X

X

X

X

X

X

1. Studiare la seguente funzione e disegnarne il grafico: $f(x) = \arctan\left(\frac{e^x}{e^x - 1}\right)$;

1) Dominio $\quad D$ atan: $(-\infty, +\infty) \Rightarrow e^x - 1 \neq 0 \rightarrow e^x \neq 1 \rightarrow \ln(e^x) = \ln(1) \rightarrow \boxed{x \neq 0}$

2) Simmetrie: $f(-x) = \operatorname{atan}\left(\frac{e^{-x}}{e^{-x}-1}\right) = \operatorname{atan}\left(\frac{1}{e^x} \cdot \frac{1}{\frac{1-e^x}{e^x}}\right) = \operatorname{atan}\left(\frac{1}{\cancel{e^x}} \cdot \frac{\cancel{e^x}}{1-e^x}\right)$

$\operatorname{atan}\left(\frac{1}{1-e^x}\right) \neq \begin{cases} f(x) \\ f(-x) \end{cases}$

3) Segno $f(x) > 0$

$\operatorname{atan}\left(\frac{e^x}{e^x-1}\right) > 0 \quad$ per $\frac{e^x}{e^x-1} > 0 \rightarrow \quad N: e^x > 0 \rightarrow \forall x \in \mathbb{R}$

$D: e^x > 1 \rightarrow x > \emptyset$

N, D, 0, +

$f(x) > 0$ per $\quad x > \emptyset$

$f(x) < 0$ per $x < 0$

$f(x) < 0 \rightarrow \quad N: e^x > 0 \ \nexists x \in \mathbb{R}$

$D: e^x < 1 \rightarrow \quad x < \emptyset$

4) Intersez:

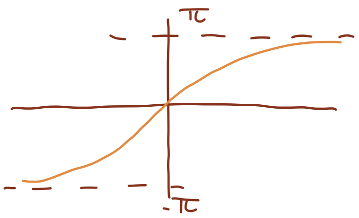
$\begin{cases} f(x) \\ x = 0 \end{cases} \rightarrow \quad y = \operatorname{atan}\left(\frac{1}{1-1}\right) \Rightarrow \underline{\nexists x \in \mathbb{R}}$

$\begin{cases} y = f(x) \\ y = 0 \end{cases} \rightarrow \operatorname{atan}\left(\frac{e^x}{e^x-1}\right) = 0 \ \cup \ \frac{e^x}{e^x-1} = 0$

$\rightarrow N: e^x = 0 \ \underline{\nexists x \in \mathbb{R}}$

5) Asintoti

$\lim_{x \to 0^+} f(x) = \operatorname{atan}\left(\frac{1}{\underbrace{e^0 - 1}_{0^+}}\right)^{+\infty}$



$\Rightarrow \operatorname{atan}(+\infty) \rightarrow \frac{\pi}{2}$

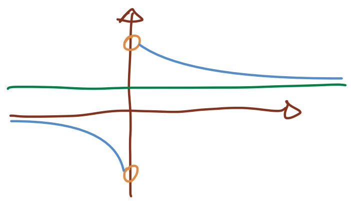
NO A

$\Rightarrow \lim_{x \to 0^-} f(x) \rightarrow -\frac{\pi}{2}$

Non interseca y ma tende a $\pm\frac{\pi}{2}$ in $x = 0$

$\lim_{x \to +\infty} \operatorname{atan}\left(\frac{e^x}{\underbrace{e^x}_{+\infty} - 1}\right)^{+\infty} \rightarrow \operatorname{atan}\left(\frac{\cancel{e^x}}{\cancel{e^x}(1-0)}\right) = \operatorname{atan}(1) \rightarrow \frac{\pi}{4}$

$\lim_{x \to -\infty} f(x) \rightarrow \quad \operatorname{atan}\left(\frac{e^{-\infty} \rightarrow \frac{1}{e^{\infty}} \rightarrow 0}{\underbrace{e^{-\infty}}_{0} - 1}\right) \rightarrow \operatorname{atan}(0) \rightarrow 0$



6) Derivate: Max Min

$D[\operatorname{atan}(x)] = \frac{1}{1+x^2} \rightarrow f'(x) = \left(\frac{1}{1 + \frac{e^x}{e^x - 1}}\right) \cdot \frac{e^x(e^x-1) - e^{2x}}{(e^x-1)^2} = \frac{1}{\frac{e^x - 1 + e^x}{e^x - 1}} \cdot \frac{\cancel{e^{2x}} - e^x - \cancel{e^{2x}}}{(e^x-1)^2}$

