3.1 Utilizzando le formule di riduzione, calcolare l'integrale doppio

$$\iint_S xy\,dx\,dy$$

dove S è il semicerchio chiuso in figura 3.1, di centro (1,0) e raggio 1, con $y \geq 0$.

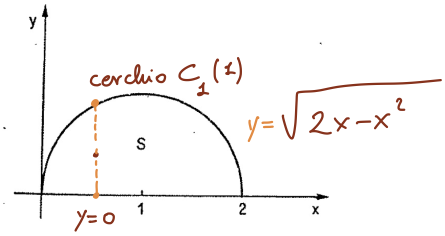



figura 3.1

$S = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le \sqrt{2x-x^2}$

cerchio: $(x-\alpha)^2 + (y-\beta)^2 = r^2$  $C = (1,0)$, $r = 1$

$\Rightarrow (x-1)^2 + y^2 = 1 \Rightarrow x^2 - 2x + \cancel{1} + y^2 = \cancel{1}$

$\Rightarrow x^2 - 2x + y^2 = 0 \Rightarrow y = \sqrt{2x-x^2}$

$\Rightarrow \iint_S xy\,dx = \int_0^2 x\,dx \int_0^{\sqrt{2x-x^2}} y\,dy \Rightarrow \int_0^2 x \left[\frac{y^2}{2}\right]_0^{\sqrt{2x-x^2}} dx$

$= \int_0^2 x \cdot \frac{2x-x^2}{2}\,dx = \int_0^2 \frac{2x^2}{2} - \int_0^2 \frac{x^3}{2}\,dx = \left[\frac{x^3}{3}\right]_0^2 - \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2$

$= \frac{8}{3} - \frac{1}{2}\cdot\frac{16}{4} - \frac{2}{3}$

3.2 Calcolare l'integrale doppio

$$\iint_A xy\,dx\,dy$$

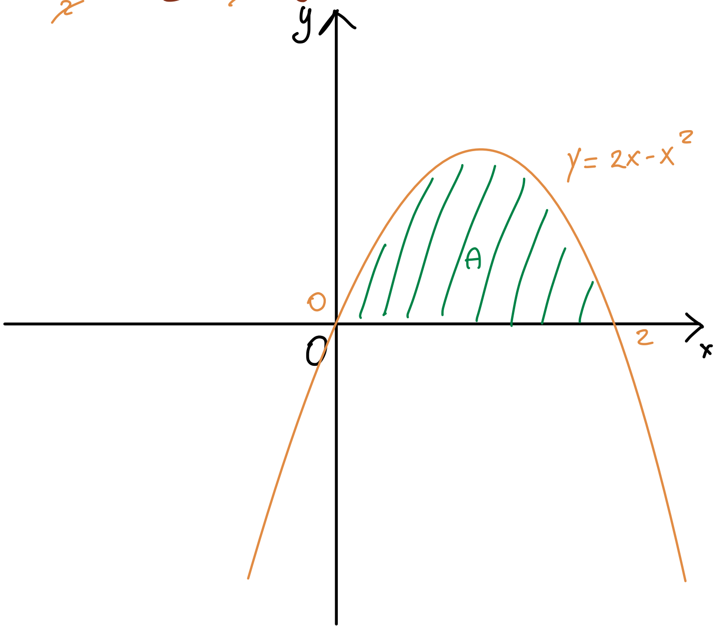
dove $A=\{(x,y)\in\mathbb{R}^2:\ 0\le x\le 2,\ 0\le y\le 2x-x^2\}$.

Non c'e' bisogno di disegnare il grafico

$\int_0^2 x\,dx \int_0^{2x-x^2} y\,dy = \int_0^2 x \cdot \left[\frac{y^2}{2}\right]_0^{2x-x^2} dx$

$= \int_0^2 x \cdot \frac{(2x-x^2)^2}{2}\,dx = \int x \cdot \left(\frac{4x^2-4x^3+x^4}{2}\right)dx = 2\int_0^2 x^3 - 2\int_0^2 x^4 + \frac{1}{2}\int_0^2 x^5\,dx = 2\left[\frac{x^4}{4}\right]_0^2 - 2\left[\frac{x^5}{5}\right]_0^2 + \frac{1}{2}\left[\frac{x^6}{6}\right]_0^2$

$= 2\,\frac{\cancel{16}^{\,4\,\cancel{8}}}{\cancel{4}_{\,\cancel{2}}} - 2\,\frac{32}{5} + \frac{1}{\cancel{2}}\,\frac{\cancel{64}^{\,32}}{6} = 8 - \frac{64}{5} + \frac{32}{6} = \frac{240-384+160}{30} = \frac{\cancel{16}^{\,8}}{\cancel{30}_{\,15}} = \frac{8}{15}$



3.4 Calcolare l'integrale doppio

$$\iint_A \frac{x}{1+y}\,dx\,dy,$$

dove A è l'insieme $A=\{(x,y)\in R^2: 0\le x\le 1, x^2\le y\le x\}$.

Dominio rispetto ad $x$

$$\int_0^1 x\,dx \int_{x^2}^{x} \frac{1}{1+y}\,dy \to \int_0^1 x\cdot\left[\ln|1+y|\right]_{x^2}^{x} dx$$

$$=\int_0^1 x\left[\ln|1+x|-\ln|1+x^2|\right]dx = \int_0^1 \underbrace{x}_{f}\,\underbrace{\ln|1+x|}_{g'}\,dx \;(a) - \int_0^1 x\ln|1+x^2|\,dx \;(b)$$ Risolvo per parti

$$\int f\cdot g\,dx = F\cdot g-\int F\cdot g'\,dx \to$$ a) $$\frac{x^2}{2}\cdot\ln|1+x|-\int_0^1 \frac{x^2}{2}\cdot\frac{1}{1+x}\,dx$$ Risolvo l'integr.

$$\to \frac{1}{2}\int_0^1 \frac{x^2}{1+x} \quad \text{pongo } t=1+x \to dx=dt \to x=1-t \to \frac{1}{2}\int_0^1 \frac{(1-t)^2}{t}dt \to \frac{1}{2}\int \frac{1-2t+t^2}{t}\,dt$$

$$=\frac{1}{2}\int_0^1 \frac{1}{t}-\int_0^1 \frac{t}{t}+\frac{1}{2}\int_0^1 \frac{t^2}{t}\,dt = \frac{x^2}{2}\cdot\ln|1+x|-\left\{\left[\frac{1}{2}\ln|t|\right]_0^1-\left[t\right]_0^1+\left[\frac{1}{2}\,\frac{t^2}{2}\right]_0^1\right\}=$$

$$=\frac{x^2}{2}\ln|1+x|-\left[\frac{1}{2}\ln|1|-1+\frac{1}{4}\right]=\frac{x^2}{2}\ln|1+x|+1-\frac{1}{4}=\frac{x^2}{2}\ln|x+1|+\frac{4-1}{4}=\frac{x^2}{2}\ln|1+x|+\frac{3}{4}$$

$$=\frac{2x^2\ln|x+1|+3}{4}$$

3.5 Si calcolino i seguenti integrali doppi con due metodi, considerando il dominio di integrazione prima normale rispetto all'asse x, poi normale rispetto all'asse y:

(a) $\iint_A x\,dx\,dy$

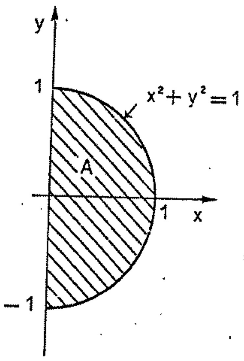



figura 3.5

$x^2+y^2=1$ circ di $r=1$ e $c:(0,0)$ $\to$ $y=\pm\sqrt{1-x^2}$

$A=\{(x,y) / 0\le x\le 1 , -\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\}$

$\Rightarrow \int_0^1 x\,dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \int_0^1 x\,[y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx = \int x\left[\sqrt{1-x^2}+\sqrt{1-x^2}\right]dx$

$2\int x\cdot\sqrt{1-x^2}\,dx = 2\int x\cdot(1-x^2)^{\frac12}$ pongo $t=1-x^2 \to x=\sqrt{1-t}$

$dx=-\frac{1}{2x}dt$

$\Rightarrow -\cancel{2}\int \cancel{x}\cdot(1-x^2)^{\frac12}\cdot\frac{1}{\cancel{2x}} = -\int t^{\frac12}dt = -\left[\frac{t^{\frac32}}{\frac32}\right]_0^1$

$=\left[-\frac23\cdot(1-x^2)^{\frac32}\right]_0^1 = -\frac23\cdot\cancel{1}^{\frac32}\ 1 - \left[-\frac23(1-1)^{\frac32}\right] = +\frac23$

Dominio rispetto ad y

$y^2+x^2=1 \to x=\pm\sqrt{1-y^2}$

$A=\{(x,y)/-1<y<1 , 0<x<+\sqrt{1-y^2}\}$

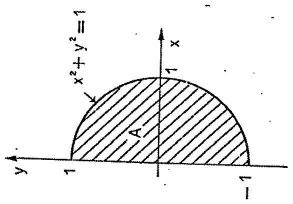



figura 3.5

$\Rightarrow \int_{-1}^{1} dy\int_0^{\sqrt{\ }} x\,dx \to \int_{-1}^{1}\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}}dy \to \int_{-1}^{1}\left[\frac{1-y^2}{2}-\frac12\right]_0^1 dy$

$\to \int_{-1}^{1} \frac{\cancel{1}-y^2\cancel{-1}}{2}\,dy = \frac12\int_{-1}^{1}1-y^2\,dy = \frac12\int dy-\frac12\int y^2\,dy$

$=\frac12\left[y-\frac{y^3}{3}\right]_{-1}^{1} = \frac12\left[\left(1-\frac13\right)-\left(-1-\frac{(-1)}{3}\right)\right]=\frac12\left[\frac{3-1}{3}-\left(\frac{-3+1}{3}\right)\right]$

$=\frac12\left[\frac23+\frac23\right]=\frac12\cdot\frac43=\frac23$

(b) $\iint_B \frac{y}{(1+x)\,(1+y^2)}\,dx\ dy$

Rispetto ~~ed~~ $x$ $\quad B = \{(x,y) / \ 0 \le x \le 1 \ , \sqrt{x} \le y \le 1\}$

Calcolo a

$$\to \overbrace{\int_0^1 \frac{1}{(1+x)}\,dx}^{b} \int_{\sqrt{x}}^1 \frac{y}{(1+y^2)}\,dy \to$$

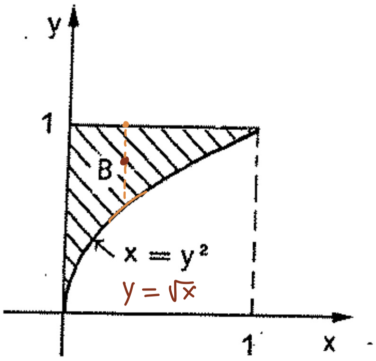



figura 3.6

a) $\int_{\sqrt{x}}^1 \frac{y}{1+y^2}\,dy$ pongo $1+y^2=t \to dy = \frac{1}{2y}\,dt$

$$\to \int_{\sqrt{x}}^1 \frac{\cancel{y}}{1+y^2}\,\frac{1}{2\cancel{y}}\,dt = \frac{1}{2}\int_{\sqrt{x}}^1 \frac{1}{t}\,dt = \frac{1}{2}\left[\ln|1+y^2|\right]_{\sqrt{x}}^1$$

$$\to \frac{1}{2}\left[\ln|2| - \ln|1+\sqrt{x}|\right]$$

Torno a b $\quad \frac{1}{2}\int_0^1 \frac{1}{1+x}\cdot \ln|2| - \frac{1}{2}\int \underbrace{\frac{1}{1+x}}_{g}\,\underbrace{\ln|1+\sqrt{x}|}_{f}\,dx$

$$\to \frac{1}{2}\ln|2| \int \frac{1}{1+x}\,dx - \frac{1}{2}\int_0^1 \frac{\ln|1+\sqrt{x}|}{1+x}\,dx$$

