Metodi di integrazione

- Integrazione per sostituzione:

$$\int f(x)\,dx = \int f[g(t)]\,g'(t)\,dt$$

- Integrazione per parti

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

Formule di integrazione

- $\int x^b\,dx = \frac{x^{b+1}}{b+1} + c$
- $\int \frac{1}{x}\,dx = \ln|x| + c$ $\qquad \int \frac{1}{f(x)} \cdot f'(x)\,dx = \ln|f(x)| + c$
- $\int \tan x\,dx = -\ln|\cos x| + c$
- $\int \frac{1}{\cos^2 x}\,dx = \tan x + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx \quad \arcsin x + c$
- $\int \frac{1}{1+x^2}\,dx \quad \arctan x + c$
- $\int a^x\,dx = \frac{a^x}{\ln a} + c$

- $\int x^3\,dx = \frac{x^4}{4} + c$
- $\int x^5\,dx = \frac{x^6}{6} + c$
- $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx \Rightarrow \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2\sqrt{x^3}}{3} + c$
- $\int x^{1/2}\,dx = \frac{2}{3}\sqrt{x^3} + c = \frac{2}{3}\sqrt{x^2 \cdot x} + c = \frac{2}{3}x\sqrt{x} + c$
- $\int x^{\frac{2}{3}}\,dx = 2\,\frac{x^{\frac{5}{3}}}{5} + c = \frac{3\sqrt[3]{x^5}}{5} = \frac{3}{5}\,x\sqrt[3]{x} + c$

- $\int x^{-3}\,dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$
- $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}} + c = \frac{2}{\sqrt{x}} + c$
- $\int \frac{1}{\sqrt{x}} + 3\,dx = \int \frac{1}{\sqrt{x}} + 3\int dx = \int x^{-\frac{1}{2}}\,dx + 3\int dx = \frac{2}{\sqrt{x}} + 3x + c$
- $\int \frac{\sqrt{x}}{\sqrt[3]{x}}\,dx = \int x^{\frac{1}{2}} \cdot x^{-\frac{1}{3}}\,dx = \int x^{\frac{1}{2}-\frac{1}{3}}\,dx = \int x^{\frac{1}{6}}\,dx = 6\,\frac{x^{\frac{7}{6}}}{7} + c = \sqrt[6]{x^7} = x\sqrt[6]{x} \cdot \frac{6}{7} + c$
- $\int \frac{x^2 + x^3}{x^5}\,dx = \int \frac{x^2}{x^5}\,dx + \int \frac{x^3}{x^5}\,dx = \int x^{-3}\,dx + \int x^{-2}\,dx = \frac{x^{-2}}{2} + x^{-1} = \frac{1}{2x^2} + \frac{1}{x} + c$
- $\int \frac{2x+1}{x^3}\,dx = \int \frac{2x}{x^3} + \int x^{-3}\,dx = -\frac{2}{x} + -\frac{1}{2x^2} + c = \frac{-4x+1}{2x^2} = \frac{1-4x}{2x^2}$
- $\int \frac{1+x^3}{2x^2}\,dx = \frac{1}{2}\int x^{-2}\,dx + \frac{1}{2}\int \frac{x^3}{x^2}\,dx = \frac{1}{2}\,\frac{x^{-1}}{-1} + \frac{1}{2}\,\frac{x^2}{2} + c = \frac{1}{2}\left(-\frac{1}{x} + \frac{x^2}{2}\right) + c$
- $\int \frac{3+2x}{\sqrt{x}}\,dx = 3\int x^{-\frac{1}{2}}\,dx + 2\int x \cdot x^{-\frac{1}{2}}\,dx = 6\sqrt{x} + \cancel{6}^{\,3}\,\frac{x^{\frac{3}{2}}}{\cancel{2}} = 6\sqrt{x} + 3x\sqrt{x} + c$
- $\int 2x^2(1 - x + 2x^3)\,dx = 2\int x^2 - 2\int x^3 + 4\int x^5\,dx = 2\,\frac{x^3}{3} - \cancel{2}\,\frac{x^4}{\cancel{4}} + 4\,\frac{x^6}{6} + c$
- $\int x^2 + \frac{1}{x^2} - \frac{2}{x^3}\,dx = \int x^2 + \int x^{-2} - 2\int x^{-3}\,dx = \frac{x^3}{3} - \frac{1}{x} - \cancel{2}\,\frac{x^{-2}}{\cancel{-2}} = \frac{x^3}{3} - \frac{1}{x} + \frac{1}{x^2} + c$

