4A. Equazioni differenziali lineari del primo ordine

Un'equazione differenziale lineare di I ordine è un'eq del tipo:

$y' = b(x)$ ed ha soluzione $y = B(x) + c$, infatti ci basta integrare per avere:

$\int y' = \int b(x)\, dx \Rightarrow y = B(x)$

Primitiva

Un altro esempio è l'eq:

$y' = y$ → Soluzioni → ce^x , con $c \in \mathbb{R}$, infatti:

$\int y' = \int y\, dx \Rightarrow y = ce^x$

Se invece $y(x)$ è una soluzione di $y' = y(x)$ otteniamo:

$y' - y(x) = 0$ Se moltiplichiamo per e^{-x} otteniamo:

$e^{-x} y'(x) - e^{-x} y(x) = 0$, ovvero $\frac{d}{dx}[e^{-x} y(x)] = 0$

Infatti: $D(e^{-x} y(x)) = -e^{-x} y(x) + e^{-x} \cdot y'(x) = e^{-x} y'(x) - e^{-x} y(x) = 0$

Ne cosegue che $e^{-x} y(x) = c \Rightarrow y(x) = ce^x$

Formula generale dell'eq lineare $y' = a(x)y + b(x)$:

$y(x) = e^{A(x)} \cdot \left[\int e^{-A(x)}\, b(x)\, dx \right]$

4.4 Risolvere l'equazione differenziale lineare omogenea $y' = 8xy$.

$y' - 8xy = 0$; $a(x)$

1) Omogenea associata: $y' - 8xy = 0$

2) Var sep: $\frac{y'}{y} - 8x = 0$; $\frac{y'}{y} = 8x$

3) Integro: $\int \frac{y'}{y} = \int 8x\,dx = \ln|y| = 4x^2 + c_1 \Rightarrow e^{\ln|y|} = e^{4x^2}\cdot e^{c_1} \Rightarrow y(x) = ce^{4x^2}$

4.5 Risolvere l'equazione differenziale lineare omogenea

$$y' = \frac{x}{x^2+1}y$$

$\frac{y'}{y} = \frac{x}{x^2+1} \Rightarrow \int \frac{y'}{y} = \int \frac{x}{x^2+1}dx \Rightarrow \ln|y| = \int \frac{x}{x^2+1}dx$

$\ln|y| = \int \frac{\cancel{x}}{x^2+1}\cdot\frac{1}{2\cancel{x}}dt = \ln|y| = \frac{1}{2}\int\frac{1}{t}dt \Rightarrow \ln|y| = \frac{1}{2}\ln|t| + c_1$

Pongo $t = x^2+1 \Rightarrow dx = \frac{1}{2x}dt \Rightarrow$

$\Rightarrow \ln|y| = \frac{1}{2}\ln|x^2+1| + c_1 \Rightarrow e^{\frac{1}{2}\ln|x^2+1|}\cdot e^{c_1} \Rightarrow y(x) = c\sqrt{x^2+1}$

4.7 Determinare l'integrale generale delle seguenti equazioni differenziali lineari *omogenee*

$y' = 3y \Rightarrow \frac{y'}{y} = 3 \Rightarrow \int\frac{y'}{y}dx = 3\int dx = \ln|y| = 3\frac{x^2}{2} + c_1 \Rightarrow e^{\ln|y|} = e^{3x+c_1}$

$\Rightarrow y(x) = ce^{3x}$

$y' = 2xy \Rightarrow \frac{y'}{y} = 2x \Rightarrow \int\frac{y'}{y}dx = \int 2x\,dx \Rightarrow \ln|y| = 2\frac{x^2}{2} \Rightarrow y(x) = e^{x^2+c_1} \Rightarrow y(x) = ce^{x^2}$

$y' = (x-1)y\frac{1}{x} \Rightarrow \frac{y'}{y} = \frac{(x-1)}{x} \Rightarrow \frac{y'}{y} = 1 - \frac{1}{x} \Rightarrow \int\frac{y'}{y}dx = \int 1 - \frac{1}{x}\,dx \Rightarrow \ln|y| = \int dx - \int\frac{1}{x}dx$

$\Rightarrow \ln|y| = x - \ln|x| + c \Rightarrow y(x) = e^{x}\cdot\frac{1}{e^{\ln x}}\cdot e^{c_1} \Rightarrow y(x) = \frac{1}{x}\cdot e^{x}\cdot c = \frac{ce^{x}}{x}$

$y' = (\cos x)y \Rightarrow \frac{y'}{y} = \cos x \Rightarrow \int\frac{y'}{y}dx = \int\cos x \Rightarrow \ln|y| = \sin x + c \Rightarrow y(x) = ce^{\sin x}$

$y' = -e^{x}y \Rightarrow \frac{y'}{y} = -e^{x} \Rightarrow y(x) = -\int e^{x}dx \Rightarrow y(x) = ce^{-e^{x}}$

$y' = 2xe^{x^2}y \Rightarrow \frac{y'}{y} = 2xe^{x^2} \Rightarrow \ln|y| = 2\int xe^{x^2}$ pongo $t = e^{x^2} \Rightarrow dx = \frac{1}{e^{x^2}2x}dt$

$\Rightarrow \ln|y| = \cancel{2}\int \cancel{x}\cancel{e^{x^2}}\cdot\frac{1}{\cancel{e^{x^2}}\cdot\cancel{2x}}dt \Rightarrow \ln|y| = \int dt \Rightarrow y(x) = ce^{t} \Rightarrow y(x) = ce^{e^{x^2}}$

$y' = (\mathrm{tg}\,x)y \Rightarrow \frac{y'}{y} = \mathrm{tg}\,x \Rightarrow \int\frac{y'}{y} = \int\mathrm{tg}\,x\,dx \Rightarrow \ln|y| = \int\frac{\sin x}{\cos x}dx$ pongo $t = \cos x \Rightarrow dx = -\frac{1}{\sin x}dt$

$\Rightarrow \ln|y| = -\int\frac{\cancel{\sin x}}{\cos x}\cdot\frac{1}{\cancel{\sin x}}dt \Rightarrow \ln|y| = -\int\frac{1}{t}dt \Rightarrow \ln|y| = c_1 - \ln|\cos x| \Rightarrow y(x) = \frac{c}{\cos x}$

$y' = -y/2x \Rightarrow \frac{y'}{y} = -\frac{1}{2x} \Rightarrow \ln|y| = -\frac{1}{2}\int\frac{1}{x}dx \Rightarrow \ln|y| = \ln|x|^{-\frac{1}{2}} + c = \frac{c}{\sqrt{x}}$

$y' = \frac{2y}{x} \Rightarrow \frac{y'}{y} = \frac{2}{x} \Rightarrow y(x) = cx^2$

$y' = -\mathrm{cotg}\,x \Rightarrow$ Come faccio senza y?

$y' = \sqrt{x}\,y \Rightarrow \frac{y'}{y} = \sqrt{x} \Rightarrow \ln|y| = \int x^{\frac{1}{2}}dx \Rightarrow y(x) = e^{\frac{2}{3}x\sqrt{x}}$

$y' = y\cdot\frac{1}{\sqrt{x+5}} = \int\frac{y'}{y} = \int\frac{1}{\sqrt{x+5}}dx \to \int(x+5)^{-\frac{1}{2}}dx \Rightarrow 2(x+5)^{\frac{1}{2}} + c = 2\sqrt{x+5} \Rightarrow y(x) = ce^{2\sqrt{x+5}}$

$y' = y\log x\cdot\frac{1}{x} = \ln|y| = \int\frac{\log x}{x}dx$ pongo $\log x = t \Rightarrow dx\ \frac{1}{\frac{1}{x}}dt \Rightarrow dx = x\,dt$

$\Rightarrow \ln|y| = \int\frac{\log x}{\cancel{x}}\cdot\cancel{x}\,dt = \int t\,dt = \frac{t^2}{2} + c \Rightarrow y(x) = ce^{\frac{1}{2}\log x^2}$

$y' = xy \cdot \frac{1}{x^2-1} \Rightarrow \ln|y| = \int \frac{x}{x^2-1}\,dx$ pongo $t = x^2-1 \to dx = \frac{1}{2x}\,dt$

$\Rightarrow \ln|y| = \frac{1}{2}\int \frac{\cancel{x}}{x^2-1}\cdot\frac{1}{\cancel{x}}\,dt = \frac{1}{2}\int \frac{1}{t}\,dt \Rightarrow \ln|y| = \ln(x^2-1)^{\frac{1}{2}}+c \Rightarrow y(x) = c\sqrt{x^2-1}$

$y' = (1+\ln x)y \Rightarrow y(x) = \int dx + \int \ln x\,dx =$ Integrazione per parti $\Rightarrow \int f(x)\cdot g'(x)\,dx = f(x)g(x) - \int f'(x)\cdot g(x)$

$\Rightarrow \int \overset{g'}{1}\cdot \overset{f(x)}{\ln(x)}\,dx \Rightarrow \ln(x)x - \int \frac{1}{\cancel{x}}\cancel{x}\,dx = x\ln x - x + c$

$\Rightarrow \ln|y| = x + x\ln x - x \Rightarrow y(x) = \cancel{e}^{c\cdot\cancel{x}}\cdot e^{x\ln x}\cdot\frac{1}{\cancel{e^x}} = c\cancel{e}^{\ln x^x} \Rightarrow y(x) = cx^x$

$y' = y\cdot\frac{1}{\sin(x+1)} \Rightarrow \ln y = \int \frac{1}{\sin(x+1)}\,dx$ pongo $t = x+1$

$\Rightarrow \ln|y| = \int \frac{1}{\sin t}\cdot\frac{\sin t}{\sin t}\,dt = \int \frac{\sin t}{\sin^2 t}\,dt \qquad \cos^2+\sin^2 = 1 \Rightarrow \sin^2 = 1-\cos^2$

$\Rightarrow \int \frac{\sin t}{1-\cos^2 t}\,dt =$ Pongo $\gamma = \cos x \Rightarrow dx = \frac{1}{-\sin t}\,d\gamma = -\int \frac{\cancel{\sin t}}{1-\cos^2 t}\cdot\frac{1}{\cancel{\sin t}}\,d\gamma = -\int \frac{1}{1-\gamma^2}\,d\gamma$

$= \int \frac{1}{\gamma^2-1^2}\,dx \Rightarrow \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\cdot\ln\left|\frac{x-a}{x+a}\right| = \frac{1}{-2}\ln\left|\frac{\gamma+1}{\gamma-1}\right|+c$

$\Rightarrow \ln|y| = -\frac{1}{2}\ln\left|\frac{\gamma+1}{\gamma-1}\right|+c_1 \Rightarrow y(x) = ce^{\ln\left|\frac{\gamma+1}{\gamma-1}\right|^{-\frac{1}{2}}} = \frac{c}{\sqrt{\frac{\gamma-1}{\gamma+1}}}$

$y' = xy\sin x \Rightarrow \ln|y| = \int \overset{f}{x}\,\overset{g'}{\sin x} = f\cdot g - \int f'\cdot g \Rightarrow -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + c$

$\Rightarrow y(x) = e^{-x\cos x}\cdot e^{\sin x}\,c = \frac{ce^{\sin x}}{e^{x\cos x}}$

$y' = -\sin 2x\; y \Rightarrow \ln|y| = -\int \sin(2x)\,dx \Rightarrow$ pongo $t = 2x \Rightarrow dx = \frac{1}{2}\,dt \Rightarrow \ln|y| = \frac{1}{2}\cos(2x)$

$\Rightarrow y(x) = ce^{\frac{1}{2}\cos(2x)}$

$y' = \arctan x\; y \Rightarrow \arctan x = \tan^{-1} x = \left(\frac{\sin x}{\cos x}\right)^{-1} = \frac{\cos x}{\sin x} \Rightarrow \ln|y| = \int \frac{\cos x}{\sin x}\,dx$ pongo $t = \sin x$

$\Rightarrow dx = \frac{1}{\cos x}\,dt \Rightarrow \int \frac{\cancel{\cos x}}{\sin x}\cdot\frac{1}{\cancel{\cos x}}\,dt = \int \frac{1}{t}\,dt = \sin x + c \Rightarrow \ln|y| = ce^{\sin x}$

Sbaglio qualcosa

$y' = y + e^x$

$y' - y = 0 \; ; \; \frac{y'}{y} = 1 \Rightarrow \ln y = \int dx = y_0(x) = e^x c$

$y' = c'e^x + ce^x \Rightarrow c'e^x + \cancel{ce^x} - \cancel{e^x c} = e^x \Rightarrow c' = \frac{e^x}{e^x} = 1 \Rightarrow c = \int dx \Rightarrow c = x$

$\Rightarrow y_p(x) = xe^x \qquad \Rightarrow y(x) = xe^x + ce^x = e^x(x+c)$

Eq diff. lin. NON omogenee di I ordine

$$y' = -\frac{2}{x}y + \frac{\operatorname{sen} 4x}{x^2}$$

1) Integrale generale dell'omogenea associata:

$$y' + \frac{2}{x}y = 0 \Rightarrow \frac{y'}{y} = -\frac{2}{x} \Rightarrow \ln|y| = -2\int \frac{1}{x}\,dx \Rightarrow$$

$$\Rightarrow \ln|y| = -2\ln|x| + c \Rightarrow ce^{-\ln|x|^2} = \frac{c}{x^2}$$

2) Integrale particolare $\quad D\left(\frac{c(x)}{x^2}\right) = \left[c'(x)\,x^2 - c(x)2x\right]\cdot \frac{1}{x^4}$

$$\frac{c'(x)x^2 - c(x)2x}{x^4} + \frac{2}{x}\cdot\frac{c}{x^2} = \frac{\sin 4x}{x^2} \Rightarrow \frac{c'x^2 \cancel{- c2x} + \cancel{2cx}}{x^4} = \frac{\sin 4x}{x^2} \Rightarrow \frac{c'\cancel{x^2}}{\cancel{x^4}} = \frac{\sin 4x}{x}$$

$\Rightarrow c' = x\sin(4x)$ $\quad$ integriamo $\Rightarrow \int c'\,dx = \int \overset{g}{x}\,\overset{f'}{\sin(4x)}\,dx$ $\quad$ per parti

$$\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g'(x) \Rightarrow \cos(4x)\,x - \int \cos(4x)\,dx = -x\cos(4x) + \sin(4x) + c$$

$$\Rightarrow c = -x\cos(4x) + \sin(4x) \qquad \Rightarrow y_p(x) = -\frac{x\cos(4x) + \sin(4x)}{x^2}$$

Sostituisco nell'eq: $\quad y(x) = y_0(x) + y_p(x)$

$$\Rightarrow y(x) = \frac{c}{x^2} - \frac{x\cos(4x) + \sin(4x}{x^2} = \frac{c - x\cos(4x) + \sin(4x)}{x^2}$$

$y' = 3y + 1$ Variazioni delle costanti

1) Trovo le primitive $\to$ $y' - 3y = 0$ $\Rightarrow$ $\frac{y'}{y} = 3$ $\to$ $\ln|y| = 3x + c$ $\Rightarrow$ $y_0(x) = ce^{3x}$

2) $y_p(x) = C(x) \cdot e^{3x}$ $\to$ Sostituisco in $y' - 3y = 1$ $\Rightarrow$ $D(C(x)e^{3x}) = C'(x)e^{3x} + C(x)3e^{3x}$

$\Rightarrow$ $c'e^{3x} + \cancel{c3e^{3x}} - \cancel{3ce^{3x}} = 1$ $\Rightarrow$ $c'e^{3x} = 1$; $c' = 1 \cdot e^{-3x}$ Integro per trovare c

$c = \int dx - \int e^{-3x} dx + c = x - \frac{1}{3}e^{3x} + \cancel{c}$

Sostituisco c: $y_p(x) = 3x + \left(x - \frac{1}{3}e^{3x}\right) = 3x + \frac{3x - e^{3x}}{3} = \frac{9x + 3x - e^{3x}}{3} = \frac{12x - e^{3x}}{3}$ $y_p(x)$

Sol eq gen data da $y(x) = y_0(x) + y_p(x) = ce^{3x} + \frac{12x - e^{3x}}{3} = \frac{3ce^{3x} + 12x - e^{3x}}{3} = \frac{e^{3x}(3c-1) + 12x}{3}$

$y' = y\sin x + \sin x$ $\to$ Trovo $y_0(x)$ $\to$ $y' - y\sin x = 0$ $\to$ $\frac{y'}{y} = \sin x$ $\Rightarrow$ $\ln|y| = \int \sin x \, dx$

$\Rightarrow$ $\ln|y| = -\cos x + c$ $\Rightarrow$ $y_0(x) = ce^{-\cos x}$

$y(x) = C(x) \cdot e^{-\cos x}$ $\to$ $D(y(x)) = c'e^{-\cos x} + ce^{-\cos x} \cdot \sin x$

$\Rightarrow$ $c'e^{-\cos x} + \cancel{c\sin x\, e^{-\cos x}} - \cancel{ce^{-\cos x}} \cdot \cancel{\sin x} = \sin x$ $\Rightarrow$ $c'e^{-\cos x} = \sin x$ $\Rightarrow$ $c' = \frac{\sin x}{e^{-\cos x}}$ $\Rightarrow$ $c' = \sin x\, e^{\cos x}$

Trovo c: $c = \int \sin x\, e^{\cos x} dx$ pongo $t = e^{\cos x}$ $\Rightarrow$ $dx = -\frac{1}{\sin e^{\cos x}} dt$ $\Rightarrow$ $c = \int \cancel{\sin x}\, \cancel{e^{\cos x}} \cdot \frac{1}{\cancel{\sin e^{\cos x}}} dt$

$\Rightarrow$ $c = \int dt$ $\Rightarrow$ $c = t$; $c = e^{\cos x}$ Sostituisco a $y_p(c) = e^{\cos x} \cdot e^{-\cos x} = 1$

$\Rightarrow$ $y(x) = y_0(x) + y_p(x) = ce^{-\cos x} + 1$ Si TROVA!

$y' - 2y = 1$ $\to$ $y' - 2y = 0$ $\Rightarrow$ $\ln y = 2\int dx$ $\Rightarrow$ $y_0(x) = ce^{2x}$

$y_0 = C(x)e^{2x}$ Varia la c $\to$ $D(y) = c'e^{2x} + 2ce^{2x}$

$\Rightarrow$ $c'e^{2x} + \cancel{2ce^{2x}} - \cancel{2ce^{2x}} = 1$ $\Rightarrow$ $c' = \frac{1}{e^{2x}}$ $\Rightarrow$ $c = \int e^{-2x} dx$ $\Rightarrow$ $c = -\frac{1}{2}e^{-2x}$

$\to$ Sostituisco c $\to$ $y_p(x) = -\frac{1}{2}\cancel{e^{-2x}}\cancel{e^{2x}}$ $\Rightarrow$ $y_p(x) = -\frac{1}{2}$

Eq generale: $y(x) = y_p(x) + y_0(x)$ $\to$ $y(x) = -\frac{1}{2} + ce^{2x}$

$y' - 2y = x^2 + x \qquad y' - 2y = 0 \Rightarrow \ln|y| = 2\int dx \Rightarrow y_0(x) = ce^{2x}$

$y = c(x)e^{2x} \Rightarrow \overset{a}{y} = \overset{b}{}\ y' = c'e^{2x} + 2ce^{2x} \Rightarrow y_p(x) = c'e^{2x} + \cancel{2ce^{2x}} - \cancel{2ce^{2x}} = x^2 + x \Rightarrow c' = \frac{x^2+x}{e^{2x}}$

z) $\Rightarrow c = \overset{a}{\int \frac{x^2}{e^{2x}}} + \overset{b}{\int \frac{x}{e^{2x}}}\, dx$

a) $\int \underset{g}{x^2} \cdot \underset{f}{e^{-2x}}\, dx = -\frac{1}{2} e^{-2x} x^2 + \frac{1}{\cancel{2}} \int e^{-2x} \cdot \cancel{2}x = -\frac{1}{2} e^{-2x} x^2 + \int \underset{f}{e^{-2x}}\, \underset{g}{x}$ per parti $= -\frac{1}{2} e^{-2x} x^2 \quad -\frac{1}{2} e^{-2x} x - \frac{1}{2} \int e^{-2x} dx$

$= -\frac{1}{2} e^{-2x} x^2 - \frac{1}{2} e^{-2x} x + \frac{1}{4} e^{-2x} + c = -\frac{x^2}{2e^{2x}} - \frac{x}{2e^{2x}} + \frac{1}{4e^{2x}} + c = -\frac{x^2+x}{2e^{2x}} + \frac{1}{4e^{2x}} + c$ a

b) $\int \underset{g}{x} \cdot \underset{f}{e^{-2x}}\, dx$ per parti $\Rightarrow -\frac{1}{2} e^{-2x} x - \frac{1}{2} \int e^{-2x} dx = -\frac{1}{2} e^{-2x} x + \frac{1}{4} e^{-2x} + c = -\frac{x}{2e^{2x}} + \frac{1}{4e^{2x}} + c = -\frac{2x-1}{4e^{2x}}$ b

z) $c = a + b = -\frac{x^2+x}{2e^{2x}} + \frac{1}{4e^{2x}} - \frac{2x-1}{4e^{2x}} = -\frac{2x^2 \cancel{+2x} + 1 \cancel{-2x} + 1}{4e^{2x}} = \frac{2x^2+1}{4e^{2x}}$ c

Sostituisco c $\quad y_p(x) = \frac{2x^2+1}{4\cancel{e^{2x}}} \cdot \cancel{e^{2x}} \Rightarrow y_p(x) = \frac{2x^2+1}{4}$

Integ. gener. $\quad y(x) = y_p(x) + y_0(x) \Rightarrow y(x) = \frac{2x^2+1}{4} + ce^{2x} = \frac{2x^2+1+4ce^{2x}}{4}$

$y' = y + x \Rightarrow y' - y = 0 \qquad \ln y = \int dx \to y_0(x) = ce^{x}$

$y = c(x)e^{x} \Rightarrow y' = c'e^{x} + ce^{x}$

$\Rightarrow c'e^{x} + \cancel{ce^{x}} - \cancel{ce^{x}} = x \Rightarrow c' = x \cdot e^{-x} \Rightarrow c = \int \underset{g}{x}\, \underset{f}{e^{-x}}\, dx$ per parti $\to -e^{-x} x - \int e^{-x} dx$

$\Rightarrow -x \underset{1}{\cancel{e^{-x}}} + \cancel{e^{-x}} \Rightarrow y_p(x) = -x$

$\Rightarrow y(x) = y_0(x) + y_p(x) = ce^{x} - x\,(-1)$?

$y' = -y + e^{-x} \to y' + y = 0 \to \ln|y| = -\int dx \Rightarrow y_0(x) = ce^{-x}$

$y = c(x)e^{-x} \to y' = c'e^{-x} - ce^{-x} \Rightarrow c'e^{-x} \cancel{-ce^{-x}} \cancel{+ce^{-x}} = e^{-x} \Rightarrow c'e^{-x} \cdot e^{x} = \cancel{e^{-x}} \cdot \cancel{e^{x}} \Rightarrow c' = 1$

$c = \int dx = x \to y_p(x) = e^{-x}$

$y(x) = e^{-x} + ce^{-x} = e^{-x}(\overset{x?}{1} + c)$

$y' + y \cdot \frac{1}{x} = \frac{1}{x} \to y' + y\frac{1}{x} = 0 \to \frac{y'}{y} = -\frac{1}{x} \qquad \ln|y| = -\int \frac{1}{x} dx \Rightarrow y_0(x) = ce^{-\ln|x|} \Rightarrow y_0(x) = \frac{c}{x}$

$y = c(x) \cdot \frac{1}{x} \Rightarrow y' = c'\frac{1}{x} + \left[-\frac{1}{x^2}\right]c \Rightarrow \frac{c'}{x} - \frac{c}{x^2} = \frac{xc' - c}{x^2}$

$\Rightarrow \frac{xc' - c}{x^2} + \frac{c}{x} \cdot \frac{1}{x} = \frac{1}{x} \Rightarrow \frac{xc' \cancel{-c} \cancel{+c}}{x^2} = \frac{1}{x} \Rightarrow \frac{\cancel{x}}{x^2} c' = \frac{1}{x} \Rightarrow c' = 1$

$\Rightarrow c = \int dx \Rightarrow c = x \Rightarrow y_p(x) = 1$

$\Rightarrow y(x) = 1 + \frac{c}{x}$

$y = \frac{y}{x} + xe^x \rightarrow y - y \cdot \frac{1}{x} = 0 \rightarrow \ln|y| = \int \frac{1}{x}dx \Rightarrow y_0(x) = cx$

$y = c(x)\,x \rightarrow y' = c'x + c \Rightarrow c'x + \cancel{c} - \cancel{c} = xe^x \Rightarrow c' = e^x$

$c = \int e^x dx \rightarrow c = e^x \xrightarrow{\text{Sostituisco}} y_p(x) = xe^x$

$\Rightarrow y(x) = cx + xe^x = x(c + e^x)$

$y' = y + e^x \rightarrow y' - y = 0 \rightarrow y_0(x) = ce^{e^x}$ $\quad y = c(x)e^{e^x} \Rightarrow y' = c'e^{e^x} + ce^{e^x} \cdot e^x$

$\Rightarrow c'e^{e^x} + ce^{e^x} \cdot e^x - ce^{e^x} = e^x$

BOH

$y' = 4y - e^{2x} \rightarrow y' - 4y = 0 \rightarrow \ln|y| = 4\int dx \Rightarrow y_0(x) = ce^{4x}$

$\Rightarrow y(x) = c(x)e^{4x} \rightarrow y' = c'e^{4x} + 4ce^{4x}$

$\Rightarrow c'e^{4x} + \cancel{4ce^{4x}} - \cancel{4ce^{4x}} = -e^{2x} \Rightarrow c' = -\frac{\cancel{e^{2x}}}{e^{4x}} \Rightarrow c' = \frac{1}{e^{2x}}$

$\rightarrow c = \int \frac{1}{e^{2x}} dx = -\frac{1}{2}e^{-2x} \xrightarrow{\text{Sostituisco}} y_0(-\frac{1}{2}e^{-2x}) = -\frac{1}{2}e^{-2x} \cdot e^{4x} \Rightarrow y_p(x) = -\frac{1}{2}e^{2x}$

$y(x) = -\frac{1}{2}e^{2x} + ce^{4x} = -\frac{e^{2x} - 2ce^{4x}}{2}$

$y' = ay + e^{bx} \rightarrow y' - ay = 0 \qquad \ln|y| = \int a\,dx \Rightarrow y_0 = ce^{a}$

$y = c(x) \cdot e^{a} \rightarrow y' = c'e^{a} + ce^{a}$

$\Rightarrow c'e^{a} + ce^{a} - ace^{a} = e^{bx} \Rightarrow c' = \frac{e^{bx} - ce^{a} + ace^{a}}{e^{a}} \Rightarrow c' = \frac{e^{bx}}{e^{a}} - c + ac$

$\Rightarrow c = \underbrace{\int \frac{e^{bx}}{e^{a}} dx}_{a)} - c\int dx + ac\int dx \qquad$ a) $\int e^{bx} \cdot e^{-a} dx \Rightarrow$ X

$y' = -2xy + xe^{-x^2} \Rightarrow y' + 2xy = 0 \qquad \ln|y| = -2\int x\,dx \Rightarrow y_0(x) = ce^{-x^2}$

$y = c(x) \cdot e^{-x^2} \rightarrow y' = c'e^{-x^2} - 2cxe^{-x^2}$

$\Rightarrow c'e^{-x^2} - \cancel{2cxe^{-x^2}} + \cancel{2cxe^{-x^2}} = xe^{-x^2} \Rightarrow c' = \frac{x\cancel{e^{-x^2}}}{\cancel{e^{-x^2}}} \rightarrow c' = x \rightarrow c = \frac{x^2}{2}$

$\Rightarrow y_p(x) = \frac{x^2}{2} \cdot e^{-x^2} \qquad \Rightarrow y(x) = \frac{x^2}{2}e^{-x^2} + ce^{-x^2} = e^{-x^2}\left(\frac{x^2}{2} + c\right)$

$y' = \left(2y\cdot\frac{1}{x}\right) + \left(\frac{x+1}{x}\right) \Rightarrow y' - \frac{2y}{x} = \frac{x+1}{x} \Rightarrow \frac{y'}{y} - \frac{2}{x} = 0 \qquad \ln|y| = 2\int \frac{1}{x}\,dx$

$\Rightarrow y_0(x) = cx^2 \qquad y = c(x)\,x^2 \Rightarrow y' = c'x^2 + 2xc \Rightarrow c'x^2 + \cancel{2xc} - \cancel{2cx} = \frac{x+1}{x}$

$\Rightarrow c' = \frac{x+1}{x^3} \Rightarrow c = \underbrace{\int \frac{1}{x^2}}_{a} + \underbrace{\int \frac{1}{x^3}\,dx}_{b} \qquad$ a) $\int x^{-2}\,dx = -\frac{1}{x} + c$

b) $\int x^{-3}\,dx = -\frac{1}{x^2} + c \Rightarrow c = -\frac{1}{x} - \frac{1}{x^2} = -\frac{x+1}{x^2} \qquad y_0(c) = -\frac{x+1}{\cancel{x^2}}\cancel{x^2} \Rightarrow y_p(x) = 1 - x$

$\Rightarrow y(x) = 1 - x + cx^2$

$-\frac{1}{2}$?

$y' = y + x^2 - 1 \Rightarrow y' - y = x^2 - 1 \Rightarrow \ln y = \int dx \Rightarrow y_0(x) = ce^x$

$\Rightarrow y = c(x)e^x \Rightarrow y' = c'e^x + ce^x \Rightarrow c'e^x + \cancel{ce^x} - \cancel{ce^x} = x^2 - 1 \Rightarrow c' = \frac{x^2-1}{e^x}$

$\Rightarrow c = \underbrace{\int \frac{x^2}{e^x}\,dx}_{a} - \int e^{-x}\,dx \rightarrow -\frac{1}{e^x} \qquad$ a) $-x^2e^{-x} - 2xe^{-x} - 2e^{-x}$

$\Rightarrow c = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} - \frac{1}{e^x} \Rightarrow y_p(x) = \left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x} - \frac{1}{e^x}\right)e^x = x^2 - 1 \qquad -x^2\cancel{e^{-x}}\cdot\cancel{e^{x}} - 2x\cancel{e^{-x}}\cancel{e^{x}} - \frac{\cancel{e^x}}{\cancel{e^x}} = x^2 - 1$

$\Rightarrow y_p(x) = -x^2 - 2x - 1 = x^2 - 1 \quad ; \quad -x^2 - x^2 - 2x \quad ; \quad -2x^2 - 2x \quad ; \quad -2x(x^2+1) = y_p(x)$

$y(x) = -2x^2 - 2x + ce^x$

$y' = e^x - \frac{y}{x} \Rightarrow y + y\cdot\frac{1}{x} = e^x \Rightarrow \ln|y| = -\int \frac{1}{x}\,dx \Rightarrow y_0(x) = ce^{x^{-1}}$

$y = c(x)e^{x^{-1}} \Rightarrow y' = c'e^{x^{-1}} + ce^{x^{-1}}\cdot(-1)x^{-2} = \frac{c'}{e^x} - \frac{c}{e^{2x}}$

$\Rightarrow \frac{c'}{e^x} - \cancel{\frac{c}{e^{2x}}} + \cancel{\frac{c}{e^x}}\cdot\frac{1}{x} = e^x \Rightarrow \frac{c'}{e^x}\cdot\frac{1}{x} = e^x \Rightarrow c' = xe^{2x}$

$\Rightarrow c = \int \underset{g}{x}\,\underset{f}{e^{2x}}\,dx = \frac{1}{2}e^{2x}x - \frac{1}{2}\int e^{2x}\,dx = \frac{1}{2}e^{2x}x - \frac{1}{4}e^{2x}$

$y_0(c) = \left(\frac{1}{2}e^{2x}x - \frac{1}{4}e^{2x}\right)\cdot e^{x^{-1}} = \frac{1}{2}\frac{e^{\cancel{2}x}}{\cancel{e^x}}x - \frac{1}{4}\frac{e^{\cancel{2}x}}{\cancel{e^x}} \Rightarrow y_p(x) = \frac{xe^x}{2} - \frac{e^x}{4}$

$y(x) = \frac{xe^x}{2} - \frac{e^x}{4} + \frac{c}{e^x} = \frac{2e^{2x}x - e^{2x} + c}{4e^x} = \frac{e^{\cancel{2}x}(2x-1) + c}{4\cancel{e^x}} = \frac{e^x(2x-1) + c}{4}$

$y' = 4y - e^{2x}$ $\qquad y' - 4y = 0$; $\ln|y| = 4\int dx \rightarrow y_0(x) = ce^{4x}$

$y' = c'e^{4x} + 4ce^{4x} \rightarrow c'e^{x} + \cancel{4ce^{4x}} - \cancel{4ce^{4x}} = -e^{2x}$; $c' = -\frac{e^{\cancel{2}x}}{\cancel{e^{x}}} \Rightarrow c' = -e^{x} \Rightarrow C = -e^{x}$

$\Rightarrow y_p(x) = -e^{x}e^{4x} = -e^{5x}$

$y(x) = ce^{4x} - e^{5x}$

Problemi di Cauchy

ES:
$$\begin{cases} y' = 3xe^{x^2} y \leftarrow \text{eq diff} \\ y(0) = 1 \leftarrow \text{Condizione} \end{cases}$$
omogenea del I ordine

1) Trovare una primitiva dell'eq diff:

$$y' - 3xe^{x^2} y = 0 \quad ; \quad \frac{y'}{y} = 3xe^{x^2} \Rightarrow \ln|y| = \frac{3}{2}\int 2x e^{x^2} dx = \frac{3}{2} e^{x^2} + c$$

$$\Rightarrow y_0(x) = c e^{3e^{x^2} \cdot \frac{1}{2}}$$

2.1) Imponiamo la condizione
$y(0) = 1 \longrightarrow$ Sostituisco $x = 0$ in $y_0(x) \longrightarrow y(0) = c e^{3 \cdot e^{0^2} \cdot \frac{1}{2}} = c e^{\frac{3}{2}} = 1$

2.2) Troviamo la c
$$c = \frac{1}{e^{3/2}} \Rightarrow c = e^{-\frac{3}{2}}$$

2.3) Sostituiamo la c all'integrale generale dell'eq omogenea

$$\Rightarrow y(x) = e^{-\frac{3}{2}} \cdot e^{3e^{x^2} \cdot \frac{1}{2}} = e^{\frac{-3+3e^{x^2}}{2}} = e^{3(e^{x^2}-1) \cdot \frac{1}{2}}$$
VITTORIA!