$\rightarrow -\frac{\cancel{e^x - 1}}{2e^x - 1} \cdot \frac{e^x}{(e^x-1)^{\cancel{2}}} = -\frac{e^x}{2e^{2x} - 2e^x - e^x + 1} = -\frac{e^x > 0 \ \forall x \in \mathbb{R}}{2e^{2x} - 3e^x + 1} > 0 \quad$ per $2e^{2x} - 3e^x + 1 < 0$

pongo $t = e^x \rightarrow 2t^2 - 3t + 1 < 0$

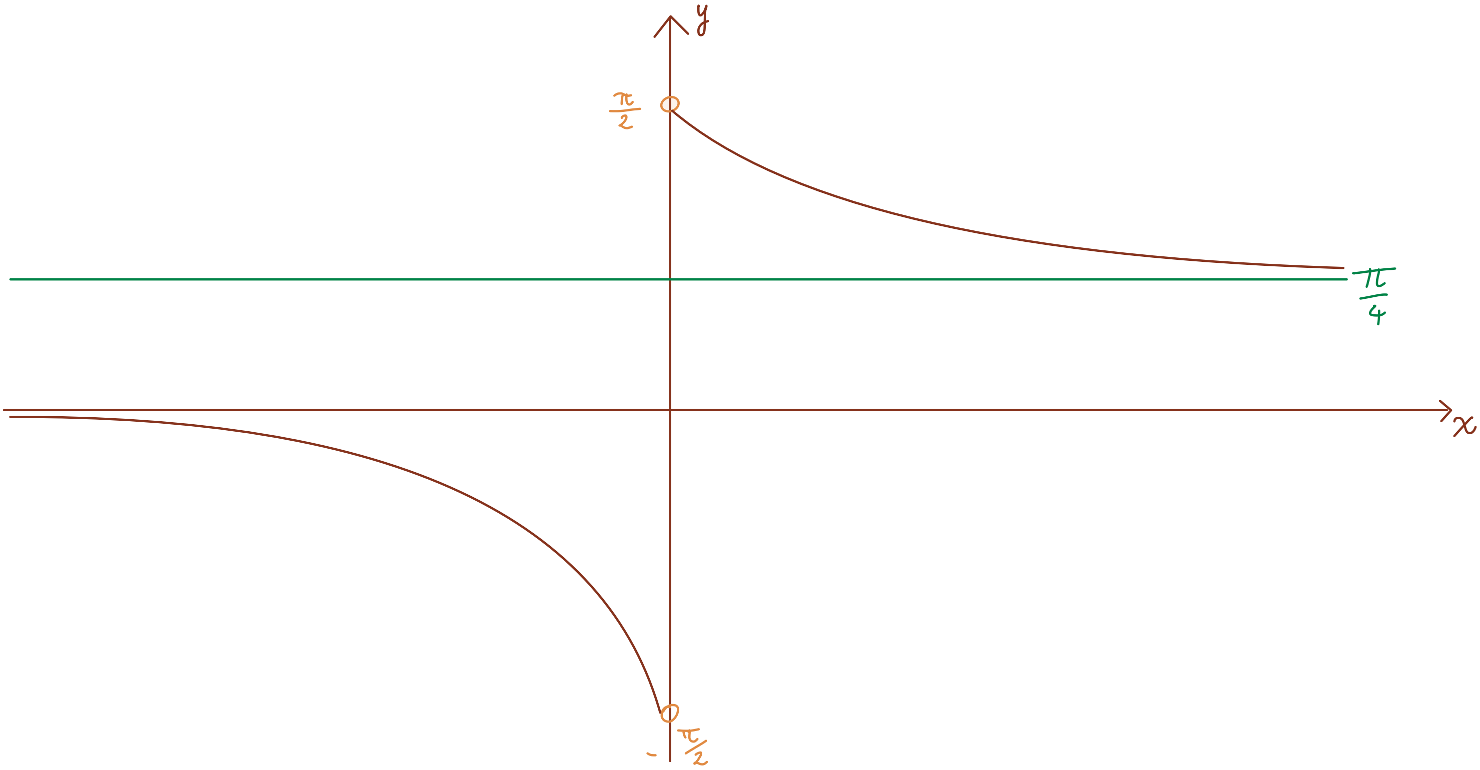
$a > 0 \ eq < 0$

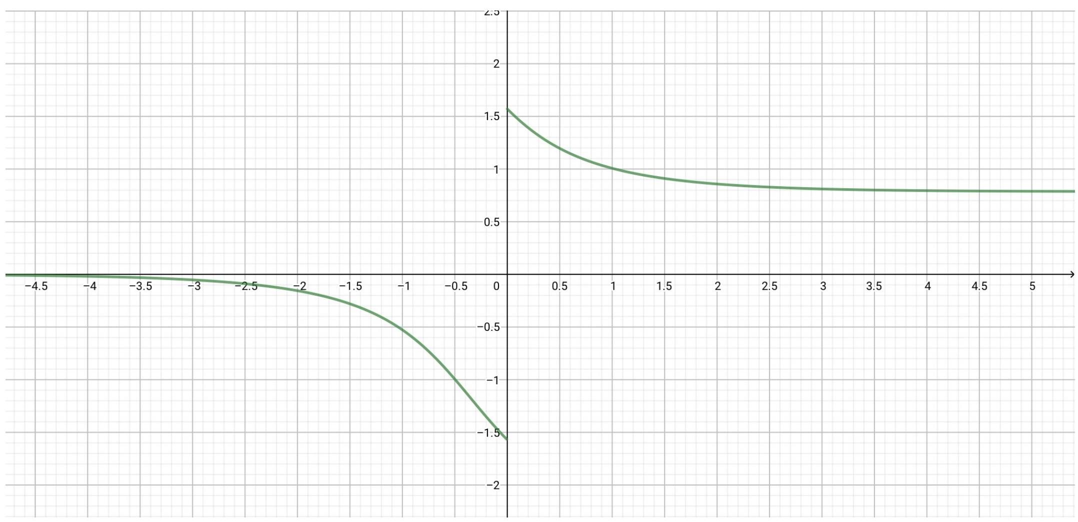
$\rightarrow$ Valori interni

$\rightarrow \Delta = 9 - 4 \cdot 2 \cdot 1 = 1 \Rightarrow t_{1,2} = \frac{3 \pm 1}{4} \begin{cases} 1 \\ \frac{1}{2} \end{cases}$

$f'(x) > 0$ per $\quad \frac{1}{2} < x < 1$

Troppo articolato e non serve


y
π/2
π/4
x
-π/2


2.5
2
1.5
1
0.5
−0.5
−1
−1.5
−2
−4.5
−4
−3.5
−3
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

2. Calcolare il seguente limite: $\lim_{x\to 0} \frac{x\ln(1+\tan 8x)}{6^{x^2}-1}$;

$$\lim_{x\to 0} \frac{x \ln(1+\tan(8x))}{6^{x^2}-1} \quad \left[\frac{0}{0}\right]$$