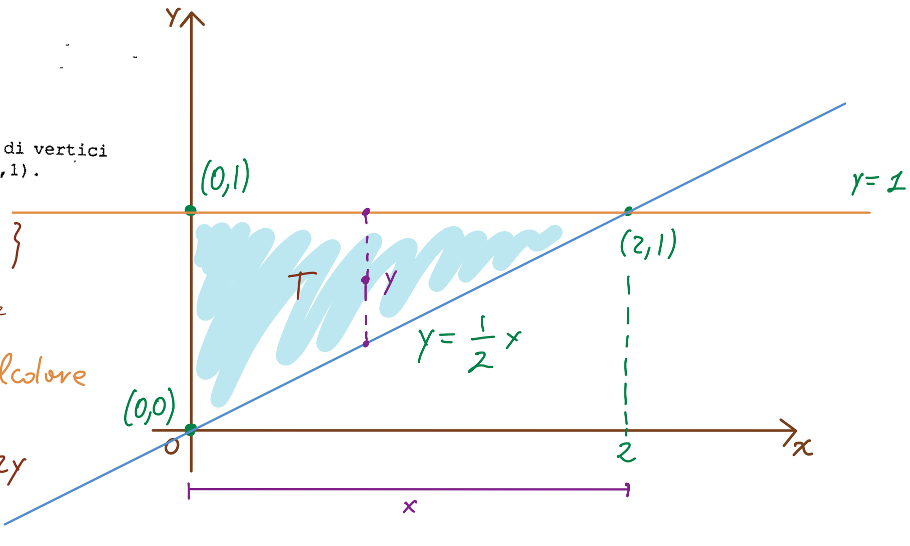
risolvo c) per parti $\int f\cdot g = F\cdot g - \int F\cdot g' \to$

$$\to \frac{1}{1+\sqrt{x}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\cdot\frac{1}{1+x} - \int \frac{1}{1+\sqrt{x}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\cdot\left(-\frac{1}{(1+x)^2}\right)dx = \frac{1}{1+\sqrt{x}}\cdot\frac{1}{2\sqrt{x}\,(1+x)}\int$$

3.6 Calcolare l'integrale doppio

$$\iint_T e^{y^2}\,dx\,dy$$

dove T è il triangolo chiuso del piano x,y di vertici nei punti di coordinate (0,0), (0,1), (2,1).




$T = \{(x,y) \,/\, 0 \le x \le 2 \;,\; \frac{1}{2}x \le y \le 1\}$

$\rightarrow \int_0^2 dx \int_{\frac{1}{2}x}^{1} e^{y^2}\,dy \rightarrow$ $D(e^{y^2}) = 2y\,e^{y^2}$

Difficile da calcolare

$y = \frac{1}{2}x \rightarrow x = 2y$

Rispetto a y

$T = \{(x,y) \,/\, 0 \le y \le 1 \;,\; 0 \le x \le 2y\}$

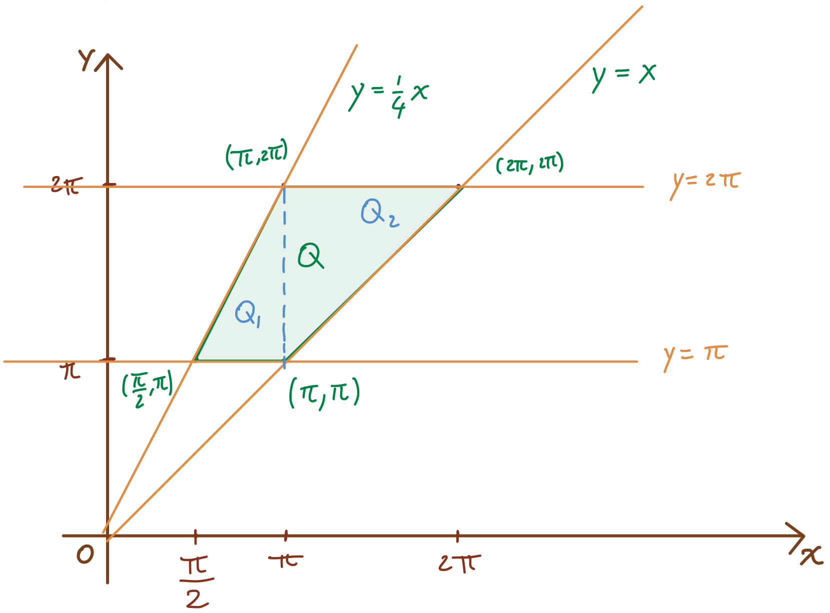
$\rightarrow \int_0^1 e^{y^2}dy \int_0^{2y} dx \rightarrow \int_0^1 e^{y^2}\cdot[x]_0^{2y} = \int_0^1 e^{y^2}\cdot \underbrace{2y}_{D(e^{y^2})} = [e^{y^2}]_0^1 = e - 1$

3.7 Calcolare l'integrale doppio

$$\iint_Q \frac{\operatorname{sen} y}{y}\,dx\,dy$$

dove Q è il quadrilatero di vertici

$$\left(\frac{\pi}{2},\ \pi\right),\ (\pi,\ \pi),\ (\pi,\ 2\pi),\ (2\pi,\ 2\pi).$$




1) $\frac{\pi}{2}\cdot x = 2\pi y \rightarrow x = \frac{4\cancel{\pi}y}{\cancel{\pi}} \rightarrow y = \frac{1}{4}x$

2) $\pi x = \pi y \rightarrow y = x$

Dominio secondo x

$Q_1 = \{(x,y) \,/\, \frac{\pi}{2} \le x \le \pi \;,\; \pi \le y \le \frac{1}{4}x\}$

$Q_2 = \{(x,y) \,/\, \pi \le x \le 2\pi \;,\; x \le y \le 2\pi\}$

$Q = Q_1 + Q_2$

Risolvo $Q_1$

$\int_{\frac{\pi}{2}}^{\pi} dx \int_{\pi}^{1/4 x} \frac{\sin y}{y}\,dy \rightarrow$ Come cavolo si calcola $\int \frac{\sin y}{y}$ ?

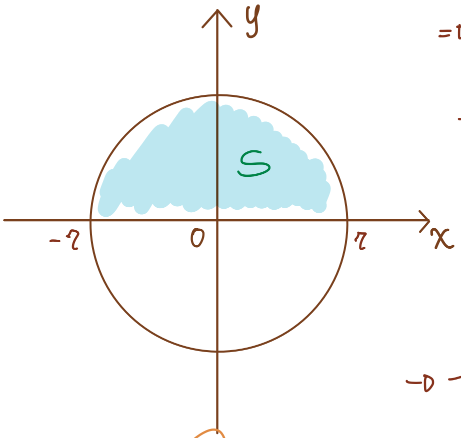
3.9 Calcolare l'integrale doppio

$$\iint_S (x-y)\,dx\,dy$$

dove S è il semicerchio di centro l'origine e raggio r, contenuto nel semipiano delle y positive.

$$C_r(0) = x^2+y^2=r^2 \to y=\pm\sqrt{r^2+x^2}$$

$$\to (x-0)^2+(y-0)^2=r$$




$$\Rightarrow S=\{(x,y)\,/\, -r\le x\le r \;,\; 0\le y\le \sqrt{r^2+x^2}\}$$

$$\to \int_{-r}^{r} x\,dx\,(-)\int_0^{\sqrt{r^2+x^2}} y\,dy = \int_{-r}^{r} x\cdot\left[-\left(\frac{y^2}{2}\right)\right]_0^{\sqrt{x^2+r^2}} dx$$

$$\to -\int_{-r}^{r} x\cdot\frac{x^2+r^2}{2}\,dx = -\frac12\int_{-r}^{r} x^3+x^2r^2\,dx$$

$$\to -\frac12\left[\frac{x^4}{4}\right]_{-r}^{r} - \frac{r^2}{2}\left[\frac{x^3}{3}\right]_{-r}^{r} = -\frac12\left[\frac{r^4}{4}-\frac{r^4}{4}\right] - \overset{?}{\frac{r^2}{2}}\left[\frac{r^3}{3}+\frac{r^3}{3}\right]$$

$$= -\frac23 r^3\cdot r^2$$ sbagliato

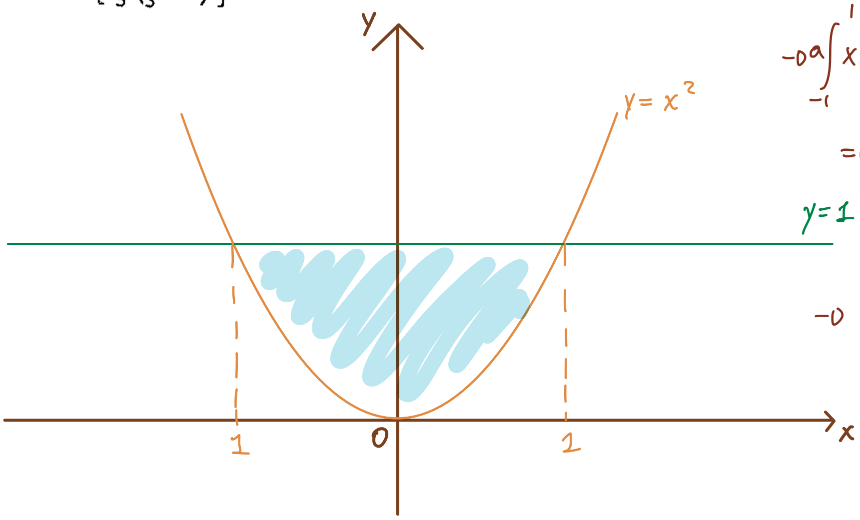
3.10 Calcolare l'integrale doppio

$$\iint_X (ax^2+by)\,dx\,dy,$$

ove $a,b\in\mathbb{R}$ e $X=\{(x,y)\in[-1,1]\times\mathbb{R}: x^2\le y\le 1\}$.

$\left[\frac45\left(\frac a3+b\right)\right]$

$$X=\{(x,y)\,/\, -1\le x\le 1 \;,\; x^2\le y\le 1\}$$




$$a\int_{-1}^{1}x^2\,dx\; b\int_{x^2}^{1} y\,dy = a\int_{-1}^{1}x^2\cdot b\left[\frac{y^2}{2}\right]_{x^2}^{1}dx$$

$$\to a\int_{-1}^{1} x^2\cdot b\left[\frac12-\frac{x^4}{2}\right]dx = a\int_{-1}^{1} x^2\cdot\left[\frac b2-\frac{bx^4}{2}\right]dx$$

$$\Rightarrow \frac{ab}{2}\int_{-1}^{1} x^2 - \frac{ab}{2}\int x^6\,dx$$

$$\to \frac{ab}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} - \frac{ab}{2}\left[\frac{x^7}{7}\right]_{-1}^{1}$$

$$\to \frac{ab}{2}\left[\frac13+\frac13\right]-\frac{ab}{2}\left[\frac17+\frac17\right]$$

$$\to \frac{ab}{2}\left[\frac23-\frac27\right]=\frac{ab}{2}\left[\frac{14-6}{21}\right]=\frac{ab}{2}\cdot\frac{8}{21}$$

3.11 Calcolare l'integrale doppio

$$\iint_A x \cos y \, dx \, dy$$

ove $A = \{(x,y) \in [0,1] \times R: 0 \leq y \leq 1-x^2\}$.

