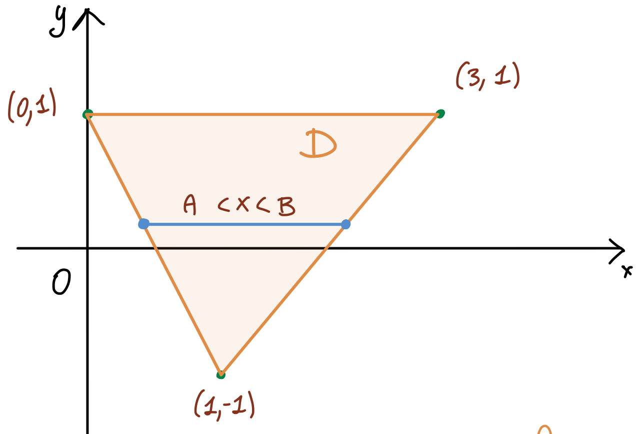
**Esercizio 1.** Calcolare

$$\iint_D \sin(x) y \, dx dy$$

Dove $D$ è il triangolo di vertici $A = (0, 1)$ $B = (1, -1)$ $C = (3, 1)$.

Dominio di int




$$A \Rightarrow \frac{x-0}{1-0} = \frac{y-1}{-1-1} \Rightarrow x = -\frac{y-1}{2} \quad A$$

$$B \Rightarrow \frac{x-1}{3-1} = \frac{y+1}{1+1} \Rightarrow \frac{x-1}{2} = \frac{y+1}{2}$$

$$\Rightarrow x - 1 = y + 1 \Rightarrow x = y+2 \quad B$$

$$\Rightarrow D_y = \left\{ (x,y) \;/\; -1 < y < 1 \;,\; -\frac{y-1}{2} < x < y+2 \right\}$$

$$\Rightarrow \int_{-1}^{1} y \, dy \int_{-\frac{y-1}{2}}^{y+2} \sin x \, dx = \Big[-\cos(x)\Big]_{-\frac{y-1}{2}}^{y+2}$$

$$= \left[-\cos(y+2) + \cos\left(-\frac{y-1}{2}\right)\right]$$

$$\Rightarrow \underbrace{-\int_{-1}^{1} y \cdot \cos(y+2)}_{a} + \underbrace{\int_{-1}^{1} y \cos\left(-\frac{y-1}{2}\right) dy}_{b}$$

a) Pongo $t = y+2 \Rightarrow dy = dt$, $y = t-2$ $\Rightarrow -\int (t-2)\cos(t)\,dt = -\int t\cos t + 2\int \cos(t)\,dt$

$$= -\left[t\sin t - \int \sin t\right] + 2\sin t = -t\sin t - \cos t + 2\sin t = \Big[\sin(t)(2-t) - \cos t\Big]_{-1}^{1}$$

$$= \Big[\sin(y+2)(\cancel{2} - y - \cancel{2}) - \cos(y+2)\Big]_{-1}^{1} = \sin(3)(-1) - \cos(3) - \Big(\sin(1)(1) - \cos(1)\Big)$$

$$= \underbrace{\sin(3) - \cos(3) - \sin(1) + \cos(1)}_{a}$$

b) $\int_{-1}^{1} y \cos\left(\frac{1-y}{2}\right) dy$ pongo $t = \frac{1-y}{2} = \frac{1}{2} - \frac{1}{2}y \Rightarrow dy = -2t\,dt \Rightarrow y = -2t - 1$

$$\Rightarrow -2\int t(-2t-1)\cdot\cos(t)\,dt = +4\int t^2\cos(t)\,dt + 2\int t\cos(t)\,dt$$

$$= 4\left[t^2\sin t - \int t\sin t\,dt\right] + 2\left[t\sin t - \int \sin t\,dt\right] = 4\left[t^2\sin t - \left(-t\cos t + \int \cos t\,dt\right)\right] + 2\Big[t\sin t + \cos t\Big]$$

$$= 4\Big[t^2\sin t + t\cos t - \sin t\Big] + 2\Big[t\sin t + \cos t\Big]$$

$$t = \frac{1-y}{2} \Rightarrow 4\left[\left(\frac{1-y}{2}\right)^2 \sin\left(\frac{1-y}{2}\right) + \left(\frac{1-y}{2}\right)\cos\left(\frac{1-y}{2}\right) - \sin\left(\frac{1-y}{2}\right)\right] + 2\left[\left(\frac{1-y}{2}\right)\sin\left(\frac{1-y}{2}\right) + \cos\left(\frac{1-y}{2}\right)\right]$$