Hôpital

$$= \lim_{x\to 0} D[f(x)] =$$

$$D[\tan(8x)+1] = \frac{8}{\cos^2(8x)}, \quad D[\ln(1+\tan(8x))] = \frac{1}{1+\tan(8x)} \cdot D[1+\tan(8x)]$$

$$\Rightarrow \frac{1}{1+\tan(8x)} \cdot \frac{1}{\cos^2(8x)} \cdot 8 = \frac{8\sec^2(8x)}{1+\tan(8x)} = \frac{8}{\cos^2(8x)} \cdot \frac{1}{1+\tan(8x)} = \frac{8}{\cos^2(8x) + \frac{\sin(8x)}{\cos(8x)} \cdot \cos^2(8x)}$$

$$\frac{1}{\cos^2(x)} = \sec^2(x) \qquad \sec x = \frac{1}{\cos x}$$

$$= \frac{8}{\cos^2(8x)+\sin(8x)\cos(8x)} \Rightarrow D[x\ln(1+\tan(8x))] = \underbrace{\frac{8x}{\cos^2(8x)+\sin(8x)\cos(8x)} + \ln(1+\tan(8x))}_{f'(x)}$$

$$D[6^{x^2}-1] = \underbrace{2\,6^{x^2}\ln(6)\,x}_{g'(x)}$$

$$\Rightarrow \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{\left[\frac{8x}{\cos^2(8x)+\sin(8x)\cos(8x)} + \ln(1+tg(8x))\right]}{2\cdot 6^{x^2}\cdot \ln(6)\cdot x}$$