• $\int 2(2x+1)^2\,dx = 2\int (2x+1)^2 dx = 2\int 4x^2+4x+1\,dx = 8\int x^2 dx + 8\int x\,dx + 2\int dx = 8\frac{x^3}{3} + 8\frac{x^2}{2} + 2x + c$

$= 2x\left(4\frac{x^2}{3} + 4\frac{x}{2} + 1\right) + c = 2x\left(\frac{8x^2+12x+1}{6}\right) = \frac{16x^3}{6} + \frac{24x^2}{6} + \frac{2x}{6}$ $/2$ $\Rightarrow \frac{8x^3}{3} + \frac{12x^2}{3} + \frac{x}{3}$

Per sostituzione pongo $t = 2x+1$ $dx = D(t)dt = 2\,dt$

$\Rightarrow \int 2(2x+1)^2\,dx = 2\int t^2 \cdot 2\,dt = 4\frac{t^3}{3} + c$ Torno a x $\frac{4}{3}(2x+1)^3 + c$

• $\int \frac{1}{(x+1)^2}\,dx =$ Per sostituzione pongo $t = x+1 \Rightarrow dx = dt$

$\Rightarrow \int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = \frac{t^{-1}}{-1} + c \Rightarrow -\frac{1}{t} + c \Rightarrow -\frac{1}{x+1} + c$ Ritorno a t

• $\int x^2\sqrt{x^3+1}\,dx$ pongo $t = x^3+1$

$dx = \frac{1}{t'}dt = \frac{1}{3x^2}dt$

$\Rightarrow \int \cancel{x^2}\sqrt{x^3+1}\,\frac{1}{3\cancel{x^2}}dt = \frac{1}{3}\int\sqrt{x^3+1}\,dt$

$\Rightarrow \frac{1}{3}\int\sqrt{t}\,dt = \frac{1}{3}\int t^{\frac{1}{2}}dt \Rightarrow \frac{1}{3}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{9}t\sqrt{t}$ Torno ad x $\frac{2}{9}(x^3+1)\sqrt{x^3+1} + c$

• $\int x\sqrt[3]{1+x^2}\,dx$ pongo $t = 1+x^2 \Rightarrow dx = \frac{1}{t'}dt = \frac{1}{2x}dt$

$\Rightarrow \int \cancel{x}\sqrt{1+x^2}\cdot\frac{1}{2\cancel{x}}dt = \frac{1}{2}\int\sqrt{t}\,dt = \frac{1}{2}\int t^{\frac{1}{2}}dt = \frac{1}{\cancel{2}}\frac{t^{3/2}}{\frac{3}{\cancel{2}}} + c = \frac{t\sqrt{t}}{3} + c$

Torno a $x \Rightarrow \frac{1}{3}\,1+x^2\sqrt{1+x^2} + c$

• $\int x\sqrt[3]{1+x^2}\,dx$ pongo $t = 1+x^2 \rightarrow dx = \frac{1}{t'}dt = \frac{1}{2x}dt$

$\Rightarrow \int \cancel{x}\sqrt[3]{1+x^2}\,\frac{1}{2\cancel{x}}dt = \frac{1}{2}\int\sqrt[3]{t}\,dt = \frac{1}{2}\frac{t^{\frac{4}{3}}}{\frac{4}{3}} = \frac{3}{8}t\sqrt{t}$ Torno a $\rightarrow x$ $\frac{3}{8}(1+x^2)\sqrt[3]{1+x^2}$

