Esempio continuità funzioni di 2 variabili

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{per } (x,y) \neq (0,0) \\ 0 & \text{per } (x,y) = (0,0) \end{cases}$$

1) deve risultare $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ Allora è continua

2) Calcolo possiamo usare le coordinate polari o prendere la retta $y - 0 = m(x-0)$ ← $y = mx$

$$\to \lim_{x\to 0} \frac{x \cdot mx}{x^2+(mx)^2} = \frac{mx^2}{x^2+m^2x^2} = \frac{\cancel{x^2}m^2}{\cancel{x^2}(1+m^2)} = \frac{m^2}{1+m^2}$$

Dipende solo da m → Non esiste

⇒ se il lim non esiste la funzione non è continua in $O(0,0)$.

- $$\begin{cases} \frac{\ln(1+x^2+y^2)}{\sqrt{x^2+y^2}} & \text{per } (x,y) \neq (0,0) \\ 0 & \text{per } (x,y) = (0,0) \end{cases}$$

⇒ Calcolo lungo $y = mx$

Cordinate polari

$$\begin{cases} x = \delta\cos\theta \\ y = \delta\sin\theta \end{cases}$$

$$\Rightarrow \lim_{\delta\to 0^+} \frac{\ln(1+\delta^2\cos^2\theta+\delta^2\sin^2\theta)}{\sqrt{\delta^2\cos^2\theta+\delta^2\sin^2\theta}} = \frac{\ln(1+\delta^2\underbrace{(\cancel{\cos^2\theta+\sin^2\theta})}_{1})}{\sqrt{\delta^2\underbrace{(\cancel{\cos^2\theta+\sin^2\theta})}_{1}}} = \frac{\ln(1+\delta^2)}{\delta}$$

$$\to \frac{\ln(1)}{0^+} \to \left[\frac{0}{0^+}\right]$$

Limite notevole

$$\lim_{\delta\to 0} \frac{\ln(1+\delta^2)}{\delta^2} = 1 \Rightarrow \lim_{\delta\to 0^+} \frac{\ln(1+\delta^2)\,\delta}{\delta^2} = \lim_{\delta\to 0^+} 1\cdot\delta = 0$$

⇒ coincide con $f(0,0) = 0$ → Continua.

- $$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Considero lungo gli assi

$$\begin{cases} f(x,0) = \frac{x^2}{x^2} = 1 \\ f(0,y) = \frac{-y^2}{y^2} = -1 \end{cases}$$

Considerando le restrizioni $f(x,0)$ e $f(0,y)$ → $\lim_{x\to 0} f(x,0) = 1$ $\quad \lim_{y\to 0} f(0,y) \to -1$

Siccome $\lim_1 \neq \lim_2$, il lim in $(x,y) = (0,0)$ NON ESISTE ⇒ NON CONTINUA

- $$f(x,y) = \begin{cases} \left(\frac{x^2 y}{x^4+y^2}\right)^2 & \text{per } (x,y) \neq (0,0) \\ 0 & \text{per } (x,y) = (0,0) \end{cases}$$

Metodo della parametrizzazione, curva $\gamma$

scegliamo il cammino

$$\gamma: \begin{cases} x = t \\ y = t^2 \end{cases} \quad \lim_{t\to 0} f(t,t^2) = \left(\frac{t^2\, t^2}{t^4+t^4}\right)^2 = \left(\frac{\cancel{t^4}}{2\cancel{t^4}}\right)^2 = \frac{1}{4} \neq 0$$

Non è continua in $(0,0)$

* esercizi dal video (7) di ing. Cerroni

- $\begin{cases} \dfrac{\sin(xy)}{x^2+2y^2} & \text{per } (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$ $\quad \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

Calcolo lungo $y = mx \to \lim_{x\to 0} \dfrac{\sin(x\cdot mx)}{x^2+2m^2x^2} = \dfrac{\sin(mx^2)}{x^2(1+2m^2)} = \left[\dfrac{0}{0}\right]$

lim notevole $\lim_{t\to 0} \dfrac{\sin t}{t} \to 1$

Siccome $\lim_{x\to 0} \dfrac{\sin(mx^2)}{x^2(1+2m^2)} = \underbrace{\dfrac{\sin(mx^2)}{mx^2}}_{\to 1} \dfrac{m}{(1+2m^2)} = 1 \cdot \dfrac{m}{1+2m^2}$ non dip. da $x$