$$\lim_{x\to 0} \frac{8x + \ln(1+tg(8x))\cdot\cos^2(8x) + \ln(1+tg(8x))\sin(8x)\cos(8x)}{\cos^2(8x)+\sin(8x)\cos(8x)} = \frac{1}{\cancel{\cos(8x)}} \cdot \frac{\cancel{\cos(8x)}[\ln(1+tg 8x)+\sin(8x)\ln(1+tg\,8x)]}{\cos(8x)+\sin(8x)}$$

$$= \left[\frac{\ln(1+tg\,8x)[1+\sin(8x)]}{\cos(8x)+\sin(8x)} + \frac{8x}{\cos(8x)+\sin(8x)}\right] \cdot \frac{1}{2\,6^{x^2}\ln(6)\,x}$$

$$= \frac{\frac{\ln(1+tg\,8x)[1+\sin(8x)]}{\cos(8x)+\sin(8x)}}{2\,6^{x^2}\ln(6)\,x} + \frac{\frac{\cancel{8}^{\,4}\cancel{x}}{\cos(8x)+\sin(8x)}}{\cancel{2}\,6^{x^2}\ln(6)\,\cancel{x}} = \frac{4}{\ln(6)}$$

3. Calcolare il seguente integrale: $\int_{\frac{2}{\pi}}^{\frac{6}{\pi}} \frac{1}{x^2}\cos^3\left(\frac{1}{x}\right)dx$;   Risolvo $\int x^{-2}\cdot\cos^3\left(\frac{1}{x}\right)dx$

pongo $\frac{1}{x} = u \Rightarrow dx = \frac{1}{-\frac{1}{x^2}}dt = -x^2dt \Rightarrow \int x^{-2}\cos^3\left(\frac{1}{x}\right)dx = -\int \cancel{x^{-2}}\cos^3\left(\frac{1}{x}\right)\cdot\cancel{x^2}\,dt$

$= -\int\cos^3(t)\,dt = -\left[\int\cos(t)\cdot\cos^2 dt\right]$ $\int\cos(t)dt = \sin(t)$

$\Rightarrow \int\cos^2(t)\,dt \Rightarrow D[\sin x] = \cos x \Rightarrow \int\cos(t)\cdot D[\sin(t)]\,dt \overset{\text{PARTI}}{=} \cos(t)\cdot\sin(t) + \int\sin^2(t)\,dt$

$\Rightarrow \sin^2(t) = 1-\cos^2(t) \Rightarrow \cos t\sin t + \int dt - \int\cos^2 dt = \int\cos^2 dt$ pongo $\int\cos^2 dt = I$

$\Rightarrow I = \frac{\cos t\sin t + t}{2}$

$\Rightarrow \int\underset{f}{\cos t}\cdot\underset{g}{\cos^2 t}\,dt = \cos t\cdot\frac{\cos t\sin t + t}{2} + \int\sin t\cdot\frac{\cos t\sin t + t}{2}\,dt$

$= \underset{b}{\frac{1}{2}\int\sin^2(t)\cos t} + \underset{a}{\frac{1}{2}\int\sin t\cdot t\,dt}$

a) $\overset{\text{Parti}}{=} \frac{1}{2}\left[-t\cos t + \int\cos t\,dt\right] = -\frac{1}{2}t\cos t + \sin t$

b) $\int\sin^2(t)\,dt = -\sin t\cos t + \int\cos^2(t)\,dt \underset{\cos^2 t = 1-\sin^2 t}{=} -\sin t\cos t + \int dt - \int\sin^2 dt = \int\sin^2 dt$

$\Rightarrow \int\sin^2 dt = \frac{-\sin t\cos t + t}{2}$

$\Rightarrow \int\sin^2(t)\cdot\cos t = -\cos^2(t)\cdot\frac{\sin t\cos t + t}{2} + \frac{1}{2}\int\sin t\cdot\sin t\cos t + t\,dt$

$= -\frac{\cos^2(t)\sin t\cos t + t}{2} + \frac{1}{2}\int\sin^2 t\cos t + \frac{1}{2}\int t\,dt$

$\Rightarrow \overset{\frac{1}{2}I}{I - \frac{1}{2}I} = -\frac{\cos^3(t)\sin t + t}{\cancel{2}} + \frac{1}{\cancel{4}\,2}t^2 \Rightarrow \int\sin^2 t\cos t\,dt = -\cos^3(t)\sin t + t + \frac{1}{2}t^2$