$$\Rightarrow 4\Big[0 + 0 - \sin(0)\Big] + 2\Big[0 + \cos(0)\Big] - \Big\{4\Big[\cancel{\sin(1)} + \cos(1) - \cancel{\sin(1)}\Big] + 2\Big[\sin(1) + \cos(1)\Big]\Big\}$$

$$= 2 - 4\cos(1) - 2\sin(1) + 2\cos(1) = \underbrace{2 - 2\cos(1) - 2\sin(1)}_{b}$$

$$a + b = \sin(3) - \cos(3) - \cancel{\sin(1)} + \cancel{\cos(1)} + 2 - \cancel{2}^{\,1}\cos(1) - 2\cancel{\sin(1)} =$$

$$= \sin(3) - \cos(3) - 3\sin(1) - \cos(1) + 2$$

**Esercizio 2.** Si determinimo i punti di massimo o minimo relativi per la funzione

$$f(x,y) = x^3 + y^3 - 6xy$$

$f'_x = 3x^2 - 6y \qquad f'_y = 3y^2 - 6x \qquad f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = f_{yx} = -6$

$\begin{cases} 3x^2 - 6y = 0 \\ 3y^2 - 6x = 0 \end{cases} \to 3x^2 = 6y \to y = \frac{3}{6}x^2 = \frac{1}{2}x^2 \Rightarrow 3 \cdot \frac{1}{4}x^4 - 6x = 0 \to 3x\left(\frac{1}{4}x^3 - 2\right) = 0 \qquad \frac{1}{4}x^3 - 2 = 0$

$\Rightarrow y = \frac{1}{2}x^2 \underset{x=0}{\to} y = 0 \Rightarrow (0,0) \in f(x)$ $P_0$ $\quad \hookrightarrow x_1 = 0 \qquad \Rightarrow x^3 = 8 \Rightarrow x = \sqrt[3]{8} = 2$, $x_{2,3} = 2$

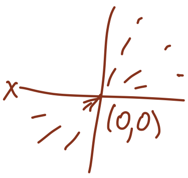
$y = \frac{1}{2}x^2 \underset{x=2}{\to} y = \frac{1}{2}4 = 2 \Rightarrow (2,2) \in f(x)$ $P_1$

$H = \begin{vmatrix} 6x & -6 \\ -6 & 6y \end{vmatrix} \Rightarrow H(P_0) = \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = 0 - 36 < 0 \quad H < 0 \Rightarrow$ Sella

$H(P_1) \begin{vmatrix} 12 & -6 \\ -6 & 12 \end{vmatrix} = 144 + 36 > 0 \qquad f_{xx}(P_1) = 12 > 0 \to P_1$ Minimo

$\to$ Non sono sicuro di $P_0$

$\Rightarrow y = mx \to f(x,mx) = x^3 + m^3x^3 - 6mx^2 \Leftarrow$ funzione in $x$ lungo $y = mx$