4.11 Risolvere i seguenti problemi di Cauchy

$$\begin{cases} y' = (1-y)/x \\ y(1) = 0 \end{cases} \qquad [y = (x-1)/x]$$

1) $y' = (1-y) \cdot \frac{1}{x}$; $\frac{y'}{1-y} = \frac{1}{x}$; Come cazzo si isola y'/y ?!

$$\begin{cases} y' = 2y + 1 \\ y(0) = 1 \end{cases} \qquad [y = (3e^{2x} - 1)/2]$$

$y' = 2y + 1$ eq omog: $y' - 2y = 0 \Rightarrow \frac{y'}{y} = 2 \Rightarrow y_0(x) = e^{2x} c$

$$\Rightarrow y' = c'e^{2x} + 2ce^{2x} \Rightarrow c'e^{2x} + \cancel{2ce^{2x}} - \cancel{2ce^{2x}} = 1 \Rightarrow c' = \frac{1}{e^{2x}} \Rightarrow c' = e^{-2x} \Rightarrow c = -2e^{-2x}$$

$$\Rightarrow y_p(x) = -2e^{-2x} \cdot e^{2x} = -2$$

$$\Rightarrow y(x) = ce^{2x} - 2 \Rightarrow y(0) = c - 2 = 1 \Rightarrow c = 3$$

Sol: $3e^{2x} - 2$

$$\begin{cases} y'+\frac{1}{x}y=x^3 \\ y(1)=1/5\end{cases} \qquad [y=x^4/5]$$

$y'+\frac{1}{x}y=x^3$; $y'+\frac{1}{x}y=0$; $\frac{y'}{y}=-\frac{1}{x}$

$\Rightarrow \ln|y| = -\int \frac{1}{x}\,dx \Rightarrow y_0(x)=cx^{-1}=\frac{c}{x}$

$y'=\frac{c'}{x}-\frac{c}{x^2} \Rightarrow \frac{c'}{x}-\cancel{\frac{c}{x^2}}+\cancel{\frac{c}{x^2}}=x^3 \Rightarrow c'=x^4 \Rightarrow c=\frac{x^5}{5}$

$\Rightarrow y_p(x)=\frac{x^{\cancel{5}\,4}}{5}\cdot\frac{1}{\cancel{x}} = y_p(x)=\frac{x^4}{5} \Rightarrow y(x)=\frac{x^4}{5}+\frac{c}{x}$

2) $y(1)=\frac{1}{5}+c=\frac{1}{5} \Rightarrow c=0$

$\Rightarrow y(x)=\frac{x^4}{5}+0 \Rightarrow y(x)=\frac{x^4}{5}$

$$\begin{cases} y'=(\mathrm{tg}\,x)y+1 \\ y(\pi)=1\end{cases} \qquad [y=\mathrm{tg}\,x-(1/\cos x)]$$

$\frac{y'}{y}=\mathrm{Tg}\,x \Rightarrow \ln y=\int \mathrm{tg}\,x = \int\frac{\sin x}{\cos x}\,dx$

pongo $\cos x=t \Rightarrow dx=\frac{1}{-\sin x}\,dt \Rightarrow -\int\frac{\cancel{\sin x}}{\cos x}\cdot\frac{1}{\cancel{\sin x}}\,dx = -\ln|\cos x|+c$

$\Rightarrow y_0(x)=c\cos^{-1}x$ $\qquad y'=c'\cos^{-1}x+c\,\mathrm{Tan}\,x\sec x$

$\Rightarrow c'\cos^{-1}x+\cancel{c\,\mathrm{Tan}\,x\sec x}-\cancel{\mathrm{tg}\,x\cdot c\cos^{-1}x}=1 \Rightarrow c'=\cos x \Rightarrow c=\sin x$

$\Rightarrow y_p(x)=\sin x\cos^{-1}x=\mathrm{Tan}\,x \Rightarrow y(x)=\mathrm{Tan}\,x+c\cos^{-1}x$

2) $y(\pi)=\mathrm{Tan}\,\pi+c\frac{1}{\cos\pi}=1$; $c=(1-\mathrm{Tan}\,\pi)\cdot\cos\pi=\cos\pi-\mathrm{Tan}\,\pi\cos\pi=-1$ c

$\Rightarrow y(x)=\mathrm{Tan}\,x-\frac{1}{\cos x}$ Soluzione

$$\begin{cases} y'=[(x+1)y/x]+x(1-x) \\ y(1)=e\end{cases} \qquad [y=(e-1)xe^{x-1}+x^2]$$

$y'=\frac{(x+1)y}{x}+x(1-x)$

eq om: $y'-\frac{(x+1)y}{x}=0 \Rightarrow \frac{y'}{y}=\frac{x+1}{x} \Rightarrow \ln|y|=\int\frac{\cancel{x}}{\cancel{x}}+\int\frac{1}{x}\,dx$

$\Rightarrow \ln|y|=x+\ln x+c \Rightarrow y_0(x)=cxe^x$

$D(xe^x)=e^x+xe^x$; $y'=c'xe^x+ce^x+cxe^x \Rightarrow c'xe^x+ce^x+cxe^x-\frac{(x+1)cxe^x}{x}=x(1-x)$

$=c'xe^x+ce^x+cxe^x-\frac{cx^2e^x-cxe^x}{x}=x(1-x)$: $\frac{c'x^2e^x+\cancel{cxe^x}+\cancel{cx^2e^x}-\cancel{cx^2e^x}-\cancel{cxe^x}}{x}=x(1-x) \Rightarrow \frac{c'x^2e^x}{\cancel{x}}=x(1-x)$

$\Rightarrow c'xe^x=x(1-x) \Rightarrow c'=\frac{\cancel{x}(1-x)}{\cancel{x}e^x} \Rightarrow c'=(1-x)\frac{1}{e^x}$

$\Rightarrow c=\underbrace{\int e^{-x}}_{a}-\underbrace{\int xe^{-x}\,dx}_{b}$ $\qquad$ a) $-e^{-x}$ $\qquad$ b) $\int \underbrace{xe^{-x}}_{gf}\,dx=-e^{-x}x+\int e^{-x}\,dx=-e^{-x}x-e^{-x}$

Torno a $c=a-b$ $\Rightarrow c=\cancel{-e^{-x}}+e^{-x}x+\cancel{e^{-x}} \Rightarrow c=xe^{-x}$

$y_p(x)=xe^{-x}\cdot xe^{x}=x^2$ $\qquad y(x)=x^2+cxe^x$

$y(1) = \underbrace{\cancel{x}^2}_{1} + c\,\underbrace{\cancel{x}}_{1}\,e^x = e \Rightarrow ce^x + 1 = e \quad ; \quad c\cancel{e^x} = \frac{e-1}{e^x} \quad ; \quad c = \frac{e-1}{e^x}$

$\Rightarrow y = x^2 + \frac{e-1}{\cancel{e^x}} \cdot x\cancel{e^x} = x^2 + xe - x \quad ; \quad x(e-1) + x^2$

Eq del tipo $y' = g(ax+by)$

ES. $y' = 1 + x^2 - 2xy + y^2$ Provo a Trovare l'eq a var sep: $y' - y^2 + 2xy = 1 + x^2$; considero l'eq omogenea

$\Rightarrow y' - y^2 + 2xy = 0$; $\dfrac{y'}{y} - y + 2x = 0$ NON FUNZIONA!

Metodo di risoluzione: $y' = 1 + x^2 - 2xy + y^2 \Rightarrow y' = 1 + \underbrace{(x-y)^2}_{g(ax+b)}$

pongo $z = x - y \Rightarrow \dfrac{d}{dx} \Rightarrow z' = 1 - y' \Rightarrow y' = 1 - z'$

Sostituisco a $y' = 1 + (x+y)^2 \longrightarrow \cancel{1} - z' = \cancel{1} + z^2$; $z' = -z^2$ EQ A VAR SEP

$\Rightarrow \dfrac{z'}{z^2} = -1 \Rightarrow \displaystyle\int \frac{z'}{z^2}\,dx = -\int dx \Rightarrow \int \underset{\text{ignoro}}{z'} \cdot \underset{\text{derivo}}{z^{-2}}\,dx = -\int dx \Rightarrow \frac{z^{-1}}{-1} - x + c = \quad \frac{1}{z} = x - c$

$\Rightarrow z = \dfrac{1}{x-c} \xrightarrow[z = x-y]{\text{Torno a}} x - y = \dfrac{1}{x-c}$; $y = -\dfrac{1}{x-c} + x$; $y = \dfrac{x^2 - cx - 1}{x-c}$

Eq. diff. lineari del II ordine a coefficienti costanti

Sono eq del Tipo: $y'' + ay + by = f(x)$ ← è l'unica funzione

4.14 Risolvere le equazioni differenziali omogenee

(a) $y'' - 6y' + 5y = 0$ Eq caratt.: $\lambda^2 - 6\lambda + 5 = 0$ ← omogenea

$\to \Delta = 36 - 4 \cdot 5 = 16 > 0$ $y_1 = e^{\lambda_1 x}$, $y = e^{\lambda_2 x}$ $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$\Rightarrow$ Trovo λ_1 e λ_2 $\Rightarrow \lambda_{1,2} = \frac{6 \pm 4}{2} < \begin{matrix} 5 \\ 1 \end{matrix}$ $\Rightarrow y(x) = c_1 e^{5x} + c_2 e^{x}$

(b) $y'' - 2y' + 2y = 0$

$\lambda^2 - 2\lambda + 2 = 0$ ← omogenea $\to \Delta = 4 - 4 \cdot 2 = -4 < 0 \Rightarrow$ 2 radici imm.

$\Rightarrow \sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} \Rightarrow \pm 2i$

$\Rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{2} < \begin{matrix} \frac{2+2i}{2} / 2 = 1+i \\ \frac{2-2i}{2} / 2 = 1-i \end{matrix}$ $\Rightarrow$ α = reale = 1, β = imm = 1 $\Rightarrow y_0(x) = e^{x}[c_1 \cos(x) + c_2 \sin(x)]$

$y_0(x) = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

4.15 Risolvere l'equazione differenziale $y'' - 2y' + y = 0$.

eq omog: $\lambda^2 - 2\lambda + 1 = 0$ ← Omogenea $\Delta = 4 - 4 = 0 \Rightarrow$ 2 rad coincidenti

$x = \frac{2}{2} = 1$ $\Rightarrow y_1 = y_2 = e^{x}$ $\Rightarrow y = c_1 e^{x} + c_2 \underline{x} e^{x}$

NON DIMENTICARE!

4.16 Risolvere le equazioni differenziali lineari omogenee

(a) $y'' - 2y' = 0$ (b) $y'' + 4y = 0$

a) $\lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0$ $\Rightarrow y_1 = e^{2x}$ $y_2 = 1$

$\to \lambda_1 = 0$

$\to \lambda_2 = 2$ $\Rightarrow y(x) = c_1 e^{2x} + c_2$ ✓

b) $y'' + 4 = 0 \Rightarrow y = \sqrt{-4} \to y = \pm 2i$ $\Rightarrow$ α = reale → 0, β = imm = 2 $\}$ $y_0(x) = e^{0}[c_1 \cos(2x) + c_2 \sin(2x)]$ ✓

1

4.17 Risolvere l'equazione $y'' + 2y' + y = 0$.

$\lambda^2 + 2\lambda + 1 = 0 \to (\lambda + 1)^2 = \lambda^2 + 2\lambda + 1$ $\Rightarrow \Delta = 0 \to$ 1 Radice

$\Rightarrow (\lambda + 1)^2 = 0$ per $\lambda = -1$ $\Rightarrow y_1 = y_2 = e^{-x}$ $\Rightarrow y_0(x) = \frac{c_1}{e^x} + \frac{x c_2}{e^x}$ ✓

4.18 Determinare l'integrale generale delle seguenti equazioni lineari omogenee

$y''-3y'+2y=0$ $\quad \lambda^2-3\lambda+2=0 \quad \Delta = 9-1=1>0 \to$ 2 Rad

$\lambda_{1,2}=\frac{3\pm1}{2} < \begin{matrix}2\\1\end{matrix} \Rightarrow y_1=e^{2x},\ y_2=e^{x} \Rightarrow y_0(x)=c_1e^{2x}+c_2e^{x}$ $\checkmark$

$y''-10y'+21y=0$

$\lambda^2-10\lambda+21=0 \quad \Delta=100-4\cdot 21=16 \quad \lambda_{1,2}=\frac{10\pm4}{2} < \begin{matrix}7 \Rightarrow y_1=e^{7}\\ 3 \Rightarrow y_2=e^{3}\end{matrix}$

$\Rightarrow y_0(x)=c_1e^{7}+c_2e^{3}$ $\checkmark$

$y''-2y'+y=0$ $\quad \lambda^2-2\lambda+1=0 \equiv (\lambda-1)^2 \to \Delta=0$ 1 RAD

$\lambda=\frac{2}{2}=1 \Rightarrow y_0(x)=c_1e^{x}+c_2xe^{x}=(c_1+c_2x)e^{x}$ $\checkmark$

$y''-10y'+25y=0$ $\quad \lambda^2-10\lambda+25=0 \to \Delta=100-4\cdot25=0 \Rightarrow$ 1 radice

$\Rightarrow \lambda=\frac{10}{2}=5 \Rightarrow y_0(x)=c_1e^{5x}+c_2xe^{5x}$ $\checkmark$

$y''+y=0$ $\quad \lambda^2+1=0 \Rightarrow \lambda=\sqrt{-1} \Rightarrow \lambda=i \quad \begin{matrix}\alpha=0\\ \beta=1\end{matrix}$

$\Rightarrow y_0(x)=e^{0}\left[c_1\cos(x)+c_2\sin(x)\right]$ $\checkmark$

$y''+3y=0$ $\quad \lambda^2+3=0 \Rightarrow \lambda=\sqrt{-3} \to \lambda=i\sqrt{3} \Rightarrow \begin{matrix}\alpha=0\\ \beta=\sqrt{3}\end{matrix}$

$\Rightarrow e^{0}\left[c_1\cos(\sqrt{3}x)+c_2\sin(\sqrt{3}x)\right]$ $\checkmark$

$y''-2y'-15y=0$ $\Rightarrow \lambda^2-2\lambda-15=0 \quad \Delta=4-4\cdot(-15)=64>0$

$\Rightarrow \lambda_{1,2}=\frac{2\pm8}{2} < \begin{matrix}5\\-3\end{matrix} \quad y_1=e^{5x} \quad y_2=e^{-3x} \Rightarrow y_0(x)=c_1e^{5x}+\frac{c_2}{e^{3x}}$

$y''-y=0$ $\to \lambda^2-1=0 \to \lambda=\pm1$

$y_0(x)=c_1e^{-x}+c_2e^{x}$ $\checkmark$

$y''-4y'+4y=0$ $\to \lambda^2-4\lambda+4=0 \quad \Delta=16-4\cdot4=0$ 1 radice

$\lambda=\frac{4}{2}=2 \Rightarrow y_0(x)=c_1e^{2x}+c_2xe^{2x}$

$y''-2y'+5y=0$ $\to \lambda^2-2\lambda+5=0 \quad \Delta=4-20=-16<0$ 2 rad imm

$\lambda_{1,2}=\frac{2\pm4i}{2} < \begin{matrix}1+2i\\1-2i\end{matrix} \Rightarrow \begin{matrix}\alpha=1\\ \beta=2\end{matrix}$

$\Rightarrow y_0(x)=e^{x}\left[c_1\cos(2x)+c_2\sin(2x)\right]$ $\checkmark$

$y''+y'+y=0$ $\to \lambda^2+\lambda+1=0 \to \Delta=1-4=-3<0 \to$ 2 rad imm

$\to \lambda_{1,2}=\frac{-1\pm\sqrt{3}i}{2} < \begin{matrix}\frac{-1+\sqrt{3}i}{2}\\ -\frac{1}{2}-\frac{\sqrt{3}i}{2}\end{matrix} \Rightarrow \left.\begin{matrix}\alpha=-\frac{1}{2}\\ \beta=\frac{\sqrt{3}}{2}\end{matrix}\right\} \Rightarrow e^{-\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$ $\checkmark$

$y'' - 4y' + 20y = 0$ $\rightarrow$ $\lambda^2 - 4\lambda + 20 = 0$ $\rightarrow$ $\Delta = 16 - 80 = -64 < 0$ 2 rad imm

$\Rightarrow \lambda_{1,2} = \frac{4 \pm 8i}{2} < \begin{matrix} 2+4i \\ 2-4i \end{matrix} \Big\} \begin{matrix} \alpha = 2 \\ \beta = 4 \end{matrix} \Big\} \rightarrow$ $y_0(x) = e^{2x}\left[C_1 \cos(4x) + C_2 \sin(4x)\right]$ ✓

$y'' + 9y = 0$ $\rightarrow$ $\lambda^2 + 9 = 0$ $\rightarrow$ $\lambda = \pm\sqrt{-9}$ $\Delta < 0 \rightarrow$ 2 rad imm

$\rightarrow$ $y_{1,2} = \pm 3i$ $\rightarrow$ $y_0(x) = e^0\left[C_1 \cos(3x) + C_2 \sin(3x)\right]$

$\alpha = 0$ $\beta = 3$

$y'' - 6y' + 10y = 0$ $\lambda^2 - 6\lambda + 10 = 0$ $\rightarrow$ $\Delta = 36 - 40 = -4 < 0 \rightarrow$ 2 rad imm

$\rightarrow$ $\lambda_{1,2} = \frac{6 \pm 2i}{2} < \begin{matrix} 3+i \\ 3-i \end{matrix} \Big\} \begin{matrix} \alpha = 3 \\ \beta = 1 \end{matrix} \Big\} \rightarrow$ $y_0(x) = e^{3x}\left[C_1 \cos(x) + C_2 \sin(x)\right]$ ✓

$y'' + \sqrt{2}y' = 0$ $\rightarrow$ $\lambda^2 + \sqrt{2}\lambda = 0$ $\rightarrow$ $\lambda(\lambda + \sqrt{2}) = 0$

$\hookrightarrow \lambda_1 = 0$

$\hookrightarrow \lambda = -\sqrt{2}$ $\Big\}$ 2 radici $\Delta > 0$

$y_0(x) = C_1 e^0 + C_2 e^{-\sqrt{2}x} = C_1 + \frac{C_2}{e^{x\sqrt{2}}}$ ✓

$y'' - 8y' + 16y = 0$ $\lambda^2 - 8\lambda + 16 = 0$ $\rightarrow$ $\underline{(\lambda - 4)^2 = \lambda^2 - 8\lambda + 16}$ $\Rightarrow$ $\lambda - 4 = 0$ $\rightarrow$ $\lambda = 4$ $\Rightarrow \Delta = 0$

$y_0(x) = C_1 e^{4x} + C_2 x e^{4x}$ ✓

$y'' - y'/2 + y/16 = 0$ $\rightarrow$ $\lambda^2 - \frac{1}{2}\lambda + \frac{1}{16} = 0$ $\Rightarrow$ $\left(\lambda - \frac{1}{4}\right)^2 = \lambda - \frac{\cancel{2}\lambda}{\cancel{4}} + \frac{1}{16}$ $\Rightarrow$ $\lambda - \frac{1}{4} = 0$ per $\lambda = \frac{1}{4}$ $\Delta = 0$

$\Rightarrow$ $y_0(x) = C_1 e^{\frac{1}{4}x} + C_2 x e^{\frac{1}{4}x} = \sqrt[4]{e}\,(C_1 + C_2 x)$

Eq. lin complete di II ordine

ESEMPIO $y'' - 4y' = xe^x$

1) Eq Omog. $y'' - 4y' = 0 \to \lambda^2 - 4\lambda = 0 \to \lambda(\lambda - 4) = 0$
$\to \lambda_1 = 0$
$\to \lambda_2 = 4$

$\Rightarrow y_0(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 + c_2 e^{4x}$

2) Eq completa $f(x) = x e^{1x}$ $\to \gamma$ $\Rightarrow \gamma = 1$ è radice dell'eq caratteristica? NO

$y_p(x) = e^x$ (Polinomio dello stesso grado) $\Rightarrow e^x \underbrace{(Ax + B)}_{q_1}$

Sost y_p nell'eq compl. $y' = e^x(Ax+B) + Ae^x$, $y'' = e^x(Ax+B) + Ae^x + Ae^x = e^x(Ax+B+2A)$

$\Rightarrow y'' - 4y' = xe^x \to \frac{e^x(Ax+B+2A)}{e^x} - \frac{4e^x(Ax+B)}{e^x} - \frac{4Ae^x}{e^x} = \frac{xe^x}{e^x} = \cancel{Ax} + \cancel{B} + \cancel{2A} - \cancel{4}_3 Ax - \cancel{4}_3 B - \cancel{4}_2 A = x$

$\Rightarrow -3Ax - 3B - 2A = x$ Sistema:
$$\begin{cases} -3A = 1 & \Rightarrow A = -\frac{1}{3} \\ -3B - 2A = 0 & \Rightarrow -3B + \frac{2}{3} = 0 \Rightarrow B = \frac{2}{9} \end{cases}$$

Sostituisco a q_1

$\Rightarrow y_p(x) = e^x\left(-\frac{1}{3}x + \frac{2}{9}\right)$ Integrale particolare

3) $y(x) = y_0(x) + y_p(x) \Rightarrow y(x) = c_1 + c_2 e^{4x} + e^x\left(-\frac{1}{3}x + \frac{2}{9}\right)$ Integrale generale.

Caso γ Non è radice ↑

4.34 Risolvere l'equazione differenziale non omogenea $y'' - 3y' + 2y = 2x^3 - x^2 + 1$.

$\lambda^2 - 3\lambda + 2 = 0 \to \Delta = 9 - 8 = 1 > 0$

$\lambda_{1,2} = \frac{3 \pm 1}{2} < \begin{matrix} 2 \\ 1 \end{matrix}$ $\Rightarrow y_0(x) = c_1 e^{2x} + c_1 e^x$

2) $f(x) = 2x^3 - x^2 + 1$ Siccome non è del tipo $e^{\alpha x} \to \gamma = 0$

$\Rightarrow y_p(x) = (Ax^3 + Bx^2 + Cx + d)$ $y' = 3Ax^2 + 2Bx + C$ $y''\ 6Ax + 2B$

$(6Ax + 2B) - 3(3Ax^2 + 2Bx + C) + 2(Ax^3 + Bx^2 + Cx + D) = 2x^3 - x^2 + 1$

$\Rightarrow 6Ax + 2B - 9Ax^2 - 6Bx + C + 2Ax^3 + 2Bx^2 + 2Cx + 2D = 2x^3 - x^2 + 1$
$2Ax^3 + 2Bx^2 - 9Ax^2 + 6Ax - 6Bx + 2Cx + 2B + C + 2D = 2x^3 - x^2 + 1$
$\Rightarrow 2Ax^3 + x^2(2B - 9A) + x(6A - 6B + 2C) + 2B + C + 2D = //$

$$\begin{cases} A = 1 \\ 2B - 9A = -1 & \Rightarrow B = 4 \\ 6A - 6B + 2C = 0 & \Rightarrow C = 9 \\ 2B + C + 2D = +1 & \Rightarrow D = 10 \end{cases}$$

$\Rightarrow y_p(x) = x^3 + 4x^2 + 9x + 10$
$\Rightarrow y(x) = c_1 e^{2x} + c_2 e^x + x^3 + 4x^2 + 9x + 10$

4.35 Risolvere l'equazione differenziale non omogenea $y'' - 4y' = x^2 + 1$.

$\lambda^2 - 4\lambda = 0 \Rightarrow \lambda(\lambda - 4) = 0$ $\rightarrow \lambda = 0$, $\rightarrow \lambda = 4$

$\Rightarrow y_0(x) = c_1 + c_2 e^{4x}$

2) $f(x) = x^2 + 1 \neq e^{\gamma x} \Rightarrow \gamma = 0 \Rightarrow$

$\Rightarrow y_p(x) = x(Ax^2 + Bx + C)$ $\qquad y' = Ax^2 + Bx + C + 2Ax^2 + Bx \qquad y'' = 2Ax + B + 4Ax + B = 6Ax + 2B$

$\Rightarrow 6Ax + 2B - 4(3Ax^2 + 2Bx + C) = x^2 + 1 \Rightarrow 6Ax + 2B - 12Ax^2 - 8Bx - 4C = x^2 + 1$

$\Rightarrow -12Ax^2 + x(6A - 8B) - 4C + 2B = x^2 + 1$

$$\begin{cases} -12A = 1 \Rightarrow A = -\frac{1}{12} \\ 6A - 8B = 0 \Rightarrow B - \frac{1}{16} \\ -4C + 2B = 1 \Rightarrow C = \left(1 + \frac{1}{8}\right) \cdot \frac{1}{4} \Rightarrow C = -\frac{9}{32} \end{cases} \Rightarrow y_p(x) = x\left(-\frac{1}{12}x^2 - \frac{1}{16}x - \frac{9}{32}\right)$$

$\Rightarrow y(x) = -\frac{1}{12}x^3 - \frac{1}{16}x^2 - \frac{9}{32}x + c_1 + c_2 e^{4x}$

4.36 Risolvere l'equazione differenziale non omogenea $y'' - 2y' - 3y = 8e^{3x}$.

$\lambda^2 - 2\lambda - 3 = 0 \rightarrow \Delta = 4 + 12 = 16 > 0$

$\lambda_{1,2} = \frac{2 \pm 4}{2} \begin{smallmatrix} 3 \\ -1 \end{smallmatrix}$ $\Rightarrow y_0(x) = c_1 e^{3x} + \frac{c_2}{e^x}$

$f x = 8e^{3x} \rightarrow \gamma = 3$ é soluzione di molteplicità $1 \rightarrow$ é una soluzione infatti $\lambda_1 = 3$, $\lambda_2 = -1$

Vedi lez 35

(ii) *se* $P(\lambda) = 0$ *e* λ *ha molteplicità* h, *allora la (2) ammette un integrale particolare del tipo*

$$x^h e^{\lambda x} q_m(x).$$

$\Rightarrow y = xe^{3x}q \Rightarrow y' = e^{3x}q + 3xe^{3x}q \qquad y'' = 3e^{3x}q + (3e^{3x}q + 9xe^{3x}q)$

$\Rightarrow$ Sostituisco nell'eq completa $\rightarrow \frac{6e^{3x}q}{e^{3x}} + \frac{9xe^{3x}q}{e^{3x}} - 2\left(\frac{e^{3x}q}{e^{3x}} + \frac{3xe^{3x}q}{e^{3x}}\right) - \frac{3xe^{3x}q}{e^{3x}} = \frac{8e^{3x}}{e^{3x}}$

$\Rightarrow \cancel{6}_{4}q + \cancel{9x}^{3xq}q - \cancel{2q} - 6xq - \cancel{3xq} = 8 \Rightarrow 4q = 8 \Rightarrow q = 2$

$\Rightarrow$ Sostituisco a $y_0(x) \rightarrow y_0(x) = 2xe^{3x} \Rightarrow y(x) = \underbrace{2xe^{3x}}_{y_p(x)} + \underbrace{c_1e^{3x} + c_2e^{-x}}_{y_0(x)} = e^{3x}(2x + c_1) + c_2e^{-x}$

$$y'' + y = x + 1$$

$\lambda^2 + 1 = 0 \implies \lambda_{1,2} = \pm i \implies \begin{matrix}\alpha = 0 \\ \beta = 1\end{matrix} \implies$

$\implies y_0(x) = c_1 \cos x - c_2 \sin x$

2) $f(x) = x + 1 \implies \gamma = 0 \implies$ NON E' RADICE $\implies y = Ax + B \qquad y' = A \qquad y'' = 0$

$\implies Ax + B = x + 1 \implies \begin{cases} A = 1 \\ B = 1 \end{cases} \implies y_p(x) = x + 1 \qquad y(x) = c_1 \cos x - c_2 \sin x + x + 1$

$$y'' - 2y' + y = x^2 + x$$

$\lambda^2 - 2\lambda + 1 = 0 \implies (\lambda - 1)^2 = 0$ per $\lambda = 1 \quad \Delta = 0$

$\implies y_1 = y_2 = e^x \implies y_0(x) = c_1 e^x + c_2 x e^x$

2) $x^2 + x \implies \gamma = 0 \implies y_p(x) = (Ax^2 + Bx + C) \implies y' = 2Ax + B \qquad y'' = 2A$

$\implies y_p(x) = 2A - 2(2Ax + B) + Ax^2 + Bx + C = x^2 + x \implies 2A - 4Ax - 2B + Ax^2 + Bx + C = x^2 + x$

$\implies \underset{\to 1}{A}x^2 + x(\underset{\to 1}{B - 4A}) + 2A + C - 2B = x^2 + x \qquad \begin{cases} A = 1 \\ B - 4A = 1 \implies B = 5 \\ 2A + C - 2B = 0 \implies C = 8 \end{cases} \implies y_p(x) = x^2 + 5x + 8$

$\implies y(x) = c_1 e^x + c_2 x e^x + x^2 + 5x + 8 = e^x (c_1 + c_2 x) + x^2 + 5x + 8$

$$y'' - 2y' + y = e^x$$

$\lambda^2 - 2\lambda + 1 = 0 \implies (\lambda - 1)^2 = 0 \implies \lambda = 1 \qquad \Delta = 0$

$\implies y_0(x) = c_1 e^x + c_2 x e^x \qquad$ II) $f(x) = e^x \implies \gamma = 1$ RADICE molteplicita' 2

$\implies x^2 e^x \cdot (Ax + B) \implies Ax^3 e^x + Bx^2 e^x =$ Deriv Troppo lunghe

$$y'' - 5y' + 6y = e^x$$

$\lambda^2 - 5\lambda + 6 = 0 \qquad \Delta = 25 - 24 = 1 > 0$

$\implies \lambda_{1,2} = \frac{+5 \pm 1}{2} \begin{matrix} \nearrow 3 \\ \searrow 2 \end{matrix} \implies y_0(x) = c_1 e^{3x} + c_2 e^{2x}$

II) $f(x) = e^x \implies \gamma = 1$ NO SOL $\implies e^x [Ax + B] \implies Axe^x + Be^x \qquad y' = Ae^x + Axe^x + Be^x$

$\implies y'' = Ae^x + Ae^x + Axe^x + Be^x = e^x(2A + Ax + B)$

$\implies 2Ae^x + Axe^x + Be^x - 5[Ae^x + Axe^x + Be^x] + 6[Axe^x + Be^x] = e^x$

$\implies 2Ae^x + Axe^x + Be^x - 5Ae^x - 5Axe^x - 5Be^x + 6Axe^x + 6Be^x$

$\implies -3Ae^x + 2Axe^x + 2Be^x = e^x \implies e^x(2B - 3A) + 2Axe^x = e^x$

$\begin{cases} 2A = 0 \implies A = 0 \\ 2B - 3A = 1 \implies 2B - 0 = 1 \implies B = \frac{1}{2} \end{cases} \implies y_p(x) = \frac{1}{2} e^x$

$\implies y(x) = \frac{1}{2} e^x + c_1 e^{3x} + c_2 e^{2x}$ ✓

$y'' - 2y' - 3y = (2x+1)e^x$ $\qquad \lambda^2 - 2\lambda - 3 = 0 \quad \to \quad \Delta = 4+12 = 16 > 0 \quad$ 2 rad

$\to \lambda_{1,2} = \frac{2 \pm 4}{2} < \begin{matrix} 3 \\ -1 \end{matrix} \Rightarrow y_0(x) = c_1 e^{3} + \frac{c_2}{e^x}$

II) $f(x) = (2x+1)e^x \to \gamma = 1$ NO SOLUZ

$\Rightarrow y = e^x[Ax+B] \to \qquad y' = Ae^x + Axe^x + Be^x \quad y'' = Ae^x + Ae^x + Axe^x + Be^x$

$\Rightarrow 2Ae^x + Axe^x + Be^x - 2[Ae^x + Axe^x + Be^x] - 3Axe^x - 3Be^x = (2x+1)e^x$

$\to \cancel{2Ae^x} + Axe^x + Be^x - \cancel{2Ae^x} - 2Axe^x - 2Be^x - 3Axe^x - 3Be^x = 2xe^x + e^x$

$\Rightarrow xe^x(Ae^x - 2Ae^x - 3Ae^x) + e^x(B - 2B - 3B) = (2x+1)e^x$

$\Rightarrow \begin{cases} -4B = 1 \Rightarrow B = -\frac{1}{4} \\ -4A = 2 \Rightarrow A = -\frac{1}{2} \end{cases} \qquad \Rightarrow y_p(x) = -\frac{1}{4}e^x x - \frac{1}{4}e^x \qquad \Rightarrow y(x) = -\frac{1}{2}e^x x - \frac{1}{4}e^x + c_1 e^{3x} + \frac{c_2}{e^x}$

$= -\frac{1}{4}e^x(2x+1) + c_1 e^{3x} + \frac{c_2}{e^x}$ ✓

$y'' - y = xe^x$ $\qquad \lambda^2 - 1 = 0 \quad \to \quad \lambda = \pm 1 \quad \to \quad \Delta > 0$

$y_0(x) = c_1 e^x + \frac{c_2}{e^x}$ $\qquad$ II) $f(x) = xe^x \Rightarrow \gamma = 1$ RADICE molt 1 $\Rightarrow y = xe^x[Ax+B]$

$y' = e^x + xe^x[Ax+B] + Axe^x = e^x + Ax^2e^x + Bxe^x = e^x[Ax^2 + Bx + 1] \qquad y'' = e^x[Ax^2+Bx+1] + e^x 2Ax + B$

$\Rightarrow e^x[Ax^2+Bx+1] + e^x 2Ax + B - Ax^2e^x + Bxe^x = xe^x \to Ax^2e^x + Bxe^x + e^x + 2Axe^x + B - Ax^2 + Bxe^x = xe^x$

$x^2(Ae^x - A) + x(2Be^x) + e^x + B = xe^x$

$\begin{cases} B = 1 \\ A = 0 \end{cases} \qquad y_p(x) = xe^x(1) \qquad \Rightarrow y(x) = xe^x + c_1 e^x + \frac{c_2}{e^x}$

$y'' - 2y' + 2y = e^{2x}$ $\qquad \lambda^2 - 2\lambda + 2 = 0 \qquad \Delta = 4 - 4\cdot 2 = -4 \quad \to$ 2 rad complesse

$\lambda_{1,2} = \frac{2 \pm 2i}{2} = \begin{matrix} 1+i \\ 1-i \end{matrix} \quad \Rightarrow y_0(x) = e^x(c_1\cos(x) + c_2\sin(x))$