$\Rightarrow \int\cos t\cdot\cos^2 t\,dt = \left[\cos t\cdot\left(\frac{\cos t\sin t}{2} + \frac{t}{2}\right)\right] - \frac{1}{2}t\cos t + \sin t \quad -\cos^3(t)\sin t + t + \frac{1}{2}t^2$

$= \frac{\cos^2 t\sin t}{2} + \cancel{\frac{t\cos t}{2}} - \cancel{\frac{1}{2}t\cos t} + \sin t - \cos^3 t\sin t + t + \frac{1}{2}t^2$

$= \frac{\cos^2 t\sin t}{2} + \frac{2\sin t - 2\cos^3 t\sin t + 2t + t^2}{2} = \frac{\cos^2 t\sin t}{2} + \frac{\cos^2 t\sin t(2-2\cos t)}{2} + \frac{2t+t^2}{2}$

$= \cos^2 t\sin t(1-\cos t) + t\left(1+\frac{1}{2}t\right)$

$\Rightarrow \left[\cos^2 t\sin t(1-\cos t) + t\left(1+\frac{1}{2}t\right)\right]_{\frac{2}{\pi}}^{\frac{6}{\pi}} = \left[\cos^2 t\sin t(1-\cos t) + t\left(1+\frac{1}{2}t\right)\right]^{\frac{6}{\pi}} - \left[\cos^2 t\sin t(1-\cos t) + t\left(1+\frac{1}{2}t\right)\right]^{\frac{2}{\pi}}$

5. Calcolare gli eventuali estremi relativi della funzione: $f(x,y) = x^3 - 7x^2 + 2xy + 2y^2 + 12x$;

$z = x^3 - 7x^2 + 2xy + 2y^2 + 12x$ $\quad f_x = 3x^2 - 14x + 2y + 12$ $\quad f_y = 2x + 4y$

$f_{xx} = 6x - 14$ $\quad f_{yy} = 4$ $\quad f_{\substack{xy\\yx}} = 2$

$\Rightarrow \begin{cases} 2x + 4y = 0 \rightarrow x = -2y \\ 3x^2 - 14x + 2y + 12 = 0 \end{cases} \rightarrow 3(-2y)^2 - 14(-2y) + 2y = -12 \rightarrow 12y^2 + 28y + 12 = 0$

$\rightarrow \Delta = 784 - 4 \cdot 12 \cdot 12 = 208 > 0 \Rightarrow \dfrac{-28 \pm \sqrt{208}}{24} \begin{cases} \dfrac{-28 + 4\sqrt{13}}{24} = \dfrac{-7 + \sqrt{13}}{6} \\ \dfrac{-28 - 4\sqrt{13}}{24} = \dfrac{-7 - \sqrt{13}}{6} \end{cases}$

$\sqrt{208} = \sqrt{16 \cdot 13}$

$\Rightarrow x = -2y \Rightarrow x = \dfrac{7 - \sqrt{13}}{3} \sim 1$, $\Rightarrow \dfrac{7 + \sqrt{13}}{3} \simeq 3{,}5$

$y = \dfrac{-7 + \sqrt{13}}{6}$ $\quad y = \dfrac{-7 - \sqrt{13}}{6}$

$\Rightarrow \underset{a}{\left(\dfrac{7 - \sqrt{13}}{3}, \dfrac{-7 + \sqrt{13}}{6} \right)}, \underset{b}{\left(\dfrac{7 + \sqrt{13}}{3}, \dfrac{-7 - \sqrt{13}}{6} \right)}$

$H = \begin{bmatrix} 6x - 14 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow H(a) = \begin{bmatrix} \cancel{6}^{2} \cdot \left(\dfrac{7 - \sqrt{13}}{\cancel{3}} \right) - 14 & 2 \\ 2 & 4 \end{bmatrix} = (\sim -7 \cdot 2 \cdot 4) - (4) < 0$ <u>sella</u>

$H(b) = \begin{pmatrix} 6 \cdot \left(\dfrac{7 + \sqrt{13}}{3} \right) - 14 & 2 \\ 2 & 4 \end{pmatrix} = (\sim +7{,}2 \cdot 4) - 4 > 0$ $\quad H(f_{xx}(b)) > 0 \Rightarrow$ PUNTO DI MINIMO