$A = \{(x,y) = 0 \leq x \leq 1 \;,\; 0 \leq y \leq -x^2\}$

$\rightarrow \int_0^1 x \, dx \int_0^{-x^2} \cos y \, dy \rightarrow \int_0^1 x \left[\sin y\right]_0^{-x^2} dx \rightarrow \int_0^1 \overset{g}{x} \overset{f}{\sin(-x^2)} \, dx \rightarrow$ pongo $t = -x^2$

$dx = -\frac{1}{2x} dt \rightarrow -\int_0^1 \cancel{x} \sin(t) \cdot \frac{1}{2\cancel{x}} dt = -\frac{1}{2}\int_0^1 \sin(t) \, dt \rightarrow \cancel{-}\frac{1}{2}\left[\cancel{-}\cos(-x^2)\right]_0^1 = \frac{1}{2}\left[\cos(-1) - \cos(0)\right]$

$= \frac{\cos(-1) - 1}{2}$

3.12 Calcolare l'integrale doppio

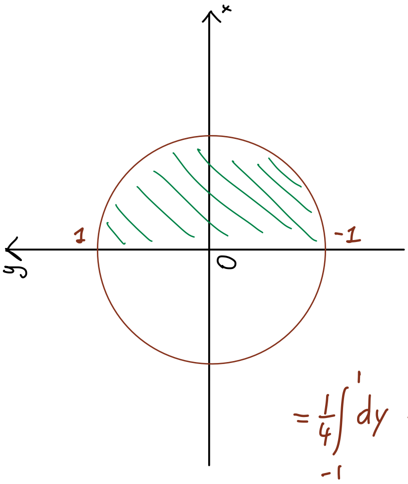
$$\iint_C x^3 \, dx \, dy$$

dove l'insieme C è costituito rispettivamente dal

(a) semicerchio $\{x^2+y^2 \leq 1, \; x \geq 0\}$

(b) cerchio $\{x^2+y^2 \leq 1\}$

} Trattore "$\gtrless$" come "="




a) $C_1(0)$ Valori interni al cerchio e $x \geq 0$

Consideriamo il dominio rispetto ad $y$ $\quad x^2+y^2 = 1 \rightarrow x = \sqrt{1-y^2}$

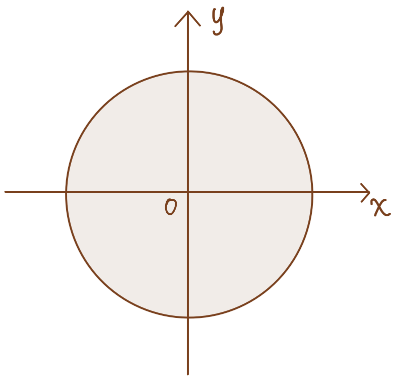
$C = \{(x,y) \,/\, -1 \leq y \leq 1 \;,\; 0 < x < +\sqrt{1-y^2}\}$

$\int_{-1}^1 dy \int_0^{\sqrt{1-y^2}} x^3 \, dx = \int_{-1}^1 \left[\frac{x^4}{4}\right]^{\sqrt{1-y^2}} dy = \int_{-1}^1 \frac{(1-y^2)^2}{4} dy =$

$= \frac{1}{4}\int_{-1}^1 dy - \frac{1}{2}\int_{-1}^1 y^2 + \frac{1}{4}\int_{-1}^1 y^4 \, dy \rightarrow \frac{1}{4}\left[y\right]_{-1}^1 - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-1}^1 + \frac{1}{4}\left[\frac{y^5}{5}\right]_{-1}^1$

$= \frac{1}{2} - \frac{1}{2}\left[\frac{1}{3}+\frac{1}{3}\right] + \frac{1}{4}\left[\frac{1}{5}+\frac{1}{5}\right] = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \frac{15-10+3}{30} = \frac{\cancel{8}^{\,4}}{\cancel{30}_{\,15}} = \frac{4}{15}$

b) Caso $x^2+y^2 \leq 1$



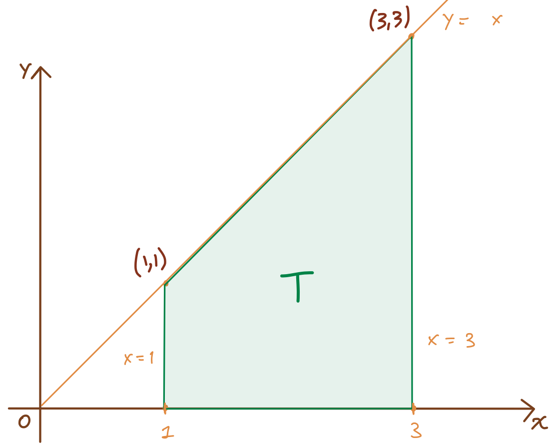

$C = \{(x,y) \,/\, -1 \leq y \leq 1 \;,\; -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\}$

$\int_{-1}^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^3 \, dx = \int_{-1}^1 \frac{\cancel{(1-y^2)^2}}{\cancel{[-(1-y^2)]^2}} dx = \int_{-1}^1 dx = 0$

3.13 Calcolare l'integrale doppio

$$\iint_T \frac{dx\,dy}{x^2+y^2},$$

dove T è il trapezio del piano x,y di vertici (1,0), (1,1), (3,0), (3,3).




$T = \{(x,y) \;/\; 1 \le x \le 3 \;,\; 0 \le y \le 2x\}$

$$\int_1^3 dx \int_0^x \frac{1}{x^2+y^2}\,dy \qquad \text{Pongo } x^2 = c$$

$$\to \int_1^3 dx \int_0^x \frac{1}{c^2+y^2}\,dy \;\to\; \int \frac{1}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\Rightarrow \int_1^3 \left[\frac{1}{x}\cdot\arctan\left(\frac{y}{x}\right)\right]_0^x dx \;\to\; \int_1^3 \left[\frac{1}{x}\arctan\left(\frac{x}{x}\right) - \frac{1}{x}\arctan(0)\right] dx = \arctan(1)\int_1^3 \frac{1}{x}\,dx$$

$$= \arctan(1)\cdot\left[\ln|3| - \overset{0}{\cancel{\ln|1|}}\right] = \arctan(1)\ln(3) = \frac{\pi}{4}\ln(3)$$

## Coordinate polari

$$\begin{cases} x = a + \rho\cos\theta \\ y = b + \rho\sin\theta \end{cases} \qquad \text{Es prec: } \iint \frac{dx\,dv}{x^2+y^2} \to \iint \frac{1}{\rho^2\cos^2\theta + \rho^2\sin^2\theta}\cdot\rho\,d\rho\,d\theta$$

$$\to 2 \le \rho \le \sqrt{(3-1)^2+(3-1)^2} \to 2 \le \rho \le 2\sqrt{2} \Rightarrow \int_2^{2\sqrt{2}} d\rho \int_0^{\frac{\pi}{4}} \frac{1}{\rho^2\cos^2\theta + \rho^2\sin^2\theta}\cdot\rho\,d\theta$$

$$= \int_2^{2\sqrt{2}} d\rho \int \frac{1}{\rho^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}}\,\cancel{\rho}\,d\theta = \int_2^{2\sqrt{2}} d\rho \int \frac{1}{\rho}\,d\theta \to \int_2^{2\sqrt{2}} \frac{1}{\rho}\,d\rho \int_0^{\frac{\pi}{4}} d\theta \to \int_2^{2\sqrt{2}} \frac{1}{\rho}\cdot\frac{\pi}{4}\,d\rho$$

$$= \frac{\pi}{4}\int_2^{2\sqrt{2}} \frac{1}{\rho}\,d\rho = \frac{\pi}{4}\ln|2\sqrt{2}| - \ln|2| =$$

L'angolo $\theta$ è sbagliato
ma il resto è giusto

3.21 Calcolare l'integrale doppio

$$\iint_C \frac{y}{x^2+y^2}\,dx\,dy$$

ove $C=\{(x,y)\in\mathbb{R}^2:\ y\geq 0,\ 1\leq x^2+y^2\leq 4\}$.

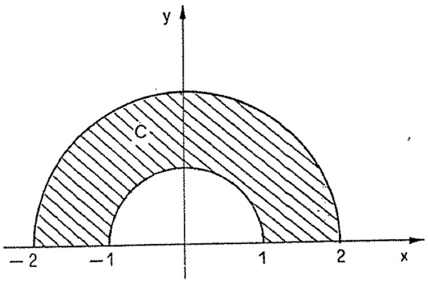



figura 3.11

$$C=\{(x,y)\ /\ -2\leq x\leq 2\ ,\sqrt{1-x^2}\leq y\leq\sqrt{4-x^2}\ \}$$

$$C_1 = C_1(0) \to x^2+y^2=1 \to y=\sqrt{1-x^2}$$
$$C_2 = C_2(0) \to x^2+y^2=4 \to y=\sqrt{4-x^2}$$

Coord. polari

$$1\leq\rho\leq 2\ ,\quad 0\leq\theta\leq\pi$$

$$\to \int_1^2 d\rho \int_0^\pi \frac{\sin\theta}{\rho^2\cos^2\theta+\rho^2\sin^2\theta}\,\rho\,d\theta \to \int_1^2 d\rho \int_0^\pi \frac{\cancel{\rho}\sin\theta}{\cancel{\rho^2}\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}}\cdot\cancel{\rho}$$

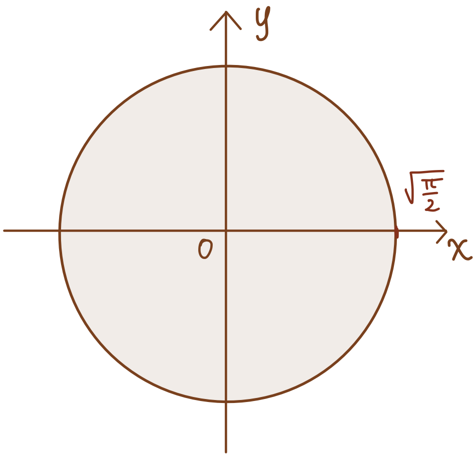
$$\to \int_1^2 d\rho \int_0^\pi \sin\theta\,d\theta = \int_1^2 [\underbrace{-\cos\pi}_{1}+\cos 0]\,d\rho = 2\int_1^2 d\rho$$

$$\to 2\,[2-1]=2$$

3.23 Calcolare l'integrale doppio

$$\iint_C |xy|\ \text{sen}\,(y^2)\cos(x^2+y^2)\,dx\,dy,$$

dove C è il cerchio di centro (0,0) e raggio $\sqrt{\pi/2}$.




$$C_{\sqrt{\frac{\pi}{2}}}(0) = x^2+y^2=\frac{\pi}{2}$$

$$\Rightarrow 0\leq\rho\leq\sqrt{\frac{\pi}{2}}\ ,\quad 0\leq\theta\leq 2\pi$$

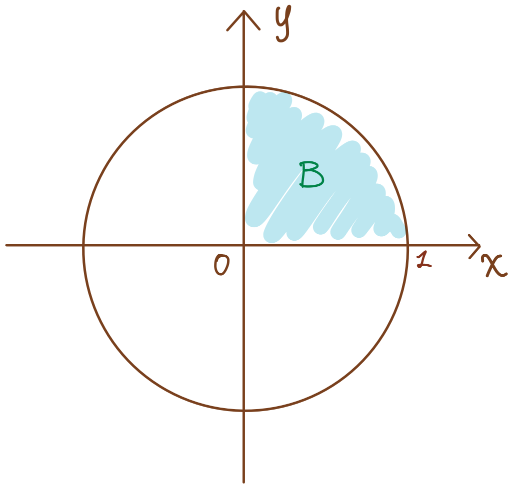
$$\to \int_0^{\sqrt{\frac{\pi}{2}}} d\rho \int_0^{2\pi} |\rho\cdot\cos\theta\cdot\rho\sin\theta|\cdot\sin(\rho^2\sin^2\theta)\cdot\cos(\rho^2\cos^2\theta+\rho^2\sin^2\theta)\cdot\rho\cdot d\theta$$

$$\to \int_0^{\sqrt{\frac{\pi}{2}}} \rho(\cos\theta\sin\theta)\cdot\sin(\rho^2\sin^2\theta)\cdot\cos(\rho)\cdot\rho\ d\theta$$

3.30 Calcolare l'integrale doppio

$$\iint_B \sqrt{x^2+y^2}\ dx\ dy$$

dove B è il settore del cerchio di centro l'origine e di raggio 1, contenuto nel primo quadrante.




$$B = \{(\delta,\theta) /\quad 0 \leq \delta \leq 1 \quad , \quad 0 \leq \theta \leq \frac{\pi}{2}\}$$

$$\int_0^1 d\delta \cdot \int_0^{\frac{\pi}{2}} \sqrt{\delta^2\cos^2\theta + \delta^2\sin^2\theta}\ \cdot \delta\ d\theta$$

$$= \int_0^1 d\delta \cdot \int_0^{\frac{\pi}{2}} \delta \cdot \delta\ d\theta \to \int_0^1 \delta^2 d\delta \cdot [\theta]_0^{\frac{\pi}{2}} d\delta \to \frac{\pi}{2}\int_0^1 \delta^2 d\delta \to \frac{\pi}{2}\left[\frac{\delta^3}{3}\right]_0^1 = \frac{\pi}{6}$$