II) $f(x) = e^{2x} \to \gamma = 2 \Rightarrow$ NON è soluzione $\to y = e^{2x}(Ax+B) \to y' = 2e^{2x}(Ax+B) + Ae^{2x}$

$y' = 2Axe^{2x} + 2Be^{2x} + Ae^{2x} \qquad y'' = 2Ae^{2x} + 2Ax2e^{2x} + 4Be^{2x} + 2Ae^{2x} = 2Ae^{2x} + 4Axe^{2x} + 4Be^{2x} + 2Ae^{2x}$

$2Ae^{2x} + 4Axe^{2x} + 4Be^{2x} + 2Ae^{2x} - 2[2Axe^{2x} + 2Be^{2x} + Ae^{2x}] + 2[Axe^{2x} + Be^{2x}] = e^{2x}$

$2Ae^{2x} + 4Axe^{2x} + 4Be^{2x} + 2Ae^{2x} - 4Axe^{2x} - 4Be^{2x} - 2Ae^{2x} + 2Axe^{2x} + 2Be^{2x} = e^{2x}$

$\begin{cases} 2A + 4B + 2A - 4B - 2A + 2B = 1 \\ 4A - 4A + 2A = 0 \end{cases} \qquad \begin{cases} 4B - 4B + 2B = 1 \\ A = 0 \end{cases} \qquad \begin{cases} B = \frac{1}{2} \\ A = 0 \end{cases} \Rightarrow y_p = e^{2x}\cdot\frac{1}{2}$

$\Rightarrow y(x) = \frac{e^{2x}}{2} + e^x(c_1\cos x + c_2\sin x)$

ESEMPIO con Sin - cos

$y''+2y'+2y = \sin x + \cos x$

I) Eq omogenea ass: $\lambda^2+2\lambda+2=0 \to \Delta = 4-8 = -4 \to$ 2 real compl

$\to \lambda_{1,2} = \frac{-2 \pm 2i}{2} \begin{cases} -1+i \\ -1-i \end{cases} \Rightarrow \begin{matrix} \lambda = -1 \\ \beta = 1 \end{matrix} \Rightarrow y_0(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$

II) Trovare l'eq particolare: Cerchiamo soluzioni del tipo: $u(x) = C\cos(x) + D\sin(x)$

Calcolo le derivate: $y' = D\cos x - C\sin x$ $\quad y'' = -D\sin x - C\cos x$

Sostituisco: $-D\sin x - C\cos x + 2[D\cos x - C\sin x] + 2[C\cos x + D\sin x] = \sin x + \cos x$

$\Rightarrow -D\sin x - C\cos x + 2D\cos x - 2C\sin x + 2C\cos x + 2D\sin x = \sin x + \cos x$

Raccolgo C e D

$D(2\cos x - \sin x + 2\sin x) + C(2\cos x - 2\sin x - \cos x) = \sin x + \cos x$

$\to D(2\cos x + \sin x) + C(\cos x - 2\sin x) = \sin x + \cos x$

$\cos x(2D+C) + \sin x(D-2C) = \sin x + \cos x$

$\begin{cases} 2D+C = 1 \to C = 1-2D \to C = \frac{5-6}{5} = -\frac{1}{5} \\ D-2C = 1 \to D-2(1-2D) = 1 \to D-2+4D = 1 \to D = \frac{3}{5} \end{cases}$

$\Rightarrow y_p(x) = -\frac{1}{5}\cos x + \frac{3}{5}\sin x \qquad y(x) = e^{-x}(c_1\cos x + c_2\sin x) - \frac{1}{5}\cos x + \frac{3}{5}\sin x$ ✓

$$y'' - y' - 2y = 2\operatorname{sen} x$$

$\lambda^2 - \lambda - 2 = 0 \to \Delta = 1+8 = 9 \quad \lambda_{1,2} = \frac{1 \pm 3}{2} \begin{cases} 2 \\ -1 \end{cases}$

$\Rightarrow y_0(x) = c_1 e^{2x} + c_2 e^{-x}$

II) $f(x) = 2\sin x \to \gamma = 0$, $\mu = 1$ NO RADICE $\to y = A\cos x + B\sin x$ $\quad y' = -A\sin x + B\cos x$ $\quad y'' = -A\cos x - B\sin x$

$\to -A\cos x - B\sin x + A\sin x - B\cos x - 2[A\cos x + B\sin x] = 2\sin x$

$\to -A\cos x - B\sin x + A\sin x - B\cos x - 2A\cos x - 2B\sin x = 2\sin x$

$\cos x(-A-B-2A) + \sin x(B+A-2B) = 2\sin x$

$\begin{cases} -B-3A = 0 \to B = -3A \to B = -\frac{3}{5} \\ A-3B = 5 \to A+3A = 2 \to A = \frac{1}{5} \end{cases} \Rightarrow y_p(x) = \frac{1}{2}\cos x - \frac{3}{2}\sin x$ ✓

$$y'' + y = \cos x$$ $\lambda^2 + 1 = 0 \to \lambda = \pm i \to 2$ rad compl

$\Rightarrow \lambda_{1,2} = \pm i \Rightarrow \begin{matrix}\alpha = 0 \\ \beta = \frac{1}{2}\end{matrix} \Rightarrow c_1 \cos(x) + c_2 \sin(x)$

II) $f(x) = \cos x \to \gamma \pm i\mu \to \begin{cases}\gamma = 0 \\ \mu = 1\end{cases} \Rightarrow \pm i \to$ NO RADICE

$\Rightarrow y_p(x) = (A\cos x + B\sin x)x \to y' = A\cos x + B\sin x + x(-A\sin x + B\cos x)$

$y'' = -A\sin x + B\cos x - A\sin x + B\cos x + x(-A\cos x - B\sin x)$

$\to -A\sin x + B\cos x - A\sin x + B\cos x + x(-A\cos x - B\sin x) + xA\cos x + xB\sin x = \cos x$

$\to -A\sin x + B\cos x - A\sin x + B\cos x - \cancel{xA\cos x} - \cancel{xB\sin x} + \cancel{xA\cos x} + \cancel{xB\sin x} = \cos x$

$\sin x(-2A) + \cos x(2B) = \cos x \to \begin{cases}-2A = 0 \to A = 0 \\ 2B = 1 \to B = \frac{1}{2}\end{cases}$

$\Rightarrow y_p(x) = \frac{x\sin x}{2} \Rightarrow y(x) = \frac{x\sin x}{2} + C_1\cos(x) + C_2\sin(x)$ ✓

$$y'' + 4y = \text{sen}\, 2x$$ $\lambda^2 + 4 = 0 \to \lambda = \pm 2i \to \underline{y_o(x) = C_1\cos(2x) + C_2\sin(2x)}$

II) $f(x) = \sin(2x) \to \gamma = 0 \to$ cerco $P(x)$ del Tipo: $x(A\cos(2x) + B\sin(2x))$

$y' = A\cos(2x) + B\sin(2x) - 2xA\sin(2x) + 2xB\cos(2x)$

$y'' = -2A\sin(2x) + 2B\cos(2x) - 2A\sin(2x) - 4xA\cos(2x) + 2B\cos(2x) - 4xB\sin(2x)$

$-2A\sin(2x) + 2B\cos(2x) - 2A\sin(2x) - \cancel{4xA\cos(2x)} + 2B\cos(2x) - \cancel{4xB\sin(2x)} + \cancel{4xA\cos(2x)} + \cancel{4xB\sin(2x)} = \sin(2x)$

$\sin(2x)(-4A) + \cos(2x)(4B) = \sin(2x)$

$\begin{cases}-4A = 1 \to A = -\frac{1}{4} \\ 4B = 0 \to B = 0\end{cases} \Rightarrow y_p(x) - \frac{x\cos(2x)}{4} \Rightarrow y(x) = -\frac{x\cos(2x)}{4} + C_1\cos(2x) + C_2\sin(2x)$

$$y'' - 2y' + 2y = \text{sen}\, x$$ $\lambda^2 - 2\lambda + 2 = 0 \to \Delta = 4 - 4\cdot 2 = -4 \to 2$ Rad imm

$\to \frac{2 \pm 2i}{2} < \begin{matrix}1+i \\ 1-i\end{matrix} \to \alpha = 1 \quad \beta = 1 \to y_o(x) = e^x[C_1\cos x + C_2\sin x]$

II) $f(x) = \sin x \to \gamma = 0 \to$ Del Tipo: $A\cos x + B\sin x \to y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

$\Rightarrow -A\cos x - B\sin x - 2[-A\sin x + B\cos x] + 2A\cos x + 2B\sin x = \sin x$

$\to -A\cos x - B\sin x + 2A\sin x - 2B\cos x + 2A\cos x + 2B\sin x = \sin x$

$\Rightarrow \cos x(-A - 2B + 2A) + \sin x(-B + 2A + 2B) = \sin x$

$\begin{cases}A - 2B = 0 \to A = 2B \to A = \frac{2}{5} \\ B + 2A = 1 \to B + 4B = 1 \to B = \frac{1}{5}\end{cases} \Rightarrow y_p(x) = \frac{2}{5}\cos x + \frac{1}{5}\sin x$

$$y'' - 2y' + y = e^{2x} \qquad \lambda^2 - 2\lambda + 1 = 0 \;\rightarrow\; (\lambda-1)^2 = 0 \quad \text{per} \quad \lambda = 1 \qquad \Delta = 0 \;\rightarrow\; 1 \text{ rad}$$

$$\Rightarrow\; y_0(x) = c_1 e^x + c_2 x e^x$$

II) $f(x) = e^{2x}$ $\quad \gamma = 2 \;\rightarrow\;$ NO SOL $\rightarrow$ cerco $P(x)$ del tipo : $e^{2x}(Ax+B) \;\rightarrow\; y' = 2e^{2x}(Ax+B) + Ae^{2x}$

$$y'' = 4e^{2x}(Ax+B) + 2e^{2x}A + 2Ae^{2x}$$

$$\Rightarrow\; 4e^{2x}Ax + 4e^{2x}B + 2e^{2x}A + 2Ae^{2x} - 2\left[2e^{2x}Ax + 2e^{2x}B + Ae^{2x}\right] + e^{2x}Ax + e^{2x}B = e^{2x}$$

$$\rightarrow\; \underline{4e^{2x}A} + \underline{4e^{2x}B} + \underline{2e^{2x}A} + \underline{2Ae^{2x}} - \underline{4e^{2x}Ax} - \underline{4e^{2x}B} - \underline{2Ae^{2x}} + \underline{e^{2x}Ax} + \underline{e^{2x}B} = e^{2x}$$

$$\rightarrow\; e^{2x}(4A + 4B + 2A + 2A - 4B - 2A + B) + xe^{2x}(-4A + A) = e^{2x}$$

$$\begin{cases} 6A + B = 1 \;\rightarrow\; B = 1 \\ -3A = 0 \;\rightarrow\; A = 0 \end{cases} \Rightarrow\; y_p(x) = e^{2x} \;\Rightarrow\; y(x) = c_1 e^x + c_2 x e^x + e^{2x} \quad \checkmark$$

$$y'' - 4y' + 4y = e^{2x} \qquad \lambda^2 - 4\lambda + 4 = 0 \;\rightarrow\; (\lambda-2)^2 = 0 \;\rightarrow\; \lambda = 2 \quad \Delta = 0 \;\rightarrow\; 1 \text{ Sol}$$

$$\Rightarrow\; y_0(x) = c_1 e^{2x} + c_2 x e^{2x}$$

II) $f(x) = e^{2x} \;\rightarrow\; \gamma = 2$ <u>Soluzione</u> ! di molteplicita' $h = 2 \;\rightarrow\; y_p(x) = x^2 e^{2x}(Ax+B)$

$$\Rightarrow\; y' = (2xe^{2x} + 2x^2e^{2x})(Ax+B) + (x^2e^{2x})\cdot A = 2x^2e^{2x}A + 2x^3e^{2x}A + 2xe^{2x}B + 2x^2e^{2x}B + x^2e^{2x}A$$

$$y'' = 4xe^{2x}A + 4x^2e^{2x}A + 6x^2e^{2x}A + 4x^3e^{2x}A + 2e^{2x}B + 4xe^{2x}B + 4xe^{2x}B + 4x^2e^{2x}B + 2xe^{2x}A + 2x^2e^{2x}A$$

$$\underline{4xe^{2x}A} + \underline{4x^2e^{2x}A} + \underline{6x^2e^{2x}A} + \underline{4x^3e^{2x}A} + \underline{2e^{2x}B} + \underline{4xe^{2x}B} + \underline{4xe^{2x}B} + \underline{4x^2e^{2x}B} + \underline{2xe^{2x}A} + \underline{2x^2e^{2x}A} - 4\left[\underbrace{2x^2e^{2x}A}_{4} + \underbrace{2x^3e^{2x}A}_{4} + \underbrace{2xe^{2x}B}_{4} + \underbrace{2x^2e^{2x}B}_{4} + \underbrace{x^2e^{2x}A}_{4}\right] +$$

$$+\, 4\left[\underbrace{x^3e^{2x}A}_{4} + \underbrace{x^2e^{2x}B}_{0}\right] = e^{2x}$$

$$\rightarrow\; xe^{2x}(4A + 4B + 4B + 2A + 8B) + x^2e^{2x}(4A + 6A + 4B + 2A - 8A - 8B - 4A + 4B) + x^3e^{2x}(4A - 8A + 4A) +$$

$$+\, 2e^{2x}B = e^{2x}$$

$$2e^{2x}B = e^{2x} \;\rightarrow\; B = 1/2 \;\Rightarrow\; \begin{cases} B = 1/2 \\ 16B + 10A = 0 \\ // \\ // \end{cases} \Rightarrow\; \frac{16}{2} + 10A = 0 \;\rightarrow\; A = -\frac{\cancel{16}^{\,8}}{2}\cdot\frac{1}{\cancel{10}_{\,5}} = -\frac{\cancel{8}}{\cancel{10}} = -\frac{4}{5}$$

$$\Rightarrow\; y_p(x) = x^2e^{2x}\left(-\frac{4}{5}x + \frac{1}{2}\right) = x^2e^{2x}\left(\frac{-8x+5}{10}\right) =$$

$y'' - y' = \cos x$ $\quad \lambda^2 - \lambda = 0 \to \lambda(\lambda-1) = 0$

$\to \lambda = 0$, $\to \lambda = 1$ $\Big\}$ $\Delta > 0 \Rightarrow y_0(x) = c_1 e^x + c_2$

I) $f(x) = \cos x \to \gamma = 0$ cerco $y(x)$ del tipo $x(A\cos x + B\sin x)$ $\quad y' = A\cos x + B\sin x - xA\sin x + xB\cos x$

$y'' = -A\sin x + B\cos x - xA\cos x - xB\sin x$

$\Rightarrow \underline{-A\sin x} + \underline{B\cos x} - xA\cos x - xB\sin x - \underline{A\cos x} + \underline{B\sin x} - xA\sin x + xB\cos x = \cos x$

$\to A(-\cancel{\sin x} - \cos x + \cancel{\sin x}) + B(\cos x) + x(-A\cos x + B\sin x - A\sin x + B\cos x) = \cos x$

$\to \cos x(B-A) + x[-A(+\cos x + \sin x) + B(\sin x + \cos x)] = \cos x \to \cos x(B-A) + x(\cos x + \sin x)(-A-B) = \cos x$

$\begin{cases} B - A = 1 \to B = 1 + A \to B = \frac{1}{2} \\ -A - B = 0 \to -1 - A - A = 0 \to A = -\frac{1}{2} \end{cases} \Rightarrow y_p(x) = x\left(-\frac{1}{2}\cos x + \frac{1}{2}\sin x\right)$ ✓

$$y'' + y' = \cos x + \operatorname{sen} x$$

$\lambda^2 + \lambda = 0 \to \lambda(\lambda+1) = 0$

$\to \lambda = 0$, $\to \lambda = -1$ $\Rightarrow y_0(x) = c_1 + \frac{c_2}{e}$

II) $f(x) = \cos x + \sin x \to \gamma = 0,\ \mu = 1$

$\to \gamma \pm i\mu \to \pm i$ NON E' soluzione

y è del tipo: $A\cos x + B\sin x \to y' = -A\sin x + B\cos x \to y'' = -A\cos x - B\sin x$

$\to -A\cos x - B\sin x - A\sin x + B\cos x = \cos x + \sin x$

$\cos x(-A+B) + \sin x(-B-A) = \cos x + \sin x$ $\quad \begin{cases} B - A = 1 \to B = 1 + A \to B = 0 \\ -B - A = 1 \to -1 - A - A = 1 \to A = -1 \end{cases}$

$\Rightarrow y_p(x) = -\cos x$ ✓

$$y'' - y = 2x\operatorname{sen} x$$

$\lambda^2 - 1 = 0 \to \lambda = \pm 1 \Rightarrow y_0(x) = c_1 e^x + c_2 e^{-x}$

II) $f(x) = 2x\sin x \to \gamma = 0,\ \mu = 1 \to \pm i$ $\underline{\text{NO SOLUZ.}}$

$\Rightarrow y$ è del tipo $(Ax+B)\cos x + (Cx+D)\sin x \to y' = A\cos x - (Ax+B)\sin x + C\sin x + (Cx+D)\cos x$

$y'' = -A\sin x - A\sin x - (Ax+B)\cos x + C\cos x + C\cos x - (Cx+D)\sin x$

$\Rightarrow -2A\sin x - Ax\cos x - B\cos x + 2C\cos x - Cx\sin x - D\sin x - [Ax\cos x + B\cos x + Cx\sin x + D\sin x] = 2x\sin x$

$\to -2A\sin x - Ax\cos x - B\cos x + 2C\cos x - Cx\sin x - D\sin x - Ax\cos x - B\cos x - Cx\sin x - D\sin x = 2x\sin x$

$\to -2A\sin x - 2Ax\cos x - 2B\cos x + 2C\cos x - 2Cx\sin x - 2D\sin x = 2x\sin x$

$(-2A - 2D)\sin x + (-2B + 2C)\cos x + x\sin x(-2C) + x\cos x(-2A) = 2x\sin x$

$\begin{cases} -2C = 2 \to C = -1 \\ -2A - 2D = 0 \to D = 0 \\ -2B + 2C = 0 \to -2B - 2 = 0 \to B = -1 \\ -2A = 0 \to A = 0 \end{cases}$ $\Rightarrow y_p(x) = -\cos x - x\sin x$ ✓

$y'' - 3y' + 2y = 2e^{3x}$ $\lambda^2 - 3\lambda + 2 = 0 \to \Delta = 9 - 4 \cdot 8 = 1 > 0 \to$ 2 real

$\lambda_{1,2} = \frac{3 \pm 1}{2} < \begin{matrix} 2 \\ 1 \end{matrix} \Rightarrow y_p(x) = c_1 e^{2x} + c_2 e^x$

II : $f(x) = 2e^{3x} \to \gamma = 3$ NO SOL $\Rightarrow y$ è del tipo $e^{3x}(Ax+B)$

$y' = 3e^{3x}(Ax+B) + Ae^{3x}$ $y'' = 9e^{3x}(Ax+B) + A3e^{3x} + 3Ae^{3x}$

$\Rightarrow A9e^{3x}x + B9e^{3x} + 6Ae^{3x} - 3[Axe^{3x} + 3Be^{3x} + Ae^{3x}] + 2[Axe^{3x} + Be^{3x}] = 2e^{3x}$

$= \cancel{Ax9e^{3x}} + \cancel{B9e^{3x}} + 6Ae^{3x} - \cancel{3Axe^{3x}} - \cancel{9Be^{3x}} - 3Ae^{3x} + \cancel{2Axe^{3x}} + 2Be^{3x} = 2e^{3x}$

$\to 2Be^{3x} + 3Ae^{3x} = 2e^{3x}$ $\begin{cases} A = 0 \\ B = 1 \end{cases} \Rightarrow y_p(x) = e^{3x}$ ✓

$y'' + y' = e^x(3\cos x + \operatorname{sen} x)$ $\lambda^2 + \lambda = 0 \to \lambda(\lambda+1) = 0$

$\hookrightarrow \lambda = 0$

$\hookrightarrow \lambda = -1$

$\Rightarrow y_0(x) = \frac{c_1}{e} + c_2$

II) $f(x) = e^x(3\cos x + \sin x)$ $\gamma = 1, \mu = 1$ $\to 1 \pm i \Rightarrow$ NO SOLUZ.

cerco y del tipo $e^x[(Ax+B)\cos x + (Cx+D)\sin x]$

$y' = e^x[(Ax+B)\cos x + (Cx+D)\sin x] + e^x[A\cos x - (Ax+B)\sin x + C\sin x + (Cx+D)\cos x]$

$y'' = e^x[(Ax+B)\cos x + (Cx+D)\sin x] + e^x[A\cos x - (Ax+B)\sin x + C\sin x + (Cx+D)\cos x] +$
$+ e^x[A\cos x - (Ax+B)\sin x + C\sin x + (Cx+D)\cos x] + e^x[-A\sin x - A\sin x + (Ax+B)\cos x + C\cos x +$
$+ C\cos x - (Cx+D)\sin x$

Mi rifiuto di svolgere oltre

EQUAZIONI DIFFERENZIALI NON LINEARI DEL PRIMO ORDINE

5A. Equazioni a variabili separabili

Si dice a *variabili separabili* un'equazione differenziale del primo ordine del tipo

(*) $$y' = f(x) \cdot g(y)\,,$$

5.1 Risolvere le equazioni differenziali a variabili separabili

(a) $y' = \dfrac{x}{y}$ (b) $y' = 1 + y^2$

a) $y' = \dfrac{x}{y} \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y} \rightarrow dy = \dfrac{x}{y}\,dx$

$\rightarrow y\,dy = x\,dx \xrightarrow{\int} \int y\,dy = \int x\,dx$

$\Rightarrow \dfrac{y^2}{2} + c_1 = \dfrac{x^2}{2} + c_2 \Rightarrow y^2 = x^2 + 2c$

b) $y' = 1 + y^2 \Rightarrow \dfrac{dy}{dx} = 1 + y^2 \rightarrow \dfrac{dy}{1+y^2} = dx \Rightarrow \int \dfrac{1}{1+y^2}\,dy = \int dx \Rightarrow \operatorname{arctg} y = x + c$

Quindi $\rightarrow \tan[\arctan(y)] = \tan(x+c) \rightarrow \underline{y = \tan(x+c)}$

Come verifico? Se $y' = 1 + y^2$ e $y = \tan(x+c)$ allora basta derivare y

$\dfrac{d}{dx}[\tan(x+c)] = \dfrac{1}{\cos^2(x+c)}$ sostituisco y all'eq iniziale $y' = 1 + y^2 \Rightarrow \dfrac{1}{\cos^2(x+c)} = 1 + \tan^2(x+c)$

$\Rightarrow \dfrac{1}{\cos^2(x+c)} = 1 + \dfrac{\sin^2(x+c)}{\cos^2(x+c)} \Rightarrow \dfrac{1}{\cos^2(x+c)} = \dfrac{\cos^2(x+c) \cdot \sin^2(x+c)}{\cos^2(x+c)} \Rightarrow \dfrac{1}{\cos^2(x+c)} = \dfrac{1}{\cos^2(x+c)}$ ✓

$=1$ Rel Fond

4.4 Risolvere l'equazione differenziale lineare omogenea $y' = 8xy$.

Primo esercizio ↑

$y' = 8xy \rightarrow \dfrac{dy}{dx} = 8xy \rightarrow \dfrac{dy}{y} = 8x\,dx \Rightarrow \int \dfrac{1}{y}\,dy = 8\int x\,dx \rightarrow \ln|y| = 8\dfrac{x^2}{2} + c$

$\Rightarrow e^{\ln|y|} = c\,e^{4x^2} \Rightarrow y = c\,e^{4x^2}$

5.2 Risolvere le equazioni differenziali a variabili separabili

(a) $y' = \cos^2 y$ (b) $y' = 2x \cos^2 y$

a) $y' = \cos^2 y \Rightarrow \dfrac{dy}{dx} = \cos^2 y \rightarrow \dfrac{dy}{\cos^2 y} = dx \rightarrow \int \dfrac{1}{\cos^2 y}\,dy = \int dx$

$\Rightarrow \tan y = x + c \Rightarrow \operatorname{atan}(\tan y) = \operatorname{atan}(x+c) \Rightarrow \underline{y_0 = \operatorname{atan}(x+c)}$

b) $y' = 2x\cos^2 y \rightarrow \dfrac{dy}{dx} = 2x\cos^2 y \rightarrow \dfrac{dy}{\cos^2 y} = 2x\,dx \rightarrow \int \dfrac{1}{\cos^2 y}\,dy = 2\int x\,dx$

$\Rightarrow \tan y = \cancel{2}\dfrac{x^2}{\cancel{2}} + c \Rightarrow \underline{y_0 = \operatorname{atan}(x^2 + c)}$

5.3 Risolvere le equazioni a variabili separabili

(a) $y' = 2xy$ (b) $y' = \frac{y}{x}$ (c) $y' = -\frac{x}{y}$

a) $y' = 2xy \Rightarrow \frac{dy}{dx} = 2xy \Rightarrow \frac{dy}{y} = 2x\,dx \Rightarrow \int \frac{1}{y}\,dy = 2\int x\,dx$

$\Rightarrow \ln|y| = x^2 + c \Rightarrow y = ce^{x^2}$

b) $y' = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{1}{x}dx \Rightarrow \ln|y| = \ln|x| + \ln|c| \Rightarrow y = cx$

c) $y' = -\frac{x}{y} \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\,dy = -x\,dx \Rightarrow \int y\,dy = -\int x\,dx$

$\Rightarrow \frac{1}{2}y^2 + c_1 = -\frac{x^2}{2} + c_2 \Rightarrow c_1 + y^2 = c_2 - x^2 \Rightarrow y = -x + c'$

Oppure come risolve il libro: $\frac{1}{2}y^2 + c_1 = -\frac{1}{2}x^2 + c_2 \Rightarrow c + y^2 = -x^2 + c \Rightarrow y^2 + x^2 = c'$

ovvero tutte le circonferenze di $r = c$ e centro in $0,0$

5.4 Risolvere le equazioni a variabili separabili

(a) $xy' = \operatorname{tg} y$ (b) $y'\operatorname{tg} x = y$

a) $xy' = \operatorname{tg} y \Rightarrow x\frac{dy}{dx} = \operatorname{tg} y \Rightarrow \frac{dy}{\operatorname{tg} y} = \frac{1}{x}dx \Rightarrow \int \frac{1}{\operatorname{tg} y}\,dy = \int \frac{1}{x}\,dx$

$= \int \frac{1}{\frac{\sin y}{\cos y}}\,dy = \int \frac{\cos y}{\sin y}\,dy$ pongo $t = \sin y \Rightarrow dy = \frac{1}{\cos y}dt \Rightarrow \int \frac{\cancel{\cos y}}{\sin y}\frac{1}{\cancel{\cos y}}\,dt = \int \frac{1}{t}\,dt$

$= \ln|\sin y| = \ln|x| + c \Rightarrow e^{\ln|\sin y|} = e^{\ln|x| \cdot c} \Rightarrow \sin y = cx \Rightarrow y = \arcsin(cx)$

b) $y'\operatorname{tg} x = y \Rightarrow \frac{dy}{dx}\operatorname{tg} x = y \Rightarrow \frac{dy}{y} = \operatorname{tg} x\,dx \Rightarrow \ln|y| = \int \operatorname{tg} x\,dx = \ln|\sin x| + c$

$\Rightarrow y = c\sin x$

5.7 Determinare tutte le soluzioni dell'equazione differenziale

$$y' = 2x\sqrt{1-y^2}$$

$\frac{dy}{dx} = 2x\sqrt{1-y^2} \Rightarrow \frac{dy}{\sqrt{1-y^2}} = 2x\,dx \Rightarrow \int \frac{1}{\sqrt{1-y^2}}\,dy = 2\int x\,dx$

$= \arcsin(y) = x^2 + c \Rightarrow y = \sin(x^2 + c)$

5.8 Risolvere l'equazione differenziale $y' = e^{x-y}\cos x$.

$\frac{dy}{dx} = e^{x-y}\cos x \Rightarrow \frac{dy}{dx} = e^x/e^y \cos x \Rightarrow e^y dy = e^x \cos x\,dx$

$\Rightarrow e^y = \int \underset{f}{e^x}\,\underset{g}{\cos x}\,dx = e^x\cos x + \int \underset{f}{e^x}\,\underset{g}{\sin x}\,dx$ re integro per parti

stessi

$\Rightarrow e^x\cos x + e^x \sin x - \int e^x \cos x\,dx \Rightarrow \int e^x\cos x\,dx = e^x\cos x + e^x\sin x - \int e^x \cos x\,dx$

$\Rightarrow 2\int e^x\cos x\,dx + \int e^x\cos x\,dx = \frac{e^x\cos x + e^x\sin x}{2} + c$

Quindi: $y = \ln\left|\frac{e^x\cos x + e^x\sin x}{2} + c\right|$

5.10 Risolvere i seguenti problemi di Cauchy

(a) $\begin{cases} xy' = 1 + y^2 \\ y(1) = 1 \end{cases}$ (b) $\begin{cases} xy' = 1 + y^2 \\ y(-1) = 1 \end{cases}$

a) $xy' = 1 + y^2 \to x\frac{dy}{dx} = 1 + y^2 \to \frac{1}{1+y^2}dy = \frac{1}{x}dx \to \int \frac{1}{1+y^2}dy = \int \frac{1}{x}dx$

$\to \arctan(y) = \ln|x| + c \to y = \tan(\ln|x| + c)$

$y(1) = \tan(\ln|1| + c) = 1 \to \tan(0 + c) = 1 \Rightarrow c = \frac{\pi}{4}$

Soluzione al problema: $y = \tan(\ln|x| + \frac{\pi}{4})$

b) $x\frac{dy}{dx} = 1 + y^2 \to y = (\ln|x| + c)$ $y(-1) = \tan(\ln|-1| + c) = 1$

$\Rightarrow \tan(0 + c) = 1 \Rightarrow c = \frac{\pi}{4} \Rightarrow y = \tan(\ln|-1| + \frac{\pi}{4}) \leftarrow$ Sol

5.11 Risolvere i seguenti problemi di Cauchy

(a) $\begin{cases} y' + 3x^2y^4 = 0 \\ y(1) = 0 \end{cases}$ (b) $\begin{cases} y' + 3x^2y^4 = 0 \\ y(1) = 1 \end{cases}$ $-\frac{1}{3}y^{-3} = \int y^{-4}\,dy = \int -3x^2\,dx = -x^3 + c$

$\frac{dy}{dx} + 3x^2y^4 = 0 \to \frac{dy}{dx} = -3x^2y^4 \to \frac{dy}{y^4} = -3x^2\,dx \to \int \frac{1}{y^4}dy = -3\int x^2dx$

$\to -\frac{1}{3}y^{-3} = -\cancel{3}\frac{x^3}{\cancel{3}} + c \to \frac{1}{3}y^{-3} = x^3 - c \to y^{-3} = 3x^3 - 3c$

Cauchy: $\underbrace{y(1) = 3 - 3c = 1}_{\text{Prima risolvo in modo semplice}} \to c = \frac{2}{3} \Rightarrow y^{-3} = 3x^3 - 2 \Rightarrow \boxed{y = \frac{1}{\sqrt[3]{3x^3 - 2}}}$ Sol

Poi ricavo y

b) $\frac{dy}{dx} = -3x^2y^4$ con $y(1) = 1$

$\Rightarrow y(1) = 3 - 3c = 0 \to c = 1 \Rightarrow y^{-3} = 3x^3 - 3 \Rightarrow y = \frac{1}{\sqrt[3]{3x^3 - 3}}$

5.12 Risolvere le equazioni a variabili separabili

(a) $y' = y\,\mathrm{cotg}\,x$ (b) $y' = (y - 3)\,\mathrm{cotg}\,x$

a) $\frac{dy}{dx} = y\,\mathrm{cotg}\,x \to \frac{dy}{y} = \mathrm{cotg}\,x\,dx \to \ln|y| = \int \mathrm{cotg}\,x\,dx$ $\mathrm{cotg} = \frac{\cos(x)}{\sin x}$

$\Rightarrow \ln|y| = \int \frac{\cos x}{\sin x}dx$ $t = \sin x \Rightarrow dx = \frac{1}{\cos x}dt \Rightarrow \ln|y| = \int \frac{\cancel{\cos x}}{\sin x}\frac{1}{\cancel{\cos x}}dx \Rightarrow \ln|y| = \int \frac{1}{\sin x}dx$

$\Rightarrow \ln|y| = \ln|\sin x| + c \Rightarrow y = \overset{e^c}{c}\sin x$

b) $\frac{dy}{dx} = (y-3)\,\mathrm{cotg}\,x \to \frac{dy}{y-3} = \mathrm{cotg}\,x\,dx \to \int \frac{1}{y-3}dy = \int \mathrm{cotg}\,x\,dx$ $\to \mathrm{cotg}\,x = \frac{\cos x}{\sin y}$

$\Rightarrow \ln|y-3| = \ln|\sin x| + c \to y = c'\sin x + 3$

5.13 Risolvere l'equazione $2x^2yy' = 1 + y^2$.

$2x^2 \cdot y \cdot y' = 1 + y^2 \;\Rightarrow\; \frac{dy}{dx} y \frac{1}{1+y^2} = \frac{1}{x^2} \;\Rightarrow\; \int \frac{2y}{1+y^2} dy = \int \frac{1}{x^2} dx$

$\ln|1+y^2| = -\frac{1}{x} + c \;\Rightarrow\; 1 + y^2 = c' e^{-\frac{1}{x}} \;\Rightarrow\; y^2 = c' e^{-\frac{1}{x}} - 1 \;\Rightarrow\; y = \sqrt{\frac{c'}{e^x} - 1}$

5.14 Risolvere l'equazione $xy' = y \log y$.

$x \frac{dy}{dx} = y \ln y \;\Rightarrow\; \frac{dy}{y \ln y} = \frac{1}{x} dx \;\Rightarrow\; \int \frac{1}{y \ln y} dy = \int \frac{1}{x} dx$

Pongo $t = \ln x \Rightarrow dy = \frac{1}{\frac{1}{y}} = y\,dt$

$\Rightarrow \int \frac{1}{y \ln y} \cdot y\,dy = \ln|x| + c \;\Rightarrow\; \int \frac{1}{\ln y} dy = \ln y + c \;\Rightarrow\; \ln|\ln y| = \ln|x| + c$

$\Rightarrow e^{\ln|y|} = c'x \;\Rightarrow\; y = e^{c'x}$

5.15 Risolvere l'equazione $4\sqrt{x^3}\,yy' = 1 - y^2$.

$4\sqrt{x^3} y y' = 1 - y^2 \;\Rightarrow\; \frac{4y\,dy}{1-y^2} = \frac{1}{\sqrt{x^3}} dx \;\; = -2\int \frac{2y}{1-y^2} = \int x^{-\frac{3}{2}} dx$