$f(b) = \underset{\sim 42{,}8}{\left(\dfrac{7 + \sqrt{13}}{3} \right)^3} - 7 \underset{\sim 12{,}5}{\left(\dfrac{7 + \sqrt{13}}{3} \right)^2} + 2 \underset{\sim 3{,}5}{\left(\dfrac{7 + \sqrt{13}}{3} \right)} \underset{\sim -1{,}7}{\left(\dfrac{-7 - \sqrt{13}}{6} \right)} + 2 \underset{-2{,}89}{\left(\dfrac{-7 - \sqrt{13}}{6} \right)^2} + 12 \underset{3{,}5}{\left(\dfrac{7 + \sqrt{13}}{3} \right)}$

$= 42{,}8 - 87{,}5 - 11{,}9 - 6 + 42 \simeq -20$

GIUSTO

6. Calcolare l'integrale del seguente problema di Cauchy: $\begin{cases} y' = \dfrac{x}{1+y^2}, \\ y(0) = 1 \end{cases}$;

$$y' = \frac{x}{1+y^2} \Rightarrow$$

$$\Rightarrow y' \cdot (1+y^2) = x \rightarrow \frac{dy}{dx}(1+y^2) = x \rightarrow \int 1+y^2\, dy = \int x\, dx$$

$$\Rightarrow \int dy + \int y^2\, dy = \int x\, dx \rightarrow y + \frac{1}{3}y^3 = \frac{1}{2}x^2 + c$$

$$y(0) = 1 \Rightarrow \begin{matrix} y = 1 \\ x = 0 \end{matrix} \Rightarrow 1 + \frac{1}{3} = c \Rightarrow c = \frac{4}{3} \Rightarrow \text{Soluzione: } y + \frac{1}{3}y^3 = \frac{1}{2}x^2 + \frac{4}{3}$$

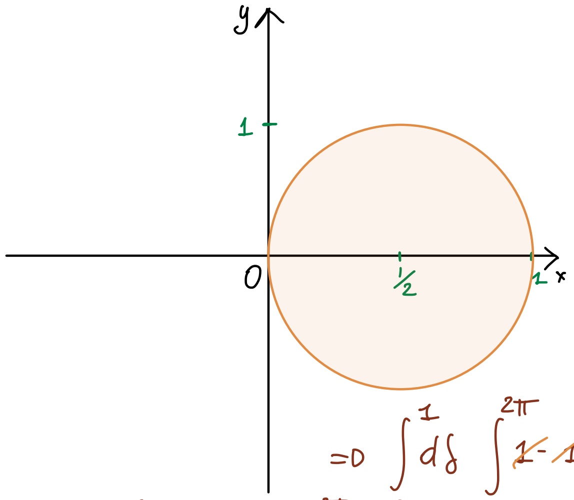
7. Calcolare il seguente integrale doppio $\displaystyle\iint_D (1 - 2x - 3y)\, dxdy$, dove

$$D = \{(x, y) : x^2 + y^2 \le x\}.$$

$$D: \{(x,y) \,/\, x^2+y^2 \le x\}$$

$$x^2+y^2 \le x \rightarrow x^2+y^2-x \le 0$$

$$y \le \sqrt{x - x^2}$$



Formule di Riduzione

$$\Rightarrow D: \{(\rho,\theta) \,/\, 0 < \rho < 1,\ 0 < \theta < 2\pi\}$$

$$\begin{cases} x = \frac{1}{2} + \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\Rightarrow \int_0^1 d\rho \int_0^{2\pi} \cancel{1} - \cancel{1} + 2\rho\cos\theta - 3\rho\sin\theta \cdot \rho\, d\theta$$

$$= \int_0^1 d\rho\, 2\int_0^{2\pi} \rho^2\cos\theta - 3\int_0^{2\pi} \rho^2\sin\theta\, d\theta = \int_0^1 d\rho\, 2\rho^2\left[\sin\theta\right]_0^{2\pi} - 3\rho^2\left[-\cos\theta\right]_0^{2\pi} =$$

$$-2\rho \quad -3\rho^2$$

$$= \int_0^1 3\rho^2\, d\rho = 3\int_0^1 \rho^2\, d\rho = 3\left[\frac{\rho^3}{3}\right]_0^1 = 1$$

Coordinate Normali $D: \{(x,y) \,/\, 0 < x < 1,\ \sqrt{x-x^2} < y <$