3.31 Sia B la circonferenza di centro l'origine e raggio 1. Verificare che risulta

$$\iint_B x^2\ e^{-(x^2+y^2)}\ dx\ dy = \frac{\pi(e-1)}{4e}.$$

$$B = \{(\delta,\theta) /\quad -1 \leq \delta \leq 1 \ , \quad 0 \leq \theta \leq 2\pi\}$$

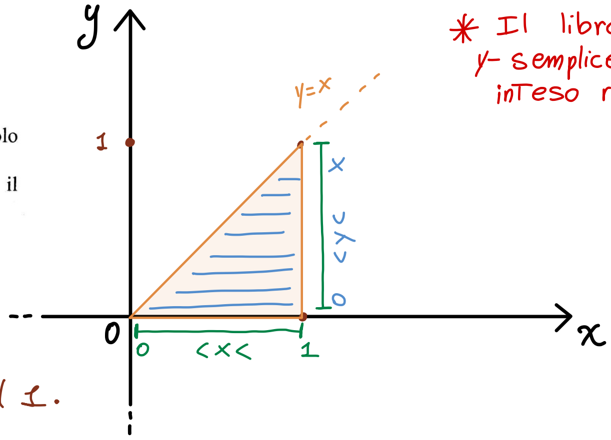
$$\int_{-1}^1 d\delta \int_0^{2\pi} \delta^2\cos^2\theta \cdot e^{(\delta^2)} \cdot \delta\ d\theta \to \int_{-1}^1 \delta^3 \cdot e^{\delta^2} d\delta \int_0^{2\pi} \cos\theta\ d\theta \to \int_{-1}^1 \delta^3 \cdot e^{\delta^2} \cdot [\underbrace{\sin(2\pi)}_{0} - \underbrace{\sin(0)}_{0}]\ d\theta$$

Nuovi esercizi

**Esempio 5.1.**

*a.* Scrivere come dominio $y$-semplice e come dominio $x$-semplice il triangolo $T$ di vertici $(0,0),(0,1),(1,1)$.

*b.* Scrivere come dominio $y$-semplice (o unione di domini di questo tipo) il triangolo di vertici $(-1,0),(0,2),(3,0)$.



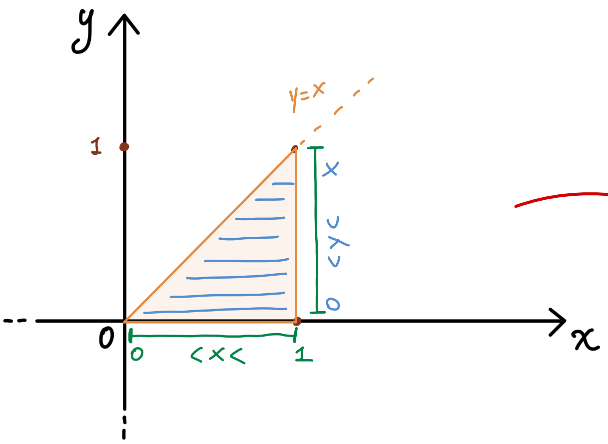

* Il libro intende come y-semplice il Dominio inteso rispetto ad $x$

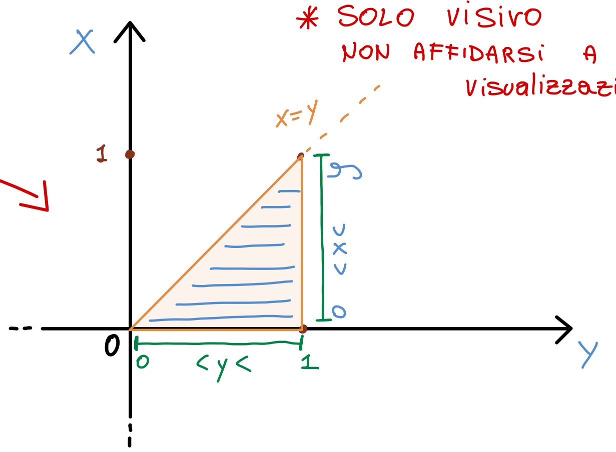
Per trovare il dominio del triangolo, abbiamo come prima cosa capire quali valori assume la $x$: x varia tra 0 ed 1.

Successivamente ci chiediamo, come varia la $y$ per ogni $x$ fissato?

$\Rightarrow$ $y$ varia da 0 ed $y=x$, o piu' semplicemente: $0 \leq y \leq x$

$\Rightarrow$ Dominio rispetto ad $y$: $D = \{(x,y) \,/\, 0 \leq x \leq 1 \,,\, 0 \leq y \leq x\}$



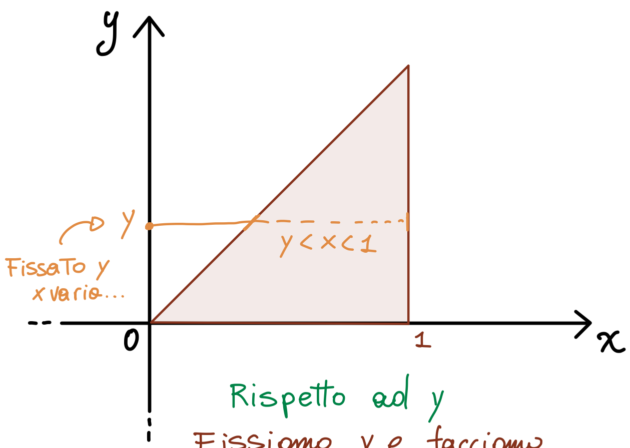




* SOLO VISIVO

NON AFFIDARSI A QUESTA visualizzazione!

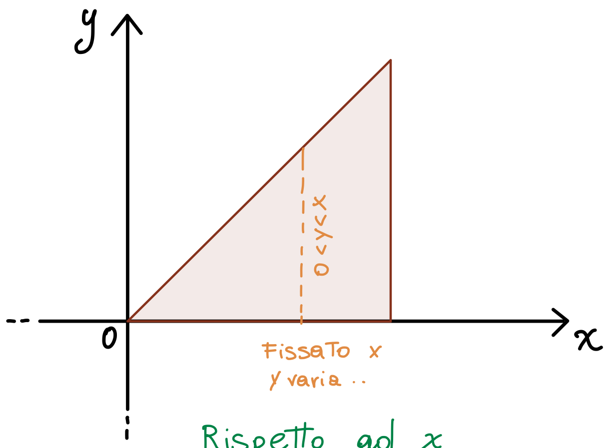
Cosa facciamo nella pratica?

Facciamo VARIARE $y$ da 0 ad 1 $\Rightarrow$ $y \in [0,1]$. Successivamente, per ogni $y$ FISSATO La retta di ordinata $y$ interseca il triangolo nel segmento $y < x < 1$:




Rispetto ad $y$

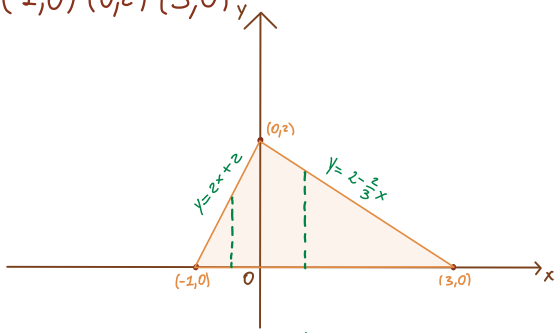
Fissiamo $y$ e facciamo variare $x$

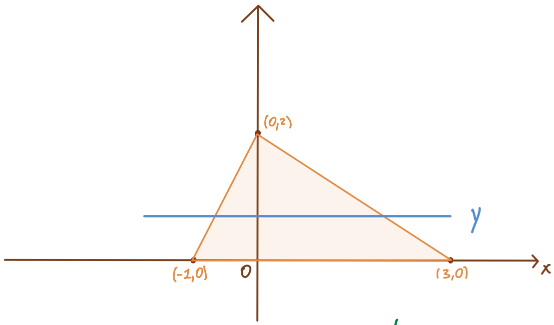



Rispetto ad $x$

Fissiamo $x$ e facciamo variare $y$

b) $(-1,0)\ (0,2)\ (3,0)$




Rispetto ad $x$

$-1 \leq x \leq 3$ , Con $y$ c'e' un Problema:

Rispetto ad $y$

$0 < y < 2$ , $y = 2x+2 \to x = \frac{1}{2}y - 1$

$y = 2 - \frac{2}{3}x \to x = \frac{3}{2}y - 3$

$\frac{1}{2}y - 1 \leq x \leq \frac{3}{2}y - 3$

Conviene "spezzare il $\overbrace{\text{triangolo}}^{\text{Dominio del}}$ in due insiemi

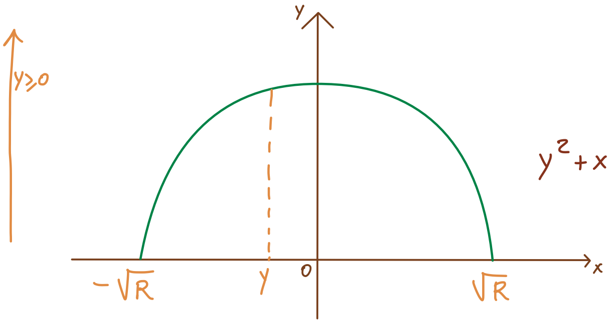
1. Per $-1 \leq x \leq 0 \to 0 \leq y \leq 2x+2$
2. Per $0 \leq x \leq 3 \to 0 \leq y \leq 2 - \frac{2}{3}x$

UNIONE $\to$ D: $D_1 = \{(x,y) / -1 \leq x \leq 0,\ 0 \leq y \leq 2x+2\} \cup$
$\cup\ D_2 = \{(x,y) / 0 \leq x \leq 3,\ 0 \leq y \leq 2 - \frac{2}{3}x\}$

**Esempio 5.2.**

*a.* Scrivere come dominio $y$-semplice e come dominio $x$-semplice il semicerchio $C$ di centro l'origine, raggio $R$, contenuto nel semipiano $y \geq 0$.

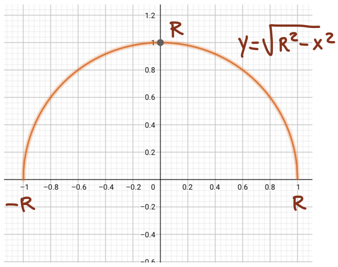
*b.* Scrivere come dominio $y$-semplice (o unione di più domini di questo tipo) il parallelogramma $P$ di vertici $(0,0), (1,2), (4,0), (5,2)$.