$\Rightarrow -\ln|1-y^2| = -\frac{x^{-\frac{1}{2}}}{\frac{1}{2}} \;\Rightarrow\; -\ln|1-y^2| = \frac{2}{\sqrt{x}} + c \;\Rightarrow\; y^2 \frac{1}{1-y^2} = c' e^{\frac{2}{\sqrt{x}}} \;\Rightarrow\; c' e^{\frac{2}{\sqrt{x}}} (1-y^2) = 1$

(in arancione: $1-y^2$, $1-y^2$)

$\Rightarrow y = \sqrt{1 + ce^{1/\sqrt{x}}}$

5.16 Risolvere i seguenti problemi di Cauchy

(a) $\begin{cases} y' = \dfrac{y^2-1}{x^2-1} \\ y(0) = 0 \end{cases}$ (b) $\begin{cases} y' = \dfrac{y^2+1}{x^2+1} \\ y(1) = 1 \end{cases}$

(c) $\begin{cases} y' = \dfrac{y^2-1}{x^2-1} \\ y(0) = 1/2 \end{cases}$ (d) $\begin{cases} y' = \dfrac{y^2+1}{x^2+1} \\ y(0) = \sqrt{3} \end{cases}$

a) $y' = \frac{y^2-1}{x-1} \;\Rightarrow\; \frac{dy}{dx} = \frac{y^2-1}{x^2-1} \;\Rightarrow\; \frac{dy}{y^2-1} = \frac{1}{x^2-1} dx \;\Rightarrow\; \int \frac{1}{y^2-1} dy = \int \frac{1}{x^2-1} dx$

$\Rightarrow (y+1)(y-1) = y^2 \cancel{-y} \cancel{+y} - 1$

$\Rightarrow \overset{a}{\int \frac{1}{(y+1)(y-1)} dy} = \overset{b}{\int \frac{1}{(x+1)(x-1)} dx}$ a) $\frac{A}{y+1} + \frac{B}{y-1} = \frac{1}{(y+1)(y-1)} \;\Rightarrow\; \frac{A(y-1) + B(y+1)}{(y+1)(y-1)} = \frac{1}{''}$

$\Rightarrow Ay - A + By + B = 1 \;\Rightarrow\; y(A+B) - A + B = 1 \;\Rightarrow\; \begin{cases} A+B=0 \Rightarrow A+A+1=0 \Rightarrow 2A=-1 \Rightarrow A = -\frac{1}{2} \\ -A+B=1 \Rightarrow B = A+1 \Rightarrow B = \frac{1}{2} \end{cases}$

$\Rightarrow \int \frac{1}{y^2-1} dy = -\int \frac{\frac{1}{2}}{y+1} + \int \frac{\frac{1}{2}}{y-1} dy = -\frac{1}{2}\int \frac{1}{y+1} + \frac{1}{2}\int \frac{1}{y-1} dy = -\frac{1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| + c$

$= \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| + c$ Stesso per x : $\int \frac{1}{x^2-1} dx = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$

$\Rightarrow \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c \;\Rightarrow\; \frac{y-1}{y+1} = \frac{(x-1)c}{x+1} \;\Rightarrow\; (y-1)(x+1) = [(x-1)(y+1)]c$

$\Rightarrow yx + y - x - 1 = [xy + x - y - 1]c \;\Rightarrow\; yx + y - x - 1 = xyc + xc - yc - c$

$\Rightarrow yx + y - xyc + yc = x + 1 + xc - c \quad \Rightarrow \quad y(\cancel{x + 1 - xc + c}) = \dfrac{x + 1 + xc - c}{-x - 1 + xc - c}$

$\Rightarrow y(0) = \dfrac{1 - c}{1 + c} = 1 - c = 0$ per $c = 1$

Quindi la soluzione è $y = x$ con $c = 1$

b) $y(1) = \dfrac{1 + 1 + \cancel{c} - \cancel{c}}{-1 - 1 + \cancel{c} - \cancel{c}} = 1 \quad -\dfrac{2}{2} = 1$ BOH

5.17 Risolvere l'equazione differenziale

$$\sqrt{x}\, y' + \sqrt{y}\,\text{sen}\,\sqrt{x} = 0$$

$\dfrac{dv}{dx}\sqrt{x} + \sqrt{y}\,\sin\sqrt{x} = 0 \quad \Rightarrow \quad \dfrac{dy}{dy}\sqrt{x} = -\sqrt{y}\,\sin\sqrt{x} \quad \Rightarrow \quad \dfrac{dy}{\sqrt{y}} = \dfrac{-\sin\sqrt{x}}{\sqrt{x}} \cdot dx$

b

$\Rightarrow \displaystyle\int \frac{1}{\sqrt{y}}\, dy = -\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$ pongo $t = \sqrt{x} \Rightarrow dx = \dfrac{1}{-\frac{1}{2}\frac{1}{\sqrt{x}}}\, dt = -2\sqrt{x}\, dt$

a

$\Rightarrow +2\displaystyle\int \frac{\sin\sqrt{x}}{\cancel{\sqrt{x}}}\, \cancel{\sqrt{x}}\, dt = +2\int \sin t\, dt = +2\cos(\sqrt{x}) + c$

b) pongo $t = \sqrt{y} \Rightarrow dx = \dfrac{1}{-\frac{1}{2}\frac{1}{\sqrt{y}}}\, dt \Rightarrow 2\sqrt{y}\, dt \Rightarrow -2\displaystyle\int dt = -2\sqrt{y} + c$

$\Rightarrow -2\sqrt{y} = 2\cos\sqrt{x} + c \Rightarrow \sqrt{y} = -\cos\sqrt{x} \Rightarrow y = \cos^2\sqrt{x} + c'$ ✓

Problema esame:

$$\begin{cases} y' = \dfrac{e^y}{x^2+4x+5} \\ y(-2) = 0 \end{cases} \to \frac{dy}{dx} = \frac{e^y}{x^2+4x+5} \to \frac{dy}{e^y} = \frac{1}{x^2+4x+5}\,dx$$

$$\to \overset{b}{\int e^{-y}\,dy} = \overset{a}{\int \frac{1}{x^2+4x+5}\,dx}$$

risolvo a: $\int \frac{1}{x^2+4x+4+1}\,dx = \int \frac{1}{(x+2)^2+1}\,dx$ pongo $t = x+2$ $\to dx = dt$

a $\Rightarrow \int \frac{1}{t^2+1}\,dx = D(\arctan(t)) = \arctan(x+2)+c$

b) $\int e^{-y}\,dy = -e^{-y}+c \Rightarrow +e^{-y} = -\arctan(x+2)+c \Rightarrow -\ln|e^{-y}| = \ln|\arctan(x+2)+c|$

$\Rightarrow y = -\ln|\arctan(x+2)|c'$

$y(-2) = -\ln|\arctan(-2+2)+c| = 0 \to \ln|c| = 0$ per $c = 1$

$\Rightarrow$ Soluzione: $y = -\ln|\arctan(x+2)+1|$

Equazioni di Bernoulli

Si dice *di Bernoulli* un'equazione differenziale del primo ordine del tipo

$$y' = a(x)y + b(x)y^{\alpha}$$

Differisce dalle lineari per $b(x)y^{\alpha}$, dove $\alpha \neq 0 \neq 1$, altrimenti avremmo un'eq lineare.

Il metodo di risoluzione è il seguente: preliminarmente si dividono entrambi i membri dell'equazione per y^{α} (così facendo si trascura la soluzione identicamente nulla, nel caso in cui α sia positivo); si ottiene

$$\frac{y'}{y^{\alpha}} = a(x)y^{1-\alpha} + b(x)\,.$$

← Processo simile alle eq a var. sep.

Si pone poi $z(x) = (y(x))^{1-\alpha}$. La derivata della nuova funzione incognita $z(x)$, per la regola di derivazione delle funzioni composte, vale

$$z'(x) = \frac{d}{dx}(y(x))^{1-\alpha} = (1-\alpha)y^{-\alpha}y' = (1-\alpha)\frac{y'}{y^{\alpha}}\,.$$

L'equazione differenziale, nell'incognita z, diviene

$$z' = (1-\alpha)a(x)z + (1-\alpha)b(x)\,;$$

Viene poi risolta come eq lineare.

5.23 Risolvere l'equazione differenziale di Bernoulli

$$y' = 2y - e^x y^2$$

$y' = 2y - e^x y^2 \to \frac{y'}{y^2} = \frac{2\cancel{y}}{y^{\cancel{2}}} - e^x$

$\to \frac{y'}{y^2} = 2y^{-1} - e^x \to y'y^{-2} = 2y^{-1} - e^x$

Pongo $z(x) = (y(x))^{-1} \to z' = -y(x)^{-2}\cdot y'(x) \to$ Rispetto a $z \to -z' = 2z - e^x$ eq lineare

$\to z' = -2z + e^x$ del tipo $z' = a(x) + b(x)$ dove $a(x) = -2$ e $b(x) = e^x$

Scelgo $A(x)$ come primitiva: $A(x) = \int -2\,dx = -2x$ Troviamo l'int. gen. con $e^{A(x)}\int e^{-A(x)}\cdot b(x)\,dx$

$\to e^{-2x}\int e^{2x}\cdot e^{x}\,dx = e^{-2x}\int e^{3x}\,dx \to e^{-2x}\left(\frac{1}{3}e^{3x} + c\right) = ce^{-2x} + \frac{1}{3}e^{-2x}e^{3x} = \frac{1}{3}e^{x} + ce^{-2x}$ z

Ricordiamo che $z' = y^{-1} \Rightarrow y = z^{-1} = \left(\frac{1}{3}e^{x} + ce^{-2x}\right)^{-1} = y_0(x)$

5.24 Risolvere le seguenti equazioni di Bernoulli

(a) $y' = \frac{y}{x} - \frac{1}{y}$ (b) $2y' = \frac{y}{x} - \frac{1}{y}$

a) $y' = \frac{y}{x} - \frac{1}{y}$ divido per y^{-1}

$\to \frac{y'}{\frac{1}{y}} = \frac{\frac{y}{x}}{\frac{1}{y}} - 1 \to y'y = \frac{y^2}{x} - 1$

pongo $z = y^2 \to z = 2yy' \to \frac{z'}{2} = \frac{z}{x} - 1$

$\to z' = \frac{2}{x}z - 2 \to a(x) = \frac{2}{x} \to A(x) = 2\ln|x| = \ln|x|^2$, $b(x) = -2$

$\Rightarrow y_0(x) = e^{\ln|x|^2}\int e^{-\ln|x|^2}\cdot(-2)\,dx = e^{\ln|x|^2}\cdot(-2)\int e^{\ln|x|^{-2}}\,dx = -2x^2\int x^{-2}\,dx = +2x^2\left(\frac{x^{-1}}{-1} + c\right)$

$= +2x^2c + \frac{2x^{\cancel{2}}}{\cancel{x}} = 2x + cx^2$ z Siccome $z = y^2 \to y = \sqrt{z}$ otteniamo $\sqrt{2x + cx^2}$

b) $2y' = \frac{y}{x} - \frac{1}{y} \to 2y'\cdot y = \frac{y^2}{x} - 1 \to$ pongo $z = y^2 \to z' = 2yy'$

$\to z' = \frac{z}{x} - 1 \to$ Risolvo

1) metodo "lungo" $\rightarrow \quad \frac{dz}{dx} = \frac{z}{x} - 1 \quad \rightarrow \quad \frac{dz}{z} = \left(\frac{1}{x} - 1\right) dx \quad \rightarrow \int \frac{1}{z}\, dx = \int \frac{1}{x}\, dx - \int dx$

$\Rightarrow \ln|z| = \ln|x| - x + c \quad \rightarrow \quad z = e^{\ln x} \cdot e^{x} c \quad \Rightarrow \quad z = x e^{x} c$

$z = y^2 \rightarrow \quad y = \sqrt{z} \quad \rightarrow \quad \sqrt{x e^{x} c}$

2) Metodo formula $\qquad a(x) = \frac{z}{x} \qquad b(x) = -1 \quad \rightarrow \quad A(x) = 2 \ln|x|$

$\Rightarrow \quad e^{2\ln|x|} \int e^{-2\ln|x|} \cdot (-1) \quad = \quad x^2 \cdot (-1) \int \frac{1}{x^2}\, dx \quad = \quad -x^2\left(-\frac{1}{x} + c\right) \quad = \quad x - cx$

$t = \cos x \quad \rightarrow \quad dx = \frac{1}{\sin x}\, dt$

Nuovi esercizi Eq. Differenziali

4.4 Risolvere l'equazione differenziale lineare omogenea $y' = 8xy$.

$\underbrace{y'}_{\text{Derivata a Sx}} = \overbrace{8xy}^{\text{Tutto il resto a Dx}}$ E' in forma Normale

$\Rightarrow$ Dividiamo per y. $\frac{y'}{y} = 8x \xrightarrow{\text{Integro}} \int \frac{y'}{y}dx = 8\int x\,dx$

$= \ln|y| = 8\frac{x^2}{2}+c$; $\ln y = 4x^2+c$; $e^{\ln y} = ce^{4x^2} \Rightarrow y = ce^{4x^2}$

~~4.~~5 Risolvere l'equazione differenziale lineare omogenea

$$y' = \frac{x}{x^2+1}y$$

$y' = \frac{x}{x^2+1}y$; $\frac{y'}{y} = \frac{x}{x^2+1}$

$\ln y = \int \frac{x}{x^2+1}dx \Rightarrow \ln y = \frac{1}{2}\int\frac{2x}{x^2+1}dx \rightarrow \ln y = \frac{1}{2}\ln|x^2+1|+c \rightarrow y = ce^{\frac{1}{2}\ln x^2+1}$

$\rightarrow y = ce^{\ln(x^2+1)^{\frac{1}{2}}} = c\sqrt{x^2+1}$

4.6 Risolvere nell'intervallo $(0, \pi)$ l'equazione differenziale omogenea

$$y' = (\cot g\, x)y\,.$$

$y' = (\cot g\, x)y \rightarrow \frac{y'}{y} = \cot g\, x$

$\rightarrow \ln|y| = \int \cot g\, x = \int\frac{\cos x}{\sin x}dx$ pongo $t = \sin x \Rightarrow dx = \frac{1}{\cos x}dt$

$\rightarrow \int \frac{\cancel{\cos x}}{\sin x}\cdot\frac{1}{\cancel{\cos x}}dx \rightarrow \int\frac{1}{t}dt = \ln|\sin x| \Rightarrow \ln y = \ln|\sin x|+c$ $0<x<\pi$

$\Rightarrow y = c\sin x$

$$y' = 3y$$
$$y' = 2xy$$
$$y' = (x-1)y/x$$
$$y' = (\cos x)y$$
$$y' = -e^x y$$
$$y' = 2xe^{x^2}y$$
$$y' = (\operatorname{tg} x)y$$

$y' = 3y \rightarrow \ln y = 3\int dx \rightarrow y = ce^{3x}$

$y' = 2xy \rightarrow y = ce^{x^2}$

$y' = \frac{(x-1)y}{x} \rightarrow \ln y = \int dx - \int\frac{1}{x}dx \rightarrow \ln y = x - \ln|x|+c$

$\rightarrow y = c\cdot e^x/e^{\ln|x|} = \frac{ce^x}{x}$

$y' = (\cos x)y \rightarrow y = ce^{\sin x}$

$y' = -e^x y \rightarrow y = ce^{-e^x}$

$y' = 2xe^{x^2}y \rightarrow \ln|y| = \int 2xe^{x^2}dx$ pongo $t = e^{x^2} \rightarrow dx = \frac{1}{2xe^{x^2}}dt$

$\rightarrow \int \cancel{2xe^{x^2}}\cdot\frac{1}{\cancel{2xe^{x^2}}}dt = e^{x^2}+c \Rightarrow y = ce^{e^{x^2}}$

$y' = \operatorname{tg}x\, y \rightarrow y = ce^{\int \operatorname{tg}x\,dx} \rightarrow \int\operatorname{tg}x\,dx = \int\frac{\sin x}{\cos x}dx$ $D\cos x = -\sin x \Rightarrow -\ln|\cos x|$

$\rightarrow y = ce^{-\ln|\cos x|} = c\cos^{-1}x = \frac{c}{\cos x}$

$$y' = -y/2x$$
$$y' = 2y/x$$
$$y' = -(\operatorname{cotg} x)$$
$$y' = (\sqrt{x})y$$
$$y' = y/\sqrt{x+5}$$
$$y' = (\log x)y/x$$
$$y' = xy/(x^2-1)$$

$y' = -\frac{y}{2x} \to \ln y = -\frac{1}{2}\int \frac{1}{x}\,dx \to y = ce^{\ln x^{-\frac{1}{2}}} = \frac{c}{\sqrt{x}}$

$y' = \frac{2y}{x} \to \ln y = 2\int \frac{1}{x}\,dx \to y = ce^{\ln x^2} = cx^2$

$y' = \frac{y}{\sqrt{x+5}}\,dx \to \ln y = \int \frac{1}{\sqrt{x+5}}\,dx = \ln y = \int (x+5)^{-\frac{1}{2}}\,dx$

$\to y = ce^{\frac{(x+5)^{\frac{1}{2}}}{\frac{1}{2}}} = ce^{2\sqrt{x+5}}$

$y' = -(\operatorname{cotg} x) \to \int y'\,dx = -\int \operatorname{cotg} x \to y = -\int \frac{\cos x}{\sin x}\,dx$

$\to y = -\ln|\sin x|\, c$

$y' = (\sqrt{x})y \to \ln y = \int x^{\frac{1}{2}}\,dx \Rightarrow y = ce^{\frac{2x\sqrt{x}}{3}}$

$y' = \frac{xy}{x^2-1} \to \ln y = \int \frac{x}{x^2-1} \to \frac{1}{2}\int \frac{2x}{x^2-1}\,dx \to \frac{1}{2}\ln|x^2-1|+c$

$\Rightarrow y = ce^{\ln(x^2-1)^{\frac{1}{2}}} = c(x^2-1)^{\frac{1}{2}}$

$y' = (\ln x)\frac{y}{x} \to \ln y = \int \frac{\ln x}{x}$ $\quad\int \ln x\,dx \to \int \ln x \cdot D[x]\,dx \to \ln x \cdot x - \int \frac{1}{x}\cdot x\,dx$

$\to t = \ln x \to dx = x\,dt \to \int \frac{\ln x}{\cancel{x}}\cdot \cancel{x}\,dt = \int t\,dt = \frac{\ln^2 x}{2} \Rightarrow y = ce^{\frac{\ln x^2}{2}}$

$$y' = (1+\log x)y$$
$$y' = y/(x\log x)$$
$$y' = y/\operatorname{sen}(x+1)$$
$$y' = xy\operatorname{sen} x$$
$$y' = -(\operatorname{sen} 2x)y$$

$y' = (1+\ln x)y \to \ln y = \int dx + \int \ln x$

$\to \ln y = x + x\ln x + x \to \ln y = 2x + x\ln x$

$\to y = ce^{x\ln x}\cdot e^{2x} = cx^x\cdot e^{2x}$

$y' = \frac{y}{x\ln x}\,dx \to \ln y = \int \frac{1}{x\ln x}\,dx$

$\to \int \frac{1}{x}\cdot\frac{1}{\ln x}\,dx \to \frac{1}{\ln x}\ln x + \int \ln x \cdot \frac{1}{\ln^2 x}\,dx$

$= 1 + \int \frac{1}{\ln x}\,dx \to$ Non esiste una primitiva!

Riprovo pongo $t = \ln x \to dx = x\,dt \to \int \frac{1}{\cancel{x}\ln x}\cdot\cancel{x}\,dt = \int \frac{1}{t}\,dt = \ln|\ln x| + c$

$\Rightarrow y = C\ln x$

$y' = \frac{y}{\sin(x+1)} = \ln y = \int \frac{1}{\sin(x+1)}\,dx \Rightarrow$ pongo $t = \frac{1}{\sin(x+1)} \Rightarrow \frac{+\cos(x+1)}{\sin^2(x+1)} \to dx = \frac{1}{\frac{\cos(x+1)}{\sin^2(x+1)}}\,dt$

$\to \int \frac{1}{\cancel{\sin(x+1)}}\cdot\frac{\sin^{\cancel{2}}(x+1)}{\cos(x+1)}\,dt = \int \frac{\sin(x+1)}{\cos(x+1)}\,dt$ $\quad \frac{1}{t} = \sin(x+1) \to x+1 = \arcsin\left(\frac{1}{t}\right)$

$x = \arcsin\left(\frac{1}{t}\right) - 1$

$\Rightarrow \int \frac{\frac{1}{t}}{\cos(x+1)}\,dt = \int \frac{\cos(\arcsin(\frac{1}{t}))}{t}\,dt = \int \frac{\frac{\sqrt{t^2-1}}{t}}{t}\,dt = \int \sqrt{t^2-1}\,dt$

credo sia sbagliato

$y' = xy \sin x \to \ln y = \int x \sin x \, dx = -x\cos x + \int \cos x \, dx = -x\cos x + \sin x$

$\to y = c e^{-x\cos x + \sin x}$

$y' = -\sin 2x \; y \to \ln y = -\int \sin 2x$ pongo $t = 2x \to x = \frac{t}{2} \Rightarrow dx = \frac{1}{2} dt$

$\to \ln y = -\frac{1}{2} \int \sin(t) \, dt \Rightarrow \ln y = +\frac{1}{2}\cos(2x) + c \to y = c e^{+\frac{1}{2}\cos(2x)}$

Eq. lineare non omogenea

Esempio

$y' - \frac{y}{tg x} = \sin x \to$ Om. ass. $\to y' = \frac{y}{tg} \to \frac{y'}{y} = \frac{1}{tg x} \to \ln y = \int \frac{1}{tg x}$

$\to \ln y = \int \frac{\cos x}{\sin x} dx \quad D[\sin x] = \cos x \Rightarrow \ln y = \ln|\sin x| + c \to y_0 = c \sin x$ Int. generale

$y_p(x) = c(x) \cdot \sin x$ Calcolo $y' = D[y_0] = c' \sin x + c \cos x \Rightarrow c' \sin x + c\cos x - \frac{c \sin x}{tg x} = \sin x$

$c' \sin x + c\cos x - c \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} = \sin x \to c' \sin x + c\cancel{\cos x} - c\cancel{\cos x} = \sin x$

$\to c' \cancel{\sin} x = \cancel{\sin x} \to c' = 1 \Rightarrow c = \int c' dx = \int dx = x$

$y_p(x) = x \cdot \sin x$ Integrale particolare

$y(x) = y_p(x) + y_0 = x \sin x + c \sin x$

$y' = -\frac{2}{x} y + \frac{\sin 4x}{x^2}$; Si considera l'omogenea associata

$$y' = -\frac{2}{x} y + \frac{\text{sen}\, 4x}{x^2}$$

$y' = -\frac{2}{x} y \to \frac{y'}{y} = -\frac{2}{x} \to \ln y = -2\int \frac{1}{x} \to \ln y = -2\ln|x| + c \to y_0 = c e^{-2\ln|x|} = \frac{c}{x^2}$

$y_p(x) = c(x) \cdot \frac{1}{x^2} \Rightarrow y' = \frac{c' x^2 - c x^2}{x^4} \Rightarrow \frac{(c' - c)}{x^2} = -\frac{2c}{x^3} + \frac{\sin(4x)}{x^2}$

$\to \frac{c' - c}{x^2} = \frac{-2c + x\sin(4x)}{x^3}$; $\frac{c' - c}{x^2} + \frac{2c}{x^3} = \frac{\sin(4x)}{x^2} \to \frac{c'x - cx + 2c}{x^3} = \frac{\sin(4x)}{x^2}$

$\to \frac{c'\cancel{x}}{x^{\cancel{3}2}} - \frac{c(x+2)}{x^3} = \frac{\sin(4x)}{x^2}$

$$\cancel{y}' = 3y + 1$$
$$\cancel{y}' = ay + b, \qquad (a, b \in \mathbb{R},\ a \neq 0)$$
$$\cancel{y}' = y + x$$
$$\cancel{y}' = -y + e^{-x}$$
$$\cancel{y}' + y/x = 1/x$$
$$y' = (y/x) + xe^{x}$$
$$y' = y + e^{x}$$
$$y' = 4y - e^{2x}$$

$y' = 3y + 1 \to \frac{y'}{y} = 3 \to \ln y = 3\int dx$

$\to y_0 = ce^{3x} \Rightarrow y' = c'e^{3x} + 3ce^{3x}$

$\Rightarrow c'e^{3x} + \cancel{3ce^{3x}} = \cancel{3ce^{3x}} + 1$

$\to c'e^{3x} = 1 \quad ; \quad c' = 1 - e^{3x}$

$\Rightarrow c = \int dx - \int e^{3x}\,dx = x - \frac{1}{3}\int e^{3x} \cdot 3$

$\to c = x - \frac{1}{3}e^{3x}$

$\Rightarrow y_p(x) = \left(x - \frac{1}{3}e^{3x}\right) \cdot e^{3x} = xe^{3x} - \frac{1}{3}e^{3x} \cdot e^{3x} = e^{3x}\left(x - \frac{1}{3}e^{3x}\right)$

$\to y(x) = xe^{3x} - \frac{1}{3}e^{6x} + ce^{3x} = e^{3x}\left(x - \frac{1}{3}e^{3x} + c\right)$

$y' = ay + b \to \ln y = a\int dx \to y_0 = ce^{ax} \Rightarrow y' = c'e^{ax} + cae^{ax}$

$\Rightarrow e^{ax}(c' + c) = ace^{ax} + b \to c'e^{ax} + \cancel{ce^{ax}} = a\cancel{ce^{ax}} + b \to c'e^{ax} = a + b; \quad c' = \frac{a+b}{e^{ax}}$

$c = a\int e^{-ax} + b\int e^{-ax} = -\frac{1}{e^{ax}} - \frac{b}{a} \Rightarrow y_p = \left(-\frac{1}{\cancel{e^{ax}}} - \frac{b}{a}\right) \cdot \cancel{e^{ax}} = -1 - \frac{b}{a}$

$y(x) = ce^{ax} - \frac{b}{a} - 1$ Boh

$y' = y + x \to \ln y = \int dx \to y_0 = ce^{x} \Rightarrow y' = c'e^{x} + ce^{x} \to c'e^{x} + \cancel{ce^{x}} = \cancel{ce^{x}} + x$

$\to c' = x - e^{x} \Rightarrow c = \int x\,dx - \int e^{x}dx = \frac{x^2}{2} - e^{x}$

$\Rightarrow y_p = \left(\frac{x^2}{2} - e^{x}\right) \cdot e^{x} = \frac{1}{2}x^2e^{x} - e^{2x}$

$y(x) = \frac{1}{2}x^2e^{x} - e^{2x} + ce^{x} = e^{x}\left(\frac{1}{2}x - e^{x} + c\right)$

$y' = -y + e^{-x} \to \ln|y| = -\int dx \Rightarrow y_0 = ce^{-x} \qquad y' = c'e^{-x} - ce^{-x}$

$c'e^{-x} - \cancel{ce^{-x}} = -\cancel{ce^{-x}} + e^{-x} \quad ; \quad c' = \frac{e^{-x}}{e^{-x}} = \frac{\frac{1}{e^{x}}}{\frac{1}{e^{x}}} = 1 \Rightarrow c = \int dx = x$

$\Rightarrow y_p = xe^{-x} \Rightarrow y(x) = xe^{-x} + ce^{-x} = e^{-x}(x + c)$ ✓

$y' + \frac{y}{x} = \frac{1}{x} \to \ln y = -\int \frac{1}{x}dx \to y_0 = -cx \Rightarrow y' = -c'x - c \to -c'x - \cancel{c} + \cancel{c} = \frac{1}{x}$

$\to c' = -\frac{1}{x^2} \Rightarrow c = -\int \frac{1}{x^2}dx = -\int x^{-2}dx = -\frac{x^{-3}}{3} = -\frac{1}{3x^3}$

$\Rightarrow y_p = +\frac{1}{3}\frac{1}{x^{\cancel{3}2}} \cdot \cancel{x} = \frac{1}{3x^2} \to y(x) = \frac{1}{3x^2} - cx = \frac{1 - 3x^3c}{3x^2} = \frac{1}{3x} - xc$

$$y' = \frac{y}{x} + xe^{x} \to y' = \frac{y}{x} \to \ln y = \int \frac{1}{x}\,dx \to y_0 = ce^{x}$$

$$y' = c'e^{x} + ce^{x} \to c'e^{x} + ce^{x} = \frac{ce^{x}}{x} + xe^{x} \Rightarrow c'e^{x} + ce^{x} = \frac{ce^{x} + x^2e^{x}}{x} \to$$

$$\to c'e^{x} = \frac{ce^{x}}{x} + xe^{x} - ce^{x} \to c'\cancel{e^{x}} = \cancel{e^{x}}\left(\frac{c}{x} + x - c\right) \to c' = \frac{\cancel{c} + x^2 - \cancel{c}}{x} \to c' = x$$

$$\Rightarrow c = \int x\,dx = \frac{x^2}{2} \Rightarrow y_p(x) = xe^{x}$$

$$y(x) = xe^{x} + ce^{x} = e^{x}(x+c)$$

$$y' = y + e^{x} \to y' = y \to \ln y = \int dx \to y_0 = ce^{x} \qquad y' = c'e^{x} + ce^{x}$$

$$\to c'e^{x} + \cancel{ce^{x}} = \cancel{ce^{x}} + e^{x} \Rightarrow c' = 1 \Rightarrow c = \int dx = x$$

$$\to y_p(x) = xe^{x} \to y(x) = xe^{x} + ce^{x} = e^{x}(x+c)$$

$$y' = 4y - e^{2x} \to y' = 4y \to \ln y = 4\int dx \to y_0 = ce^{4x} \Rightarrow y' = c'e^{4x} + c4e^{4x}$$

$$\to c'e^{4x} + \cancel{c4e^{4x}} = \cancel{4ce^{4x}} - e^{2x} \to c' = -\frac{1}{e^{2x}} \Rightarrow c = -\int \frac{1}{e^{2x}} = \frac{1}{2}\int e^{-2x}\cdot 2\,dx = \frac{1}{2}e^{-2x} = \frac{1}{2e^{2x}}$$

$$\Rightarrow y_p(x) = \frac{1}{2\cancel{e^{2x}}}e^{\cancel{4x}\,2x} = \frac{1}{2}e^{2x} \Rightarrow y(x) = \frac{1}{2}e^{2x} + ce^{4x} = e^{2x}\left(\frac{1}{2} + ce^{2x}\right)$$

$$y' = ay + e^{bx} \qquad (a \neq b)$$
$$y' = -2xy + xe^{-x^2}$$
$$y' = (2y/x) + (x+1)/x$$
$$y' = y + x^2 - 1$$
$$y' = e^{x} - (y/x)$$

$$y' = ay + e^{bx} \to \ln y = a \to y_0 = ce^{ax}$$

$$y' = c'a^{ex} + cea^{ex} \to c'a^{ex} + \cancel{cea^{ex}} = \cancel{ace^{ax}} + e^{bx}$$

$$\to c' = \frac{e^{bx}}{a^{ex}} \to c = \int \frac{e^{bx}}{a^{ex}}\,dx \to \text{BOH?}$$

$$y' = -2xy + xe^{-x^2} \to \ln y = -2\int x\,dx \to y_0 = ce^{-x^2} \to y' = c'e^{-x^2} + \left[-2xe^{-x^2}c\right]$$

$$c'e^{-x^2} - \cancel{2cxe^{-x^2}} = -\cancel{2xce^{-x^2}} + xe^{-x^2} \to c' = \frac{x\cancel{e^{-x^2}}}{\cancel{e^{-x^2}}} \to c' = x \Rightarrow c = \frac{x^2}{2}$$

$$\Rightarrow y_p = \frac{x^2}{2}e^{-x^2} = \frac{x^2}{2e^{x^2}} \to y(x) = \frac{x^2}{2e^{x^2}} + ce^{-x^2} = \frac{x^2 + c}{2e^{x^2}}$$

$$y' = \frac{2y}{x} + \frac{x+1}{x} \to \ln y = 2\int \frac{1}{x}\,dx \to y_0 = ce^{2\ln x} = cx^2 \qquad y' = c'x^2 + c2x$$

$$\to c'x^2 + \cancel{c2x} = \cancel{\frac{2cx^2}{x}} + 1 + \frac{1}{x} \to c' = \frac{1}{x^2} + \frac{1}{x^3} \Rightarrow c = \int \frac{1}{x^2}\,dx + \int \frac{1}{x^3}\,dx$$

$$\to c = -\frac{1}{2x^2} - \frac{1}{3x^3} = \frac{-3x-2}{6x^3} \Rightarrow y_p = -\frac{3x-2}{6x^3}x^2 = -\frac{1}{2} - \frac{1}{3x}$$

$$y(x) = -\frac{1}{2} - \frac{1}{3x} + cx^2$$

Problemi di Cauchy

$$\begin{cases} y' + \frac{1}{x}y = x^3 \\ y(1) = 1/5 \end{cases}$$

1) Risolviamo l'eq. diff $\to$ $y' + \frac{1}{x}y = x^3 \to \ln y = -\frac{1}{x} \Rightarrow \underbrace{y_0 = ce^{-\ln x} = \frac{c}{x}}$

$\to y' = \frac{c'x - c}{x^2} \to \frac{c'x - c}{x^2} + \frac{c}{x^2} = x^3 \to \frac{c'}{\cancel{x}} - \cancel{\frac{c}{x^2}} + \cancel{\frac{c}{x^2}} = x^{3\,4}$

$\to c' = x^4 \Rightarrow c = \int x^4 dx = \frac{x^5}{5} \Rightarrow y_p(x) = \frac{x^{\cancel{5}\,4}}{5\cancel{x}} = \frac{x^4}{5}$

$\to y(x) = \frac{x^4}{5} + \frac{c}{x} = \frac{x^5 + 5c}{5x}$

2) Soddisfiamo l'eq $y(1) = \frac{1}{5}$

$\to \frac{1^5 + 5c}{5 \cdot 1} = \frac{5c+1}{5} = \frac{1}{5} \to c + \cancel{\frac{1}{5}} = \cancel{\frac{1}{5}} \Rightarrow c = 0$ TEST $\frac{1+0}{5} = \frac{1}{5}$ ✓

3) Scriviamo l'integrale particolare (unica soluzione)