• $\int \sin(x+1)\cos x\,dx$

Esercizi dal libro:

Integrali immediati: Altre formule utili

- $\int [f(x)]^b f'(x)\,dx = \frac{[f(x)]^{b+1}}{b+1} + c$
- $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$
- $\int e^{f(x)} \cdot f'(x)\,dx = e^{f(x)} + c$

4.4)

$\int \cot g\, x\,dx = \int \frac{\cos x}{\sin x}\,dx =$ uso $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c \Rightarrow \ln|\sin x| + c$

$\int tg\,x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{\overbrace{-\sin x}^{f'(x)}}{\underbrace{\cos x}_{f(x)}}\,dx = -\ln|\cos x| + c$

• $\int \sin(ax)\,dx$ = pongo $t = ax \Rightarrow dx = \frac{1}{t'} = \frac{1}{a}dt \Rightarrow \int \sin(t)\frac{1}{a}dt \Rightarrow \frac{1}{a} - \cos(t) + c$
Torno ad $x \to -\frac{1}{a}\cos(ax) + c$

• $\int \cos(ax)\,dx$ pongo $t = ax \Rightarrow dx = \frac{1}{t'}dt = \frac{1}{a}dt \Rightarrow \frac{1}{a}\int \cos(t)\,dt = \frac{1}{a}\sin(ax) + c$

• $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = 2\frac{x^{\frac{3}{2}}}{3} = \frac{2}{3}x\sqrt{x} + c$

• $\int \frac{1}{\sqrt{x+a}}\,dx = \int (x+a)^{-\frac{1}{2}}\,dx$ = pongo $t = x+a$ $\quad dx = 1 = dt \Rightarrow \int t^{-\frac{1}{2}}\,dx = 2t^{\frac{1}{2}} + c$

Sostituire non era necessario

$= 2\sqrt{t} + c$ Torno ad $x = 2\sqrt{x+a} + c$

• $\int \frac{x}{\sqrt{a+x^2}}\,dx$ = Pongo $t = \sqrt{a+x^2} \Rightarrow dx = \frac{1}{t'} = \frac{1}{\frac{1}{2}(a+x^2)^{-\frac{1}{2}}}dt = \frac{1}{\frac{1}{2}\cdot\frac{2}{\sqrt{a+x^2}}\cdot 2x}dt = \frac{1}{\frac{1}{\cancel{2}\sqrt{a+x^2}}\cdot\cancel{2}x}dt$

$\Rightarrow \int \frac{\cancel{x}}{\cancel{\sqrt{a+x^2}}}\cdot\frac{1}{\frac{\cancel{x}}{\cancel{\sqrt{a+x^2}}}}\,dt = \int dt = t + c$

Torno ad $x = \sqrt{a+x^2} + c$

• $\int \frac{x}{\sqrt{a-x^2}}\,dx$ pongo $t = \sqrt{a-x^2} \Rightarrow dx = \frac{1}{\frac{1}{2}\frac{1}{\sqrt{a-x^2}}\cdot(-2x)}dt = \frac{1}{-\frac{\cancel{2}x}{\cancel{2}\sqrt{a-x^2}}}dt = \frac{1}{-\frac{x}{\sqrt{a-x^2}}} = -\frac{\sqrt{a-x^2}}{x}dt$

$\Rightarrow -\int \frac{\cancel{x}}{\cancel{\sqrt{a-x^2}}}\cdot\frac{\cancel{\sqrt{a-x^2}}}{\cancel{x}}\,dt = -\int dt = -t + c \to \sqrt{a-x^2} + c$

ES 4.5)

• $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ Regola: $\int \frac{f'(x)}{\sqrt{1-f^2(x)}}\,dx = \arcsin(f(x)) + c$