$\Rightarrow$ Non Esiste il lim

$\Rightarrow$ Non continua

- $f(x,y) = \begin{cases} \dfrac{x^3y}{x^6+y^2} & \text{per } (x,y)\neq(0,0) \\ 0 & \text{per } (x,y)=(0,0) \end{cases}$

1) proviamo a considerare la funz. lungo gli assi

$\left.\begin{array}{l} f(x,0) = \dfrac{x^3\cdot 0}{x^6} = \dfrac{0}{x^6}, \ \lim_{x\to 0^+} \dfrac{0}{x^6} \to 0 \\ f(0,y) = \dfrac{0}{y^2} \to \lim_{y\to 0^+} \dfrac{0}{y^2} = 0 \end{array}\right\}$ Sarebbe continua

2) Se consideriamo la curva $\gamma: \begin{cases} x = t \\ y = t^3 \end{cases} \to f(t,t^3) = \dfrac{t^3t^3}{t^6+t^6} = \dfrac{t^6}{2t^6} = \dfrac{1}{2} \neq 0$

$\Rightarrow$ NON CONTINUA

- $f(x,y) = \begin{cases} \dfrac{x^4y}{x^2+|y|} & \text{per } (x,y)\neq 0,0 \\ 0 & \text{per } (x,y)=(0,0) \end{cases}$

Coord polari

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \to f(\rho,\theta) = \dfrac{\rho^4\cos^4\theta\,\rho\sin\theta}{\rho^2\cos^2\theta + |\rho\sin\theta|}$

$0 \leq \dfrac{|\rho^5\cos^4\theta\sin\theta|}{\rho^2\cos\theta+\rho|\sin\theta|} \leq \rho^4\cos^4\theta \leq \rho^4 \to 0 \text{ per } \rho\to 0^+ \Rightarrow \lim(\rho,\theta)=0 \Rightarrow$ Continua

- $\begin{cases} \dfrac{x^3y^2}{(x^2+y^2)^2} & (x,y)\neq 0 \\ 0 & (x,y)=0 \end{cases}$

$\boxed{|xy| \leq \frac{1}{2}(x^2+y^2)}$

DISUGUAGLIANZA

$\left|\dfrac{x^3y^2}{(x^2+y^2)^2}\right| = \dfrac{|x|\underbrace{|x^2y^2|}_{>0}}{\underbrace{|(x^2+y^2)^2|}_{>0}}$

$\to \quad 0 \leq \dfrac{x^3y^2}{(x^2+y^2)^2} \leq |x|\dfrac{\cancel{(x^2+y^2)^2}}{4\cancel{(x^2+y^2)^2}} \leq \dfrac{|x|}{4} \to 0$ per $x\to 0 \Rightarrow f(x,y)\to 0 = 0 \Rightarrow$ CONTINUA

In coordinate polari

$f(\rho,\theta) = \dfrac{\rho^3\cos^3\theta\,\rho^2\sin^2\theta}{(\rho^2\underbrace{\cancel{\cos^2\theta}+\rho^2\cancel{\sin^2\theta}}_{1})^2} = \dfrac{\rho^5\cos^3\theta\sin^2\theta}{\rho^4} = \rho\cos^3\theta\sin^2\theta$

$\lim_{\rho\to 0^+} \rho\cos^3\theta\sin^2\theta \to 0$ per $\rho\to 0 \Rightarrow$ CONTINUA

Lungo gli assi $\Rightarrow f(x,0) = \dfrac{0}{(x^2)^2} = 0$, $f(0,y) = \dfrac{0}{(y^2)^2} = 0$

$f(x,0) = f(0,y) = 0 \to$ CONTINUA

# Differenziabilita'

- $\begin{cases} \frac{x\sqrt{y}}{\sqrt{x^2+y^2}} & \text{per } (x,y)\neq(0,0) \\ 0 & \text{per } (x,y)=(0,0) \end{cases}$

$$\lim_{P\to P_0} \frac{f(P)-f(P_0)-f'_x(P_0)(x-x_0)-f'_y(P_0)(y-y_0)}{\underbrace{\overline{PP_0}}_{\text{Distanza}}}$$

$$f_x(P_0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{(0+\Delta x)\sqrt{0}}{\sqrt{(0+\Delta x)^2+0}}-0}{\Delta x}\to\frac{0}{\Delta x}=0$$

$f_y(P_0)=\frac{0}{\Delta y}=0$ $\quad\exists$ e sono nulle le deriv parz in $P_0$ $\Rightarrow$ possiamo applicare la formula