$y = y_p + y_0 = \frac{\cancel{c}^{\,0}}{x} + \frac{x^4}{5} + \frac{\cancel{c}^{\,0}}{x} = \frac{1}{5}x^4$

4.10 Risolvere il problema di Cauchy

$$\begin{cases} y' = 3xe^{x^2}y \\ y(0) = 1 \end{cases}$$

$y' = 3xe^{x^2}y \to \ln y = 3\int xe^{x^2} = \frac{3}{2}\int 2xe^{x^2}$

$\Rightarrow \ln y = \frac{3}{2}e^{x^2} \to y = ce^{\frac{3}{2}e^{x^2}}$

$y' = 2x\frac{3}{2}e^{x^2}e^{\frac{3}{2}e^{x^2}}c + c'e^{\frac{3}{2}e^{x^2}}$

$\Rightarrow \cancel{3xe^{x^2} \cdot e^{\frac{3}{2}e^{x^2}}c} + c'e^{\frac{3}{2}e^{x^2}} = \cancel{3xe^{x^2}ce^{\frac{3}{2}e^{x^2}}} \Rightarrow c' = \frac{1}{e^{\frac{3}{2}e^{x^2}}} \Rightarrow c = \int \frac{1}{e^{\frac{3}{2}e^{x^2}}} = \int e^{-\frac{3}{2}e^{x^2}}$

$$\begin{cases} y' = (1-y)/x \\ y(1) = 0 \end{cases}$$

$[y = (x-1)/x]$

$y' = \frac{1-y}{x} = \frac{1}{x} - \frac{y}{x} \Rightarrow y' = -\frac{y}{x}$

$\Rightarrow \ln y = -\int \frac{1}{x}dx \to \underbrace{y_0 = ce^{-\ln x} = \frac{c}{x}} \Rightarrow y' = \frac{c'x - c}{x^2} \Rightarrow \frac{c'x - c}{x^2} = \frac{1}{x} - \frac{\left(-\frac{c}{x}\right)}{x}$

$\to \frac{c'}{x} - \cancel{\frac{c}{x^2}} = \frac{1}{x} - \cancel{\frac{c}{x^2}} \to c' = 1 \Rightarrow c = x \Rightarrow y_p = 1$

$y(x) = 1 + \frac{c}{x} \Rightarrow y(1) = 1 + c = 0 \Rightarrow c = -1$

$\Rightarrow y_p(x) = -\frac{1}{x} + 1 = \frac{x-1}{x}$ ✓

$$\begin{cases} y' = 2y + 1 \\ y(0) = 1 \end{cases}$$

$[y = (3e^{2x} - 1)/2]$

$y' = 2y \to \ln y = 2\int dx \to y = ce^{2x} \to y' = c'e^{2x} + c2e^{2x}$

$c'e^{2x} + \cancel{c2e^{2x}} = \cancel{2ce^{2x}} + 1 \to c' = \frac{1}{e^{2x}} \Rightarrow c = \int e^{-2x}dx = -\frac{1}{2}\int -2e^{-2x}dx \Rightarrow c = -\frac{1}{2}e^{-2x}$

$\Rightarrow y_p = -\frac{1}{2}\cancel{e^{-2x}} \cdot \cancel{e^{2x}} = -\frac{1}{2} \Rightarrow y(x) = -\frac{1}{2} + ce^{2x}$

$y(0) = -\frac{1}{2} + ce^{0} = 1 \Rightarrow c = \frac{3}{2} \Rightarrow y = -\frac{1}{2} + \frac{3}{2}e^{2x} = \frac{-1 + 3e^{2x}}{2}$ ✓

$$\begin{cases} y' = ay + b \\ y(0) = 0 \end{cases} \quad (a, b \in \mathbb{R},\ a \neq 0) \qquad [y = (e^{ax} - 1)b/a]$$

$y' = ay \to y = ce^{ax} \qquad y' = c'e^{ax} + cae^{ax}$

$\to c'e^{ax} + \cancel{cae^{ax}} = \cancel{ace^{ax}} + b \to c' = \frac{b}{e^{ax}} \Rightarrow c = b\int e^{-ax}dx = -\frac{b}{a}\int a e^{-ax}dx = -\frac{b}{a}e^{-ax}$

$\Rightarrow y_p(x) = -\frac{b}{a}\cancel{e^{-ax}}\,\cancel{e^{ax}} = -\frac{b}{a} \Rightarrow y(x) = ce^{ax} - \frac{b}{a}$

$y(0) = c - \frac{b}{a} = 0 \Rightarrow c = \frac{b}{a} \qquad \Rightarrow y = \frac{b}{a}e^{ax} - \frac{b}{a} = \frac{b}{a}(e^{ax} - 1)$

$$\begin{cases} y' + \frac{1}{x}y = x^3 \\ y(1) = 1/5 \end{cases} \qquad [y = x^4/5]$$

$y' = -\frac{1}{x}y \to y = \frac{c}{x} \Rightarrow y' = \frac{c'x - c}{x^2}$

$\frac{c'x - c}{x^2} + \frac{1}{x}\frac{c}{x} = x^3 \to \frac{c'}{x} - \cancel{\frac{c}{x^2}} + \cancel{\frac{c}{x^2}} = x^3 \to c' = x^4 \Rightarrow c = \int x^4 dx = \frac{x^5}{5}$

$\to y_p = \frac{x^5}{5} + \frac{c}{x} \to y(1) = \frac{1}{5} + c = \frac{1}{5} \Rightarrow c = \emptyset \qquad \Rightarrow y = \frac{x^4}{5}$

$$\begin{cases} y' = (\operatorname{tg} x)y + 1 \\ y(\pi) = 1 \end{cases} \qquad [y = \operatorname{tg} x - (1/\cos x)]$$

$y' = (\operatorname{tg} x)y \to \ln y = \int \operatorname{tg} x \to \ln y = \int \frac{\sin x}{\cos x}$

$D[\cos x] = -\sin x \Rightarrow \ln y = -\ln(\cos x) \Rightarrow y_0 = \frac{c}{\cos x} \Rightarrow y' = \frac{c'\cos x + c\sin x}{\cos^2 x}$

$\Rightarrow \frac{c'}{\cos x} + \frac{c\sin x}{\cos^2 x} = \operatorname{tg} x \cdot \frac{c}{\cos x} + 1 \to \frac{c'}{\cos x} + \cancel{c\frac{\sin x}{\cos^2 x}} = \cancel{\frac{\frac{\sin x}{\cos x}c}{\cos x}} + 1 \to c' = \cos x$

$\Rightarrow c = \int \cos x\, dx = \sin x \Rightarrow y_p = \frac{\sin x}{\cos x} = \operatorname{tg} x \Rightarrow y(x) = \operatorname{tg} x + \frac{c}{\cos x}$

$y(\pi) = \operatorname{tg} x + \frac{c}{\cos x} = 1 \to 0 + \frac{c}{-1} = 1 \Rightarrow -c = 1 \Rightarrow c = -1$

$\Rightarrow y = \operatorname{tg} x - \frac{1}{\cos x}$

$$\left[\frac{3}{10}-\left(\frac{8}{9}-\frac{5}{6}\right)\cdot\frac{6}{5}\right]:x=x:\left[\left(\frac{1}{4}+\frac{1}{5}\right)\cdot\frac{4}{3}-\frac{3}{7}\right]$$

$$\to \frac{3}{10}-\left(\frac{16-15}{18}\right)\cdot\frac{6}{5}\Bigg]:x=x:\left[\left(\frac{5+4}{20}\right)\cdot\frac{4}{3}-\frac{3}{7}\right]$$

$$\to \frac{3}{10}-\frac{1}{\cancel{18}\,3}\cdot\frac{\cancel{6}^{\,1}}{5}\quad :x=x\left[\frac{\cancel{9}^{\,3}}{\cancel{20}\,5}\cdot\frac{\cancel{4}}{\cancel{3}}-\frac{3}{7}\right]$$

$$\to \frac{9-2}{30}\ :x=x\left[\frac{21-15}{35}\right]\to\frac{7}{30}\ :x=x:\ \frac{6}{35}$$

$$\to \frac{\cancel{7}^{\,1}}{\cancel{30}\,5}\cdot\frac{\cancel{6}^{\,1}}{\cancel{35}\,5}=\frac{1}{25}\Rightarrow\sqrt{\frac{1}{25}}=\frac{1}{\sqrt{25}}=\frac{1}{5}$$

$$\begin{cases} y' = [(x+1)y/x] + x(1-x) \\ y(1) = e \end{cases}$$

$$y'=\frac{(x+1)y}{x}+x(1-x)$$

$$\to \frac{y'}{y}=\frac{x+1}{x}\to \ln y=\int dx+\int\frac{1}{x}\,dx$$

$$\to \ln y = x+\ln x+c \to y=ce^{x}+ce^{\ln x}\to y=c(e^{x}+x)\Rightarrow y'=c'(e^{x}+x)+ce^{x}$$

$$\Rightarrow c'(e^{x}+x)+ce^{x}=\left[\frac{x+1}{x}\right]\cdot(ce^{x}+cx)+x(1-x)\to c'e^{x}+c'x+ce^{x}=\frac{x+1}{x}ce^{x}+\frac{x+1}{\cancel{x}}c\cancel{x}+x(1-x)$$

$$\to c'e^{x}+c'x+\cancel{ce^{x}}=\cancel{ce^{x}}+\frac{ce^{x}}{x}+cx+c+x-x^{2}\to c'e^{x}+c'x=\frac{ce^{x}}{x}+cx+c+x-x^{2}$$

$$\to c'e^{x}+c'x=c\left(\frac{e^{x}}{x}+x+1\right)+x-x^{2}\to c'e^{x}+c'x=x\,(ce^{x}$$

Eq. Diff. lineari omogenee a coeff costanti — Di II ordine

$$\underbrace{a}_{}y''+\underbrace{b}_{}y'+\underbrace{c}_{}y=f(x)$$
$\in\mathbb{R}$ e costanti

La differenza con le equazioni precedenti è che è presente un termine derivato 2 volte.

Se $f(x)\equiv 0$ si dice omogenea, altrimenti non omogenea.

Qual è la soluzione?

Per soluz. o integrale particolare, si intende una $y=y(x)$ tale che $y''(x)+ay'(x)+by=f(x)$ quindi una $y(x)$ che derivata 1 e 2 volte, e sostituita ad y, y' e y'' ci dia $f(x)$.

Osservazione

L'insieme delle soluzioni è uno spazio vettoriale di dimensione 2 $\Rightarrow$

- se moltiplichiamo una soluzione per una costante otteniamo un'altra soluzione.
- Se prendiamo 2 soluzioni qualsiasi e le sommiamo (e/o moltiplicate per k), il risultato è ancora soluzione.

Quindi: $y(x)=\underbrace{c_1}_{}y_1(x)+\underbrace{c_2}_{}y_2(x)$ $\Longrightarrow$ Se troviamo y_1 e y_2 che formano una base, abbiamo risolto il problema.

parametri liberi
↓
Determiniamo c_1 e c_2 solo quando dobbiamo risolvere un prob di Cauchy

Base dello spazio delle soluzioni

Metodo di risoluzione

Risolviamo l'eq. caratteristica: $az^2+bz+c=0$ (stessi parametri)

Siccome abbiamo un'eq di II° abbiamo 3 casi possibili:

1) $\lambda_1\neq\lambda_2 \Rightarrow e^{\lambda_1 x}, e^{\lambda_2 x} \Rightarrow y(x)=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$

2) $\lambda_1=\lambda_2 \Rightarrow e^{\lambda x}, xe^{\lambda x} \Rightarrow y(x)=c_1e^{\lambda x}+c_2xe^{\lambda x}$

3) $\lambda_{1,2}=\alpha\pm i\beta$ (complesse) $\Rightarrow e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x) \Rightarrow y(x)=c_1e^{\alpha x}\cos(\beta x)+c_2e^{\alpha x}\sin(\beta x)$

ES: $y''-5y'+4y=0 \rightarrow z^2-5z+4=0 \rightarrow \Delta=25-4\cdot4=9$

$\Rightarrow \lambda_{1,2}=\frac{5\pm3}{2}$ $<$ 4, 1

BASE $\rightarrow e^{4x}, e^{x} \Rightarrow$ soluzione $y(x)=c_1e^{4x}+c_2e^{x}$

ES, Cauchy:
$$\begin{cases} y''+2y'+2y=0\\ y(0)=1\\ y'(0)=1\end{cases}$$

$z^2+2z+2=0 \rightarrow \Delta=4-6=-2$

$\Rightarrow \lambda_{1,2}=\frac{-2\pm\sqrt{-2}}{2}$ $<$ $\frac{-2+\sqrt{2}i}{2}=-1+i$ λ_1 ; $\frac{-2-\sqrt{2}i}{2}=-1-i$ λ_2 (α, β)

$\Rightarrow$ III caso $\rightarrow e^{-x}\cos(x), e^{-x}\sin(x) \Rightarrow \underline{y(x)=c_1e^{-x}\cos(x)+c_2e^{-x}\sin(x)}$
sol generale

$y(0)=c_1e^{-x}\cos x+c_2e^{-x}\sin x=1 \rightarrow c_1+0=1$

$y'(0)=+c_1e^{-x}\cos x\cdot c_1e^{-x}\sin x-c_2e^{-x}\sin x\cdot c_2e^{-x}\cos(x)=+c_1e^{-x}\cos x\, c_1e^{-x}\sin x-c_2e^{-x}\sin x\, c_2e^{-x}\cos x$

$\Rightarrow -e^{-x}(c_1\cos x + c_2\sin x) + e^{-x}(-c_1\sin x + c_2\cos x) = 1 \Rightarrow -c_1 + c_2 = 1$

Siccome $c_1 = 1 \Rightarrow -1 + c_2 = 1 \Rightarrow c_2 = 2$

$\Rightarrow$ Sol cauchy $\Rightarrow y(x) = e^{-x}\cos x + 2e^{-x}\sin x$

4.~~1~~4 Risolvere le equazioni differenziali omogenee

(a) $y'' - 6y' + 5y = 0$ (b) $y'' - 2y' + 2y = 0$

a) $y'' - 6y' + 5y = 0$ $\quad z^2 - 6z + 5 = 0 \Rightarrow \Delta = 36 - 20 = 16 > 0$ $\quad \lambda_{1,2} = \frac{6 \pm 4}{2} \left< \begin{matrix} 5 \\ 1 \end{matrix} \right.$

$\Rightarrow e^{x}, e^{5x} \Rightarrow y(x) = c_1 e^{x} + c_2 e^{5x}$

b) $y'' - 2y' + 2y = 0 \Rightarrow z^2 - 2z + 2 = 0 \Rightarrow \Delta = 4 - 4 \cdot 2 = -2 < 0 \Rightarrow$

$\lambda_{1,2} = \frac{2 \pm \sqrt{2}i}{2} \left< \begin{matrix} 1+i \\ 1-i \end{matrix} \right.$ α β $\Rightarrow e^{x}\cos(-x), e^{x}\sin(x) \Rightarrow c_1 e^{x}\cos(-x) + c_2 e^{x}\sin(x)$

4.~~1~~5 Risolvere l'equazione differenziale $y'' - 2y' + y = 0$. $\quad z^2 - 2z + z = 0 \Rightarrow \Delta = 4 - 4 = 0$

$\Rightarrow \lambda = \frac{2 \pm 0}{2} = 1 \Rightarrow e^{x}, xe^{x} \Rightarrow y(x) = c_1 e^{x} + c_2 x e^{x}$

4.~~1~~6 Risolvere le equazioni differenziali lineari omogenee

(a) $y'' - 2y' = 0$ (b) $y'' + 4y = 0$

$z^2 - 2z = 0 \Rightarrow z(z-2) = 0$
$\hookrightarrow z = 0$
$\hookrightarrow z = 2$

$\Rightarrow 1, e^{2x} \Rightarrow c_1 + c_2 e^{2x}$

b) $z^2 + 4 = 0 \Rightarrow z = \sqrt{-4} \Rightarrow z = \pm 2i \Rightarrow c_1 \cancel{e}^{0}\cos(2x) + c_2 \cancel{e}^{0}\sin(-2x)$

4.~~1~~7 Risolvere l'equazione $y'' + 2y' + y = 0$.

$z^2 + 2z + 1 = 0 \Rightarrow \Delta = 4 - 4 = 0 \Rightarrow \lambda = \frac{-2}{2} = -1 \Rightarrow e^{-x}, xe^{-x} \Rightarrow c_1 e^{-x} + c_2 x e^{-x}$

$y'' - 3y' + 2y = 0$	$[y = c_1 e^{x} + c_2 e^{2x}]$
$y'' - 10y' + 21y = 0$	$[y = c_1 e^{3x} + c_2 e^{7x}]$?
$y'' - 2y' + y = 0$	$[y = (c_1 + c_2 x)e^{x}]$
$y'' - 10y' + 25y = 0$	$[y = (c_1 + c_2 x)e^{5x}]$

$z^2 - 3z + 2 = 0 \Rightarrow 9 - 4 \cdot 2 = 1 > 0$

$\lambda_{1,2} = \frac{3 \pm 1}{2} \left< \begin{matrix} 2 \\ 1 \end{matrix} \right. \Rightarrow y(x) = c_1 e^{x} + c_2 e^{2x}$

b) $z^2 - 10z + 21 = 0 \Rightarrow \Delta = 100 - 4\ 21 = 100 - 64 = 36 \Rightarrow \lambda_{1,2} = \frac{10 \pm 6}{2} \left< \begin{matrix} 8 \\ 2 \end{matrix} \right.$

$\Rightarrow y(x) = c_1 e^{2x} + c_2 e^{8x}$

c) $z^2 - 2z + 1 = 0$ $\quad \Delta = 4 - 4 = 0 \Rightarrow \lambda = \frac{2}{2} = 1 \Rightarrow y(x) = c_1 e^{x} + c_2 x e^{x}$

d) $= z^2 - 10z + 25 = 0$ $\quad \Delta = 100 - 4 \cdot 25 = 0 \Rightarrow \lambda = \frac{10}{2} = 5 \Rightarrow y(x) = c_1 e^{5x} + c_2 x e^{5x}$

$y'' + y = 0$
$y'' + 3y = 0$
$y'' - 2y' - 15y = 0$
$y'' - y = 0$
$y'' - 4y' + 4y = 0$

$z^2 + 1 = 0 \to z = \pm i \to y(x) = e^0 \cos(x) + e^0 \sin(-x)$

b) $z^2 + 3 = 0 \to z = \pm 3i \to y(x) = c_1 \cos(3x) + c_2 \sin(-3x)$

c) $z^2 - 2z - 15 = 0 \quad \Delta = 4 - 4\cdot(-15) = 64 > 0 \quad z_{1,2} = \frac{2 \pm 8}{2} \begin{array}{l} 5 \\ -2 \end{array}$

$\to y(x) = c_1 e^{5x} + c_2 e^{-2}$

d) $z^2 - 1 = 0$
$\to z = \pm 1 \to y(x) = c_1 e^{x} + c_1 e^{-x}$

$y'' - 4y' + 4y = 0$ $\quad [y = (c_1 + c_2 x)e^{2x}]$
$y'' - 2y' + 5y = 0$ $\quad [y = e^x(c_1 \cos 2x + c_2 \mathrm{sen}\, 2x)]$
$y'' + y' + y = 0$ $\quad [y = e^{-x/2}[c_1 \cos(\sqrt{3}x/2) + c_2 \mathrm{sen}(\sqrt{3}x/2)]]$
$y'' - 4y' + 20y = 0$ $\quad [y = e^{2x}(c_1 \cos 4x + c_2 \mathrm{sen}\, 4x)]$
$y'' + 9y = 0$ $\quad [y = c_1 \cos 3x + c_2 \mathrm{sen}\, 3x]$
$y'' - 6y' + 10y = 0$ $\quad [y = e^{3x}(c_1 \cos x + c_2 \mathrm{sen}\, x)]$

$z^2 - 4z + 4 = 0 \quad \Delta = 16 - 16 = 0 \Rightarrow \lambda = \frac{4}{2} = 2 \Rightarrow y(x) = c_1 e^{2x} + c_2 x e^{2x}$

b) $z^2 - 2z + 5 = 0 \to \Delta = 4 - 20 = -16 < 0 \Rightarrow \lambda_{1,2} = \frac{2 \pm 4i}{2} \begin{array}{l} 1+2i \\ 1-2i \end{array}$

$\Rightarrow y(x) = c_1 e^{x} \cos(2x) + c_2 e^{x} \sin(-2x)$

c) $z^2 + z + 1 = 0 \to \Delta = -3 < 0 \to \lambda_{1,2} = \frac{-1 \pm 3i}{2} \begin{array}{l} -\frac{1}{2} + \frac{3}{2}i \\ -\frac{1}{2} - \frac{3}{2}i \end{array}$

$\to y(x) = c_1 \frac{1}{\sqrt{e^x}} \cos(\frac{3}{2}x) + c_2 \frac{1}{\sqrt{e^x}} \sin(-\frac{3}{2}x)$

d) $z^2 - 4z + 20 = 0 \to \Delta = 16 - 80 = -64 < 0 \to \lambda_{1,2} = \frac{4 \pm \sqrt{64}\, i}{2} \begin{array}{l} \frac{4 + 8i}{2} = 2 + 4i \\ \frac{4 - 8i}{2} = 2 - 4i \end{array}$

$\Rightarrow y(x) = c_1 e^{2x} \cos(4x) + c_2 e^{2x} \sin(-4x)$

e) $z^2 + 9 = 0 \to z = \pm 3i \to y(x) = c_1 e^{x} \cos(3x) + c_2 e^{x} \sin(-3x)$

f) $z^2 - 6z + 10 = 0 \quad \Delta = 36 - 40 = -4 < 0 \Rightarrow \lambda_{1,2} = \frac{6 \pm 2i}{2} \begin{array}{l} 3+i \\ 3-i \end{array} \Rightarrow y(x) = c_1 e^{3x} \cos(x) + c_2 e^{3x} \sin(-x)$

4.31 Determinare la soluzione del problema di Cauchy

$$\begin{cases} y'' - 2y' - y = 0 \\ y(0) = 0 \\ y'(0) = 2\sqrt{2} \end{cases}$$

$z^2 - 2z - 1 = 0 \to \Delta = 4 + 4 = 8 > 0$
$\lambda_{1,2} = \frac{2 \pm \sqrt{8}}{2} \begin{array}{l} 1+\sqrt{2} \\ 1-\sqrt{2} \end{array}$

$\Rightarrow y(x) = c_1 e^{x + x\sqrt{2}} + c_2 e^{x - x\sqrt{2}} \qquad y'(x) = c_1' e^{x + x\sqrt{2}} + c_1 [1+\sqrt{2}] e^{x + x\sqrt{2}} + c_2' e^{x - x\sqrt{2}} + c_2 [1-\sqrt{2}] e^{x - x\sqrt{2}}$

$\to y(0) = c_1 e^{x + x\sqrt{2}} + c_2 e^{x - x\sqrt{2}} = 0 = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$y'(0) = c_1' + c_2' + c_1 + \sqrt{2} c_2 + c_2 - \sqrt{2} c_1 = 2\sqrt{2} \overset{c_1 = -c_2}{\Rightarrow} c_1' + c_2' - \cancel{c_2} + \cancel{\sqrt{2} c_2} + \cancel{c_2} - \cancel{\sqrt{2} c_2} = 2\sqrt{2}$

$c_1' + c_2' = 2\sqrt{2} \Rightarrow c_1' = -2c_2\sqrt{2} \Rightarrow$

4.32 Determinare la soluzione di ciascuno dei seguenti problemi di Cauchy

(a) $\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 0 \\ y'(0) = 3 \end{cases}$ (b) $\begin{cases} y'' - 6y' + 10y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

(c) $\begin{cases} y'' - 10y' + 25y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$ (d) $\begin{cases} y'' - 2y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

a) $z^2 - z - 2 = 0 \qquad \Delta = 1 + 8 = 9 > 0$

$\lambda_{1/2} = \dfrac{1 \pm 3}{2} \begin{matrix} 2 \\ -1 \end{matrix}$

$\Rightarrow y(x) = c_1 e^{2x} + c_2 e^{-x}$

$y'(x) = c_1' e^{2x} + c_1 2 e^{2x} + c_2' e^{-x} - c_2 e^{-x}$

$\rightarrow c_1 = -c_2 \overset{y'}{\Rightarrow} c_1' \underset{1}{\cancel{e^{2x}}} - c_2 2 \underset{1}{\cancel{e^{2x}}} + c_2' \underset{1}{\cancel{e^{-x}}} - c_2 \underset{1}{\cancel{e^{-x}}} = 3 \rightarrow c_1' - 2c_2 + c_2' - c_2 = 3$

b) $z^2 - 6z + 10 = 0 \quad \rightarrow \quad \Delta = 36 - 40 = -16 < 0 \Rightarrow \lambda_{1/2} = \dfrac{6 \pm 4i}{2} \begin{matrix} 3+i \\ 3-i \end{matrix}$

$\rightarrow y(x) = c_1 e^{3x} \cos(x) + c_2 e^{3x} \sin(x)$

$y'(x) = c_1' e^{3x} \cos x + c_1 [3e^{3x} \cos x - e^{3x} \sin x] + c_2' e^{3x} \sin x + c_2 [3e^{3x} \sin x + e^{3x} \cos x]$

$y(0) = c_1 + 0 = 1 \Rightarrow c_1 = 1 \qquad \dfrac{d}{dx} 1 = 0$

$y'(0) = c_1' + 3 + c_2 \rightarrow 0 + 3 + c_2 \rightarrow c_2 = -3$

$\Rightarrow y(x) = e^{3x} \cos x - 3 e^{3x} \sin x$

a) $z^2 - z - 2 = 0 \quad \rightarrow \quad \Delta = 1 + 4 \cdot 2 = 9 > 0 \quad \rightarrow \quad \lambda_{1/2} = \dfrac{1 \pm 3}{2} \begin{matrix} 2 \\ -1 \end{matrix}$

$\rightarrow y(x) = c_1 e^{2x} + c_2 e^{-x} \quad \rightarrow \quad y'(x) = c_1' e^{2x} + c_1 2 e^{2x} + c_2' e^{-x} - c_2 e^{-x}$

$y(0) = c_1 + c_2 = 0 \rightarrow c_1 = -c_2 \rightarrow c_1' = \dfrac{d}{dx} c_1 = -c_2'$

$y'(0) = c_1' + 2c_1 + c_2' = 3 \overset{\substack{c_1 = -c_2 \\ c_1' = -c_2'}}{\longrightarrow} \cancel{-c_2'} - 2c_2 + \cancel{c_2'} = 3 \Rightarrow c_2 = -\dfrac{3}{2}$

$\Rightarrow y(x) = \dfrac{3}{2} e^{2x} - \dfrac{3}{2} e^{-x}$

c) $z^2 - 10z + 25 = 0 \quad \rightarrow \quad \Delta = 100 - 100 = 0 \quad \rightarrow \quad \lambda = \dfrac{10}{2} = 5$

$\Rightarrow y(x) = c_1 e^{5x} + c_2 x e^{5x} \quad \rightarrow \quad y' = c_1' e^{5x} + c_1 5 e^{5x} + c_2' x e^{5x} + c_2 [e^{5x} + 5x e^{5x}]$

$y(0) = c_1 = 0 \qquad y'(0) = c_1' + 5c_1 + + c_2 = 1 \rightarrow 0 + 0 + c_2 = 1 \Rightarrow c_2 = 1$

$\Rightarrow y(x) = x e^{5x}$

d) $z^2 - 2z + 5 = 0 \Rightarrow \Delta = 4 - 4 \cdot 5 = -16 < 0 \Rightarrow \lambda_{1,2} = \frac{2 \pm 4i}{2} < \begin{matrix} 1+2i \\ 1-2i \end{matrix}$

$\Rightarrow y(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$

$y'(x) = c_1' e^x \cos(2x) + c_1 \left[e^x \cos(2x) - e^x 2 \sin(2x) \right] + c_2' e^x \sin(2x) + c_2 \left[e^x \sin(2x) + 2 e^x \cos(2x) \right]$

$y(0) = c_1 = 1$ $\qquad$ $y'(0) = c_1' + c_1 + 2c_2 = 0 \Rightarrow 0 + c_1 + 2c_2 \Rightarrow 0 + 1 + 2c_2 = 0$

$\Rightarrow c_2 = -\frac{1}{2}$

$c_1' = \frac{d}{dx} 1 = 0$

$y(x) = e^x \cos(2x) - \frac{1}{2} e^x \sin(2x)$

il libro ha un risultato diverso
ma questo è giusto

Eq. a coefficienti costanti non omogenea

4.34 Risolvere l'equazione differenziale non omogenea $y'' - 3y' + 2y = 2x^3 - x^2 + 1$.

1) Considero l'equazione caratteristica dell'omogenea associata

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \Delta = 9 - 4\cdot 2 = 1 > 0 \qquad \lambda = \frac{3 \pm 1}{2} \begin{cases} 2 \\ 1 \end{cases}$

$\Rightarrow y(x) = c_1 e^{2x} + c_2 e^{x}$

2) Esaminiamo il termine noto: $2x^3 - x^2 + 1$ è del tipo $e^{\gamma x} P(x) \Rightarrow e^{\gamma x} = 1 \Rightarrow \gamma = 0$

$\Rightarrow$ Controlliamo se $\gamma = 0$ è soluzione dell'omogenea ass.

Siccome le radici dell'om. ass sono $\lambda_1 = 2$ e $\lambda_2 = 1$, $\gamma = 0$ NON è soluzione.

3) Siccome $e^{0x} \cdot \overset{n}{P}(x)$ è di grado $n = 3$, la soluzione dell'eq completa è un polinomio del tipo: $y_p(x) = e^{\gamma x} \cdot \overset{n}{q}(x)$

$V_0(x) = b_0 x^3 + b_1 x^2 + b_2 x + b_3 \Rightarrow V_0'(x) = 3b_0 x^2 + 2b_1 x + b_2 \Rightarrow V_0''(x) = 6b_0 x + 2b_1$

$\Rightarrow$ Sostituisco $y^{(n)}(x) = V_0^{(n)}(x) \Rightarrow (6b_0 x + 2b_1) - 3(3b_0 x^2 + 2b_1 x + b_2) + 2(b_0 x^3 + b_1 x^2 + b_2 x + b_3) = 2x^3 - x^2 + 1$

$\Rightarrow 6b_0 x + 2b_1 - 9b_0 x^2 - 6b_1 x - 3b_2 + 2b_0 x^3 + 2b_1 x^2 + 2b_2 x + 2b_3 = 2x^3 - x^2 + 1$

$\Rightarrow x^3 2b_0 + x^2(-9b_0 + 2b_1) + x(6b_0 - 6b_1 + 2b_2) + 2b_1 - 3b_2 + 2b_3 = 2x^3 - x^2 + 1$

$$\begin{cases} 2b_0 = 2 & \Rightarrow b_0 = 1 \\ -9b_0 + 2b_1 = -1 & \Rightarrow b_1 = 4 \\ 6b_0 - 6b_1 + 2b_2 = 0 & \Rightarrow b_2 = 9 \\ 2b_1 - 3b_2 + 2b_3 = 1 & \Rightarrow b_3 = 10 \end{cases}$$

$\Rightarrow y(x) = y_0(x) + y_p(x) = c_1 e^{2x} + c_2 e^{x} + x^3 + 4x^2 + 9x + 10$

4.35 Risolvere l'equazione differenziale non omogenea $y'' - 4y' = x^2 + 1$.

$y'' - 4y' = 0 \to \lambda^2 - 4\lambda = 0$

$\to \lambda(\lambda-4) = 0$ $\Rightarrow y_0 = c_1 + c_2 e^{4x}$
$\quad \hookrightarrow \lambda = 0$
$\quad \hookrightarrow \lambda = 4$

Esamino $x^2+1 \to \gamma = 0$
$\alpha = 0$ e' soluzione e $P(x)$ e' di grado 2
$\alpha = 0$ e' di molteplicita' 1.