$= \int \frac{\overset{f'(x)}{1}}{\sqrt{a^2(1-\underbrace{\frac{x^2}{a^2}}_{f^2(x)})}}\,dx = \int \frac{1}{a\sqrt{1-(\frac{x}{a})^2}} = \frac{1}{a}\int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}\,dx = \frac{1}{a}\arcsin\left(\frac{x}{a}\right) + c$

• $\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2[1+(\frac{x^2}{a^2})]}\,dx = \frac{1}{a^2}\int \frac{1}{1+(\frac{x}{a})^2}\,dx = \arctg\left(\frac{x}{a}\right) + c$

• 4.6) $\int \frac{1}{(3+5x)^6}\,dx$ pongo $t = 3+5x \Rightarrow dx = \frac{1}{5}\,dt$

$\Rightarrow \frac{1}{5}\int \frac{1}{t^6}\,dt = \frac{1}{5}\,\frac{t^{-5}}{-5} + c = -\frac{t^{-5}}{25} + c = -\frac{(3+5x)^{-5}}{25} + c = -\frac{1}{25(3+5x)^5} + c$

4.7) $\int \sqrt{x+2}\,dx =$ pongo $t = x+2 \Rightarrow dx = 1\,dt \Rightarrow \int \sqrt{t}\,dt = \frac{2t\sqrt{t}}{3} + c \rightarrow \frac{2x+4\sqrt{2x+2}}{3}$

4.8) $\int \frac{1}{\sqrt[4]{2x+1}}\,dx = \frac{1}{2}\int (2x+1)^{-\frac{1}{4}}\,D(2x)\,dx = \frac{\overset{2}{4}}{\cancel{2}}\,\frac{(2x+1)^{3/4}}{3} = \frac{2\sqrt[4]{(2x+1)^3}}{3}$

4.9) $\int x\sqrt{3-2x^2}\,dx = -\frac{1}{4}\int (-4)\,x\,(3-2x^2)^{\frac{1}{2}}\,dx = -\frac{2}{12}\,(3-2x^2)^{\frac{3}{2}} = -\frac{1}{6}(3-2x^2)^{\frac{3}{2}} = -\frac{1}{6}\sqrt{(3-2x^2)^3} + c$

4.10) $\int \frac{x}{\sqrt{2-3x^2}}\,dx = -\frac{1}{6}\int (6)\,x\cdot(2-3x^2)^{-\frac{1}{2}}\,dx = -\frac{1}{\cancel{6}\,3}\cdot\frac{(2-3x^2)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} = -\frac{1}{3}\,\frac{1}{\sqrt{2-3x^2}} + c$

4.11) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{1}{x\sqrt{x^2\left(1-\left(\frac{1}{x}\right)^2\right)}}\,dx = \int \frac{1}{x^2\sqrt{1-\left(\frac{1}{x^2}\right)^2}}\,dx = \int x^{-2}\cdot\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\,dx$

Boh! arcsin $\left(\frac{1}{x}\right)$

4.12) $\int \sin^5 x\cdot\cos x\,dx =$ pongo $t = \sin x \Rightarrow dx = \frac{1}{\cos x}\,dt$

$= \int \cos^5 t\cdot\cancel{\cos x}\,\frac{1}{\cancel{\cos x}}\,dt = \int t^5\,dt = \frac{1}{6}t^6 + c \longrightarrow \frac{1}{6}\sin^6(x) + c$

4.13) $\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx =$ pongo $t = \arcsin^2 x \Rightarrow dx = \frac{1}{\frac{1}{\sqrt{1-x^2}}}\,dt = \sqrt{1-x^2}\,dt$

$\Rightarrow \int \frac{\arcsin^2 x}{\cancel{\sqrt{1-x^2}}}\cdot\cancel{\sqrt{1-x^2}}\,dt = \int t^2\,dt = \frac{1}{3}t^3 + c = \frac{1}{3}\arcsin^3 x + c$

