightarrow B = -1 \\ 2B + C = -1 \quad -2 + C = -1 \Rightarrow C = 1 \end{cases}$$

$$\Rightarrow y_p(x) = x\left(\frac{1}{3}x^2 - x + 1\right)$$

$$\Rightarrow y(x) = c_1 + c_2 e^{-x} + x\left(\frac{1}{3}x^2 - x + 1\right)$$

Cauchy

$$y(0) = c_1 + c_2 \underbrace{e^{-x}}_{1} + \cancel{x\left(\frac{1}{3}x^2 - x + 1\right)}_{0} = 0 \Rightarrow c_1 + c_2 = 0 \quad \begin{matrix} c_1 = -c_2 \Rightarrow c_1 = 0 \\ c_2 = -c_1 \end{matrix}$$

$$y'(0) = -c_2 \underbrace{e^{-x}}_{1} + \cancel{\frac{1}{3}x^2} - \cancel{x} + 1 + \cancel{\frac{2}{3}x^2} - \cancel{x} = 1 \Rightarrow c_2 = 0$$

$$\Rightarrow \text{Soluzione. } y(x) = \frac{1}{3}x^3 - x^2 + x = x\left(\frac{1}{3}x^2 - x + 1\right)$$

$$\int_0^2 dy \int_y^{\frac{1}{2}y+1} xy\, dx = \int_0^2 y \left[\frac{x^2}{2}\right]_y^{\frac{1}{2}y+1} dy$$

$$= \int_0^2 y \cdot \left\{\left[\frac{(\frac{1}{2}y+1)^2}{2}\right] - \left[\frac{y^2}{2}\right] dy = \int_0^2 y \left[\frac{\frac{1}{4}y^2 + y + 1 - y^2}{2}\right] dy\right.$$

$$= \int_0^2 y \left[\frac{-\frac{3}{4}y^2 + y + 1}{2}\right] dy = \underbrace{\int_0^2 \frac{-\frac{3}{4}y^3}{2} dy}_{a} + \frac{1}{2}\int_0^2 \underbrace{y^2 dy}_{b} + \frac{1}{2}\int_0^2 \underbrace{y\, dy}_{c}$$

Integrale secondo $y$ probabilmente sbagliato

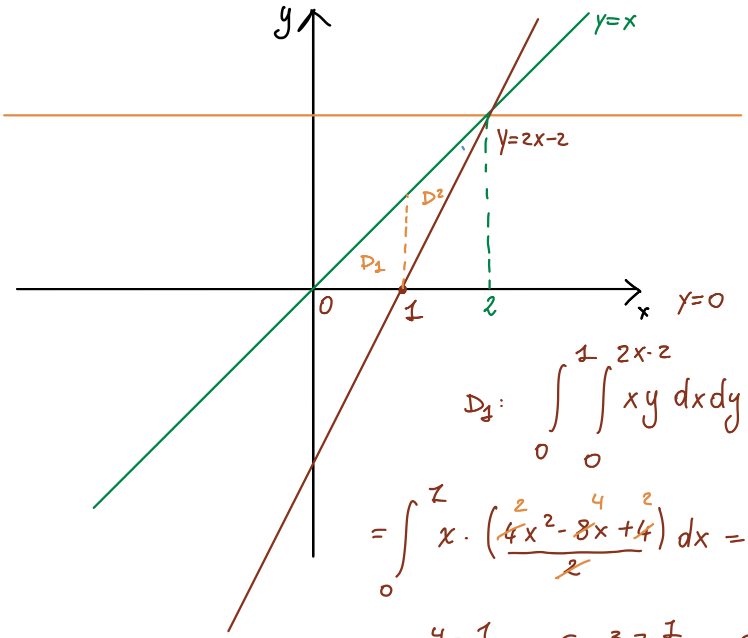
(a) $\frac{-\frac{3y}{4}}{2} = -\frac{6}{4}y = -\frac{3}{2}y \Rightarrow -\frac{3}{2}\int_0^2 y^3 dy = -\frac{3}{2}\left[\frac{y^4}{4}\right]_0^2 = -\frac{3}{2}\left[\frac{\cancel{16}^{4}}{\cancel{4}}\right] = -6$

b) $\frac{1}{2}\int_0^2 y^2 dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^2 = \frac{1}{\cancel{2}}\left[\frac{\cancel{8}^{4}}{3}\right] = \frac{4}{3}$ c) $\frac{1}{2}\int_0^2 y\, dy = \frac{1}{2}\left[\frac{y^2}{2}\right]_0^2 = \frac{1}{\cancel{2}}\left[\frac{\cancel{4}^{2}}{\cancel{2}}\right] = 1$

$$\Rightarrow \iint f(x,y)\, dx\, dy = -6 + \frac{4}{3} + 1 = \frac{-18 + 4 + 3}{3} = -\frac{11}{3}$$