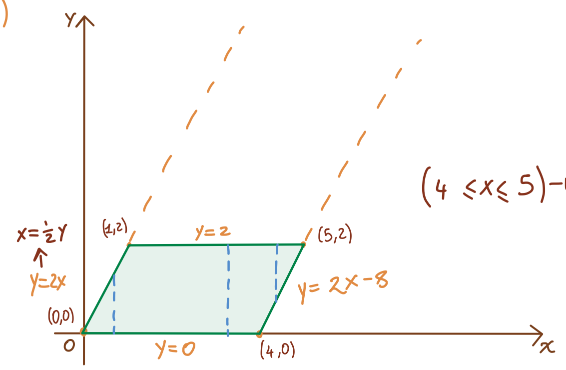

$R \leq x \leq R$ , $0 \leq y \leq \sqrt{R - x^2}$

$y^2 + x^2 = R^2 \to x = \sqrt{R - y^2}$

$y = \sqrt{R - x^2}$




b)



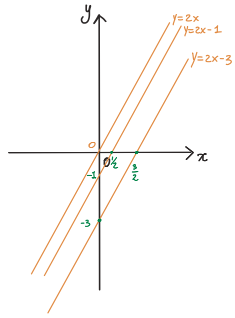

Secondo $x$

$(0 \leq x \leq 5)$ insieme Totale $x$

$(4 \leq x \leq 5) \to (2x-8 \leq y \leq 2)$

$(1 \leq x \leq 4) \to (0 \leq y \leq 2)$

$(0 \leq x \leq 1) \to (0 \leq y \leq 2x)$



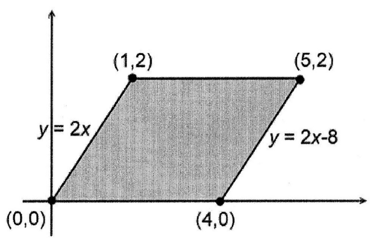



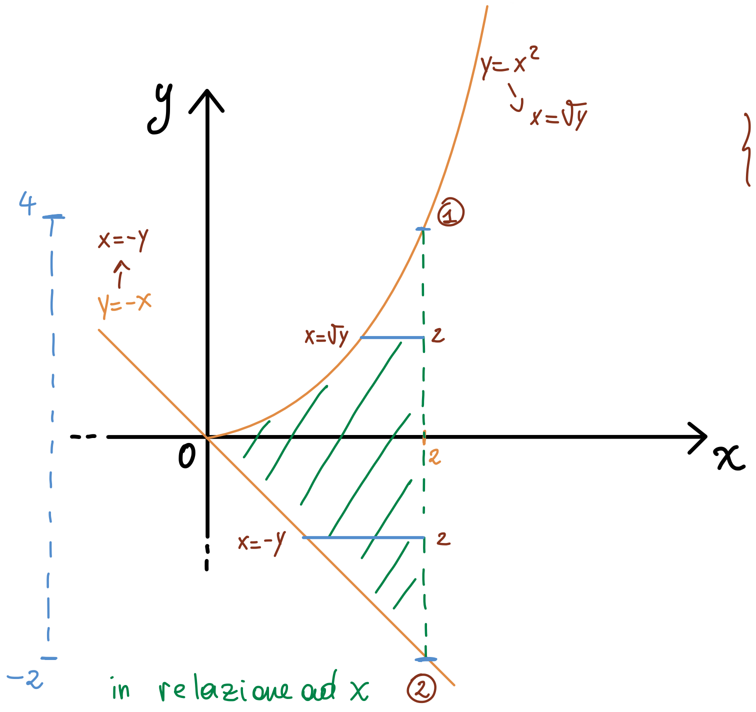

Fig.5.5.

In sostanza si è rappresentato il parallelogramma come unione di due triangoli e un rettangolo.

**Esempio 5.3.** Dato l'insieme $y$-semplice —▷ in relazione ad $x$

$$E = \{(x, y) : 0 \leq x \leq 2, -x \leq y \leq x^2\}$$

rappresentarlo come insieme $x$-semplice (o unione di insiemi $x$-semplici).




Trovo intersezioni

$\begin{cases} y = x^2 \\ x = 2 \end{cases} \Rightarrow y = 4 \to (2,4)$ ①

$\begin{cases} y = -x \\ x = 2 \end{cases} \to y = -2 \to (2,-2)$ ②

Secondo $y$ —▷ $-2 \leq y \leq 4$

$y = 0$

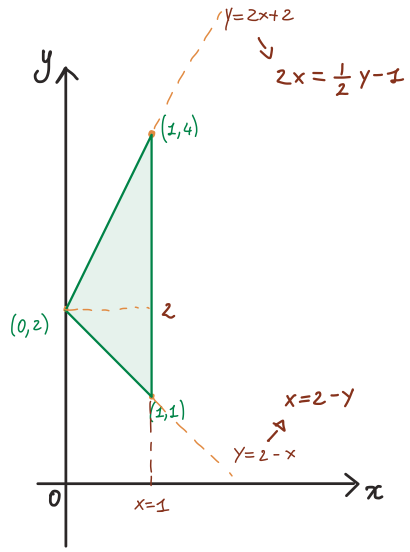
$\left[(-2 \leq y \leq 0) \to (-y \leq x \leq 2)\right]$ $\left[(0 \leq y \leq 4) \to (\sqrt{y} \leq x \leq 2)\right]$

### Esercizi

**5.1.** Rappresentare come insiemi $x$-semplici o $y$-semplici o unione di più insiemi di questo tipo (nel modo che sembra più semplice) i seguenti insiemi del piano $(x, y)$:

*a.* Il triangolo $T$ di vertici $(1, 1), (0, 2), (1, 4)$.
*b.* Il rombo $R$ di vertici $(\pm a, 0), (0, \pm b)$ con $a, b > 0$.
*c.* Il quarto di cerchio $C$ racchiuso nel 2° quadrante dalla circonferenza di centro l'origine e raggio $R$.
*d.* Il quadrilatero $Q$ di vertici $(0, 0), (2, -1), (2, 2), (6, 3)$.

a)



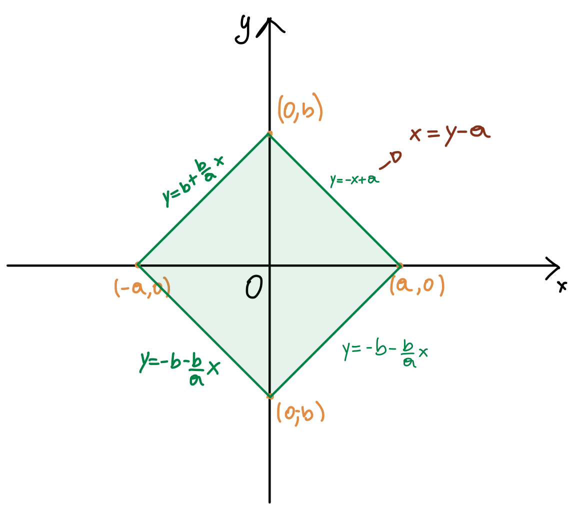

Secondo $y$ —▷ $1 \leq y \leq 4$

$\begin{pmatrix} 1 \leq y \leq 2 \\ 2-y \leq x \leq 2 \end{pmatrix}$ $y=2$

$\begin{pmatrix} 2 \leq y \leq 4 \\ 2 \leq x \leq \frac{1}{2}y - 1 \end{pmatrix}$ $y=2$

✓

b)



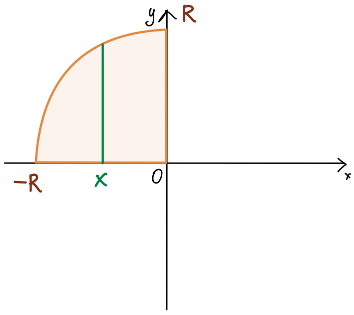

Rispetto ad $x$

$(-a \leq x \leq 0)$
$\left(-b-\frac{b}{a}x \leq y \leq b+\frac{b}{a}x\right)$

$(0 \leq x \leq a)$
$\left(-b+\frac{b}{a}x \leq y \leq b-\frac{b}{a}x\right)$

✓

c)

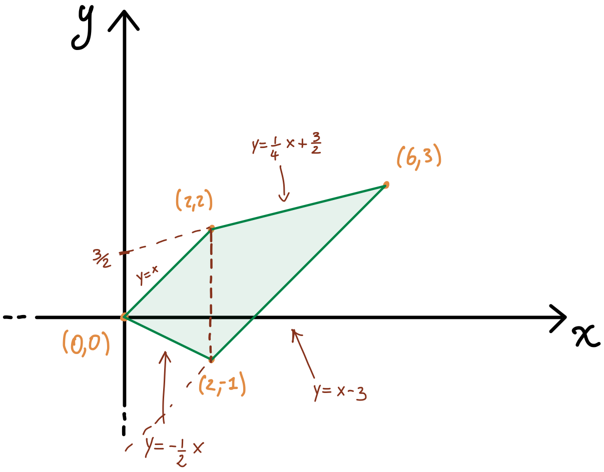



Circ = $x^2+y^2=R^2 \to y=\sqrt{R^2-x^2}$

$(-R \le x \le 0) \to 0 \le y \le \sqrt{R^2-x^2}$

✓

D) (0,0) (2,-1) (2,2) (6,3)




$(0 \le x \le 2) \to -\frac{1}{2}x \le y \le x$

$(2 \le x \le 6) \to x-3 \le y \le \frac{1}{4}x+\frac{3}{2}$

✓

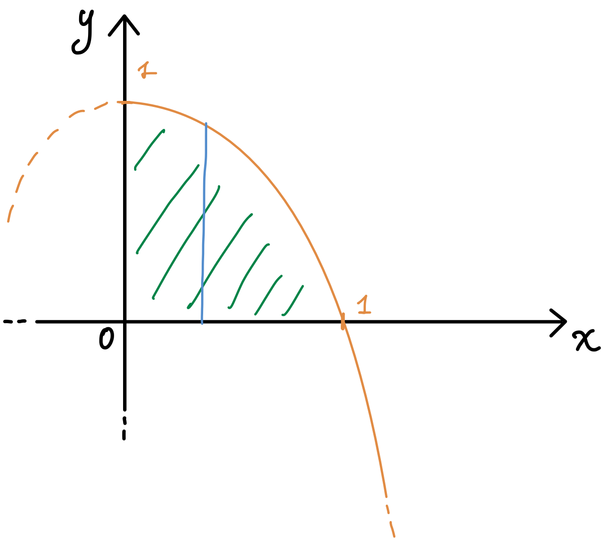
**5.2.**

*a.* Dato l'insieme $y$-semplice

$$E=\{(x,y): 0 \le x \le 1, 0 \le y \le 1-x^3\},$$

rappresentarlo come insieme $x$-semplice (o unione di insiemi $x$-semplici).

$y=1-x^3 \to -x^3=1-y \to x=\sqrt[3]{1-y}$




$(0 \le y \le 1) \to 0 \le x \le \sqrt[3]{1-y}$

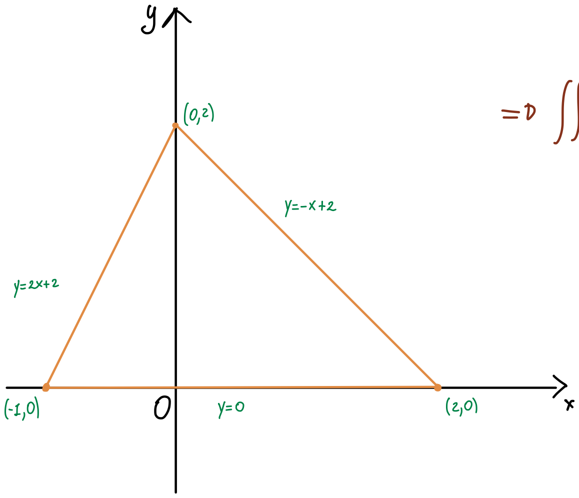
**Esempio 5.5.** Calcolare l'integrale doppio:

$$\iint_T \frac{y}{1+x}\,dxdy$$

dove $T$ è il triangolo di vertici $(-1,0),(2,0),(0,2)$.