4.14) $\int \frac{1}{\arcsin x\cdot\sqrt{1-x^2}}\,dx \quad \int \frac{\arcsin^{-1} x}{\sqrt{1-x^2}}\,dx$ pongo $t = \arcsin x \Rightarrow dx = \frac{1}{\frac{1}{\sqrt{1-x^2}}}\,dt = \sqrt{1-x^2}\,dt$

$\Rightarrow \int \frac{\arcsin^{-1} x}{\cancel{\sqrt{1-x^2}}}\cdot\cancel{\sqrt{1-x^2}}\,dt = \int t^{-1}\,dt = \int \frac{1}{t}\,dt = \ln|t| + c = \ln|\arcsin x| + c$

4.15) $\int \frac{1}{(1+x^2)\,\mathrm{arcTg}\,x}\,dx = \int \frac{\mathrm{arcTg}^{-1} x}{(1+x^2)}\,dx$ pongo $t = \mathrm{arcTg}\,x \Rightarrow dx = \frac{1}{\frac{1}{1+x^2}}\,dt = 1+x^2\,dt$

$\Rightarrow \int \frac{\mathrm{arcTg}^{-1} x}{\cancel{(1+x^2)}}\cdot\cancel{(1+x^2)}\,dt = \int \frac{1}{t}\,dt = \ln|\mathrm{arcTg}\,x| + c$

4.16) $\int \frac{1}{x\ln x}\,dx \quad \int \frac{\ln x}{x}\,dx$ pongo $t = \ln x \Rightarrow dx = \frac{1}{\frac{1}{x}}\,dt = x\,dt$

$\Rightarrow \int \frac{\ln x^{-1}}{\cancel{x}}\cdot\cancel{x}\,dt = \int \frac{1}{t}\,dt = \ln|\ln(x)| + c$

4.17) $\int \frac{\ln(x)}{x}\,dx \Rightarrow$ pongo $t = \ln x \Rightarrow dx = x\,dx \Rightarrow \int t\,dt = \frac{1}{2}\ln^2(x) + c$

4.18) $\int \frac{\ln(x)^n}{x}\,dx =$ Pongo $t = \ln(x) \Rightarrow dx = x\,dt \Rightarrow \int \frac{t^n}{\cancel{x}}\cdot\cancel{x}\,dt = \frac{t^{n+1}}{n+1} + c = \frac{\ln(x)^{n+1}}{n+1} + c$

4.19) $\int \frac{1}{x(\ln x)^n}\,dx = \int \frac{(\ln x)^{-n}}{x}\,dx$ pongo $t = \ln x \Rightarrow dx = x\,dt \Rightarrow \int t^{-n}\,dt = \frac{t^{-n+1}}{-n+1} + c = \frac{\ln^{1-n} x}{1-n} + c$

4.20) $\int 3x\,e^{x^2}\,dx = \frac{3}{2}\int 2\cdot x\,e^{x^2} = \frac{3}{2}\int e^{x^2}\cdot D(e^{x^2}) = \frac{3}{2}e^{x^2} + c \leftarrow$ usando $\int e^{f(x)}\cdot f'(x)\,dx = e^{f(x)} + c$

$D(e^{x^2}) = 2x\,e^{x}$

Verifico con sostituzione pongo $t = e^{x^2} \Rightarrow dx = \frac{1}{e^{x^2}\cdot 2x}\,dt \Rightarrow \frac{3}{2}\int \cancel{e^{x^2}\cdot 2x}\cdot\frac{1}{\cancel{e^{x^2}\cdot 2x}}\,dx = \frac{3}{2}\int dt =$

$= \frac{3}{2}t + c \rightarrow \frac{3}{2}e^{x^2} + c$

4.21) $\int \frac{1}{\sin(x+a)}\,dx =$ pongo $t = x+a \Rightarrow dx = dt \Rightarrow \int \frac{1}{\sin(t)} \cdot \frac{\sin(t)}{\sin(t)}\,dt = \int \frac{\sin(t)}{\sin^2(t)}\,dt$