$\Rightarrow y_p(x)$ e' del tipo : $x^n \cdot e^{\gamma x} \cdot q_n(x) \Rightarrow x \cdot [b_0 x^2 + b_1 x + b_2]$

$\Rightarrow V_0(x) = b_0 x^3 + b_1 x^2 + b_2 x \to V_0'(x) = 3b_0 x^2 + 2b_1 x + b_2 \to V_0''(x) = 6b_0 x + 2b_1$

$\to$ Sostituisco $\to 6b_0 x + 2b_1 - 4[3b_0 x^2 + 2b_1 x + b_2] = x^2 + 1$

$\to 6b_0 x + 2b_1 - 12b_0 x^2 - 8b_1 x - 4b_2 = x^2+1 \to x^2(-12b_0) + x(6b_0 - 8b_1) + 2b_1 - 4b_2 = x^2+1$

$$\begin{cases} -12b_0 = 1 \to b_0 = -\frac{1}{12} \\ 6b_0 - 8b_1 = 0 \to b_1 = -\frac{1}{16} \\ 2b_1 - 4b_2 = 1 \to b_2 = -\frac{1}{32} \end{cases} \Rightarrow y_p(x) = x\left(-\frac{1}{12}x^2 - \frac{1}{16}x - \frac{1}{32}\right)$$

$\Rightarrow y(x) = c_1 + c_2 e^{4x} + x\left(-\frac{1}{12}x^2 - \frac{1}{16}x - \frac{1}{32}\right)$

4.36 Risolvere l'equazione differenziale non omogenea $y'' - 2y' - 3y = 8e^{3x}$.

$y'' - 2y' - 3y = 0$; $\lambda^2 - 2\lambda - 3 = 0$

$\Delta = 4 - 4\cdot(-3) = 16 > 0 \to \lambda_{1,2} = \frac{2 \pm 4}{2} \begin{cases} 3 \\ -1 \end{cases}$ $\to \boxed{y(x) = c_1 e^{3x} + c_2 e^{-x}}$

$P(x) = 8e^{3x}$ con $\gamma = 3 \to$ soluzione di molteplicita' $h = 1$, $P^{(n)}(x) \to n = 1$

$\Rightarrow y_p(x) = x^h \cdot e^{\gamma x} \cdot q^{(n)}(x) \to V_0(x) = xe^{3x} \cdot [b_0] = b x e^{3x}$

$\to V_0'(x) = be^{3x} + bx3e^{3x} \to V_0''(x) = 3be^{3x} + b3e^{3x} + bx9e^{3x}$

$\to$ SOST $\to 3be^{3x} + b3e^{3x} + bx9e^{3x} - 2[be^{3x} + bx3e^{3x}] - 3bxe^{3x} = 8e^{3x}$

$\to 3be^{3x} + b3e^{3x} + bx9e^{3x} - 2be^{3x} - 2bxe^{3x} - 3bxe^{3x} = 8e^{3x}$

$\to e^{3x}(3b + b - 2b) + e^{3x}x(b - 2b - 3b) = 8e^{3x}$

$$\begin{cases} 3b + b - 2b = 8 \to b = 4 \\ b + 2b - 3b = 0 \to 4 + 8 - 12 = 0 \ \checkmark \end{cases}$$ $\Rightarrow \boxed{y_p(x) = 4xe^{3x}}$

$\to y(x) = c_1 e^{3x} + c_2 e^{-x} + 4xe^{3x}$

4.38 Tenendo presente l'esercizio precedente, determinare l'integrale generale dell'equazione $y'' - 3y' + 2y = 2x^3 + 1 - x^2 + e^{3x}$

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \Delta = 9 - 8 = 1$

$\Rightarrow \lambda_{1,2} = \frac{3 \pm 1}{2} \begin{matrix} 2 \\ 1 \end{matrix} \Rightarrow \underbrace{y_0(x) = c_1 e^{2x} + c_2 e^{x}}$

$P^{(n)}(x) = 2x^3 - x^2 + e^{3x} + 1 \quad = \quad e^{\gamma x} \cdot P(x) \quad = \quad e^{3x} \cdot \left[\frac{2x^3}{e^{3x}} - \frac{x^2}{e^{3x}} + 1 + \frac{1}{e^{3x}} \right]$

$\Rightarrow \gamma = 3 \quad , \quad P^{(n)}(x) \Rightarrow (n) = 3 \; .$ $\quad \gamma = 3$ è soluzione? NO

$\Rightarrow e^{\gamma x} \cdot P^{(n)}(x) \Rightarrow V_0(x) = e^{3x} \cdot \left[b_0 x^3 + b_1 x^2 + b_2 x + b_3 \right] \Rightarrow V_0'(x) = 3e^{3x} \cdot \left[b_0 x^3 + b_1 x^2 + b_2 x + b_3 \right]$
$+ e^{3x} \left[3b_0 x^2 + 2b_1 x + b_2 \right]$

$\Rightarrow V_0''(x) = 9e^{3x} \left[b_0 x^3 + b_1 x^2 + b_2 x + b_3 \right] + 3e^{3x} \left[3b_0 x^2 + 2b_1 x + b_2 \right] + 3e^{3x} \left[3b_0 x^2 + 2b_1 x + b_2 \right] +$
$+ e^{3x} \left[6b_0 x + 2b_1 \right]$

$\Rightarrow \cancel{9e^{3x} b_0 x^3} + 9e^{3x} b_1 x^2 + 9e^{3x} b_2 x + 9e^{3x} b_3 + \cancel{9e^{3x} b_0 x^2} + 6e^{3x} b_1 x + 3e^{3x} b_2 + 9e^{3x} b_0 x^2 + 6e^{3x} b_1 x + 3e^{3x} b_2 +$
$+ 6e^{3x} b_0 x + 2e^{3x} b_1 - \cancel{9e^{3x} b_0 x^3} - 9e^{3x} b_1 x^2 - 9e^{3x} b_2 x - 9e^{3x} b_3 - \cancel{9e^{3x} b_0 x^2} - 6e^{3x} b_1 - 3e^{3x} b_2 \; +$
$+ 2e^{3x} b_0 x^3 + 2e^{3x} b_1 x^2 + 2e^{3x} b_2 x + 2e^{3x} b_3 = 2x^3 - x^2 + e^{3x} + 1$

4.39 Risolvere l'equazione differenziale non omogenea $y'' - 2y' - 3y = \cos 2x$.

$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \Delta = 4 - 4(-3) = 16 > 0$

$\Rightarrow \lambda_{1,2} = \frac{2 \pm 4}{2} \begin{matrix} 3 \\ -1 \end{matrix} \Rightarrow \underbrace{y_0(x) = c_1 e^{3x} + c_1 e^{-x}}$

$P(x) = \cos 2x \quad \Rightarrow \gamma = 0 \Rightarrow$ NON è soluzione $\Rightarrow V_0(x) = b_0 \cos(2x) + b_1 \sin(2x)$

$\Rightarrow V_0'(x) = -2b_0 \sin(2x) + 2b_1 \cos(2x) \quad \Rightarrow V_0''(x) = -4 b_0 \cos(2x) - 4b_1 \sin(2x)$

$\Rightarrow -4 b_0 \cos(2x) - 4 b_1 \sin(2x) - 2\left[-2b_0 \sin(2x) + 2b_1 \cos(2x) \right] - 3\left[b_0 \cos(2x) + b_1 \sin(2x) \right]$
$= \cos(2x)$

$\Rightarrow \cos(2x)\left(-4b_0 - 4b_1 - 3b_0 \right) + \sin(2x)\left(-4b_1 + 4b_0 - 3b_1 \right) = \cos(2x)$

$\begin{cases} -4b_0 - 4b_1 - 3b_0 = 1 \Rightarrow -7b_0 - 4b_1 = 1 \Rightarrow b_0 = -\frac{4}{7} b_1 - \frac{1}{7} \\ -4b_1 + 4b_0 - 3b_1 = 0 \Rightarrow -4b_1 - \frac{16}{7} b_1 - \frac{4}{7} - 3b_1 = 0 \Rightarrow \frac{-28-16-21}{7} b_1 = \frac{4}{7} \end{cases}$

$\Rightarrow b_1 = -\frac{4}{7} \frac{}{65}$

Sinceramente non mi appare

4.40 Risolvere l'equazione differenziale non omogenea $y'' - 2y' + y = xe^x$.

$\lambda^2 - 2\lambda + 1 = 0$

$\Delta = 4 - 4 \cdot 1 = 0 \Rightarrow \lambda = \frac{2}{2} = 1 \Rightarrow$ $\boxed{y_0(x) = c_1 e^x + c_2 x e^x}$

II) $P(x) = xe^x \to \gamma = 1$ Radice di molt. $h = 2$

$\Rightarrow V_0(x) = x^h \cdot e^{\gamma x} \cdot P^{(n)}(x) \to n = 1 \Rightarrow V_0(x) = x^2 e^x \cdot (bx + c)$

$\Rightarrow V_0'(x) = 2xe^x(bx+c) + x^2[e^x(bx+c) + e^x b] = 2xe^x(bx+c) + x^2e^x(bx+c) + x^2e^x b$

$V_0''(x) = 2e^x(bx+c) + 2x[e^x(bx+c) + e^x b] + 2xe^x(bx+c) + x^2[e^x(bx+c) + e^x b] + 2xe^x b + x^2e^x b$

$\Rightarrow 2e^x(bx+c) + 2xe^x(bx+c) + 2xe^x b + 2xe^x(bx+c) + x^2e^x(bx+c) + x^2e^x b + 2xe^x + x^2e^x b -$
$- 4xe^x(bx+c) + 2x^2e^x(bx+c) + 2x^2e^x b + x^2e^x(bx+c) = xe^x$

$2e^x bx + 2ce^x + 2x^2e^x b + 2xe^x c + 2xe^x b + 2x^2e^x b + 2xe^x c + x^3e^x b + x^2e^x c + x^2e^x b +$
$+ 2xe^x + x^2e^x b - 4x^2e^x b - 4xe^x c + 2x^3e^x b + 2x^2e^x c + 2x^2e^x b + x^3e^x b + x^2e^x c = xe^x$

$e^x[2bx + 2c + \cancel{2x^2b} + \cancel{2xc} + 2xb + \cancel{2x^2b} + \cancel{2xc} + x^3b + x^2c + \cancel{x^2b} + 2x + \cancel{x^2b} - \cancel{4x^2b}$
$- \cancel{4x^2b} - \cancel{4xc} + 2x^3b + 2x^2c + \cancel{2x^2b} + x^3b + x^2c] = xe^x$

$\to e^x[4bx + 2c + 2x + 2bx^3 + 4x^2c + 2x^3b] = xe^x$

$y'' + y = x + 1$ $\quad [y = c_1 \cos x + c_2 \operatorname{sen} x + x + 1]$
$y'' - 2y' + y = x^2 + x$ $\quad [y = (c_1 + c_2 x)e^x + x^2 + 5x + 8]$
$y'' - 2y' + y = e^x$ $\quad [y = (c_1 + c_2 x)e^x + x^2 e^x/2]$
$y'' - 5y' + 6y = e^x$ $\quad [y = c_1 e^{2x} + c_2 e^{3x} + e^x/2]$
$y'' - 2y' - 3y = (2x+1)e^x$ $\quad [y = c_1 e^{-x} + c_2 e^{3x} - (2x+1)e^x/4]$

a) $y'-2y=1 \rightarrow y'=2y+1 \rightarrow \frac{dy}{2y+1}=dx$

$\rightarrow \int \frac{1}{2y+1} = \int dx \rightarrow \ln|2y+1| = x+c \rightarrow 2y+1 = ce^{x} \rightarrow y=\frac{ce^{x}-1}{2}$

b) $\underset{y'}{y'}+\underset{a(x)y}{y} = \underset{f(x)}{e^{x}} \rightarrow a=1,\ f=e^{x}$ omogeneea: $y'+y=0$

$\rightarrow y'=-y \rightarrow \frac{dy}{y} = -dx \rightarrow \ln y = -x+c \rightarrow y_p = ce^{-x}$

$y' = c'e^{-x} - ce^{-x} \Rightarrow c'e^{-x} \cancel{-ce^{-x}} + \cancel{ce^{-x}} = e^{x} \Rightarrow c'e^{-x} = e^{x} \Rightarrow c' = e^{2x}$

$\Rightarrow c = \frac{1}{2}\int e^{2x}\cdot 2\,dx = \frac{1}{2}e^{2x}$

$\Rightarrow y_p = \frac{1}{2}e^{2x}\cdot e^{-x} = \frac{1}{2}e^{x}$ $\qquad y(x) = \frac{1}{2}e^{x} + ce^{-x}$

c) $y'-2y = x^2+x \rightarrow y'-2y=0 \rightarrow \frac{dy}{2y} = dx \rightarrow \frac{1}{2}\int \frac{2}{2y}\,dy = \int dx$

$\frac{1}{2}\ln|2y| = x+c \rightarrow \sqrt{2y} = ce^{x} \rightarrow 2y = ce^{2x} \rightarrow y_p = \frac{1}{2}ce^{2x}$

$\Rightarrow y_p'(x) = \frac{1}{2}c'e^{2x} + ce^{2x}$

$\rightarrow \frac{1}{2}c'e^{2x} \cancel{+ce^{2x}} \cancel{-ce^{2x}} = x^2+x \rightarrow c' = \frac{2x^2+2x}{e^{2x}}$

$\Rightarrow c = 2\int \frac{x^2}{e^{2x}} + 2\int \frac{x}{e^{2x}}\,dx$

PARTI

1) $2\int x^2\cdot e^{-2x}dx = 2\left[-\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}dx\right] = 2\left\{-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx\right\}$

$= 2\left[-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right] = -x^2e^{-2x} - xe^{-2x} - \frac{1}{2}e^{-2x}$ (1)

2) $2\int xe^{-2x}dx = 2\left[-\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx\right] = 2\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right] = -xe^{-2x} - \frac{1}{2}e^{-2x}$ (2)

$\Rightarrow c = \underbrace{-x^2e^{-2x} - xe^{-2x} - \frac{1}{2}e^{-2x}}_{(1)} + \underbrace{-xe^{-2x} - \frac{1}{2}e^{-2x}}_{(2)} = e^{-2x}\left(-x^2 - x - \frac{1}{2} - x - \frac{1}{2}\right) = \underbrace{e^{-2x}(-x^2-2x-1)}_{c}$

$\Rightarrow y_p(x) = \frac{1}{2}\cdot \cancel{e^{-2x}}(-x^2-2x-1)\cancel{e^{2x}} = \frac{1}{2}(-x^2-2x-1)$

$\Rightarrow y(x) = \frac{1}{2}(-x^2-2x-1) + ce^{2x} = -\frac{1}{2}(x^2+2x+1) + ce^{2x}$ Integrale generale

Esempio 1.1.

a. Determinare *tutte* le soluzioni dell'equazione differenziale

$$y' = (1-y)(2-y)x$$

$y' = (1-y)(2-y)x$ E' una eq a var sep.

eq del tipo $y'(x) = a(x) \cdot b(x)$

$\to$ $a(x) = x$, $b(x) = (1-y)(2-y)$

$\to$ Dividiamo per $b(x)$ $\to$ $\frac{y'(x)}{b(x)} = a(x)$ $\to$ $\frac{y'}{(1-y)(2-y)} = x$

$\to$ $\int \frac{dy}{(1-y)(2-y)} = \int x\,dx$ Divido $\frac{y'}{b(x)}$

$\to$ $\frac{1}{(1-y)(2-y)} = \frac{A}{(1-y)} + \frac{B}{(2-y)}$ $\to$ $\frac{2A-Ay+B-By}{(1-y)(2-y)} = \frac{y(-A-B)+2A+B}{//} = \frac{1}{//}$

$\Rightarrow$ $\begin{cases} -A-B=0 \to A=-B \Rightarrow A=1 \\ 2A+B=1 \Rightarrow -2B+B=1 \to B=-1 \end{cases}$ $\Rightarrow$ $\frac{1}{(1-y)(2-y)} = \frac{1}{1-y} - \frac{1}{2-y}$

$\Rightarrow$ $\int \frac{dy}{()()} = \int x\,dx$ $\to$ $-\int \frac{-1}{1-y}\,dy + \int \frac{-1}{2-y}\,dy = \int x\,dx$

$\Rightarrow$ $-\ln|1-y| + \ln|2-y| = \frac{x^2}{2} + c$ $\to$ $\ln\left|\frac{2-y}{1-y}\right| = \frac{x^2}{2} + c$

$\to$ $\frac{2-y}{1-y} = ce^{\frac{x^2}{2}}$ $\to$ $2-y = ce^{\frac{x^2}{2}} \cdot (1-y)$ $\to$ $2-y = ce^{\frac{x^2}{2}} - yce^{\frac{x^2}{2}}$

$\to$ $-y + yce^{\frac{x^2}{2}} = ce^{\frac{x^2}{2}} - 2$ $\to$ $y\left(ce^{\frac{x^2}{2}} - 1\right) = ce^{\frac{x^2}{2}} - 2$ $\to$ $y = \frac{ce^{\frac{x^2}{2}} - 2}{ce^{\frac{x^2}{2}} - 1}$

Integrale generale

$\to$ TUTTE LE SOL.

b. Risolvere il problema di Cauchy:

$$\begin{cases} y' = (1-y)(2-y)x \\ y(0) = 3, \end{cases}$$

$y' = (1-y)(2-y)x$ $\to$ $y = \frac{ce^{\frac{x^2}{2}} - 2}{ce^{\frac{x^2}{2}} - 1}$

$\to$ $y(0) = \frac{c-2}{c-1} = 3$ $\to$ $c-2 = 3c-3$ $\to$ $\frac{\cancel{c}}{\frac{\cancel{3}c}{2}} = -3+2$ $\to$ $c = +\frac{1}{2}$

$\Rightarrow$ $y_p(x) = \frac{\frac{1}{2}e^{\frac{x^2}{2}} - 2}{\frac{1}{2}e^{\frac{x^2}{2}} - 1} = \frac{\frac{e^{\frac{x^2}{2}} - 4}{\cancel{2}}}{\frac{e^{\frac{x^2}{2}} - 2}{\cancel{2}}} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{\cancel{b}} \cdot \frac{\cancel{d}}{c}$ $\Rightarrow$ $y_p(x) = \frac{e^{\frac{x^2}{2}} - 4}{e^{\frac{x^2}{2}} - 2}$

Come trovare l'insieme di definizione?

$\to$ La soluzione e' definita sul piu' grande intervallo in cui il denom non si annulla:

$e^{\frac{x^2}{2}} - 2 \neq 0$ $\to$ $e^{\frac{x^2}{2}} = 2$ $\to$ $\ln\left(e^{\frac{x^2}{2}}\right) = \ln(2)$ $\to$ $\frac{x^2}{2} = \ln(2)$ $\to$ $x^2 = 2\ln(2)$

$\to$ $x^2 = 2\ln(2)$ $\to$ $x = \pm\sqrt{2\ln(2)}$

Esempio 1.2.

a. Scrivere l'integrale generale dell'equazione

$$y' = 2xy + 3x^3.$$

$y' = 2xy + 3x^3$ Eq. Lineare del I ordine

$y'(x) + a(x)y = f(x)$

Siccome $a(x) = -2x \rightarrow \int y' = -2\int x\,dx \rightarrow A(x) = -x^2$ Primitiva

$y_0(x) = c e^{-\int a(x)\,dx} = c e^{x^2} \rightarrow y'(x) = c' e^{x^2} + c2x e^{x^2}$

Calcoliamo la primitiva

$\rightarrow c' e^{x^2} + \cancel{c2x e^{x^2}} = \cancel{2xc e^{x^2}} + 3x^3 \rightarrow c' e^{x^2} = 3x^3 \rightarrow c' = \frac{3x^3}{e^{x^2}}$

$\Rightarrow c = 3\int e^{-x^2} x^3$ pongo $t = x^2 \rightarrow dx = \frac{1}{2x} dt \rightarrow \frac{3}{2}\int e^{-t} t\, dt$

$\rightarrow$ PARTI $\rightarrow \frac{3}{2}\left[-e^{-t}\cdot t + \int e^{-t} dt\right] = \frac{3}{2}\left[-te^{-t} + e^{-t}\right] = \frac{3}{2}\left[e^{-t}(1-t)\right]$

$\rightarrow \frac{3}{2} e^{-x^2}(1-x^2) = -\frac{3}{2} e^{-x^2}(x^2+1) = y_p(x)$

Primitiva $y(x) = y_0(x) + y_p(x) = c e^{x^2/2} - \frac{3}{2} e^{-x}(x^2+1)$ Integrale generale

b. Risolvere il problema di Cauchy:

$$\begin{cases} y' = 2xy + 3x^3 \\ y(0) = 1. \end{cases}$$

$y' = 2xy + 3x^3 \rightarrow y = c e^{x^2/2} - \frac{3}{2} e^{-x}(x^2+1)$

$y(0) = c - \frac{3}{2} = 1 \rightarrow c = \frac{3}{2} + 1 \rightarrow c = \frac{5}{2} \Rightarrow y(x) = \frac{5}{2} e^{\frac{x^2}{2}} - \frac{3}{2} e^{-x}(x^2+1)$

Insieme di def: più ampio intervallo che contiene

$\Rightarrow D: \mathbb{R}$

Esempio 1.3. Per ciascuna delle seguenti equazioni differenziali del prim'ordine, dire se è lineare, se è a variabili separabili, se è entrambe le cose, o nessuna delle due (*non* si chiede di risolverle):

(*a*) $y' + y\sin x + e^{-x} = 0$ (*b*) $y' + \sin(xy) = 0$

(*c*) $\frac{y'}{y} - \sin y\cos x = 0$ (*d*) $e^x y' = 2ye^{-x^2}$

a) $y' + y\sin x + e^{-x} = 0 \rightarrow y' + y\sin x = -e^{-x}$

$a(x) + b(x) = f(x)$

Lineare del I ordine

No var sep.

b) $y' + \sin(xy) = 0$ $\rightarrow$ No lineare, No var sep

c) $\frac{y'}{y} - \sin y \cos x = 0 \rightarrow \frac{y'}{y} = \sin y \cos x \rightarrow \frac{y'}{y\sin y} = \cos x$

$a(x)$ $f(x)$

Non lineare a var sep.

d) $e^x y' = 2ye^{-x^2} \rightarrow \frac{y'}{y} = \frac{2e^{-x^2}}{e^x} \rightarrow \frac{y'}{y} = 2e^{-2x^2}$ Lineare a var sep.

1.1.★

a. Determinare *tutte* le soluzioni dell'equazione differenziale

$$y' = y^3\sin 2x$$

$y' = y^3 \sin 2x \;\to\; \dfrac{y'}{y^3} = \sin 2x$

$\to \displaystyle\int \frac{dv}{y^3} = \int \sin(2x)\, dx$

$\to \displaystyle\int y^{-3} dy = \int \sin(2x)\, dx \;\to\; -\frac{y^{-2}}{2} = \int \sin(2x)\, dx$ pongo $t = 2x \to dx = \frac{1}{2} dt$

$\to -\frac{1}{2} y^{-2} = \frac{1}{2}\displaystyle\int \sin(t)\, dt \;\to\; -\frac{1}{2} y^{-2} = -\frac{1}{2}\cos(2x) + c \;\to\; \frac{1}{y^2} = \cos(2x) + c$

$\to y^2 = \dfrac{1}{\cos(2x)} + \dfrac{1}{c} \;\to\; y = \sqrt{\dfrac{1}{\cos(2x)} + \dfrac{1}{c}}$ Tutte le soluzioni

Soluzione banale: $y^3 \sin 2x = 0 \to \boxed{y = 0}$

b. Risolvere il problema di Cauchy:

$$\begin{cases} y' = y^3\sin 2x \\ y(\frac{\pi}{4}) = \sqrt{2}, \end{cases}$$

$y_0(x) = \sqrt{\dfrac{1}{\cos(2x) + C}}$

$\Rightarrow y\left(\frac{\pi}{4}\right) = \sqrt{\dfrac{1}{\cos\left(\frac{\pi}{2}\right) + c}} = \sqrt{2}$

$\to \sqrt{\dfrac{1}{C}} = \sqrt{2} \;\Rightarrow\; C = \dfrac{1}{2}$

Il risultato è giusto ma il libro porta un altro risultato

$y = \sqrt{\dfrac{1}{\cos(2x) + \frac{1}{2}}} = \sqrt{\dfrac{1}{\cos(2x)} + 2}$

Intervallo: $\cos(2x) + \frac{1}{2} \neq 0 \to \cos(2x) = -\frac{1}{2} \to 2x = \dfrac{\arccos(-\frac{1}{2})}{2}$

$\wedge \;\dfrac{1}{\cos(2x)} + 2 > 0 \to \dfrac{1}{\cos(2x)} > -2 \to \cos(2x) > -\frac{1}{2} \to x > \dfrac{\arccos(-\frac{1}{2})}{2} + 2k\pi$

1.2.★

a. Scrivere l'integrale generale dell'equazione

$$y' + y\tan x = 3\sin 2x$$

$\underbrace{y'}_{y'(x)} + \underbrace{y \tan x}_{a(x)y} = \underbrace{3\sin(2x)}_{f(x)}$

$\to y' = -y\tan x + 3\sin(2x) \;\Rightarrow\; a(x) = -\tan x \;\Rightarrow\; A(x) = -\displaystyle\int \tan x = -\int \frac{\sin x}{\cos x} dx$

$\to A(x) = -\ln|\cos x|$

Guarda pg succ !

$y = e^{\ln(\cos x)} \left(C + \displaystyle\int \frac{6 \sin x \cos x}{\cos x} dx\right) = \cos x \left(C + 6\int \sin x\, dx\right) = \cos x\,(C - 6\cos x)$

$\to y = C\cos x - 6\cos^2 x$

può supporre $\cos x > 0$).

b. Risolvere il problema di Cauchy:

$$\begin{cases} y' + y\tan x = 3\sin 2x \\ y(0) = 1 \end{cases}$$

$y = C\cos x - 6\cos^2 x$

$y(0) = C - 6\cos^2 x = 1 \to C = 1 + 6\cos^2 x$

$\to C = 1 + 6 = 7$

$\Rightarrow y = 7\cos x - 6\cos^2 x$

1.3.

a. Determinare tutte le soluzioni dell'equazione

$$xy' + \frac{y-1}{x} = 0.$$

$xy' + \frac{y-1}{x} = 0 \to xy' = -\frac{y-1}{x}$

$\to \frac{dy}{dx}x = -\frac{y-1}{x} \to \frac{dy}{y-1} = -\frac{1}{x^2}dx$

$\Rightarrow \int \frac{1}{y-1}dy = -\int \frac{1}{x^2}dx \to \ln|y-1| = -\int x^{-2}dx \to \ln|y-1| = +\frac{1}{x} + c$

$\Rightarrow y-1 = ce^{\frac{1}{x}} \to y_0 = ce^{1/x}+1$

1.2.★

a. Scrivere l'integrale generale dell'equazione

$$y' + y\tan x = 3\sin 2x$$

$\underbrace{y'}_{y'} + \underbrace{y\tan x}_{a(x)y} = \underbrace{3\sin 2x}_{f(x)}$

$\to A(x) = \int a(x)\,dx = \int \tan x \to -\int \frac{-\sin x}{\cos x}dx \quad D[\cos x] = -\sin x \Rightarrow \int \tan x = -\ln|\cos x| + c$

$\Rightarrow y = ce^{-\int a(x)} = ce^{\ln|\cos x|} = c\cos x \to y'(x) = c'\cos x - c\sin x$

$\to c'\cos x - \cancel{c\sin x} + \cancel{c\cos x \cdot \frac{\sin x}{\cos x}} = 3\sin(2x) \to c' = \frac{6\sin x \cancel{\cos x}}{\cancel{\cos x}}$

$\Rightarrow c = 6\int \sin x = -6\cos x$

$\Rightarrow y_p(x) = -6\cos^2 x \Rightarrow y(x)\ c\cos x - 6\cos^2 x$

b. Risolvere il problema di Cauchy:

$$\begin{cases} xy' + \frac{y-1}{x} = 0 \\ y(-1) = 2 \end{cases}$$

$xy' + \frac{y-1}{x} = 0 \to y_0 = ce^{1/x}+1$

$\Rightarrow y(-1) = ce^{-1}+1 = 2 \to c = e$

$\to y = e^{2/x}+1$

Insieme di def $\quad \frac{2}{x} \neq 0 \to x \neq 0$

$\Rightarrow$ Def $(-\infty, 0)$, Non è def in $(0, +\infty)$ perché per $x \in (0,+\infty)$ $y(-1)=2$ non si verifica.

1.4.

a. Scrivere l'integrale generale dell'equazione

$$y' + 3y = 2x.$$

$\underbrace{y'}_{y'} + \underbrace{3y}_{b(x)y} = \underbrace{2x}_{f(x)} \Rightarrow A(x) = \int a(x)\,dx = 3\int dx$

$\Rightarrow A(x) = 3x + c$

$\Rightarrow y = ce^{-\int a(x)\,dx} = ce^{-3x} \to y' = c'e^{-3x} - c3e^{-3x}$

$\to c'e^{-3x} \cancel{-c3e^{-3x}} + \cancel{3ce^{-3x}} = 2x \to c' = \frac{2x}{e^{-3x}} = \frac{2x}{\frac{1}{e^{3x}}} = e^{3x}2x$

$c = 2\int e^{3x}x$ parti $\to 2\left[x \cdot \frac{1}{3}e^{3x} - \frac{1}{3}\int e^{3x}dx\right]$

$= 2\left[\frac{1}{3}xe^{3x} - \frac{1}{3}\cdot\frac{1}{3}e^{3x}\right] = 2\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right] = \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} = e^{3x}\left(\frac{2}{3}x - \frac{2}{9}\right)$

$\Rightarrow y_p(x) = \cancel{e^{3x}}\left(\frac{2}{3}x - \frac{2}{9}\right)\cancel{e^{-3x}} = \frac{2}{3}x - \frac{2}{9} \Rightarrow \underline{y = ce^{-3x} + \frac{2}{3}x - \frac{2}{9}}$

b. Risolvere il problema di Cauchy:

$$\begin{cases} y' + 3y = 2x \\ y(0) = \frac{7}{9}. \end{cases}$$

$$y(x) = c e^{-3x} + \frac{2}{3}x - \frac{2}{9}$$

$\to$

$$\Rightarrow y = c - \frac{2}{9} = \frac{7}{9} \to c = \frac{7}{9} + \frac{2}{9}$$

$$\Rightarrow c = 1 \Rightarrow y = e^{-3x} + \frac{2}{3}x - \frac{2}{9}$$

1.5.★

a. Scrivere l'integrale generale dell'equazione

$$y' = xy^2.$$

$$y' = xy^2 \to \frac{y'}{y^2} = x$$

$$\to \frac{dy}{y^2} = x\,dx$$

$$\Rightarrow \int \frac{1}{y^2}\,dy = \int x\,dx \Rightarrow \int y^{-2}\,dy = \frac{x^2}{2} + c \to -\frac{1}{y} = \frac{x^2}{2} + c \to -\frac{1}{y} = \frac{x^2+2c}{2}$$

$$\to y_0 = -\frac{2}{x^2+2c}$$

b. Risolvere il problema di Cauchy:

$$\begin{cases} y' = xy^2 \\ y(0) = 1 \end{cases}$$

$$y_0 = -\frac{2}{x^2+2c} \Rightarrow$$

$$\Rightarrow y(0) = -\frac{2}{2c} = 1 \to -2 = 2c$$

$$\Rightarrow c = -1$$

$$\Rightarrow y = -\frac{2}{x^2-2}$$

Ins. di Def: $x^2 - 2 \neq 0 \to x = \pm\sqrt{2}$

$$\Rightarrow I = (-\sqrt{2}, \sqrt{2})$$

1.6.★

a. Scrivere l'integrale generale dell'equazione

$$y' + 2xy = x\sin(x^2).$$

$$\underbrace{y'}_{y'(x)} + \underbrace{2xy}_{a(x)y} = \underbrace{x\sin(x^2)}_{f(x)}$$

$$\Rightarrow A(x) = \int a(x) = 2\int x\,dx = x^2 + c$$

$$\Rightarrow y_0(x) = c e^{-x^2} \Rightarrow y_p(x) = e^{-x^2} \underbrace{\int x\sin(x^2)\, e^{x^2}\,dx}_{\text{Risolvo}}$$

pongo $t = x^2 \Rightarrow dx = \frac{1}{2x}\,dt$
$\Rightarrow x = \sqrt{t}$

$$\Rightarrow \int \cancel{x} \cdot \sin(x^2)\, e^{x^2} \cdot \frac{1}{2\cancel{x}}\,dx$$

$$\to \frac{1}{2}\int \sin(t)\, e^t\,dt = \text{PARTI} = \frac{1}{2}\left[\sin(t)\, e^t - \int \cos(t)\, e^t\,dt\right] = \frac{1}{2}\left[\sin t\, e^t - \left(\cos t\, e^t + \int \sin t\, e^t\,dt\right)\right]$$

$$\Rightarrow \frac{1}{2}\left[\sin t\, e^t - \cos t\, e^t - \int \sin t\, e^t\,dt\right] = \frac{1}{2}\left[\int \sin t\, e^t\,dt\right]$$

pongo $\int \sin t\, e^t\,dt = A$

$$\Rightarrow \left[\sin t\, e^t - \cos t\, e^t - A\right] = A \Rightarrow 2A = \sin t\, e^t - \cos t\, e^t \to A = \left[\frac{\sin t\, e^t}{2} - \frac{\cos t\, e^t}{2}\right] \cdot \frac{1}{2}$$

$$\Rightarrow \int \sin t\, e^t = \frac{\sin t\, e^t}{4} - \frac{\cos t\, e^t}{4} \xrightarrow{t = x^2} \frac{\sin(x^2)\, e^{x^2}}{4} - \frac{\cos x^2\, e^{x^2}}{4}$$

$$\to \underbrace{A(x) = \frac{1}{4} e^{x^2}(\sin x^2 - \cos x^2)}$$

$$\Rightarrow y_p(x) = \cancel{e^{-x^2}} \cdot \frac{1}{4}\cancel{e^{x^2}}(\sin x^2 - \cos x^2) = \frac{1}{4}\sin x^2 - \frac{1}{4}\cos x^2$$

$$y(x) = y_0(x) + y_p(x) = \frac{1}{4}\sin x^2 - \frac{1}{4}\cos x^2 + c\, e^{-x^2}$$

b. Risolvere il problema di Cauchy:

$$\begin{cases} y' + 2xy = x\sin(x^2) \\ y(0) = \frac{3}{4}. \end{cases}$$

$y = \frac{1}{4}(\sin x^2 - \cos x^2) + ce^{-x^2}$

$y(0) = \frac{1}{4}[(\sin(0) - \cos(0)] + c = \frac{3}{4}$ ($\sin(0) \to 0$, $\cos(0) \to 1$)

$\to -\frac{1}{4} + c = \frac{3}{4} \to c = 1$

1.7.★ Si consideri l'equazione differenziale:

$$y' = \frac{2y + y^2}{x}.$$

~~a.~~ Determinare tutte le soluzioni dell'equazione.
~~b.~~ Risolvere il problema di Cauchy per l'equazione precedente con la condizione iniziale $y(-1) = 2$.
~~c.~~ Precisare qual è il più ampio intervallo su cui la soluzione del problema di Cauchy è definita.

a) $\underset{y'}{y'} = \frac{2y+y^2}{x} = \underset{a(x)}{(2y+y^2)} \cdot \underset{b(x)}{\frac{1}{x}}$

$\Rightarrow \frac{y'}{2y+y^2} = \frac{1}{x}$

$\Rightarrow \frac{dy}{2y+y^2} = \frac{1}{x}dx$

$\to \underset{1)}{\int \frac{1}{2y+y^2}dy} = \int \frac{1}{x}dx$

1) $\frac{1}{2y+y^2} = \frac{1}{y(y+2)} = \frac{A}{y} + \frac{B}{y+2} = \frac{Ay + 2A + By}{y(y+2)} = 1$

$\Rightarrow y(A+B) + 2A = 1 \quad \Rightarrow \begin{cases} A+B=0 \to A=-B \Rightarrow B = -\frac{1}{2} \\ 2A = 1 \Rightarrow A = \frac{1}{2} \end{cases} \quad \Rightarrow \frac{1}{y(y+2)} = \frac{1}{2}\frac{1}{y} - \frac{1}{2}\frac{1}{y+2}$

$\Rightarrow \int \frac{1}{2y+y^2}dy = \frac{1}{2}\int \frac{1}{y}dy - \frac{1}{2}\int \frac{1}{y+2}dy = \frac{1}{2}\ln|y| - \frac{1}{2}\ln|y+2| = \ln\left|\frac{y}{y+2}\right|^{\frac{1}{2}}$

2) $\int \frac{1}{x}dx = \ln|x| \quad \Rightarrow \ln\left|\frac{y}{y+2}\right|^{\frac{1}{2}} = \ln|x| + c \to \left(\frac{y}{y+2}\right)^{\frac{1}{2}} = cx \to \frac{y}{y+2} = cx^2$

$\to y - ycx^2 = 2cx^2 \to y(1-cx^2) = 2cx^2 \to y_0 = \frac{2cx^2}{1-cx^2}$ TUTTE LE SOL.

b) Cond: $y(-1) = 2 \to y(-1) = \frac{2c}{1-c} = 2 \Rightarrow 2c = 2 - 2c \to 2c + 2c = 2$
$\to 4c = 2 \to c = \frac{1}{2}$

$\to$ Sol cauchy: $y(x) = \frac{2\frac{1}{2}x^2}{1-\frac{1}{2}x^2} = \frac{x^2}{\frac{2-x^2}{2}} = \frac{2x^2}{2-x^2}$

$\to$ Insieme di Def: $\underset{\text{soluzione}}{2-x^2} \neq 0 \to x^2 = 2 \to x = \pm\sqrt{2} \quad \to I = (-\sqrt{2}, \sqrt{2})$

$\cap \underset{\text{eq di partenza}}{\frac{2y+y^2}{x}} \neq 0 \to x \neq 0 \Rightarrow I = (-\sqrt{2}, 0) \cup \underbrace{(0, \sqrt{2})}_{2 \notin (0,\sqrt{2})} \Rightarrow I = (-\sqrt{2}, 0)$

Piu' ampio intervallo in cui la soluzione, l'eq di partenza ed i coefficienti di essa sono def

1.8.★

a. Scrivere l'integrale generale dell'equazione

$$y' + (1-2x)y = xe^{x^2}.$$

a) $\underbrace{y'}_{y'} + \underbrace{(1-2x)y}_{a(x)\,y} = \underbrace{xe^{x^2}}_{f(x)}$ eq. diff I ordine

$\Rightarrow A(x) = \int 1-2x\,dx = \int dx - 2\int x\,dx = x - x^2 + c \Rightarrow y_0(x) = ce^{-x+x^2}$

$y(x) = ce^{-x} \cdot e^{x^2} \Rightarrow y'(x) = c'e^{-x}e^{x^2} + c\left[-e^{-x}e^{x^2} + e^{-x}2xe^{x^2}\right] = c'e^{-x+x^2} - ce^{-x+x^2} + ce^{-x+x^2}2x$

$\Rightarrow c'e^{-x+x^2} \cancel{-ce^{-x+x^2}} \cancel{+ce^{-x+x^2}2x} + \cancel{ce^{-x+x^2}} - \cancel{2xce^{-x+x^2}} = xe^{x^2} \Rightarrow c' = \frac{xe^{x^2}}{e^{-x+x^2}} = \frac{x\cancel{e^{x^2}}}{e^{-x}\cdot\cancel{e^{x^2}}}$

$\Rightarrow c' = \frac{x}{\frac{1}{e^x}} = e^x x \Rightarrow c = \int e^x x\,dx = \text{PARTI} = xe^x - \int e^x\,dx = xe^x - e^x = \underline{e^x(x-1)}$

$\Rightarrow y_p(x) = e^x(x-1)e^{-x+x^2} = \cancel{e^x}(x-1)\frac{1}{\cancel{e^x}}e^{x^2} = e^{x^2}(x-1)$

$\Rightarrow y(x) = ce^{-x+x^2} + e^{x^2}(x-1)$

b) Cauchy : $y(0) = 2 \Rightarrow y(0) = c + \overset{0}{\cancel{x}} - 1 = 2 \Rightarrow c = 2+1 = 3$

$\Rightarrow \text{Sol} = \underline{3e^{-x+x^2} + e^{x^2}(x-1)}$

1.9. Si consideri l'equazione differenziale:

$$y' = \sqrt[3]{x}\,y^2.$$

a. Determinare tutte le soluzioni dell'equazione.

b. Risolvere il problema di Cauchy per l'equazione precedente con la condizione iniziale $y(0) = 2$.

c. Precisare qual è il più ampio intervallo su cui la soluzione del problema di Cauchy è definita.

a) $y' = \sqrt[3]{x}\,y^2$ Var sep.