6. Calcolare l'integrale doppio $\displaystyle\iint_D xy\,dx\,dy$, dove $D = \left\{(x,y) : 0 \le y \le 2,\ y \le x \le \frac{y}{2}+1\right\}$.

$$\begin{cases} y = 2x-2 \\ y = 0 \end{cases} \Rightarrow 2x-2=0 \Rightarrow x = 1$$




Rispetto ad $x$: $D_1: \{(x,y) / 0<x<1,\ 0<y<2x-2\}$

$D_2: \{(x,y) / 1<x<2,\ x<y<2x-2\}$

$$D_1: \int_0^1\int_0^{2x-2} xy\,dx\,dy = \int_0^1 x\int_0^{2x-2} y\,dy\,dx = \int_0^1 x\cdot\left[\frac{y^2}{2}\right]_0^{2x-2} dx$$

$$= \int_0^1 x\cdot\left(\frac{\cancel{4}^{\,2}x^2-\cancel{8}^{\,4}x+\cancel{4}^{\,2}}{\cancel{2}}\right)dx = 2\int_0^1 x^3\,dx - 4\int_0^1 x^2\,dx + 2\int_0^1 x\,dx$$

$$= 2\left[\frac{x^4}{4}\right]_0^1 - 4\left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^2}{2}\right]_0^1 = \cancel{2}\cdot\frac{1}{\cancel{4}} - 4\,\frac{1}{3} + \cancel{2}\,\frac{1}{\cancel{2}} = \frac{1}{2} - \frac{4}{3} + 1 = \frac{3-8+6}{6} = \frac{1}{6}$$

$$D_2: \int_1^2\int_x^{2x-2} xy\,dx\,dy = \int_1^2 x\int_x^{2x-2} y\,dy\,dx = \int_1^2 x\left[\frac{y^2}{2}\right]_x^{2x-2} dx = \int_1^2 x\left[\frac{\cancel{4}^{\,2}x^2-\cancel{8}^{\,4}x+\cancel{4}^{\,2}}{\cancel{2}}\right]^{2x-2}_x dx$$

$$D_2: \int_1^2 x\int_x^{2x-2} y\,dy\,dx \Rightarrow \int_1^2 x\left[\frac{y^2}{2}\right]_x^{2x-2} dx = \int_1^2 x\left\{\left[\frac{4x^2-8x+4}{2}\right]-\left[\frac{x^2}{2}\right]\right\}dx = \int_2^2 x\left[\frac{\cancel{4x^2}^{\,3x^2}-8x+4-\cancel{x^2}}{2}\right]dx$$

$$= \frac{3}{2}\int_1^2 x^3\,dx - 4\int_1^2 x^2\,dx + 2\int_1^2 x\,dx = \frac{3}{2}\left[\frac{x^4}{4}\right]_1^2 - 4\left[\frac{x^3}{3}\right]_1^2 + 2\left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{3}{2}\left[4-\frac{1}{4}\right] - 4\left[\frac{8}{3}-\frac{1}{3}\right] + 2\left[2-\frac{1}{2}\right] = \frac{3}{2}\,\frac{15}{4} - 4\cdot\frac{7}{3} + \cancel{2}\cdot\frac{3}{\cancel{2}} = \frac{45}{8} - \frac{28}{3} + 3 = -\frac{17}{24}$$

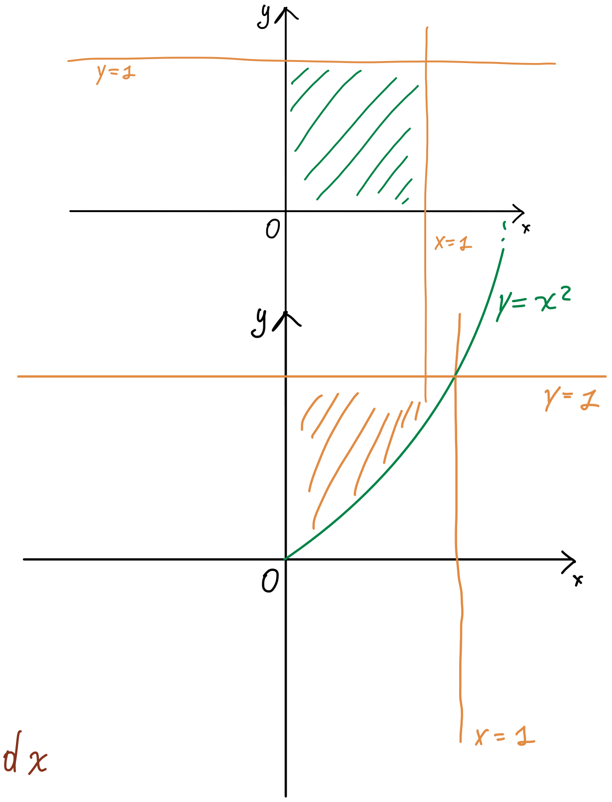
$$\Rightarrow \iint_D xy\,dx\,dy = \frac{1}{6} - \frac{17}{24} = -\frac{13}{24} \sim -0{,}5$$