$\iint_T \frac{y}{1+x}\,dx\,dy$ $\qquad D=(-1,0)(2,0)(0,2)$

$D_x=\{(x,y)\ /\ -1<x<0\ ,\ 0<y<2x+2\}\cup\{0<x<2\ ,\ 0<y<-x+2\}$




$$\Rightarrow \iint_T \frac{y}{1+x}\,dx\,dy = \underbrace{\int_{-1}^{0} dx\left(\int_0^{2x+2}\frac{y}{1+x}\,dy\right)}_{a} + \underbrace{\int_{0}^{2} dx\left(\int_0^{2-x}\frac{y}{1+x}\,dy\right)}_{b}$$

Risolvere prima dy

a) $\int_{-1}^{0}\frac{1}{1+x}\,dx\cdot\int_0^{2x+2}y\,dy = \int_{-1}^{0}\frac{1}{1+x}\,dx\cdot\left[\left(\frac{y^2}{2}\right)_0^{2x+2}\right]=$

$=\int_{-1}^{0}\frac{1}{1+x}\,dx\left[\frac{\cancel{4}^{2}x^2+\cancel{8}^{4}x+\cancel{4}^{2}}{\cancel{2}}-0\right]=\int_{-1}^{0}\frac{2x^2+4x+2}{1+x}\,dx = 2\int_{-1}^{0}\frac{(x+1)^{\cancel{2}}}{\cancel{1+x}}\,dx$

$=2\int_{-1}^{0}x+2\int_{-1}^{0}dx = 2\left[\frac{x^2}{2}+x\right]_{-1}^{0} = 2\left\{\left[\,0\,\right]-\left[\frac12-1\right]\right\}= +1$

b) $\int_0^2\frac{1}{1+x}\int_0^{2-x}y\,dy = \int_0^2\frac{1}{1+x}\cdot\left[\frac12 y^2\right]_0^{2-x}dx = \int_0^2\frac{1}{1+x}\,dx\cdot\left[\left(\frac12(x^2-4x+4)\right)-(2)\right]$

$=\int_0^2\frac{1}{1+x}\cdot\frac12x^2-2x\,dx = \int_0^2\frac{\frac12x^2-2x}{1+x}\,dx =$

$$\begin{array}{l|l}\frac12\cancel{x^2}-2x & x+1\\ \hline \frac12\cancel{x^2}+\frac12x & \frac12x-\frac52\\ -\frac52\cancel{x} & \\ -\frac52\cancel{x}-\frac52 & \end{array}$$

$\Rightarrow \frac12x-\frac52-\frac{5/2}{x+1}$

$\Rightarrow \frac12\int_0^2x\,dx-\frac52\int_0^2dx-\frac52\int\frac{1}{x+1}\,dx=$

$=\left[\frac14x^2\right]_0^2-\frac52\left[x\right]_0^2-\frac52\left[\ln|x+1|\right]_0^2 = \left[\left(\frac14 4\right)-(0)\right]-\frac{5}{\cancel{2}}\left[\cancel{2}\right]-\frac52\left[\ln|3|-\ln|1|\right]=1-5-\frac52\ln3$

$1 \qquad 5 \qquad \ln\left|\frac31\right| \qquad \ln 3$

$\Rightarrow \iint_T\frac{y}{1+x}\,dx\,dy = -3-\frac52\ln 3$

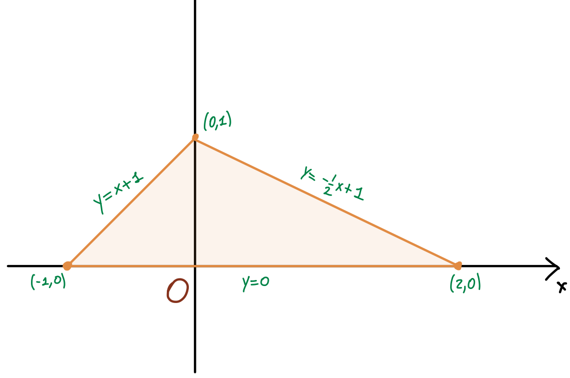
**5.3.★** Calcolare il seguente integrale doppio:

$$\iint_T |x|\,dxdy$$

dove $T$ è il triangolo di vertici $(-1,0),(0,1),(2,0)$.

$\iint_T |x|\,dx\,dy$

$T=\{(x,y)/-1<x<0\ ,\ 0<y<x+1\}\cup\{(x,y)/0<x<2\ ,\ 0<y<1-\frac12x\}$




$\Rightarrow \iint_T|x|\,dx\,dy = \int_{-1}^{0}\int_0^{x+1}|x|\,dx\,dy+\int_0^2\int_0^{1-\frac12x}|x|\,dx\,dy$

$\Rightarrow \int_{-1}^{0}|x|\,dx\int_0^{x+1}dy+\int_0^2|x|\,dx\int_0^{1-\frac12x}dy = \int_{-1}^{0}|x|\,dx\left[y\right]_0^{x+1}+\int_0^2|x|\,dx\left[y\right]_0^{1-\frac12x} = \underbrace{\int_{-1}^{0}|x|\cdot\left[x+1\right]dx}_{a}+\underbrace{\int_0^2|x|\left[1-\frac12x\right]dx}_{b}$

a) $\int_{-1}^{0}|x^2|+\int_{-1}^{0}|x|\,dx \Rightarrow \left[\frac13|x^3|\right]_{-1}^{0}+\left[\frac12|x^2|\right]_{-1}^{0} = \left(0-\frac13\right)+\left(0+\frac12\right)=\frac{-2+3}{6}=\frac16$

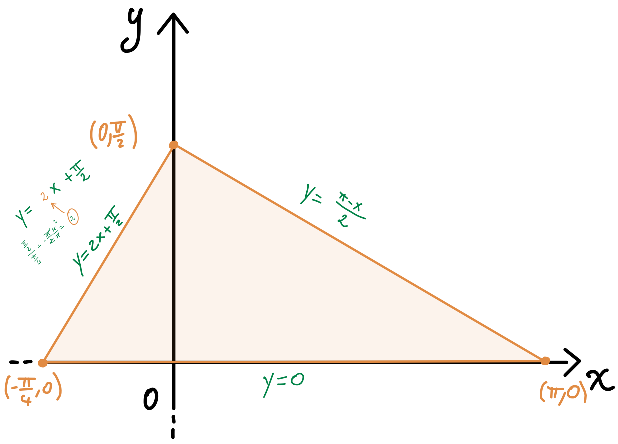
b) $\int_0^2x-\frac12\int_0^2x^2\,dx = \left[\frac12x^2\right]_0^2-\frac12\left[\frac13x^3\right]_0^2 = 2-\frac12\cdot\frac83=2-\frac43=\frac{6-4}{3}=\frac23$

$\frac16+\frac23=\frac{1+4}{6}=\frac56$

**5.5.★** Calcolare il seguente integrale doppio:

$$\int\int_T \cos y \, dx dy$$

dove $T$ è il triangolo di vertici $(-\frac{\pi}{4},0), (0,\frac{\pi}{2}), (\pi,0)$.




$\to T = \{(x,y) / -\frac{\pi}{4} < x < 0 \ , \ 0 < y < 2x+\frac{\pi}{2}\} \cup$

$\cup \{(x,y) / 0 < x < \pi \ , \ 0 < y < \frac{\pi - x}{2}\}$

$\to \int_{-\frac{\pi}{4}}^{0} dx \int_{0}^{2x+\frac{\pi}{2}} \cos y \, dy \to \int_{-\frac{\pi}{4}}^{0} [\sin y]_0^{2x+\frac{\pi}{2}} dx = \int_{-\frac{\pi}{4}}^{0} \sin(2x+\frac{\pi}{2}) \, dx$

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \Rightarrow \frac{x}{\pi} = \frac{y-\frac{\pi}{2}}{0-\frac{\pi}{2}} \to$

$\to \frac{x}{\pi} = \frac{\frac{2y-\pi}{2}}{-\frac{\pi}{2}} \Rightarrow \frac{x}{\pi} = -\frac{2y-\pi}{\cancel{2}} \cdot \frac{\cancel{2}}{\pi}$

$\to \frac{x}{\pi} = -\frac{2y}{\pi} + 1 \to y = -\frac{x}{2} + \frac{\pi}{2}$

$\to$ pongo $t = 2x+\frac{\pi}{2} \to dx = \frac{1}{2} dt$

$\to \frac{1}{2}\int \sin(t) dt = \left[-\frac{1}{2}\cos(2x+\frac{\pi}{2})\right]_{-\frac{\pi}{4}}^{0} = \left[-\frac{1}{2}\cos(\frac{\pi}{2})\right] - \left[-\frac{1}{2}\cos(-\frac{\pi}{2}+\frac{\pi}{2})\right] = 0 + \frac{1}{2} = \frac{1}{2}$

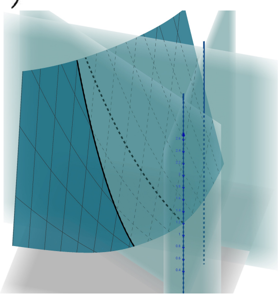
b) $\int_0^{\pi} dx \int_0^{\frac{\pi}{2}-\frac{x}{2}} \cos y \, dy = \int_0^{\pi} [\sin y]_0^{\frac{\pi - x}{2}} dy = \int_0^{\pi} \sin\frac{\pi-x}{2} \, dx$ = pongo $t = \frac{\pi - x}{2} \to dx = \frac{1}{-\frac{1}{2}} dt$

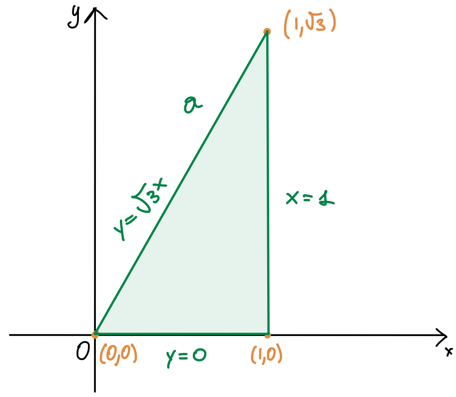
$= \frac{1}{2}\int_0^{\pi} \sin(t) dt \to \frac{1}{2}\left[-\cos(\frac{\pi-x}{2})\right]_0^{\pi} \to \frac{1}{2}\left[\underbrace{-\cos(\frac{\pi-\pi}{2})}_{-1}\right] - \frac{1}{2}\left[\underbrace{-\cos(\frac{\pi}{2})}_{0}\right] \to -\frac{1}{2}$

**5.6.★** Calcolare l'integrale doppio

$$\int\int_T xy\sqrt{x^2+y^2} dx dy$$

dove $T$ è il triangolo di vertici $(0,0), (1,0), \left(1, \sqrt{3}\right)$.






$T = \{(x,y) / \ 0 < x < 1 \ , \ 0 < y < \sqrt{3}x\}$

$\iint xy\sqrt{x^2+y^2} \, dx dy = \int_0^1 dx \int_0^{\sqrt{3}x} xy\sqrt{x^2+y^2} \, dy$

$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$

$\int \underset{\text{cost}}{xy}\sqrt{x^2+y^2} \, dy = t = \sqrt{x^2+y^2} \to dy = \frac{1}{\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y} dt$

$(x^2+y^2)^{\frac{1}{2}}$

$\Rightarrow a) \ x = \frac{y}{\sqrt{3}} \to y = \sqrt{3}x$

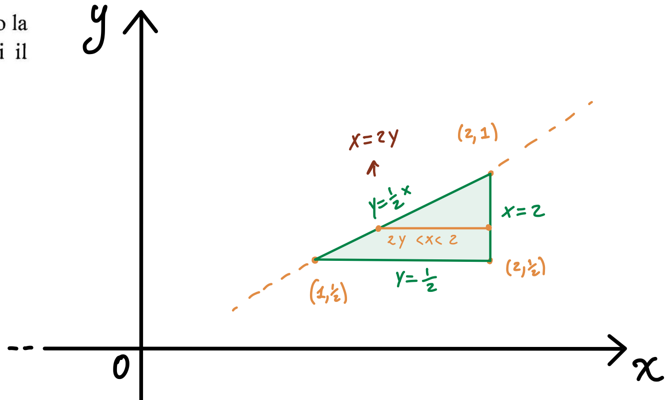
$\to \int x\cancel{y} \cdot t \cdot \frac{1}{\cancel{2y}} \cdot \cancel{2}\underbrace{\sqrt{x^2+y^2}}_{\to t} \to x\int t^2 dt = x\frac{t^3}{3} = \frac{1}{3}x(x^2+y^2)\sqrt{x^2+y^2}$

$\to \int_0^1 \left[\frac{1}{3}x(x^2+y^2)\sqrt{x^2+y^2}\right]_0^{\sqrt{3}x} dx = \int_0^1 \left[\frac{1}{3}x(\overset{4x^2}{\cancel{x^2+3x^2}})\underbrace{\sqrt{x^2+3x^2}}_{\sqrt{4x^2}=2x}\right] - \left[\frac{1}{3}x(x^2)(\sqrt{x^2})\right] dx$

$= \int_0^1 \frac{1}{3} 8x^4 - \frac{1}{3}x^4 \, dx = \frac{7}{3}\int_0^1 x^4 \, dx = \frac{7}{3}\left[\frac{x^5}{5}\right]_0^1 = \frac{7}{3}\left[\left(\frac{1}{5}\right)\right] = \frac{7}{15}$ ✓