$\Rightarrow \frac{y'}{y^2} = \sqrt[3]{x} \Rightarrow \frac{1}{y^2}dy = x^{\frac{1}{3}}dx$

$\Rightarrow y_0(x) = \int y^{-2}dy = \int x^{\frac{1}{3}}dx \Rightarrow y_0(x) = -\frac{1}{y} = \frac{3x^{4/3}}{4} + c \Rightarrow y = -\frac{1}{\frac{3}{4}x^{4/3} + c} = -\frac{4}{3}\frac{1}{x^{4/3}+c}$

b) $y(0) = 2 \Rightarrow y(0) = -\frac{1}{\frac{3}{4}0 + c} = 2 \Rightarrow c = -\frac{1}{2} \Rightarrow y = -\frac{1}{\frac{3}{4}x^{4/3} - \frac{1}{2}}$

c) $y = -\frac{1}{\frac{3}{4}x^{4/3} - \frac{1}{2}} = \frac{1}{\frac{1}{2} - \frac{3}{4}x^{4/3}} \Rightarrow \frac{1}{2} - \frac{3}{4}x^{4/3} = 0$ per $\frac{\cancel{3}}{\cancel{4}}x^{4/3} = \frac{\cancel{1}\,4}{\cancel{2}\,6} \Rightarrow x^{4/3} = \frac{4}{6} \Rightarrow x^4 = \frac{4^3}{6^3}$

$\Rightarrow x = \frac{4^{3/4}}{6^{3/4}} \Rightarrow \cup\ \sqrt[3]{x}\,y^2 \neq 0\ \forall x \in \mathbb{R} \Rightarrow I = \left[-\left(\frac{4}{6}\right)^{3/4}, \left(\frac{4}{6}\right)^{3/4}\right]

1.10.★ Risolvere il problema di Cauchy:

$$\begin{cases} y' + \dfrac{xy}{x^2-1} = 3x \\ y(0) = 3, \end{cases}$$

precisando l'intervallo più ampio su cui la soluzione è definita.

$$\underset{y'}{y'} + \underset{a(x)y}{\frac{xy}{x^2-1}} = \underset{f(x)}{3x} \qquad \text{Eq lin I°}$$

$$\Rightarrow A(x) = \int \frac{x}{x^2-1}\,dx - D[x^2-1] = 2x \Rightarrow \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \frac{1}{2}\ln|x^2-1| + c$$

$$\Rightarrow y_0(x) = c\,e^{\frac{1}{2}\ln|x^2-1|} = c\sqrt{x^2-1}$$

$$y(x) = e^{-\frac{1}{2}\ln|x^2-1|}\left(c + \int 3x \cdot e^{\frac{1}{2}\ln|x^2-1|}\,dx\right) = \frac{1}{(x^2-1)^{\frac{1}{2}}}\left(c + 3\int x \cdot (x^2-1)^{\frac{1}{2}}\,dx\right)$$

pongo $t = x^2-1 \Rightarrow dx = \frac{1}{2x}\,dt$

$$\Rightarrow y(x) = \frac{1}{(x^2-1)^{\frac{1}{2}}}\left(c + 3\int \cancel{x}\,(t)^{\frac{1}{2}} \cdot \frac{1}{2\cancel{x}}\,dx\right) = \frac{1}{(x^2-1)^{\frac{1}{2}}}\left(c + \frac{\cancel{3}}{\cancel{2}}\,\frac{(x^2-1)^{\frac{3}{2}}}{\cancel{\frac{3}{2}}}\right)$$

$$\Rightarrow y(x) = \frac{c}{(x^2-1)^{\frac{1}{2}}} + \frac{1}{\cancel{(x^2-1)^{\frac{1}{2}}}}\,(x^2+1)^{\cancel{\frac{3}{2}}} = \frac{c}{\sqrt{x^2-1}} + x^2-1$$

1.11.★

a. Determinare tutte le soluzioni dell'equazione differenziale:

$$y' = xe^{-x}(y-1)^2.$$

b. Risolvere quindi il problema di Cauchy:

$$\begin{cases} y' = xe^{-x}(y-1)^2 \\ y(0) = 3 \end{cases}$$

$$y' = xe^{-x}(y-1)^2 \qquad \text{separabili}$$

$$\frac{y'}{(y-1)^2} = xe^{-x} \Rightarrow \frac{1}{(y-1)^2}\,dy = xe^{-x}\,dx$$

$$\int \frac{1}{(y-1)^2}\,dy = \int xe^{-x}\,dx \;\text{–> PARTI –>}\; -\int e^{-x}\cdot 1\,dx = -e^{-x} \Rightarrow -xe^{-x} - \int e^{-x}\,dx = -xe^{-x} + e^{-x} + c$$

$$\frac{1}{e^x}(-x+1) + c \quad (a)$$

$$\Rightarrow = \int (y-1)^{-2}\,dy \qquad \text{pongo } y-1 = t \Rightarrow dx = dt \Rightarrow \int (t)^{-2}\,dt = -\frac{1}{t} = -\frac{1}{y-1}$$

Sbagliato

$$\Rightarrow -\frac{1}{y-1} = -xe^{-x} + e^{-x} + c \Rightarrow \frac{1}{y-1} = xe^{-x} - e^{-x} - c \Rightarrow y - \cancel{1} = \frac{1}{xe^{-x} - e^{-x} - c} + 1$$

uso la forma (a)

$$\Rightarrow y(0) = \quad -\frac{1}{y-1} = e^{-x}(1+x) + c \Rightarrow \frac{1}{y-1} = -e^{-x}(1+x) - c \Rightarrow y - 1 = \frac{1}{e^{-x}(1+x) - c}$$

$$\Rightarrow y(0) = \frac{1}{1-c} + 1 = 3 \Rightarrow \frac{1}{1-c} = 2 \Rightarrow 1 - c = \frac{1}{2} \Rightarrow c = -\frac{1}{2}$$

$$\Rightarrow \text{SOL} = \frac{1}{e^{-x}(1+x) - \frac{1}{2}} + 1$$

1.12.

a. Scrivere l'integrale generale dell'equazione

$$y' + (1-x)y = xe^{-x}.$$

b. Risolvere il problema di Cauchy:

$$\begin{cases} y' + (1-x)y = xe^{-x} \\ y(1) = 0. \end{cases}$$

$\underset{y}{y'} + (\underset{a(x)}{1-x})y = \underset{f(x)}{xe^{-x}}$ Lineare I ordine

$\to A(x) = \int 1 - x \, dx = x - \frac{x^2}{2}$

$\Rightarrow y_P(x) = e^{-x+\frac{x^2}{2}}\left(C + \underbrace{\int x e^{-x} \cdot e^{x-\frac{x^2}{2}} dx}\right)$

$\Rightarrow \int x \cdot \cancel{\frac{1}{e^x}} \cdot \cancel{e^x} \cdot \frac{1}{e^{\frac{x^2}{2}}} dx = \int x \cdot e^{-\frac{x^2}{2}}$ $D\left[e^{-\frac{x^2}{2}}\right] = -2x$ $\Rightarrow -\int - x e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}}$

$\Rightarrow y_P(x) = \frac{c\, e^{\frac{x^2}{2}}}{e^x} - \frac{\cancel{e^{\frac{x^2}{2}}}}{e^x} \cdot \cancel{e^{-\frac{x^2}{2}}} = ce^{-x+\frac{x^2}{2}} - e^{-x}$

Cauchy: $y(1) = \frac{c\, e^{\frac{1}{2}}}{e} - \frac{1}{e} = 0 \to \frac{c\, e^{\frac{1}{2}}}{\cancel{e}} = \frac{1}{\cancel{e}} \to c = \frac{1}{e^{\frac{1}{2}}}$

$\Rightarrow y(x) = \frac{1}{e^{\frac{1}{2}}} e^{-x+\frac{x^2}{2}} - e^{-x}$

1.13.★ Risolvere il problema di Cauchy

$$\begin{cases} y' = \frac{xy^3}{\sqrt{x^2-1}} \\ y\left(\sqrt{2}\right) = 2 \end{cases}$$

$y' = \frac{xy^3}{\sqrt{x^2-1}}$ Var sep.

$\frac{y'}{y^3} = \frac{x}{\sqrt{x^2-1}} \to \frac{1}{y^3} dy = \frac{x}{\sqrt{x^2-1}} dx$

$\Rightarrow \int \frac{1}{y^3} dy = \underbrace{\int \frac{x}{\sqrt{x^2-1}} dx}$ $\to$ Pongo $t = x^2 - 1 \Rightarrow x = \pm (t+1)^{\frac{1}{2}} \Rightarrow dx = \frac{1}{2} \frac{1}{\sqrt{t+1}} dt$, $x = (t+1)^{\frac{1}{2}}$

$\Rightarrow \frac{1}{2}\int \frac{\cancel{(t+1)^{\frac{1}{2}}}}{\sqrt{t}} \cdot \frac{1}{\cancel{\sqrt{t+1}}} = \cancel{\frac{1}{2}} \cdot \cancel{2}\sqrt{t} = \sqrt{x^2+1}$

$\Rightarrow -\frac{1}{2y^2} = \sqrt{x^2-1} + c \to \frac{1}{y^2} = -2\sqrt{x^2-1} + 2c \equiv c_1 \Rightarrow y^2 = \frac{1}{-2\sqrt{x^2-1} + c_1}$

Se $y(\sqrt{2}) = 2 \Rightarrow y^2(\sqrt{2}) = 4 = \frac{1}{-2\sqrt{2-1} + c_1} = \frac{1}{-2 + c_1} \Rightarrow -2 + c_1 = \frac{1}{4} \to c_1 = \frac{1}{4} + 2 = \frac{1+8}{4} = \frac{9}{4}$

$\Rightarrow$ Soluz $=$ $y^2 = \frac{1}{-2\sqrt{x^2-1} + \frac{9}{4}}$

- Eq diff del secondo ordine

$\to$ Eq a coefficienti costanti

Esempio 1.8.

a. Scrivere l'integrale generale dell'equazione

$$y'' + 2y' + 4y = 0.$$

b. Risolvere il problema di Cauchy:

$$\begin{cases} y'' + 2y' + 4y = 0 \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$y''+2y'+4y=0$ Metodo "vecchio"

$\to \lambda^2+2\lambda+4=0 \to \Delta = 4-4\cdot 4 = -12 < 0$

$\Rightarrow \lambda_{1/2} = \dfrac{-2 \pm i\sqrt{12}}{2}$ $\dfrac{-2+i2\sqrt{3}}{2} = -1+i\sqrt{3}$, $\dfrac{-2-i2\sqrt{3}}{2} = -1-i\sqrt{3}$

$\Rightarrow y_0(x) = e^{-x}\left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right]$

b) $\begin{cases} y''+2y'+4y=0 \\ y(0)=2 \\ y'(0)=1 \end{cases}$ $y'(x) = -e^{-x}\left[c_1\cos(x\sqrt{3}) + c_2\sin(x\sqrt{3})\right] + e^{-x}\left[c_1'\cos(x\sqrt{3}) - c_1\sin(x\sqrt{3})\sqrt{3} + c_2'\sin(x\sqrt{3}) + c_2\cos(x\sqrt{3})\sqrt{3}\right]$

$\Rightarrow y(0) = \cancel{e^{-x}}^{\to 1}\left[c_1\cancel{\cos(\sqrt{3}x)}^{\to 1} + c_2\cancel{\sin(\sqrt{3}x)}^{\to 0}\right] \Rightarrow c_1 = 2 \Rightarrow c_1' = 2\int dx = 2x$

$\Rightarrow y'(0) = -1\left[c_1\right] + \left[c_1' + c_2' + c_2\sqrt{3}\right] = 1 = -c_1 + c_1' + c_2' + c_2\sqrt{3} = 1$

$\to -2 + \cancel{2x}^{\to 0} + c_2' + c_2\sqrt{3} = 1 \Rightarrow c_2' + c_2\sqrt{3} = 3$

Esempio 1.9.

a. Scrivere l'integrale generale dell'equazione

$$y'' + 6y' + 9y = 0.$$

b. Risolvere il problema di Cauchy:

$$\begin{cases} y'' + 6y' + 9y = 0 \\ y(0) = 2 \\ y'(0) = 1. \end{cases}$$

$\lambda^2+6\lambda+9=0 \to \Delta = 36-4\cdot 9 = 0 \Rightarrow$

$\lambda = \dfrac{-6}{2} = -3 \Rightarrow \underbrace{y_0(x) = c_1 e^{-3x} + c_2 x e^{-3x}}$

$y'(x) = c_1' e^{-3x} - c_1 3 e^{-3x} + c_2' x e^{-3x} + c_2\left[e^{-3x} - x3e^{-3x}\right] = e^{-3x}\left[c_1' - 3c_1 + c_2' + c_2 - 3x\right]$

$y(0) = \cancel{e^{-3x}}^{\to 1}(c_1 + c_2\cancel{x}^{\to 0}) = 2 \Rightarrow c_1 = 2$

$y'(0) = \cancel{e^{-3x}}^{\to 1}\left[c_1' - 3c_1 + c_2' + c_2 - \cancel{3x}^{\to 0}\right] = 1 \Rightarrow -3c_1 + c_2' + c_2 = 1 \xrightarrow{c_1=2} -6 + c_2' + c_2 = 1$

$\Rightarrow c_2' + c_2 = 7 \to y' + y = 7 \to y' = 7 - y \to \dfrac{y'}{7-y} = 1$

$\to \dfrac{dy}{7-y} = dx \to \int \dfrac{1}{7-y}\,dy = \int dx \to -\ln|7-y| = x + c$

$\Rightarrow \dfrac{1}{7-y} = ce^{x} \to 7 - y = \dfrac{1}{ce^x} \to y = -\dfrac{1}{ce^x} + 7$ c_2

$\Rightarrow$ Sol $\Rightarrow \underline{y(x) = 2e^{-3x} + 7xe^{-3x}}$

1.61. $\begin{cases} y'' - 2y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 4 \end{cases}$

$\lambda^2 - 2\lambda + 5 = 0 \Rightarrow \Delta = 4 - 20 = -16$

$\Rightarrow \lambda_{1,2} = \dfrac{2 \pm 4i}{2} \begin{cases} 1+2i \\ 1-2i \end{cases}$

$\Rightarrow y_0(x) = e^x\left[c_1 \cos(2x) + c_2 \sin(2x)\right]$

$\Rightarrow y(0) = c_1 = 1$, $y'(x) = e^x\left[c_1\cos(2x) + c_2 \sin(2x)\right] + e^x\left[c_1'\cos(2x) - c_1 2\sin(2x) + \right.$
$\left. + c_2' \sin(2x) + c_2 2\cos(2x)\right]$

$c_1' = \int dx = x$

$\Rightarrow y'(0) = c_1 + c_1' + c_2' + 2c_2 = 4 \xrightarrow[c_1=1]{c_1'=x} 1 + x + c_2' + 2c_2 = 4 \to c_2' + 2c_2 = 3$

$c_2' + 2c_2 = 3 \to y' + 2y = 3 \Rightarrow \dfrac{dy}{dx} = 3 - 2y \to \dfrac{dy}{3-2y} = dx$

$= -\frac{1}{2}\int \frac{-2\, dy}{3-2y} dx = \int dx \to -\frac{1}{2} \ln|3-2y| = x \to (3-2y)^{-\frac{1}{2}} = e^x \Rightarrow 3 - 2y = \frac{1}{e^{2x}}$

$\to y = \left(-\frac{1}{e^{2x}} + 3\right)\frac{1}{2} = \overset{c}{\frac{3}{2}} - \frac{1}{e^{2x}} \Rightarrow c_2 = \frac{3}{2}$

$\Rightarrow$ Soluzione $= y(x) = e^x\left[\cos(2x) + \frac{3}{2}\sin(2x)\right]$ ✓

1.62. $\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 1 \\ y'(0) = 4 \end{cases}$

$\lambda^2 + 2\lambda + 1 = 0 \quad \Delta = 4 - 4 = 0$

$\Rightarrow \lambda = \frac{-2}{2} = -1$

$\Rightarrow y_0(x) = c_1 e^{-x} + c_2 x e^{-x}$

$y'(x) = c_1' e^{-x} - c_1 e^{-x} + c_2' x e^{-x} + c_2\left[e^{-x} - x e^{-x}\right]$

$y(0) = \underset{1}{\cancel{e^{-x}}}(c_1 + \overset{0}{\cancel{c_2 x}}) = 1 \to c_1 = 1$

$y'(0) = \overset{1}{\cancel{e^{-x}}}(c_1' - c_1 + \overset{0}{\cancel{c_2' x}} + c_2 - \overset{0}{\cancel{c_2 x}})$

$\to y'(0) = c_1' - c_1 + c_2 = 4 \xrightarrow{c_1=1} c_1' + c_2 = 5 \Rightarrow c_1' = \int dx = x$

$\Rightarrow c_2 = 5 \to$ SOL $y(x) = e^{-x} + 5xe^{-x}$

1.63.

$$\begin{cases} 2y'' - y' - y = 0 \\ y(0) = 3 \\ y'(0) = 0 \end{cases}$$

$2\lambda^2 - \lambda - 1 = 0 \Rightarrow 1 - 4\cdot 2\cdot(-1) = 9 > 0$

$\lambda_{1,2} = \frac{1 \pm 3}{4} \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$

$\Rightarrow y_0(x) = c_1 e^{x} + c_2 e^{-\frac{1}{2}x} \rightarrow y'(x) = c_1' e^{x} + c_1 e^{x} + c_2' e^{-\frac{1}{2}x} - c_2 \frac{1}{2} e^{-\frac{1}{2}x} =$

$y(0) = c_1 + 1 = 3 \rightarrow c_1 = 2$

$y(0) = \cancel{e^{x}}(c_1' + c_1) + \cancel{e^{-\frac{1}{2}x}}(c_2' - \frac{1}{2}c_2) = 0 \rightarrow$ $c_1 = 2$, $c_1' = 2x$ $\cancel{2x} + 2 + c_2' - \frac{1}{2}c_2 = 0$

$\Rightarrow c_2' - \frac{1}{2}c_2 = -2 \Rightarrow y' = -2 + \frac{1}{2}y = \frac{-4 + y}{2} \Rightarrow \frac{dy}{\frac{-4+y}{2}} = dx$

$2\int \frac{1}{y-4}\,dy = \int dx \rightarrow 2\ln|y-4| = x \rightarrow (y-4)^2 = ce^{x} \rightarrow y - 4 = ce^{\frac{1}{2}x}$

$\rightarrow y = ce^{\frac{1}{2}x} + 4 \; c_2 \Rightarrow$ Sol $2e^{x} + 4e^{-\frac{1}{2}x}$

1.64.

$$\begin{cases} y'' + 2y' + 4y = 0 \\ y(0) = \sqrt{3} \\ y'(0) = 0 \end{cases}$$

$\lambda^2 + 2\lambda + 4 = 0 \rightarrow 4 - 4\cdot 4 = 4 - 16 = -12 < 0$

$\Rightarrow \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{3}}{2} \begin{cases} -1 + i\sqrt{3} \\ -1 - i\sqrt{3} \end{cases}$

$\Rightarrow y_0(x) = e^{-x}\left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right]$

$y'(x) = -e^{-x}\left[c_1 \cos(x\sqrt{3}) + c_2 \sin(x\sqrt{3})\right] + e^{-x}\left[c_1' \cos(x\sqrt{3}) - c_1 \sin(x\sqrt{3})\sqrt{3} + c_2' \sin(\sqrt{3}x) + c_2 \cos(\sqrt{3}x)\sqrt{3}\right]$

$\Rightarrow y(0) = \cancel{e^{-x}}\left[c_1 \cancel{\cos(\sqrt{3}x)} + c_2 \cancel{\sin(\sqrt{3}x)}\right] = \sqrt{3} \rightarrow \boxed{c_1 = \sqrt{3}}$

$y'(0) = \cancel{e^{-x}}\left[-c_1 \cancel{\cos(\sqrt{3}x)} - \cancel{c_2 \sin(\sqrt{3}x)} + c_1' \cancel{\cos(\sqrt{3}x)} - \cancel{c_1 \sin(\sqrt{3}x)\sqrt{3}} + \cancel{c_2' \sin(\sqrt{3}x)} + c_2 \cancel{\cos(\sqrt{3}x)}\sqrt{3}\right]$

$\Rightarrow y'(0) = -c_1 + c_1' + \sqrt{3}c_2 = 0 \quad \xrightarrow{c_1 = \sqrt{3} \Rightarrow c_1' = x\sqrt{3}} \quad -\sqrt{3} + \cancel{x\sqrt{3}} + \sqrt{3}c_2 = 0 \rightarrow c_2 = \frac{\sqrt{3}}{\sqrt{3}} = 1$

$\Rightarrow y(x) = e^{-x}\left[\sqrt{3}\cos(\sqrt{3}x) + \sin(\sqrt{3}x)\right]$

1.65. $\begin{cases} y'' - 6y' + 9y = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$ $\lambda^2 - 6\lambda + 9 = 0 \to \Delta = 36-4\cdot 9 = 0$

$\lambda = \frac{6}{2} = 3 \Rightarrow y_0(x) = c_1 e^{3x} + c_2 x e^{3x}$

$\to y'(x) = c_1' e^{3x} + c_1 3e^{3x} + c_2' e^{3x} x + c_2 (e^{3x} + 3x e^{3x})$

$y(0) = \cancel{e^{3x}}_{1}(c_1 + c_2 \cancel{x}_{0}) = 1 \Rightarrow c_1 = 1$ $\quad y'(0) = \cancel{e^{3x}}_{1}(c_1' + 3c_1 + c_2' \cancel{x}_{0} + c_2 + 3x\cancel{c_2}_{0}) = 2$

$\to c_1' + 3c_1 + c_2 = 2 \xrightarrow{c_1 = 1 \Rightarrow c_1' = x} \cancel{x}_{0} + 3 + c_2 = 2 \Rightarrow c_2 = -1$

$\Rightarrow y(x) = e^{3x} - x e^{x}$

1.66. $\begin{cases} 3y'' - 2y' - y = 0 \\ y(0) = 4 \\ y'(0) = 0 \end{cases}$ $3\lambda^2 - 2\lambda - 1 = 0 \to \Delta = 4 - 4\cdot 3\cdot(-1) = 16 > 0$

$\frac{2 \pm 4}{6} \begin{cases} 1 \\ -\frac{1}{3} \end{cases}$

$\Rightarrow y_0(x) = c_1 e^{x} + c\, e^{-\frac{1}{3}x} \Rightarrow y'(x) = c_1' e^{x} + c_1 e^{x} + c_2' e^{-\frac{1}{3}x} - \frac{1}{3} c_2 e^{-\frac{1}{3}x} \to y'(x) = e^{x}(c_1' + c_1) + e^{-\frac{1}{3}x}(c_2' - \frac{1}{3}c_2)$

$y(0) = c_1 e^{x} + c_2 e^{-\frac{1}{3}x} = 4 \to c_1 + c_2 = 4 \to c_1 = 4 - c_2, \; c_2 = 4 - c_1$

$c_1 = 4 - c_2 \Rightarrow c_1' = 4\int dx - c_2 \int dx = 4x - c_2 x$

$y'(0) = \cancel{e^{x}}_{1}(c_1' + c_1) + \cancel{e^{-\frac{1}{3}x}}_{1}(c_2' - \frac{1}{3}c_2) = 0 \to c_1' + c_1 + c_2' - \frac{1}{3}c_2 = 0 \to 4\cancel{x}_{0} - c_2\cancel{x}_{0} + 4 - c_2 + c_2' - \frac{1}{3}c_2 = 0$

$\Rightarrow 4 - c_2 + c_2' - \frac{1}{3}c_2 = 0 \to \frac{-3c_2 - c_2}{3} \Rightarrow c_2' - \frac{4}{3}c_2 = -4 \to c_2' = \frac{4}{3}c_2 - 4$

$\Rightarrow y' = \frac{4}{3}y - 4 \to \int \frac{1}{\frac{4}{3}y - 4}\, dy = \int dx \to \frac{4}{3}\ln|y - 4| = x + c \to (y-4)^{\frac{4}{3}} = ce^{x}$

$\to y - 4 = (ce^{x})^{\frac{3}{4}} \to y = (ce^{x})^{\frac{3}{4}} + 4$ c_2

$\Rightarrow c_1 = 4 - 4 = 0$

$\Rightarrow y(x) = 4e^{-\frac{1}{3}x}$

1.67.

$$\begin{cases} y'' + 2y' - 3y = 0 \\ y(0) = 0 \\ y'(0) = 2e \end{cases}$$

$\lambda^2 + 2\lambda - 3 = 0 \to 4 - 4\cdot(-3) = 16 > 0$

$\to \lambda_{1,2} = \dfrac{-2 \pm 4}{2} \begin{cases} 1 \\ -3 \end{cases}$

$\Rightarrow \underbrace{y_0(x) = c_1 e^{x} + c_2 e^{-3x}} \to y'(x) = c_1' e^{x} + c_1 e^{x} + c_2' e^{-3x} - 3c_2 e^{-3x} = e^{x}(c_1' + c_1) + e^{-3x}(c_2' - 3c_2)$

$y(0) = c_1 \cancel{e^{x}}^{\to 1} + c_2 \cancel{e^{-3x}}^{\to 1} = 0 \to c_1 = -c_2 \ , \ c_2 = -c_1$

$y'(0) = \cancel{e^{x}}_{\to 1}(c_1' + c_1) + \cancel{e^{-3x}}_{\to 1}(c_2' - 3c_2) = 2e \to c_1' + c_1 + c_2' - 3c_2 = 2e$

$c_1 = -c_2 \Rightarrow c_1' = -\int c_2 = -c_2 x \Rightarrow \cancel{-c_2 x}_{\to 0} - c_2 + c_2' - 3c_2 = 2e$

$\Rightarrow c_2' - 4c_2 = 2e \to y' - 4y = 2e \to y' = 2e - 4y \to \dfrac{dy}{2e-4y} = dx$

$\to \int \dfrac{1}{2e-4y}\, dy = \int dx = -\dfrac{1}{4} \int \dfrac{-2}{e-2y}\, dy = x + C \to -\dfrac{1}{4} \ln|e-2y| = x + C$

$\to (e-2y)^{-\frac{1}{4}} = ce^{x} \to \dfrac{1}{e-2y} = (ce^{x})^4 \to e - 2y = \dfrac{1}{ce^{4x}} \to y = -\dfrac{1}{2}\dfrac{1}{ce^{4x}} - \dfrac{e}{2}$

$\Rightarrow y = \dfrac{-1 - ce^{6x}}{2ce^{4x}} = -\dfrac{1}{2ce^{4x}} - \dfrac{\cancel{c}e^{\cancel{6x}\,2}}{2\cancel{c}\cancel{e^{4x}}} \to -\dfrac{1}{2ce^{4x}} - \dfrac{1}{2}e^{2x}$ c_2

$\Rightarrow c_1 = -c_2 \Rightarrow c_1 = \dfrac{1}{2}e^{2x}$

$\Rightarrow y(x) = \dfrac{1}{2}e^{3x} - \dfrac{1}{2}e^{-x}$

Equazioni non omogenee — metodo di somiglianza

Esempio 1.10. Scrivere l'integrale generale dell'equazione:

$$y'' + 6y' + 12y = 4\sin 2x.$$

1) Integrale generale : $y''+6y'+12y = 4\sin(2x) \Rightarrow \lambda^2+6\lambda+12=0 \quad \Delta = 36-4\cdot(12) = -12<0$

$\Rightarrow \lambda_{1,2} = \dfrac{-6 \pm i2\sqrt{3}}{2} \begin{cases} -3+i\sqrt{3} \\ -3-i\sqrt{3} \end{cases} \Rightarrow y_0(x) = e^{-3x}\left(c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right)$

2) Soluzione Particolare

$f(x) = 4\sin(2x) \Rightarrow$ Il termine noto $f(x)$ è del tipo $A\sin(\omega x)$

$\Rightarrow$ la soluzione sarà del tipo: $y(x) = A\cos(2x) + B\sin(2x)$ ⓐ

2.1) Calcoliamo le derivate della ⓐ

$y'(x) = -2A\sin(2x) + 2B\cos(2x) \qquad y''(x) = -4A\cos(2x) - 4B\sin(2x)$

2.2) Sostituiamo nella eq iniziale : $y''+6y'+12y = 4\sin(2x)$

$\Rightarrow -4A\cos(2x) - 4B\sin(2x) - 12A\sin(2x) + 12B\cos(2x) + 12A\cos(2x) + 12B\sin(2x) = 4\sin(2x)$

$\Rightarrow \sin(2x)\left[-4B-12A+12B\right] + \cos(2x)\left[-4A+12B+12A\right] = 4\sin(2x)$

$\begin{cases} -4B-12A+12B = 4 \Rightarrow +\frac{8}{3}A - 12A - 8A = 4 \Rightarrow \frac{8A-36A-24A}{3} = 4 \Rightarrow -\frac{52}{3}A = 4 \Rightarrow A = -\frac{3}{13} \\ -4A+12B+12A = \emptyset \Rightarrow B = -\frac{2}{3}A \Rightarrow B = -\frac{2}{\cancel{3}}\cdot\left(-\frac{\cancel{3}}{13}\right) = \frac{2}{13} \end{cases}$

2.3) Soluz. Particolare : Sostituisco A e B nella ⓐ

$y_p(x) = -\frac{3}{13}\cos(2x) + \frac{2}{13}\sin(2x)$

3) Integrale generale : ottento sommando $y_p(x)$ e $y_0(x)$

$\Rightarrow y(x) = e^{-3x}\left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right] - \frac{3}{13}\cos(2x) + \frac{2}{13}\sin(2x)$

Esempio 1.11. Determinare l'integrale generale dell'equazione

$$y'' - 3y' + 2y = 3xe^{-x}.$$

1) $\lambda^2 - 3\lambda + 2 = 0 \to \Delta = 9 - 4 \cdot 2 = 1 > 0$

$\to \lambda_{1,2} = \frac{3 \pm 1}{2} \begin{cases} 2 \\ 1 \end{cases}$

$\Rightarrow y_0(x) = c_1 e^{2x} + c_2 e^{x}$

2) $f(x) = 3xe^{-x} \Rightarrow$ del tipo $\underbrace{P^{(n)}(x)}_{\text{grado } 1} \cdot e^{-x} \Rightarrow$ Soluzione : $\underline{y(x) = (ax+b) \cdot e^{-x}}$

$\to y(x) = axe^{-x} + be^{-x}$

$\Rightarrow y'(x) = ae^{-x} - axe^{-x} - be^{-x}$

$y''(x) = -ae^{-x} - ae^{-x} + axe^{-x} + be^{-x}$

$\Rightarrow -ae^{-x} - ae^{-x} + axe^{-x} + be^{-x} - 3ae^{-x} + 3axe^{-x} + 3be^{-x} + 2axe^{-x} + 2be^{-x} = 3xe^{-x}$