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-\cancel{f(0,0)}^{0}-\cancel{f'_x(0,0)(x-x_0)}^{0}-\cancel{f'_y(0,0)(y-y_0)}^{0}}{\underbrace{\sqrt{(x-0)^2+(y-0)^2}}_{\text{Dist Tra 2 pti}}}=\lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{x^2+y^2}}=\frac{\frac{x\sqrt{y}}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

$=\lim_{(x,y)\to(0,0)}\frac{x\sqrt{y}}{x^2+y^2}$ prendiamo la retta generica $y-y_0=m(x-x_0)\to\boxed{y=mx}$

$$\to\lim_{x\to 0}\frac{x\sqrt{mx}}{x^2+m^2x^2}=\frac{\cancel{x}\sqrt{mx}}{x^{\cancel{2}}(1+m^2)}=\frac{\sqrt{mx}}{x(1+m^2)}=\frac{\sqrt{x}}{x}\frac{\sqrt{x}}{\sqrt{x}}=\frac{x}{x\sqrt{x}}=\frac{1}{\sqrt{x}}\to\underbrace{\frac{1}{\sqrt{x}}}_{+\infty}\cdot\underbrace{\frac{\sqrt{m}}{1+m^2}}_{C}\to+\infty$$

$\Rightarrow\lim_{(x,y)\to(0,0)}\boxed{\text{FORMULA (1)}}=+\infty\neq 0$ NON DIFF.

$$\lim_{P\to P_0}\frac{f(P)-f(P_0)-f_x(P_0)(x-x_0)-f_y(P_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

# Piano Tangente

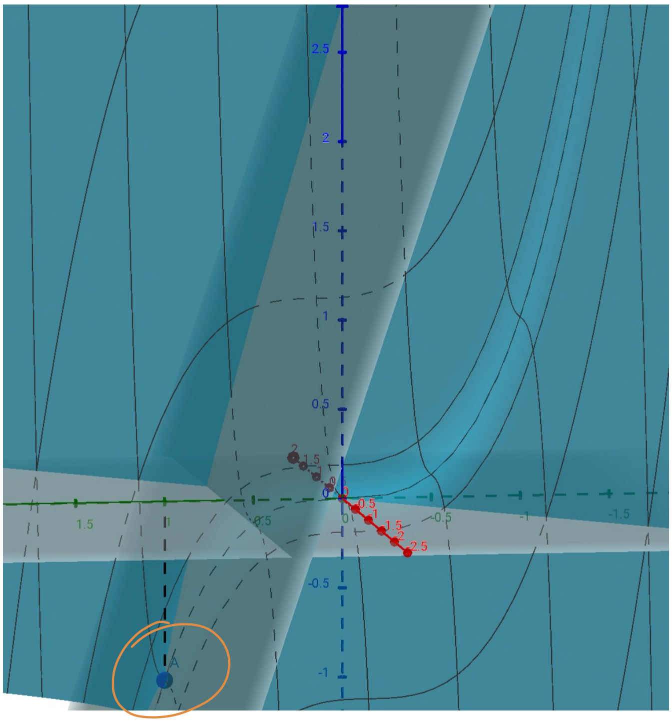
$f(x,y) = x^3 - y^3$ in $(0,1,-1)$

$f(0,1) = -1$

$f_x(0,1) = 3x^2 - 0 \quad \emptyset \qquad f_y' = -3y^2 - 0 \quad -3$

$P_t = z = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1)$

$= z = -1 + 0 + (-3)(y-1) = -1 - 3y + 3 = -3y + 2$



- $f(x,y) = \dfrac{\sin(x^2|y|)}{x^2+y^2}$ D. $x^2+y^2 \neq 0 \Rightarrow \mathbb{R}^2 - \{(x,y) = (0,0)\}$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2|y|)}{x^2+y^2}$$ Verifico lungo gli assi

0 esatto, non Tendente

$$\Rightarrow f(x,0) \to \text{lungo A. x} = \lim_{x\to 0} \frac{\sin(x^2 \cdot 0)}{x^2+0} = \frac{\sin(0)}{x^2} \to \frac{0}{0^+} = \emptyset$$

$$\lim_{y\to 0} f(0,y) = \frac{\sin(0)}{0+y^2} = \frac{\emptyset}{0^+} = \emptyset$$

Lungo $y = mx \to \lim_{x\to 0} f(x,mx) = \dfrac{\sin(x^2|mx|)}{x^2+m^2y^2} = \dfrac{\sin(mx^3)}{x^2(1+m^2)}$

$$= \underbrace{\frac{\sin(mx^3)}{x^2 \cdot mx}}_{1} \frac{mx}{1+m^2} = \frac{mx}{1+m^2} \to \emptyset^{\pm} \Rightarrow$$ Tende a $\emptyset$ lungo gli assi e la bisettrice $\Rightarrow$ discontinuità eliminabile

$$\Rightarrow \bar{f}(x,y) = \begin{cases} \dfrac{\sin(x^2|y|)}{x^2+y^2} & \text{per } (x,y) \neq (0,0) \\ 0 & \text{per } (x,y) = (0,0) \end{cases}$$

Differenziabile?