**5.8.★** Sia $T$ il triangolo di vertici $(1, \frac{1}{2}), (2, \frac{1}{2}), (2, 1)$. Dopo averne scritto la rappresentazione analitica come dominio $x$-semplice e $y$-semplice, si calcoli il seguente integrale doppio, iterando nell'ordine più conveniente:

$$\iint_T \frac{e^{\frac{x}{y}}}{y^3} dx dy.$$




$T_x = \{(x,y) / 1 < x < 2, \ \frac{1}{2} < y < \frac{1}{2}x\}$

$T_y = \{(x,y) / \frac{1}{2} < y < 2, \ 2y < x < 2\}$

$\Rightarrow \int_{1/2}^{2} dy \int_{2y}^{2} \frac{e^{x/y}}{y^3} dx \to \int_{1/2}^{2} \left(\left(\int_{2y}^{2} e^{x/y} dx\right)_{a}\right) \frac{1}{y^3} dy$

a) $\int e^{x/y} dx \Rightarrow D[e^{\frac{x}{a}}] = \frac{1}{a} e^{\frac{x}{a}} \Rightarrow$

$\Rightarrow \int e^{\frac{x}{a}} dx = a e^{\frac{x}{a}} \Rightarrow \int_{2y}^{2} e^{\frac{x}{y}} dx = \left[y e^{\frac{x}{y}}\right]_{2y}^{2} = \left[\left(y e^{\frac{2}{y}}\right) - \left(y e^{\frac{2\cancel{y}}{\cancel{y}}}\right)\right]$

$\to y e^{\frac{2}{y}} - y e^{2} = y\left(e^{\frac{2}{y}} - e^{2}\right)$

$\Rightarrow \int_{1/2}^{2} y e^{\frac{2}{y}} - y e^{2} \cdot \frac{1}{y^3} dy = \int_{1/2}^{2} e^{2/y} \cdot y^{-2} - e^{2}\int_{1/2}^{2} y^{-2} dy \to -e^{2}\left[-\frac{1}{y}\right]_{1/2}^{2} = \frac{1}{2}e^{2} + 2e^{2} = \frac{3}{2}e^{2}$

$\to$ pongo $\frac{2}{y} = t \to dx = \frac{1}{-\frac{2}{y^2}} = -\frac{1}{2}y^2 dt \quad \Rightarrow -\frac{1}{2}\int_{1/2}^{2} e^{t} \cdot \frac{1}{\cancel{y^2}} \cancel{y^2} dt = -\frac{1}{2}\left[e^{\frac{2}{y}}\right]_{1/2}^{2} = -\frac{1}{2}\left[(e) - (e^4)\right]$

$= -\frac{1}{2}e + \frac{1}{2}e^{4} = \frac{1}{2}e(e^3 - 1) \quad \Rightarrow \quad \frac{3}{2}e^{2} + \frac{1}{2}e^{4} - \frac{1}{2}e = \frac{1}{2}e(3e + e^3 - 1) \sim 37$

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \to \frac{x-1}{2-1} = \frac{y-\frac{1}{2}}{1-\frac{1}{2}}$

$\to x-1 = \frac{y-\frac{1}{2}}{\frac{1}{2}} \Rightarrow x-1 = \frac{\frac{2y-1}{2}}{\frac{1}{2}}$

$\to x-\cancel{1} = \frac{2y\cancel{-1}}{\cancel{2}} \cancel{-2} \to y = \frac{1}{2}x$

# Cambio di Variabile

Coordinate CARTESIANE $\to$ POLARI

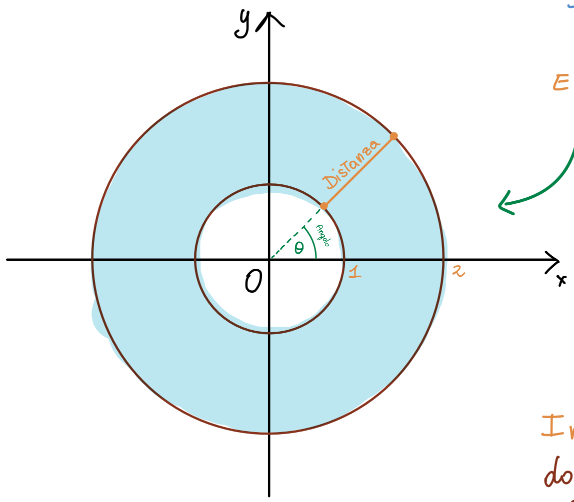
Cambiando le coordinate da cartesiane a Polari, anche la forma del "Dominio" cambia:

$D = \{(x,y) \,/\, f_1(y) < x < f_2(y),\ f_1(x) < y < f_2(x)\}$ ← Dominio a cui siamo abituati

$\Phi: (\delta,\theta) \in T \to \{x(\delta,\theta),\ y(\delta,\theta)\} \in D$ ← Nuovo dominio

In poche parole: $T = \{(\delta,\theta) \,/\, a \leq \delta \leq b,\ K_1\pi < \theta < K_2\pi\}$

Distanza   Angolo




ES: $T = \{(\delta,\theta) \,/\, 1 \leq \delta \leq 2,\ 0 \leq \theta \leq \pi\}$

Inoltre, dobbiamo anche Trasformare $x$ ed $y$:

$$\begin{cases} x = x(\delta,\theta) = \delta\cos\theta \\ y = y(\delta,\theta) = \delta\sin\theta \end{cases}$$

Infine, quando effettuiamo il cambiamento di variabile dobbiamo anche moltiplicare per il Jacobiano, che non e' altro che "$\delta$".

Teorema del cambiamento delle variabili

$$\iint_D f(x,y)\,dx\,dy = \iint_{\underbrace{\Phi(D)}_{\text{Dominio Polare}}} \underbrace{f(\delta\cos\theta, \delta\sin\theta)}_{\text{basta Sostituire}} \cdot \underbrace{\delta}_{\text{Jacobiano}}\ \overbrace{d\delta\, d\theta}^{\text{Integro per le nuove variabili}}$$

Quindi:

$T = \{(\delta,\theta) \,/\, a \leq \delta \leq b,\ K_1\pi < \theta < K_2\pi\}$ $\Rightarrow$ $\iint_D f(x,y)\,dx\,dy = \int_a^b \left( \int_{K_1\pi}^{K_2\pi} f(\delta,\theta)\, d\theta \right) d\delta$

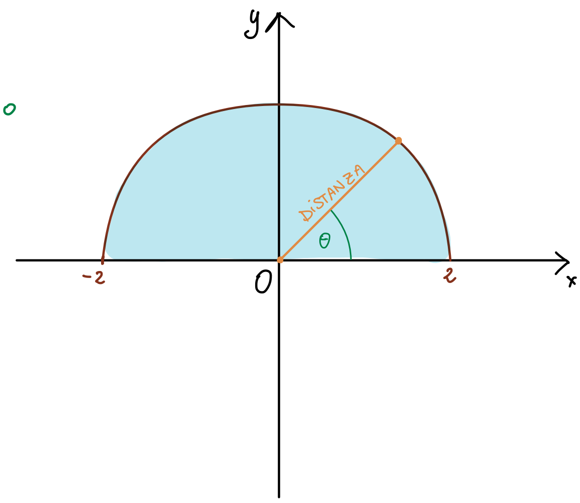
Distanza   Angolo

**Esempio 5.7.** Calcolare il seguente integrale doppio:

$$\iint_D \frac{x^2 y}{x^2+y^2}\,dxdy$$

dove $D = \{(x,y) : x^2+y^2 \leq 4, y \geq 0\}$.

y>0




$\iint_D \frac{x^2 y}{x^2+y^2}\,dx\,dy$   $\Phi = \{(\delta,\theta) \,/\, 0 < \delta < 2,\ 0 \leq \theta \leq \pi\}$

$$\Rightarrow \int_0^2 d\delta \int_0^{\pi} \frac{\delta^2\cos^2\theta \cdot \delta\sin\theta}{\delta^2\cos^2\theta + \delta^2\sin^2\theta} \cdot \delta\, d\theta$$

$$\frac{\delta^2\cos^2\theta \cdot \delta\sin\theta}{\delta^2(\underbrace{\cos^2\theta + \sin^2\theta}_{\to 1})} = \frac{\delta^{\cancel{3}\,2}\cos^2\theta\sin\theta}{\cancel{\delta}} = \delta^2\cos^2\theta\sin\theta$$

$$\Rightarrow \int_0^2 \underbrace{\delta^2\, d\delta}_{\frac{\delta^3}{3}} \int_0^{\pi} \cos^2\theta\sin\theta\, d\theta$$

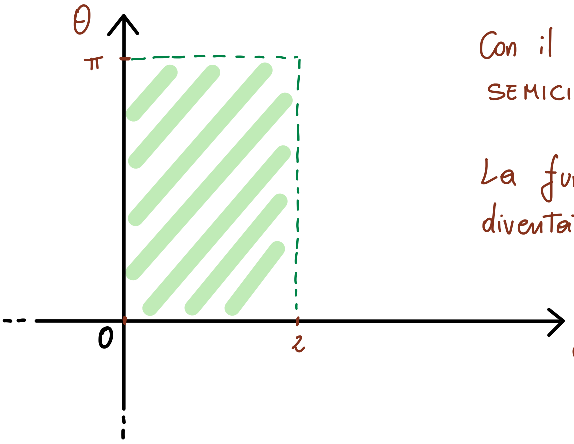
$\to \int \cos^2 x \sin x\, dx =$   $t = \cos x \to dx = \frac{1}{-\sin x}\,dt$

$$\Rightarrow \int \cos^2 x\ \cancel{\sin x} \cdot \frac{1}{-\cancel{\sin x}}\,dt = -\int t^2 x\,dt = -\frac{t^3}{3} = -\frac{\cos^3 x}{3} \Rightarrow -\frac{\cos^3\theta}{3}$$

$$\Rightarrow \left[\frac{\delta^3}{3}\right]_0^2 \cdot \left[-\frac{\cos^3\theta}{3}\right]_0^\pi = \left[\left(\frac{8}{3}\right)\right]\cdot\left[\left(\frac{1}{3}\right)+\frac{1}{3}\right] = \frac{8}{3}\cdot\frac{2}{3} = \frac{16}{9}$$

## Un grande particolare

$\Phi = \{(\delta,\theta) / 0<\delta<2 \ , \ 0\le\theta\le\pi\}$ Proviamo a rappresentare il dominio sul piano cartesiano:




Con il cambiamento delle coordinate, il dominio che prima era SEMICIRCOLARE, ora è un semplice rettangolo.

La funzione integranda, ovvero un'omogenea di grado 1 è diventata una funzione a variabili separate.

Come risultato pratico, quindi, abbiamo che un integrale Doppio si è Trasformato nel prodotto di due integrali unidimensionali.

**Esempio 5.8.**

*a.* Si dimostri il seguente risultato, dovuto ad Archimede[1]:
"L'area racchiusa dal primo giro di una spirale "di Archimede" è pari a un terzo dell'area del cerchio circoscritto".