# Qualche integrale doppio

$$\iint_A xy\,dx\,dy \qquad A=\{(x,y) / 0<x<1,\ 0<y<1\}$$

$$\to \int_0^1 x \int_0^1 y\,dy = \int_0^1 x\left[\frac{y^2}{2}\right]_0^1 dx = \int_0^1 x\,\frac12\,dx = \frac12\int_0^1 x\,dx$$

$$= \frac12\left[\frac{x^2}{2}\right]_0^1 = \frac12\cdot\frac12 = \frac14$$




$$\iint_A 2x^2+3y\,dx\,dy \qquad A=\{(x,y)/ 0<x<1,\ x^2<y<1\}$$

$$\int_0^1\int_{x^2}^1 2x^2+3y\,dx\,dy = \int_0^1 2x^2\int_{x^2}^1 dy + 3\int_{x^2}^1 y\,dy\,dx$$

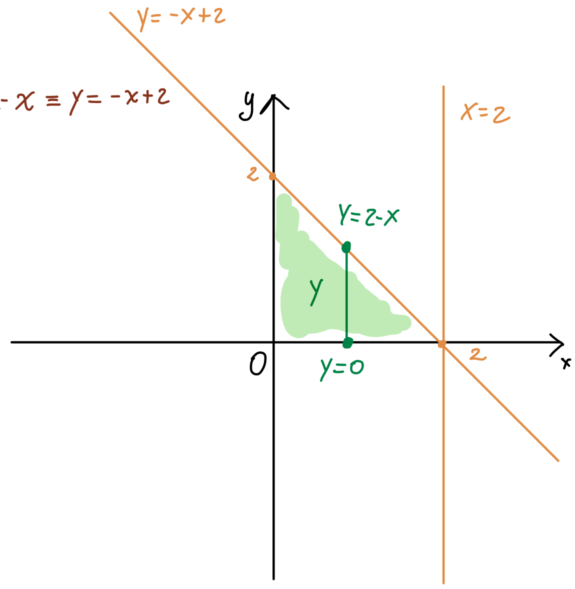
$$= \int_0^1 \left[2x^2y+\frac32 y^2\right]_{x^2}^1 dy = \int_0^1 2x^2+\frac32 - 2x^4 - \frac32 x^4)\,dx$$

$$= -\frac72\int_0^1 x^4 + 2\int_0^1 x^2 + \frac32\int_0^1 dx = -\frac72\left[\frac{x^5}{5}\right]_0^1 + 2\left[\frac{x^3}{3}\right]_0^1 + \frac32\left[x\right]_0^1 = -\frac72\,\frac15 + \frac23 + \frac32 = -\frac{7}{10}+\frac23+\frac32$$

$$= \frac{-21+20+45}{30} = \frac{44}{30} = \frac{22}{15}$$

$$\to y=2-x \equiv y=-x+2$$

$$\iint_D x-2y\,dx\,dy \qquad D:\{(x,y)/ 0<x<2,\ 0<y<2-x\}$$




$$\int_0^2\int_0^{2-x} x-2y\,dx\,dy = \int_0^2 x\int_0^{2-x}dy - 2\int_0^{2-x} y\,dy\,dx$$

$$= \int_0^2 x\left[y\right]_0^{2-x} - 2\left[\frac{y^2}{2}\right]_0^{2-x}dx = \int_0^2 x(2-x) - \cancel{2}\,\frac{(x^2-4x+\cancel{4})}{\cancel{2}}\,dx$$

$$= 2\int_0^2 x - \int_0^2 x^2dx - \int_0^2 x^2 dx + 4\int x\,dx - 4\int dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^2 \underbrace{- \left[\frac{x^3}{3}\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2}_{2\left[\frac{x^3}{3}\right]_0^2} + 4\left[\frac{x^2}{2}\right]_0^2 - 4\left[x\right]_0^2 = \cancel{2}\,\frac{4}{\cancel{2}} - \frac83\cdot 2 + \overset{2}{\cancel{4}}\,\frac{4}{\cancel{2}} - 4\cdot 2 = 4-\frac{16}{3}+\cancel{8}-\cancel{8} =$$

$$= \frac{12-16}{3} = -\frac43$$