$P_0 = (x_0, y_0) \Rightarrow \lim_{P\to P_0} \dfrac{f(P) - f'_x(P_0)(x-x_0) - f'_y(P_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = f(P_0) = \emptyset$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \left\{ \frac{\sin(x^2|y|)}{x^2+y^2} - \left[ \frac{\cancel{\cos(x^2|y|)}^{1} \cdot \cancel{2|y|x}^{0} \cdot (\cancel{x^2+y^2})^{0} - \cancel{\sin(x^2|y|)}^{0}(2x)}{(\cancel{x^2+y^2})^2 \ \emptyset\,?} \right] \cancel{(x-x_0)} - \right.$$

$$\left. - \left[ \frac{\cos(\cancel{x^2|y|})^{0} \cdot x^2 (\cancel{x^2+y^2})^{0} - (\cancel{\sin}(x^2|y|) \cdot (2y)}{(\cancel{x^2+y^2})^2 \ 0?} \right] (y-y_0) \right\} \cdot \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

Troppo lungo

• $\begin{cases} \dfrac{x^3 y}{x^6 + y^2} & \text{per } x,y \neq (0,0) \\ \emptyset & x,y = (0,0) \end{cases}$ $\quad \mathbb{D}: \ x^6 + y^2 \neq 0 \Rightarrow$ buco in $(0,0)$

passo a coord polari

$\begin{cases} x = \delta \cos\theta \\ y = \delta \sin\theta \end{cases} \Rightarrow f(\delta,\theta) = \dfrac{\delta^4 \cos^2\theta \sin^2\theta}{\delta^6 \cos^6\theta + \delta^2 \sin^2\theta} = \dfrac{\delta^{\cancel{4}\,2} \cos^2\theta \sin^2\theta}{\cancel{\delta^2}(\delta^4 \cos^4\theta + \sin^2\theta)}$

$= \dfrac{\delta^2 \cos^2\theta \sin^2\theta}{\delta^4 \cos^4\theta + \sin^2\theta} \Rightarrow \lim_{\delta \to 0} f(\delta,\theta) \to$ Dipende da $\theta$ $\Rightarrow$ Non ha senso passare a coord Polari

$\Rightarrow \lim_{x \to 0} f(x,y) = \dfrac{\cancel{x^3}^{\,0} y}{\cancel{x^6}_{\,0} + y^2} \to \emptyset \qquad \lim_{y \to 0} f(x,y) \to \emptyset$

Lungo gli assi

$\lim_{x \to 0} f(x,0) = \dfrac{\cancel{x^3 y}^{\,0}}{x^6 + y^2} = \lim_{y \to 0} (0,y) = \emptyset$

Lungo fascio di rette $\quad y = mx \Rightarrow \lim_{x \to 0} f(x,mx) = \dfrac{m x^4}{x^6 + m^2 x^2} = \dfrac{m x^{\cancel{4}\,2}}{\cancel{x^2}(m^2+1)} = \dfrac{m}{m^2+1} \cdot x^2$

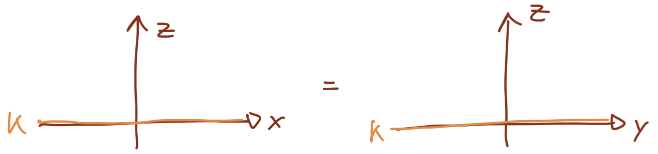
$\Rightarrow \lim_{x \to 0} f(x,mx) \to \emptyset$ Sembrerebbe continua

Lungo $y = x^3 \Rightarrow \lim_{x \to 0} \dfrac{x^4}{x^6 + x^6} = \dfrac{\cancel{x^6}}{\cancel{x^6}(2)} \to \dfrac{1}{2}$ Non dipende da $x \Rightarrow$ Non continua