$\Rightarrow \sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1-\cos^2 x \Rightarrow \int \frac{\sin(t)}{1-\cos^2 t}\,dt =$ pongo $u = \cos t \Rightarrow dt = \frac{1}{\sin t}\,du$

$\Rightarrow \int \frac{\cancel{\sin(t)}}{1-u^2} \cdot \frac{1}{\cancel{\sin(t)}}\,du = \int \frac{1}{1-u^2}\,du =$ BOH

4.22 $\int \frac{1}{\cos x}\,dx \cdot \frac{\cos x}{\cos x} = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$ pongo $t = \sin x \Rightarrow dx = -\frac{1}{\cos x}\,dt$

$\Rightarrow \int \frac{\cancel{\cos x}}{1-t^2} \cdot \frac{1}{\cancel{\cos x}}\,dt = \int \frac{1}{1-t^2}\,dt = \int \frac{1}{(1-t)(1+t)}\,dt =$

4.26 $\int \operatorname{sen}^n x \cos x\,dx$ pongo $t = \sin x \Rightarrow dx = \frac{1}{\cos x}\,dt \Rightarrow \int \sin^n x\, \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}}\,dt$

$= \int t^n\,dt = \frac{t^{n+1}}{n+1} + c = \frac{\sin^{n+1} x}{n+1} + c$

4.27 $\int \cos^n x \operatorname{sen} x\,dx =$ pongo $t = \cos x \Rightarrow dx = -\frac{1}{\sin x}\,dt \Rightarrow -\int \cos^n x \frac{\cancel{\sin x}}{\cancel{\sin x}}\,dt = -\int t^n\,dt$

$= -\frac{\cos^{n+1} x}{n+1} + c$

4.28 $\int \frac{1}{\operatorname{sen} x \cos x}\,dx = \int \frac{1}{\sin x} \cdot \frac{1}{\cos x}\,dx$ BOH

4C. Integrazione per decomposizione in somma

4.34 $\int (hx+k)\,dx = h\int x + k\int dx = \frac{h}{2}x^2 + kx + c$

4.36 $\int \frac{x}{1+x}\,dx = \int \frac{x+1-1}{1+x}\,dx = \int \frac{\cancel{x+1}}{\cancel{x+1}} - \int \frac{1}{x+1}\,dx = \int dx - \int \frac{1}{x+1}\,dx = x - \ln|x+1| + c$

4.37 $\int \frac{3x+2}{4x+5}\,dx = \int \frac{3x}{4x+5}\,dx + \int \frac{2}{4x+5}\,dx = 3 \int \frac{x}{4x+5}\,dx + 2\int \frac{1}{4x+5}\,dx$

$\Rightarrow$ pongo $t = 4x+5 \Rightarrow dx = \frac{1}{4}\,dt \Rightarrow \frac{1}{4}\int \frac{t-5}{4t}\,dt = \frac{1}{4}\int \frac{t}{4t}\,dt - \frac{1}{4}\int \frac{5}{4t}\,dt = \frac{1}{8}\int dt - \frac{5}{8}\int \frac{1}{t}\,dt$

$\Rightarrow x = \frac{t-5}{4}$

$\Rightarrow \frac{1}{8}t - \frac{5}{8}\ln|t| + c$

$\Rightarrow \frac{3}{8}t - \frac{15}{8}\ln|t| + 2\ln|4x+5| + c \rightarrow$ Torno ad x $\rightarrow \frac{3}{8}\cdot(4x+5) - \frac{15}{8}\ln|4x+5| + 2\ln|4x+5| + c$

$= \frac{3}{2}x + \frac{15}{8} - \frac{15}{8}\ln|4x+5| + \ln|4x+5|^2 + c$