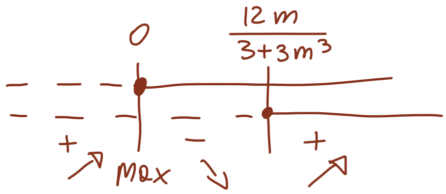



$f(0) = 0 + 0 - 0 = 0 \in D_{f(x,mx)} \Rightarrow (0,0) \in f(x,mx)$

$f'_1 = 3x^2 + 3m^3x^2 - 12mx > 0$ per $x(3x + 3m^3x - 12m) > 0$

$\hookrightarrow \underline{x > 0}$

$x(3 + 3m^3) > 12m \to x > \frac{12m}{3+3m^3}$



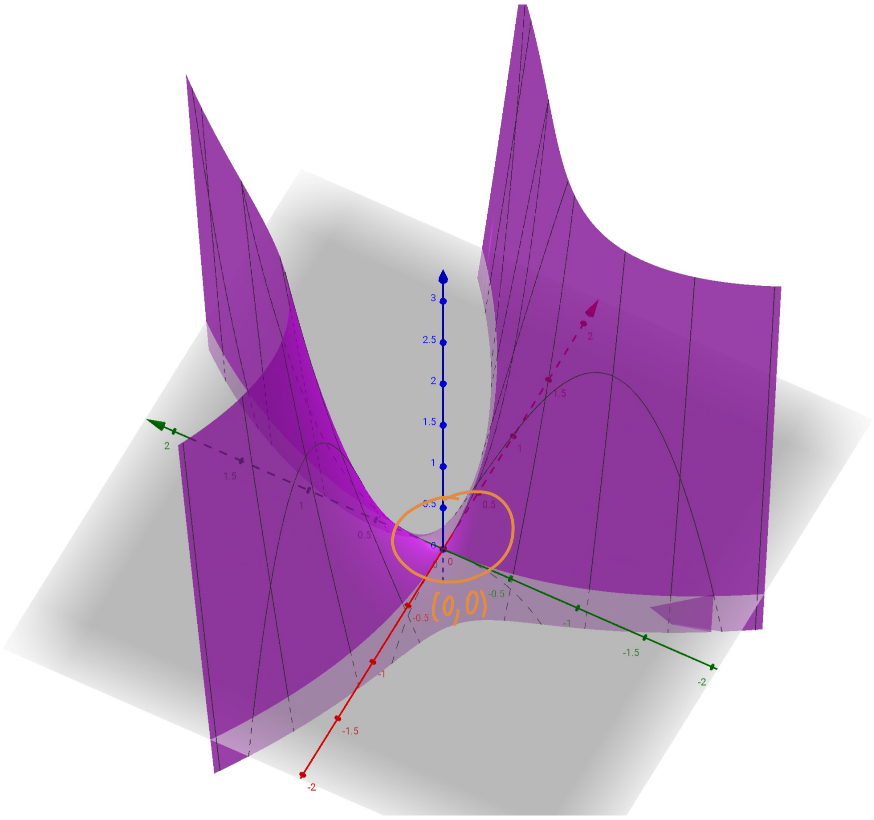

in $(0,0)$ c'e' un max lungo $y = mx$

Controllo lungo $y = 0$ ovvero $f(x,y)$ lungo asse $x$

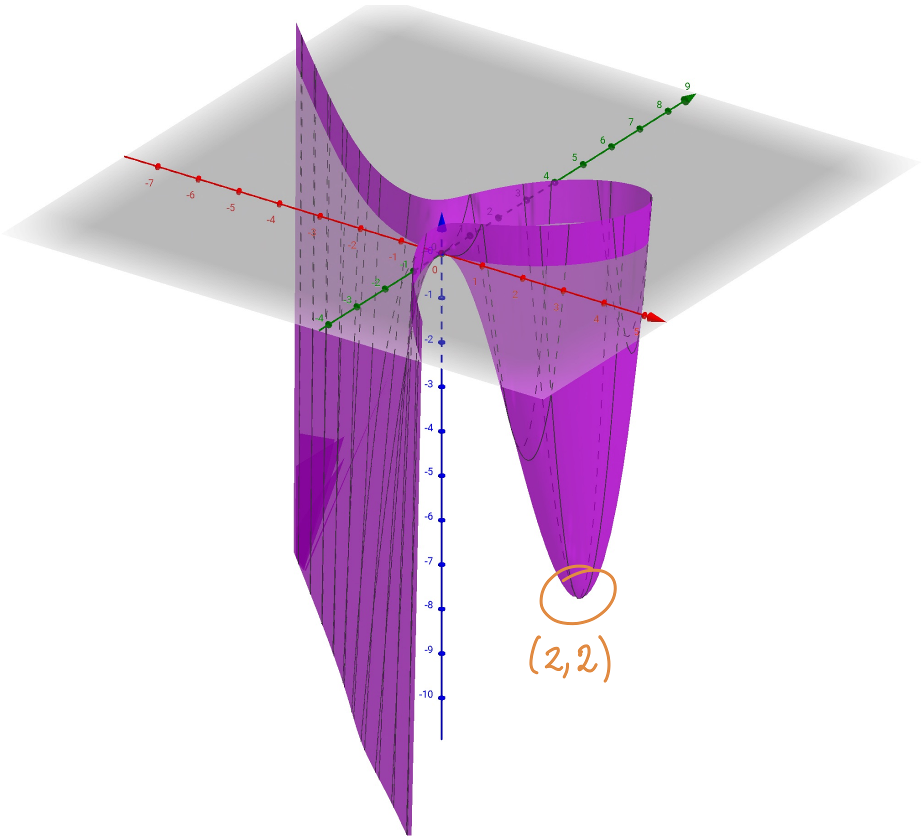
$f(x,0) = x^3 \qquad f'_{(x,0)} = 3x^2 > 0$ per $\underline{x > 0}$