Cerco di capire la funzione

lungo $x$

$\to f(x,0) = \emptyset \Rightarrow$



• $f(x,y) = \begin{cases} \dfrac{y^2|xy|(x^3-y^2)}{\sqrt{x^2+y^2}} & \text{per } x,y \neq (0,0) \\ \varnothing & \text{per } x,y = (0,0) \end{cases}$

Coord polari $\to$ $\begin{cases} x = \delta\cos\theta \\ y = \delta\sin\theta \end{cases}$ $\Rightarrow$ $f(\delta,\theta) = \dfrac{\delta^2\cos^2\theta\,|\delta^2\cos\theta\sin\theta|\,(\delta^3\cos^3\theta - \delta^3\sin^3\theta)}{\sqrt{\delta^2\cos\theta + \delta^2\sin\theta}}$

$$= \frac{}{\delta\underbrace{\sqrt{\delta^2(\cos^2\theta+\sin^2\theta)}}_{1}} = \underbrace{\delta\cos^2\theta}_{0}\,|\underbrace{\delta^2\cos\theta\sin\theta}_{0}|\,(\underbrace{\delta^3}_{0}(\underbrace{\cos\theta\cdot\cos^2\theta+\sin\cdot\sin^2\theta}_{\text{Boh}}))$$

$\lim_{\delta\to 0} \delta\cos^2\theta\,|\delta^2\cos\theta\sin\theta|(\dots) \to \varnothing = f(x_0,y_0) = 0 \Rightarrow$ continua

Derivabilità

Deve esistere finito il lim: $\lim_{h\to 0} \dfrac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$ ; $\lim_{h\to 0} \dfrac{f(x_0,y_0+h) - f(x_0,y_0)}{h}$

$$\lim_{h\to 0} \frac{\dfrac{0|h\cdot 0|(h^3-0)}{\sqrt{h^2+0}}}{h} = \varnothing \qquad \lim_{h\to 0} \frac{\dfrac{h^2|0h|(0+h^3)}{\sqrt{0+h^2}}}{h} \to \varnothing$$

$\Rightarrow f(x,y)$ deriv parz in $(0,0)$ ; $f'_x(0,0) = 0$ ; $f'_y(0,0) = 0$

Differenz.

$$f(0,0) = \lim_{x,y\to 0,0} \frac{f(x,y) - \varnothing - \varnothing - \varnothing}{\sqrt{x^2+y^2}} = \frac{\dfrac{y^2|xy|(x^3-y^2)}{\cancel{\sqrt{x^2+y^2}}}}{\underbrace{\cancel{\sqrt{x^2+y^2}}}_{(x^2+y^2)}} = \lim_{(x,y)\to(0,0)} \frac{y^2|xy|(x^3-y^2)}{x^2+y^2}$$

lungo il fascio $y = mx$

$$\lim_{x\to 0} \frac{m^2x^2|x^2m|(x^3-m^2y^2)}{x^2+m^2x^2} = \frac{\cancel{x^2}m^2|x^2m|(x^3-m^2y^2)}{\cancel{x^2}(m^2+1)} \text{ per } 0^+ \ \frac{m^3x^2(x^3-m^2x^2)}{m^2+1}$$

$$= \frac{m^3x^5 - m^6x^4}{m^2+1} = \frac{m^3}{m^2+1}\underbrace{(x^5 - x^4m^3)}_{\text{Non dipende da } m} \to \varnothing$$

per $0^-$ $\Rightarrow$ $\dfrac{m^2(-x^2m)(x^3-m^2y^2)}{m^2+1} = \dfrac{-m^3x^2(x^3-m^2y^2)}{m^2+1} = \dfrac{-m^3x^5 - m^5x^4}{m^2+1}$

$= \dfrac{-m^3(x^5+m^2x^4)}{m^2+1}$ Non dip da m $\to \varnothing$ $\Rightarrow$ Diff in $O = (0,0)$

Opzione 2: Coord polari

$\begin{cases} x = \delta\cos\theta \\ y = \delta\sin\theta \end{cases}$ $\Rightarrow$ R.I.$(\delta,\theta) = \dfrac{\cancel{\delta^2}\sin\theta\,|\delta^2\cos\theta\sin\theta|\,(\delta^3(\cos^3\theta+\sin^3\theta))}{\cancel{\delta^2}(\cos^2\theta+\sin^2\theta)}$

$$\Rightarrow \lim_{\delta \to 0} \sin\theta \, |\delta^2 \cos \sin\theta| \, \delta^3 (\cos^3\theta + \sin^3\theta) \to 0$$

non dipende da $\theta$