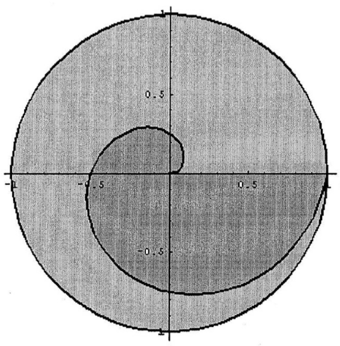


Fig. 5.8. L'area in grigio più scuro è un terzo dell'area del cerchio

Arco di spirale: $\delta_R = \frac{R\theta}{2\pi}$ per $\theta \in [0,2\pi]$

$$\Rightarrow D = \left\{(\delta,\theta) / \ \delta < \frac{R\theta}{2\pi} \ , \ 0<\theta<2\pi\right\}$$

La spirale di Archimede ha equazioni:

Parametriche:
$$\begin{cases} x(t) = a\,t\cos(t) \\ y(t) = a\,t\sin(t) \end{cases}$$

Polari:
$$r(\theta) = a\theta$$
il raggio Dipende dall'angolo

Spirale: $$\iint_D dx\,dy = \int_0^{2\pi} d\theta \int_0^{\frac{R\theta}{2\pi}} \delta\, d\delta = \int_0^{2\pi} d\theta \cdot \left[\frac{\delta^2}{2}\right]_0^{\frac{R\theta}{2\pi}} = \int_0^{2\pi}\left[\frac{\frac{R^2\theta^2}{4\pi^2}}{2}\right] d\theta = \frac{R^2}{8\pi^2}\int_0^{2\pi}\theta^2\, d\theta = \frac{R^2}{8\pi^2}\left[\frac{\theta^3}{3}\right]_0^{2\pi}$$

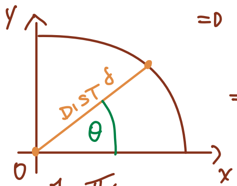
$\frac{R^2\theta^2}{8\pi}$

$$\Rightarrow \frac{R^2}{\cancel{8\pi^2}}\left[\frac{\cancel{8\pi^3}}{3}\right] = \frac{1}{3}R^2\pi$$ il cerchio ha area $\pi R^2$

Cerchio: $$T_c = \{(\delta,\theta)/ 0<\delta<R \ , \ 0<\theta<2\pi\} \Rightarrow \int_0^R \delta\,d\delta\int_0^{2\pi} d\theta = 2\pi\int_0^R\delta\,d\delta = \cancel{2}\pi\left[\frac{\delta^2}{\cancel{2}}\right]_0^R = \pi R^2$$

5.13) $\iint_D \frac{|xy|}{x^2+y^2}\,dx\,dy$ D: cerchio di centro l'origine e raggio R

Siccome abbiamo $|xy|$ la funzione è simmetrica alle bisettrici ⇒ possiamo studiarne "solo un quarto".




$$\Rightarrow D = \left\{(\delta,\theta)/0<\delta<R, 0<\theta<\frac{\pi}{2}\right\}$$

$$\Rightarrow 4\int_0^1 \delta\left(\int_0^{\pi/2}\frac{xy}{x^2+y^2}\right)d\delta\,d\theta \Rightarrow \int_0^1\delta\int_0^{\pi/2}\frac{\delta\cos\theta\,\delta\sin\theta}{\delta^2\cos^2\theta\,\delta^2\sin^2\theta}\,d\delta\,d\theta \Rightarrow \frac{\cancel{\delta^2}\cos\theta\sin\theta}{\cancel{\delta^2}(\underbrace{\cancel{\cos^2\theta\sin^2\theta}}_{1})}$$

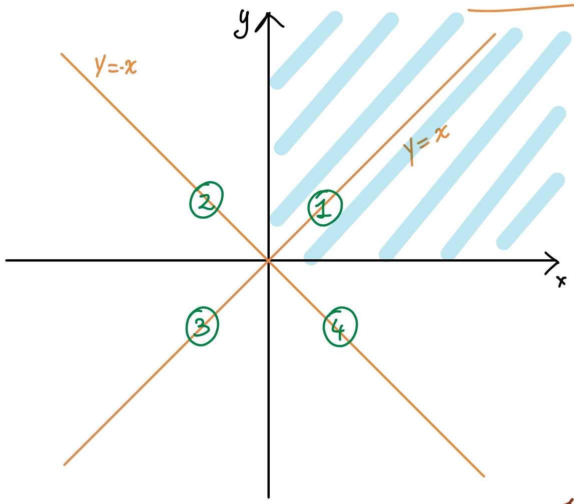
$$\Rightarrow 4\int_0^1\delta\int_0^{\pi/2}\cos\theta\sin\theta\,d\delta\,d\theta \Rightarrow \int\cos x\sin x\,dx = \quad t=\cos x \Rightarrow dx = \frac{1}{-\sin x}dt \Rightarrow -\int \overset{t}{\cancel{\cos x}}\,\cancel{\sin x}\cdot\frac{1}{\cancel{\sin x}}\,dt$$

$$\Rightarrow 4\int_0^R\delta\left[-\frac{1}{2}\cos^2\theta\right]_0^{\frac{\pi}{2}}d\delta = 4\int_0^R\delta\left[+\frac{1}{2}\right]d\delta = \frac{1}{2}\cdot4\left[\frac{\delta^2}{2}\right]_0^R = \frac{\cancel{4}^{2}}{\cancel{2}}\left[\frac{R^2}{\cancel{2}}\right] = R^2$$

Perché è simmetrica? $\frac{|xy|}{x^2+y^2}$ Dominio: $x^2+y^2>0$ eq del cerchio

tutto $\mathbb{R}^2$

$$|xy| \to \begin{cases} xy & \text{se } xy>0 \to y=-x \\ -xy & \text{se } xy<0 \to y=x \end{cases}$$




Studiamo 1/4 di piano

Studiamo senza simmetrie

$$\iint \frac{|xy|}{x^2+y^2}\,dx\,dy$$

$$\begin{cases} \iint \frac{xy}{x^2+y^2}\,dxdy & \text{se } xy>0 \\ -\iint \frac{xy}{x^2+y^2}\,dxdy & \text{se } xy<0 \end{cases}$$

$$D=\{(\rho,\theta) \,/\, 0<\rho<1,\ 0<\theta<2\pi\} \Rightarrow$$

$$\int_0^1 \rho\,d\rho \int_0^{2\pi} \frac{\cancel{\rho}\cos\theta\,\cancel{\rho}\sin\theta}{\underbrace{\cancel{\rho^2}\cos^2\theta\ \cancel{\rho^2}\sin^2\theta}_{1}}\,d\theta = \int_0^1 \rho\,d\rho \int_0^{2\pi} \cos\theta\sin\theta\,d\theta$$

$$-\int_0^1 \rho\,d\rho \int_0^{2\pi} \cos\theta\sin\theta\,d\theta$$

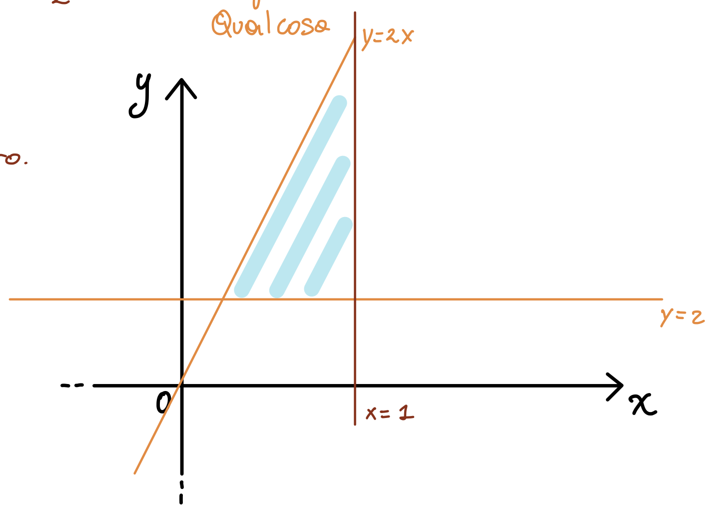
$$\to \int_0^R \rho\cdot\left[-\frac{1}{2}\cos^2\theta\right]_0^{2\pi} = \int_0^1 \rho\,d\rho\left[-\frac{1}{2}-\frac{1}{2}\right] = -\int_0^R \rho\,d\rho = -\left[\frac{\rho^2}{2}\right]_0^R = -\frac{R^2}{2}$$

$$-\int_0^R \rho\,d\rho\left[-\frac{1}{2}\cos^2\theta\right]_0^{2\pi} = \int_0^1 \rho\,d\rho = \frac{1}{2}R^2$$

Sbagliato qualcosa

Baricentro $D=\{(x,y)\,/\,0<x<1,\ 2x<y<2\}$

Battezziamo $(x_g, y_g)$ le coordinate del baricentro.




$$x_g = \frac{1}{\text{Area}(D)}\iint_D x\,dx\,dy$$

$$y_g = \frac{1}{\text{Area}(D)}\iint_D y\,dx\,dy$$

Dove $\text{Area}(D) = \iint_D dx\,dy$

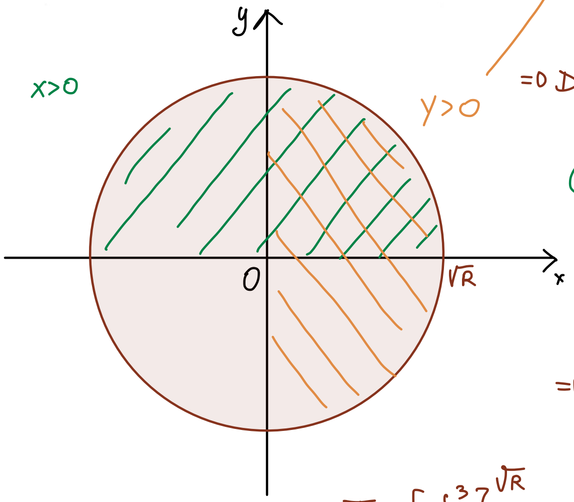
**5.14.★** Calcolare il baricentro di una lamina piana rappresentata dal quarto di cerchio

$$D = \{(x, y) : x^2 + y^2 < R, x \geq 0, y \geq 0\}$$

avente densità $f(x, y) = a + b\sqrt{x^2 + y^2}$.

1) Massa Totale

$$M = \iint_D f(x,y)\, dx\, dy = \int_0^{\sqrt{R}} \delta\, d\delta \int_0^{\pi/2} (a+b)\sqrt{\delta^2\cos^2\theta + \delta^2\sin^2\theta}\, d\theta$$




$\Rightarrow D = \{(\delta,\theta) / 0 < \delta < \sqrt{R},\ 0 < \theta < \frac{\pi}{2}\}$

a

$\Rightarrow a\int \delta\, d\theta + b\int \delta\, d\theta = a\delta[\theta]_0^{\pi/2} + b\delta[\theta]_0^{\pi/2} = a\delta\frac{\pi}{2} + b\delta\left(\frac{\pi}{2}\right)$

$= \delta\frac{\pi}{2}(a+b)$

$\Rightarrow \int_0^{\sqrt{R}} \delta \cdot \delta\frac{\pi}{2}(a+b)\, d\delta = \frac{\pi}{2}a\int_0^{\sqrt{R}} \delta^2 + \frac{\pi}{2}b\int_0^{\sqrt{R}} \delta^2\, d\delta =$

$= \frac{\pi}{2}a\left[\frac{\delta^3}{3}\right]_0^{\sqrt{R}} + \frac{\pi}{2}b\left[\frac{\delta^3}{3}\right]_0^{\sqrt{R}} = \frac{\pi}{2}a\frac{R\sqrt{R}}{3} + \frac{\pi}{2}b\frac{R\sqrt{R}}{3} = \frac{\pi}{2}\frac{R\sqrt{R}}{3}(a+b)$

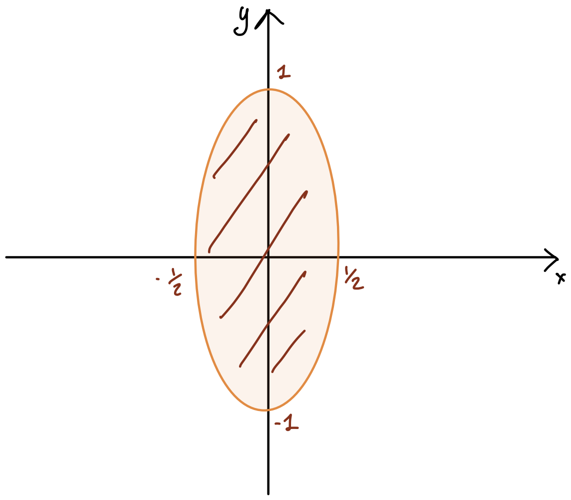
$\int_0^{\sqrt{R}} a\rho\, d\rho + \int_0^{\sqrt{R}} b\rho^2\, d\rho = a\left[\frac{\rho^2}{2}\right]_0^{\sqrt{R}} + b\left[\frac{\rho^3}{3}\right]_0^{\sqrt{R}} = a\frac{R}{2} + b\frac{R\sqrt{R}}{3} = \frac{3aR + 2bR\sqrt{R}}{6}$

Niente, Non riesco

**5.16.★** Calcolare il seguente integrale doppio

$$\iint_{4x^2+y^2\leq 1} |x|y^2\, dx\, dy$$

con un opportuno cambio di coordinate.




Da equazione a grafico - Circonferenza

$\begin{cases} 4x^2+y^2-1=0 \\ x=0 \end{cases}$ $\quad y^2-1=0 \Rightarrow y = \pm 1$

intersezioni su y

$\begin{cases} 4x^2+y^2-1=0 \\ y=0 \end{cases}$ $\Rightarrow 4x^2 = 1 \Rightarrow x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$ x

$D = \{(\delta,\theta) / 0 < \delta < 1,\ 0 \leq \theta \leq 2\pi\}$