<u>Sella</u>

Tempo ~ 12 min




Sella in (0,0,0)




Minimo in (2,2)

**Esercizio 4.** Si consideri la seguente forma differenziale

$$\omega = \frac{1}{1+y^2}\,dx + \left(y + \frac{2xy}{(1+y^2)^2}\right)dy.$$

Si dica se essa è esatta e, in caso positivo, si calcoli una primitiva.

$$A: \begin{cases} 1+y^2 \neq 0 \\ (1+y^2)^2 \neq 0 \end{cases} \Rightarrow 1+y^2 \neq 0$$

$$\Rightarrow y^2 \neq -1 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow A: \mathbb{R}^2$$

$$X = \frac{1}{1+y^2} \rightarrow X'_y = \frac{-(2y)}{(1+y^2)^2} = -\frac{2y}{(1+y^2)^2}$$

$$Y = y + \frac{2xy}{(1+y^2)^2}$$

$$Y'_x = D\left[y + \frac{2}{(1+y^2)^2}\cdot xy\right] = +\frac{2y}{(1+y^2)^2}$$

c c c

uguali

La F.D. è ~~chiusa ed esatta~~

7'

le derivate parziali sono diverse!

2) Primitiva 1° metodo (7) + 7' con passaggi che ho dovuto rifare

$$\int \frac{1}{1+y^2}\,dx = \frac{1}{1+y^2}\int dx = \frac{x}{1+y^2} + C(y)$$

$$\Rightarrow D_y\left[\frac{x}{1+y^2} + C(y)\right] = x\cdot(1+y^2)^{-1} + C(y) = C'(y) - x(1+y^2)^{-2}\cdot(2y)$$

$$= C'(y) - \frac{2xy}{(1+y^2)^2} = y + \frac{2xy}{(1+y^2)^2} \rightarrow C'(y) - y = \frac{2xy}{(1+y^2)^2} + \frac{2xy}{(1+y^2)^2}$$

$$\rightarrow C'(y) = \frac{4xy}{(1+y^2)^2} + y \quad \rightarrow \text{integro} \rightarrow C(y) = 4\int\frac{xy}{(1+y^2)^2}\,dy + \int y\,dy \rightarrow \boxed{\frac{y^2}{2}}$$

$$4x\int\frac{y}{(1+y^2)^2}\,dy \qquad t = (1+y^2) \rightarrow dy = \frac{1}{2y}\,dt$$

$$\rightarrow 2x\int\frac{\cancel{y}}{(1+y^2)^2}\cdot\frac{1}{\cancel{y}}\,dt = 2x\int\frac{1}{t^2}\,dt = 2x\int t^{-2}\,dt = 2x\left[\frac{t^{-1}}{-1}\right] = \frac{-2x}{t}$$

$$t = 1+y^2 \Rightarrow \frac{-2x}{1+y^2} = C(y)$$

Sol $\Rightarrow$ $$\frac{x}{1+y^2} - 2\frac{x}{1+y^2} = -\frac{x}{1+y^2} = F(x,y)$$

**Esercizio 5.** Si risolva il seguente problema di Cauchy

$$\begin{cases} y' = \frac{y\ln(y)}{x^2+x} \\ y(1) = e \end{cases}$$

$$y' = \frac{y \ln(y)}{x^2 - x}$$

$$\Rightarrow \frac{y'}{y\ln(y)} = \frac{1}{x^2+x}$$

$$\Rightarrow \underbrace{\int \frac{1}{y\ln(y)}\,dy}_{a} = \underbrace{\int \frac{1}{x^2+x}\,dx}_{b}$$

a) $t = \ln(y) \Rightarrow D_y[\ln y] = \frac{1}{y} \Rightarrow dy = \frac{1}{\frac{1}{y}} = y\,dt \Rightarrow \int \frac{1}{\cancel{y}\ln(y)} \cancel{y}\,dt$

$$= \int \frac{1}{t}\,dt = \ln(\ln(y))$$

b) $\int \frac{1}{x^2+x}\,dx \quad \frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{Ax + A + Bx}{x(x+1)} = 1$

$$\Rightarrow \begin{cases} A + B = 0 \Rightarrow B = -1 \\ A = 1 \end{cases} \Rightarrow \frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$$

$$\Rightarrow \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx = \ln|x| - \ln|x+1| + \bar{c} = \ln\left|\frac{x}{x+1}\right| + \bar{c}$$

$$\Rightarrow \ln(\ln y) = \ln\left|\frac{x}{x+1}\right| + \bar{c} \Rightarrow \ln(y) = \frac{x}{x+1} \cdot c \Rightarrow y = e^{\frac{cx}{x+1}}$$

$$e^{\bar{c}} = c$$

Cauchy $\quad y(1) = e^{\frac{c}{2}} = e \Rightarrow \frac{c}{2} = 1 \Rightarrow c = 2$

Tempo ~ 6'

$$\Rightarrow \text{Sol} \quad y(x) = e^{\frac{2x}{x+1}} \Rightarrow y(1) = e$$

**Esercizio 1.** Calcolare

$$\lim_{x \to 0} \frac{\ln(1 - \ln(1-x))\sin(x)}{1 - cos(x)}.$$

**Esercizio 2.** Stabire se la serie numerica

$$\sum_{n=1}^{\infty} \frac{2^n(n^3+\sin(n))}{5^n}$$

è convergente.

$$\lim_{n\to+\infty} \frac{2^n(n^3+\sin(n))}{5^n} = \left(\frac{2}{5}\right)^n \cdot [n^3+\sin(n)]$$

= Siccome $\sin(n)$ per $n \to \infty$ è oscillante è trascurabile rispetto $n^3 \to +\infty$

$$\Rightarrow \lim_{n\to+\infty} \left(\frac{2}{5}\right)^n \cdot n^3 = \frac{2^n n^3}{5^n} = \quad 2^n \gg n^3 \Rightarrow \sim \frac{2^n}{5^n} \sim \left[\frac{2}{5}\right]^n \to \emptyset$$

$\Rightarrow$ la serie potrebbe convergere

$$= \left(\frac{2}{5}\right)^n \cdot (n^3+\sin(n)) \qquad \lim_{n\to+\infty} \frac{\left(\frac{2}{5}\right)^{n+1}\left[(n+1)^3+\sin(n+1)\right]}{\left(\frac{2}{5}\right)^n\left[n^3+\sin(n)\right]} = \frac{\frac{2}{5}\left[(n+1)^3+\sin(n+1)\right]}{n^3+\sin n}$$

$$= \frac{2}{5}\lim_{n\to+\infty} \frac{(n+1)^3+\sin(n+1)}{n^3+\sin(n)} = \quad (n+1)^2(n+1) = (n^2+2n+1)(n+1) = n^3+n^2+2n^2+2n+n+1$$

$$\Rightarrow \frac{2}{5}\left[\frac{\cancel{n^3}\left(\underset{0}{\cancel{\frac{1}{n}}}+\underset{0}{\cancel{\frac{2}{n}}}+\underset{0}{\cancel{\frac{2}{n^2}}}+\underset{0}{\cancel{\frac{1}{n^2}}}+\underset{0}{\cancel{\frac{1}{n^3}}}+\underset{0}{\frac{\sin(n+1)}{\cancel{n^3}}}+1\right)}{\cancel{n^3}\left(1+\underset{0}{\frac{\sin(n)}{\cancel{n}}}\right)}\right] = \frac{2}{5}\lim 1 \to \frac{2}{5} < 1$$

Per il criterio del rapporto se $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \ell < 1 \Rightarrow a_n$ CONVERGE

Criterio di RAABE

$$\lim_{n\to+\infty} n \cdot \left[\frac{\left(\frac{2}{5}\right)^n \cdot (n^3+\sin(n))}{\left(\frac{2}{5}\right)^{n+1}\left[(n+1)^3+\sin(n+1)\right]} - 1\right] = n\left[\frac{n^3+\sin(n)}{\frac{2}{5}\left[(n+1)^3+\sin(n+1)\right]}\right] =$$

$$= \frac{5}{2}\lim n\left[\frac{\cancel{n^3}\left(1+\underset{0}{\cancel{\frac{\sin(n)}{n^3}}}\right)}{\cancel{n^3}\left(\underset{0}{\cancel{\frac{1}{n}}}+\underset{0}{\cancel{\frac{2}{n}}}+\underset{0}{\cancel{\frac{2}{n^2}}}+\underset{0}{\cancel{\frac{1}{n^2}}}+\underset{0}{\cancel{\frac{1}{n^3}}}+\underset{0}{\frac{\sin(n)}{\cancel{n}}}+1\right)} - 1\right] = \frac{5}{2}\lim_{n\to+\infty} n \cdot \emptyset \quad ?$$