$\to xe^{-x}(a + 3a + 2a) + e^{-x}(-2a + b - 3a + 3b + 2b) = 3xe^{-x}$

$\begin{cases} a + 3a + 2a = 3 \to 6a = 3 \to a = \frac{1}{2} \\ -2a + b - 3a + 3b + 2b = 0 \to -1 - \frac{3}{2} + 6b = 0 \to b = \frac{5}{12} \end{cases}$

$\to$ Sostituisco

$y_p(x) = \left(\frac{1}{2}x + \frac{5}{12}\right) e^{-x}$

$\Rightarrow y(x) = c_1 e^{2x} + c_2 e^{x} + \frac{1}{2}xe^{-x} + \frac{5}{12}e^{-x}$

Esempio 1.12. Determinare una soluzione particolare dell'equazione

$$y'' - 2y' + 4y = e^{-x}\cos x.$$

$y'' - 2y' + 4y = e^{-x}\cos x \to \lambda^2 - 2\lambda + 4 = 0$

$\Delta = 4 - 4 \cdot 4 = -12 < 0 \Rightarrow \lambda_{1,2} = \frac{+2 \pm 2i\sqrt{3}}{2} \begin{cases} 1 + i\sqrt{3} \\ 1 - i\sqrt{3} \end{cases} \Rightarrow y_0(x) = e^{x}\left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right]$

$f(x) = e^{-x}\cos x = \underbrace{P^{(n)}(x)}_{\text{goniometrico}} \cdot e^{\gamma x}$ $\gamma = -1$ $\Rightarrow$ Sol : $e^{-x} \cdot [a\cos(x) + b\sin(x)] = y(x)$

$\to y(x) = ae^{-x}\cos(x) + be^{-x}\sin(x)$

$\Rightarrow y'(x) = -ae^{-x}\cos x - ae^{-x}\sin(x) - be^{-x}\sin(x) + be^{-x}\cos x$

$y''(x) = ae^{-x}\cos x + ae^{-x}\sin x + ae^{-x}\sin x - ae^{-x}\cos x + be^{-x}\sin x - be^{-x}\cos x - be^{-x}\cos x + be^{-x}\sin x$

$e^{-x}\cos x(a - a - b - b) + e^{-x}\sin x(a + a + b + b) + 2ae^{-x}\cos x + 2ae^{-x}\sin x + 2be^{-x}\sin x - 2be^{-x}\cos x +$
$+ 4ae^{-x}\cos x + 4be^{-x}\sin x = e^{-x}\cos x$

$\to e^{-x}\cos x(-2b + 2a - 2b + 4a) + e^{-x}\sin x(2a + 2b + 2a + 4b) = e^{-x}\cos x$

$\Rightarrow \begin{cases} -2b + 2a - 2b + 4a = 1 \Rightarrow 6a - 4b = 1 \Rightarrow a = \frac{4b+1}{6} \Rightarrow \\ 2a + 2b + 2a + 4b = 0 \Rightarrow 4a + 6b = 0 \end{cases}$

$\Rightarrow \frac{16b + 4}{6} + 6b = 0 \to \frac{16b + 36b}{6} = -\frac{4}{6}$

$\to \frac{52}{\cancel{6}}b = -\frac{4}{\cancel{6}} \to b = -\frac{1}{13}$

$a = \frac{-4 \cdot \frac{1}{13} + 1}{6} = \frac{\frac{-4 + 13}{13}}{6} = \frac{3}{26}$

SOL $y = e^{-x}\left[\frac{3}{26}\cos(x) - \frac{1}{13}\sin(x)\right]$

Esempio 1.13. Risolvere il problema di Cauchy:

$$\begin{cases} y''(x) + 3y'(x) = 2x \\ y(0) = 0 \\ y'(0) = \frac{1}{3}. \end{cases}$$

$y'' + 3y' = 2x \rightarrow \lambda^2 + 3\lambda = 0$

$\Rightarrow \lambda(\lambda+3) = 0$
$\hookrightarrow \lambda = 0$
$\hookrightarrow \lambda = -3$

$\Rightarrow y_0(x) = c_1 + c_2 e^{-3x}$

$f(x) = 2x \Rightarrow \gamma = 0$, mul $1 \Rightarrow$ Sol: $y(x) = x(ax+b) = ax^2 + bx$

$\rightarrow y'(x) = 2ax + b$, $y''(x) = 2a$

$\Rightarrow 2a + 6ax + 3b = 2x \Rightarrow x(6a) + 2a + 3b = 2x$

$\Rightarrow \begin{cases} 6a = 2 \rightarrow a = \frac{1}{3} \\ 2a + 3b = 0 \Rightarrow b = -\frac{2}{9} \end{cases} \rightarrow y_P(x) = \frac{1}{3}x^2 - \frac{2}{9}x$

$\Rightarrow y(x) = c_1 + c_2 e^{-3x} + \frac{1}{3}x^2 - \frac{2}{9}x$

Cauchy $y(0) = c_1 + c_2 \underbrace{\cancel{e^{-3x}}}_{1} + \underbrace{\cancel{\frac{1}{3}x^2}}_{0} - \underbrace{\cancel{\frac{2}{9}x}}_{0} = 0 \rightarrow c_1 + c_2 = 0 \rightarrow c_1 = -c_2$, $c_2 = -c_1$

$y'(x) = c_1' + c_2' \underbrace{\cancel{e^{-3x}}}_{1} - 3c_2 \underbrace{\cancel{e^{-3x}}}_{1} + \underbrace{\cancel{\frac{2}{3}x}}_{0} 2 - \frac{2}{9} \Rightarrow y'(0) = c_1' + c_2' - 3c_2 = \frac{1}{3} + \frac{2}{9}$

$\xrightarrow[c_1' = -c_2 x]{c_1 = -c_2} -\underbrace{\cancel{c_2 x}}_{0} + c_2' - 3c_2 = \frac{5}{9} \Rightarrow y' - 3y = \frac{5}{9} \rightarrow y' = \frac{5}{9} + 3y$

$\Rightarrow \frac{1}{3}\int \frac{1 \cdot 3}{\frac{5}{9} + 3y}\,dy = \int dx \rightarrow \frac{1}{3}\ln\left|\frac{5}{9} + 3y\right| = x + c \rightarrow \left(\frac{5}{9} + 3y\right)^{\frac{1}{3}} = ce^{x} \rightarrow \frac{5}{9} + 3y = ce^{3x}$

$\rightarrow y = \frac{1}{3}ce^{3x} \underbrace{- \frac{5}{27}}_{c_2} \Rightarrow \frac{5}{27} - \frac{5}{27}e^{-3x} + \frac{1}{3}x^2 - \frac{2}{9}x = y(x)$

$\Rightarrow c_1 = -c_2 = \frac{5}{27}$

☆ Tratto c_1 e c_2 Non come costanti, ma come variabili; per questo motivo mi ritrovo con un'altra eq differenziale alla fine; risolvendola ottengo proprio la c che mi servirà.

Se invece tratto c_1 e c_2 come costanti:

$y(x) = c_1 + c_2 \underbrace{\cancel{e^{-3x}}}_{1} + \underbrace{\cancel{\frac{1}{3}x^2}}_{0} - \underbrace{\cancel{\frac{2}{9}x}}_{0} \Rightarrow y'(x) = -3c_2 e^{-3x} + \frac{2}{3}x - \frac{2}{9}$

$\Rightarrow y''(x) = 9c_2 e^{-3x} + \frac{2}{3} \rightarrow y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$y'(0) = -3c_2 - \frac{2}{9} = \frac{1}{3} \rightarrow -3c_2 = \frac{1}{3} + \frac{2}{9} \rightarrow c_2 = -\frac{5}{27}$

$\Rightarrow c_1 = \frac{5}{27}$

1.73. Scrivere l'integrale generale dell'equazione:

$$y'' + 2y' - 3y = 4e^{2x}.$$

$\lambda^2 + 2\lambda - 3 = 0 \to \Delta = 4 + 12 = 16 > 0$

$\lambda_{1,2} = \frac{-2 \pm 4}{2} \nearrow 1 \searrow -3$

$y_0(x) = c_1 e^{x} + c_2 e^{-3x}$

$f(x) = 4e^{2x} \quad \gamma = 2 \to$ NO Radice $\to$ Sol $= y(x) = e^{2x} \cdot a$

$\Rightarrow y'(x) = a2e^{2x}, \; y''(x) = 4ae^{2x} \to 4ae^{2x} + 4ae^{2x} - 3ae^{2x} = 4e^{2x}$

$\to \cancel{e^{2x}}(4a + 4a - 3a) = 4\cancel{e^{2x}} \Rightarrow 5a = 4 \to a = \frac{4}{5}$

$\Rightarrow y_p(x) = \frac{4}{5}e^{2x} \to y(x) = c_1 e^{x} + c_2 e^{-3x} + \frac{4}{5}e^{2x}$

1.74.★

a. Scrivere l'integrale generale dell'equazione

$$y'' - 4y' + 4y = 0.$$

a) $\lambda^2 - 4\lambda + 4 = 0 \to \Delta = 16 - 4 \cdot 4 = 0$

$\to \lambda = \frac{4}{2} = 2$

$\Rightarrow y_0(x) = c_1 e^{2x} + c_2 x e^{2x}$ int generale

b. Determinare una soluzione particolare dell'equazione

$$y'' - 4y' + 4y = xe^{-x}.$$

$f(x) = xe^{-x} \to \gamma = -1$ NO SOL

$\Rightarrow$ Sol: $y(x) = P^{(n)}(x) \cdot e^{\gamma x} = (ax + b) \cdot e^{-x}$

$\Rightarrow y(x) = axe^{-x} + be^{-x}$

$\to y'(x) = ae^{-x} - axe^{-x} - be^{-x} \to y''(x) = -ae^{-x} - ae^{-x} + axe^{-x} + be^{-x}$

$\Rightarrow -ae^{-x} - ae^{-x} + axe^{-x} + be^{-x} - 4ae^{-x} + 4axe^{-x} + 4be^{-x} + 4axe^{-x} + 4be^{-x} = xe^{-x}$

$\to e^{-x}(-a - a + b - 4a + 4b + 4b) + xe^{-x}(a + 4a + 4a) = xe^{-x}$

$\begin{cases} -2a + b - 4a + 4b + 4b = 0 \\ 9a = 1 \Rightarrow a = \frac{1}{9} \end{cases} \xrightarrow{a = \frac{1}{9}} -\frac{2}{9} + b - \frac{4}{9} + 4b + 4b = 0 \to 9b = \frac{6}{9}, \; b = \frac{2}{27}$

$\to y_p(x) = \left(\frac{1}{9}x + \frac{2}{27}\right)e^{-x}$ int Part

1.76.★ Scrivere l'integrale generale dell'equazione

$$y'' + 3y' - 4y = 2\sin x.$$

$\lambda^2 + 3\lambda - 4 = 0 \to 9 + 16 = 25$

$\to \lambda_{1,2} = \frac{-3 \pm 5}{2} \nearrow 1 \searrow -4$

$y_0(x) = c_1 e^{x} + c_2 e^{-4x}$

$f(x) = 2\sin x \Rightarrow \gamma = 0$, NO RAD $\Rightarrow y(x) = \cancel{e^{0x}}(a\cos(x) + b\sin(x))$ 1

$\Rightarrow y'(x) = -a\sin(x) + b\cos(x), \quad y''(x) = -a\cos x - b\sin(x)$

$\to -a\cos x - b\sin x - 3a\sin x + 3b\cos x - 4a\cos x - 4b\sin(x) = 2\sin x$

$\Rightarrow \cos x(-a + 3b - 4a) + \sin x(-b - 3a - 4b) = 2\sin x$

$\begin{cases} -a + 3b - 4a = 0 \\ -b - 3a - 4b = 2 \end{cases} \to -5a + 3b = 0 \to a = \frac{3b}{5}$

$a=\frac{3}{5}b$
$\longrightarrow \quad -b-\frac{9}{5}b-4b=2 \quad \longrightarrow \frac{-5b-9b-20b}{5}=2 \quad \longrightarrow \quad -\frac{34}{5}b=2 \quad \longrightarrow \quad b=-\frac{10}{34}=-\frac{5}{17}$

$\longrightarrow \quad a=-\frac{3}{\cancel{5}}\cdot\frac{\cancel{5}}{17}=-\frac{3}{17}$

$\Longrightarrow \quad y_p(x)=\frac{3}{17}\cos(x)+\frac{5}{17}\sin(x)$ int part

1.75.★
a. Scrivere l'integrale generale dell'equazione

$$y''+4y'+4y=0.$$

b. Determinare una soluzione particolare dell'equazione

$$y''+4y'+4y=e^{-3x}\cos x.$$

$\lambda^2+4\lambda+4=0 \quad \longrightarrow \quad \Delta=16-16=0$
$\lambda=\frac{-4}{2}=-2$
$\Longrightarrow y_0(x)=c_1e^{-2x}+c_2xe^{-2x}$
$f(x)=e^{-3x}\cdot\cos x \quad , \gamma=-3$ NO RAD

$\Longrightarrow \quad y(x)=e^{-3x}\cdot[a\cos x+b\sin x] = ae^{-3x}\cos x+be^{-3x}\sin x$

$y'(x)=-3ae^{-3x}\cos x-ae^{-3x}\sin x-3be^{-3x}\cdot\sin x+be^{-3x}\cos x$

$y''(x)=9ae^{-3x}\cos x+3ae^{-3x}\sin x+3ae^{-3x}\sin x-ae^{-3x}\cos x+9be^{-3x}\sin x-3be^{-3x}\cos x-3be^{-3x}\cos x-be^{-3x}\sin x$

$e^{-3x}\cos x\,(9a-a-3b-3b)+e^{-3x}\sin x\,(3a+3a-a+9b-b)+e^{-3x}\cos x\,(-12a+4b)+e^{-3x}\sin x\,(-4a-12b)+$
$+e^{-3x}\cos x\,(4a)+e^{-3x}\sin x\,(4b)=e^{-3x}\cos x$

$\longrightarrow \cancel{e^{-3x}}\cos x\,[9a-a-3b-3b-12a+4b+4a]+e^{-3x}\sin x\,[3a+3a-a+9b-b-4a-12b+4b]=\cancel{e^{-3x}}\cos x$

$\Longrightarrow \begin{cases} 9a-a-3b-3b-12a+4b+4a=1 & \longrightarrow \quad -2b=1 \quad \Longrightarrow \quad b=-\frac{1}{2} \\ 3a+3a-a+9b-b-4a-12b+4b=0 & \longrightarrow \quad a=0 \end{cases}$

$\Longrightarrow \quad y_p(x)=-\frac{1}{2}e^{-3x}\sin x$ ✓

1.77.
a. Scrivere l'integrale generale dell'equazione

$$y''+6y'+9y=0.$$

b. Determinare una soluzione particolare dell'equazione

$$y''+6y'+9y=2xe^{-x}.$$

$\lambda^2+6\lambda+9=0 \quad \longrightarrow \quad \Delta=36-4\cdot9=\emptyset$
$\Longrightarrow \quad \lambda=\frac{-6}{2}=-3$
$\Longrightarrow \quad y_0(x)=c_1e^{-3x}+c_2xe^{-3x}$

$f(x)=2xe^{-x} \quad \longrightarrow \quad \gamma=-1$ NO RAD $\Longrightarrow$ sol $y(x)=e^{-x}(ax+b)=e^{-x}ax+e^{-x}b$

$\longrightarrow \quad y'(x)=-ae^{-x}x+ae^{-x}-e^{-x}b \qquad \longrightarrow \quad y''(x)=ae^{-x}x-ae^{-x}-ae^{-x}+be^{-x}$

$\longrightarrow \quad e^{-x}(-a-a+b)+xe^{-x}(a)+e^{-x}(6a-6b)+xe^{-x}(-6a)+e^{-x}(9b)+xe^{-x}(9a)=2xe^{-x}$

$\Longrightarrow \quad e^{-x}(-2a+b+6a-6b+9b)+xe^{-x}(a-6a+9a)=2xe^{-x}$

$\begin{cases} -2a+b+6a-6b+9b=0 \quad \overset{a=\frac{1}{2}}{\Longrightarrow} \quad 4a+4b=0 \quad \longrightarrow \quad 4b=-2 \quad \longrightarrow \quad b=-\frac{1}{2} \\ a-6a+9a=2 \quad \longrightarrow \quad 4a=2 \quad \longrightarrow \quad a=\frac{1}{2} \end{cases}$

$\Longrightarrow \quad y_p(x)=\frac{1}{2}e^{-x}x-\frac{1}{2}e^{-x}$

1.78.★ Risolvere il problema di Cauchy:

$$\begin{cases} y''(x) + 9y(x) = 3\sin 2x \\ y(0) = 2 \\ y'(0) = 6. \end{cases}$$

$\lambda^2 + 9 = 0 \to \Delta = -4 \cdot 9 = -36$

$\to \lambda_{1,2} = \pm \frac{6i}{2} < \begin{matrix} 3i \\ -3i \end{matrix}$

$\Rightarrow y_0(x) = c_1 \cos(3x) + c_2 \sin(3x)$

$f(x) = 3\sin(2x) \to y(x) = a\cos(\omega x) + b\sin(\omega x)$

$\to y'(x) = -2a\sin(2x) + 2b\cos(2x) \to y''(x) = -4a\cos(2x) - 4b\sin(2x)$

$\to -4a\cos(2x) - 4b\sin(2x) + 9a\cos(2x) + 9b\sin(2x) = 3\sin(2x)$

$\to \cos(2x)[-4a + 9a] + \sin(2x)[-4b + 9b] = 3\sin(2x)$

$\Rightarrow \begin{cases} -4b + 9b = 3 \to b = \frac{3}{5} \\ 4a + 9a = 0 \to a = 0 \end{cases} \Rightarrow \underline{y_p(x) = \frac{3}{5}\sin(2x)}$

1.79.

a. Scrivere l'integrale generale dell'equazione

$$y''(x) - y'(x) - 6y(x) = 0.$$

b. Determinare una soluzione particolare dell'equazione

$$y''(x) - y'(x) - 6y(x) = xe^{-3x}.$$

$\lambda^2 - \lambda - 6 = 0 \quad \Delta = 1 + 4 \cdot 6 = 25$

$\to \lambda_{1,2} = \frac{1 \pm 5}{2} < \begin{matrix} 3 \\ 2 \end{matrix}$

$\Rightarrow y_0(x) = c_1 e^{3x} + c_2 e^{2x}$

$f(x) = xe^{-3x} \to \gamma = -3$ <u>NO RAD</u> $\to$ SOL: $y(x) = e^{-3x}(ax + b) = e^{-3x}ax + be^{-3x}$

$y'(x) = -3ae^{-3x}x + ae^{-3x} - 3be^{-3x} \to y''(x) = 9ae^{-3x}x - 3ae^{-3x} - 3ae^{-3x} + 9be^{-3x}$

$e^{-3x}(-3a - 3a + 9b) + xe^{-3x}(9a) + e^{-3x}(-a + 3b) + xe^{-3x}(3a) + e^{-3x}(-6b) + xe^{-3x}(-6a) = xe^{-3x}$

$\to e^{-3x}(-3a - 3a - 9b - a + 3b - 6b) + xe^{-3x}(9a + 3a - 6a) = xe^{-3x}$

$\Rightarrow \begin{cases} 9a + 3a - 6a = 1 \to a = \frac{1}{6} \\ -6a - 9b - a + 3b - 6b = 0 \xrightarrow[a=\frac{1}{6}]{} -7a - 12b = 0 \to -\frac{7}{6} - 12b = 0 \to b = \frac{7}{72} \end{cases}$

$\Rightarrow y_p(x) = e^{-3x}\left(\frac{1}{6}x + \frac{7}{72}\right)$

1.80.★

a. Scrivere l'integrale generale dell'equazione

$$y''(t) - y'(t) - 2y(t) = 6e^{2t}.$$

$\lambda^2 - \lambda - 2 = 0 \quad \Delta = 1 + 8 = 9 > 0$

$\lambda_{1,2} = \frac{1 \pm 3}{2} < \begin{matrix} 2 \\ -1 \end{matrix}$

$\Rightarrow y_0(x) = c_1 e^{2x} + c_2 e^{-x}$

$f(x) = 6e^{2x} \Rightarrow \gamma = 2$ RADICE di mul 1 $\Rightarrow y(x) = x^{h \to 1} \cdot ae^{2x}$

$y'(x) = ae^{2x} + 2xae^{2x} \to y''(x) = 2ae^{2x} + 2ae^{2x} + 4ae^{2x}x$

$\Rightarrow xe^{2x}(4a) + e^{2x}(2a + 2a) + xe^{2x}(-2a) + e^{2x}(-a) + xe^{2x}(-2a) = 6e^{2x}$

$\to \cancel{xe^{2x}(4a - 2a - 2a)} + e^{2x}(4a - a) = 6e^{2x}$

$\Rightarrow 3a = 6 \to a = 2 \Rightarrow \underline{y_p(x) = 2xe^{2x}}$

b. Risolvere il problema di Cauchy per l'equazione precedente con condizioni iniziali

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$$

$y(0) = c_1 \cancel{e^{2x}} + c_2 \cancel{e^{-x}} = 0 \Rightarrow \boxed{\begin{matrix} c_1 = -c_2 \\ c_2 = -c_1 \end{matrix}}$

$y'(x) = 2c_1 e^{2x} - c_2 e^{-x} \rightarrow y'(0) = 2c_1 - c_2 = 1 \xrightarrow{c_1 = -c_2} -2c_2 - c_2 = 1 \rightarrow c_2 = -\frac{1}{3}$

$c_1 = -c_2 \Rightarrow c_1 = \frac{1}{3} \rightarrow$ Sol: $y(x) = \frac{1}{3} e^{2x} - \frac{1}{3} e^{-x} + 2x e^{2x}$

1.81.★ Scrivere l'integrale generale dell'equazione

$$y''(t) + 2y'(t) - 3y(t) = e^{-2t}\text{cost}.$$

$\lambda^2 + 2\lambda - 3 = 0 \qquad \Delta = 4 + 4\cdot 3 = 16 > 0$

$\lambda_{1,2} = \frac{-2 \pm 4}{2} \begin{matrix} \nearrow 1 \\ \searrow -3 \end{matrix}$

$\rightarrow y_0(x) = c_1 e^{x} + c_2 e^{-3x}$

$f(x) = e^{-2x} \cdot \cos(x) \rightarrow \gamma = -2$ NO RAD

$\Rightarrow y(x) = e^{-2x}[a\cos(x) + b\sin(x)]$

$y'(x) = -2e^{-2x}[a\cos(x) + b\sin(x)] + e^{-2x}[-a\sin(x) + b\cos(x)]$

$y''(x) = 4e^{-2x}[a\cos(x) + b\sin(x)] - 2e^{-2x}[-a\sin x + b\cos(x)] - 2e^{2x}[-a\sin(x) + b\cos(x)] + e^{-2x}[-a\cos x - b\sin x]$

$\rightarrow e^{-2x}\cos x[4a - 2b - 2b - a] + e^{-2x}\sin x[4b + 2a + 2a - b] + e^{-2x}\cos x[-4a + 2b] + e^{-2x}\sin x[-4b - 2a] +$

$+ e^{-2x}\cos x[-3a] + e^{2x}\sin x[-3b] = e^{-2x}\cos t$

$\Rightarrow e^{-2x}\cos x[\cancel{4a} - 4b - a - \cancel{4a} + 2b - 3a] + e^{-2x}\sin x[\cancel{4b} + \cancel{2a} + 2a - b - \cancel{4b} - \cancel{2a} - 3b] = e^{-2x}\cos x$

$\begin{cases} -4a - 2b = 0 \rightarrow b = -2a \Rightarrow b = -\frac{1}{5} \\ 2a - 4b = 1 \rightarrow 2a + 8a = 1 \rightarrow a = \frac{1}{10} \end{cases}$

$\Rightarrow y_p(x) = e^{-2x}\left[\frac{1}{10}\cos x - \frac{1}{5}\sin x\right]$

1.82.

a. Scrivere l'integrale generale dell'equazione

$$y'' + 3y' - 4y = 0.$$

b. Determinare una soluzione particolare dell'equazione

$$y'' + 3y' - 4y = 5e^{-4x}.$$

$\lambda^2 + 3\lambda - 4 = 0 \qquad \Delta = 9 + 4\cdot 4 = 25$

$\rightarrow \lambda_{1,2} = \frac{-3 \pm 5}{2} \begin{matrix} \nearrow 1 \\ \searrow -4 \end{matrix}$

$y_0(x) = c_1 e^{x} + c_2 e^{-4x}$

$f(x) = 5e^{-4x}$, $\gamma = -4$ RADICE MUL $h = 1 \Rightarrow y(x) = x \cdot a e^{-4x}$

$\Rightarrow y'(x) = a e^{-4x} - 4ax e^{-4x} \rightarrow y''(x) = -a e^{-4x} + 16a e^{-4x} x - 4a e^{-4x}$

$\rightarrow x e^{-4x}(16a) + e^{-4x}(-a - 4a) + x e^{-4x}(-12a) + e^{-4x}(3a) + x e^{-4x}(-4a) = 5e^{-4x}$

$\rightarrow x e^{-4x}(\cancel{16a - 12a - 4a}) + e^{-4x}(-a - 4a + 3a) = 5e^{-4t}$

$\rightarrow -2a = 5 \rightarrow a = -\frac{5}{2} \qquad \Rightarrow y_p(x) = -\frac{5}{2} x e^{-4x}$

1.88.

a. Scrivere l'integrale generale dell'equazione:

$$y'' + 2y' - 3y = 0.$$

b. Risolvere il problema di Cauchy per l'equazione precedente con le condizioni:

$$y(0) = 1;\ y'(0) = 0.$$

c. Determinare una soluzione particolare dell'equazione

$$y'' + 2y' - 3y = \sin x.$$

$\lambda^2 + 2\lambda - 3 = 0 \qquad \Delta = 4 + 12 = 16 > 0$

$\lambda_{1,2} = \frac{-2 \pm 4}{2} \begin{cases} 1 \\ -3 \end{cases}$

$\rightarrow y_0(x) = c_1 e^{x} + c_2 e^{-3x}$

b) $y(0) = c_1 \cancel{e^{x}} + c_2 \cancel{e^{-3x}} = 1 \rightarrow c_1 = 1 - c_2 \ ,\ c_2 = 1 - c_1 \Rightarrow c_1 = \frac{3}{4}$

$y'(x) = c_1 e^{x} - 3c_2 e^{-3x} \rightarrow y'(0) = c_1 - 3c_2 = 0 \xrightarrow{c_1 = 1 - c_2} 1 - c_2 - 3c_2 = 0 \rightarrow c_2 = \frac{1}{4}$

$\rightarrow y_p(x) = \frac{3}{4} e^{x} + \frac{1}{4} e^{-3x}$

c) $f(x) = \sin x \Rightarrow y(x) = a\cos x + b\sin x \rightarrow y'(x) = -a\sin x + b\cos x \rightarrow y''(x) = -a\cos x - b\sin x$

$\Rightarrow \cos x(-a) + \sin x(-b) + \cos x(2b) + \sin x(-2a) + \cos x(-3a) + \sin x(-3b) = \sin x$

$\cos x(-a + 2b - 3a) + \sin x(-b - 2a - 3b) = \sin x$

$\begin{cases} 2b - 4a = 0 \rightarrow b = 2a \rightarrow b = -\frac{1}{5} \\ -2a - 4b = 1 \rightarrow -2a - 8a = 1 \rightarrow a = -\frac{1}{10} \end{cases}$

$\Rightarrow y(x) = -\frac{1}{10}\cos x - \frac{1}{5}\sin x$

1.91. $\qquad 2y'' - y' - y = e^{-x}\sin 2x.$

$2\lambda^2 - \lambda - 1 = 0 \rightarrow \Delta = 1 + 4 \cdot 2 = 9 > 0 \rightarrow \lambda_{1,2} = \frac{1 \pm 3}{4} \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$

$y_0(x) = c_1 e^{x} + c_2 e^{-\frac{1}{2}x} \qquad f(x) = e^{-x} \cdot \sin(2x) \Rightarrow \gamma = -1$ NO RAD

$\Rightarrow$ SOL $y(x) = e^{-x}[a\cos(2x) + b\sin(2x)] \rightarrow y'(x) = -e^{-x}[a\cos(2x) + b\sin(2x)] + e^{-x}[-2a\sin(2x) + 2b\cos 2x)]$

$y''(x) = e^{-x}[a\cos(2x) + b\sin(2x)] - e^{-x}[-2a\sin(2x) + 2b\cos(2x)] - e^{-x}[-2a\sin(2x) + 2b\cos(2x)] + [-4a\cos(2x) - 4b\sin(2x)]$

$\Rightarrow e^{-x}\cos(2x)[2a - 2b - 4b - 8a] + e^{-x}\sin(2x)[2b + 4a + 4a - 8b] + e^{-x}\cos(2x)[a - 2b] + e^{-x}\sin(2x)[b - 2a] + e^{-x}\cos(2x)[-a] + e^{-x}\sin(2x)[-b] = e^{-x}\sin(2x)$

$\rightarrow e^{-x}\cos(2x)[2a - 2b - 4b - 8a \cancel{+a} - 2b \cancel{-a}] + e^{-x}\sin(2x)[2b + 4a + 4a - 8b + b - 2a - b] = e^{-x}\sin(2x)$

$\rightarrow \begin{cases} 2a - 16b = 0 \rightarrow a = 8b \Rightarrow a = \frac{4}{21} \\ -6b + 6a = 1 \rightarrow -6b + 48b = 1 \rightarrow b = \frac{1}{42} \end{cases}$

$\rightarrow y_p(x) = e^{-x}\left[\frac{4}{21}\cos(2x) + \frac{1}{42}\sin(2x)\right]$

1.92. $y'' + 4y' + 4y = x^2 + 3x + 1.$

$\lambda^2 + 4\lambda + 4 = 0 \qquad \Delta = 16 - 16 = 0$

$\lambda = \frac{-4}{2} = -2$

$y_0(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

$f(x) = x^2 + 3x + 1 \qquad \gamma = 0$ NO Rad $\Rightarrow$ SOL: $y(x) = ax^2 + bx + c$

$y'(x) = 2ax + b \qquad y''(x) = 2a \Rightarrow 2a + 8ax + 4b + 4ax^2 + 4bx + 4c = x^2 + 3x + 1$

$\Rightarrow x^2(4a) + x(8a + 4b) + 2a + 4c + 4b = x^2 + 3x + 1$

$$\Rightarrow \begin{cases} 4a = 1 \Rightarrow a = 1/4 \\ 8a + 4b = 3 \Rightarrow b = \frac{1}{4} \\ 2a + 4b + 4c = 1 \Rightarrow c = 1 - \frac{1}{2} - 1 = \frac{2-1-2}{2} = -\frac{1}{2} \cdot \frac{1}{4} = -\frac{1}{8} \end{cases}$$

$\Rightarrow y_p(x) = \frac{1}{4}x^2 + \frac{1}{4}x - \frac{1}{8}$

1.99.★ Risolvere il problema di Cauchy:

$$\begin{cases} y'y'' = 1 \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

$y'' \cdot y' = 1$ Procedo per sostituzione

pongo $y' = u$, $y'' = u'u$

$\Rightarrow y''y' = 1 \Rightarrow u^2 u' = 1$ siccome $u' = \frac{du}{dy}$

$\Rightarrow u^2\, du = 1\, dy \Rightarrow \int u^2\, du = \int dy \Rightarrow \frac{u^3}{3} = y + c$

Trovo u: $u^3 = 3 \cdot (y + c) \Rightarrow u = \sqrt[3]{3} \cdot \sqrt[3]{y+c}$

Siccome $u = y' \Rightarrow y' = \sqrt[3]{3}\sqrt[3]{y+c}$ siccome $y' = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{\sqrt[3]{y+c}} = \frac{\sqrt[3]{3}\sqrt[3]{y+c}}{\sqrt[3]{y+c}}\, dx \Rightarrow (y+c)^{-\frac{1}{3}} dy = \sqrt[3]{3}\, dx \Rightarrow \int (y+c)^{-\frac{1}{3}} dy = \sqrt[3]{3} \int dx$

$\Rightarrow \frac{3}{2}(y + c_1)^{\frac{2}{3}} = \sqrt[3]{3}\, x + c_2 \Rightarrow (y + c_1)^2 = \left(\frac{2\sqrt[3]{3}}{3} x + c_2\right)^3$

$\Rightarrow y + c_1 = \left(\frac{2}{3}\sqrt[3]{3}\, x + c_2\right)^{\frac{3}{2}} \Rightarrow y = \left(\frac{2}{3}\sqrt[3]{3}\, x + c_2\right)^{\frac{3}{2}} - c_1$

$\Rightarrow y(0) = (c_2)^{\frac{3}{2}} - c_1 = 0 \Rightarrow$

$c_1 = (c_2)^{\frac{3}{2}}$

$y'(x) = \frac{3}{2}\left(\frac{2}{3}\sqrt[3]{3}\, x + c_2\right)^{\frac{1}{2}}\left(\frac{2}{3}\sqrt[3]{3}\right) \Rightarrow y'(0) = \frac{3}{2}(c_2)^{\frac{1}{2}} \cdot \frac{2}{3}\sqrt[3]{3} = 1$

$\Rightarrow (c_2)^{\frac{1}{2}} = \frac{\cancel{3}}{\cancel{2}\sqrt[3]{3}} \cdot \frac{\cancel{2}}{\cancel{3}} \Rightarrow c_2^{\frac{1}{2}} = \frac{1}{\sqrt[3]{3}} \Rightarrow c_2 = \left(\frac{1}{\sqrt[3]{3}}\right)^{\frac{1}{2}}$

1.100. Risolvere il problema di Cauchy:

$$\begin{cases} y'' + 2y' + 4y = 3\sin(2x) \\ y(0) = 0 \\ y'(0) = \frac{1}{4} \end{cases}$$

$\lambda^2 + 2\lambda + 4 = 0 \rightarrow \Delta = 4 - 16 = -12 < 0$

$\lambda_{1,2} = \dfrac{-2 \pm 2i\sqrt{3}}{2} < \begin{matrix} -1 + i\sqrt{3} \\ -1 - i\sqrt{3} \end{matrix}$

$y_0(x) = e^{-x}\left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right]$

$f(x) = 3\sin(2x) \Rightarrow \gamma = 0$ NO RAD $\Rightarrow$ $y_p(x) = a\cos(2x) + b\sin(2x)$

$y_p'(x) = -2a\sin(2x) + 2b\cos(2x)$ $\qquad y_p''(x) = -4a\cos(2x) - 4b\sin(2x)$

$\rightarrow \cos(2x)\left(\cancel{-4a} + 4b + \cancel{4a}\right) + \sin(2x)\left(\cancel{-4b} - 4a + \cancel{4b}\right) = 3\sin(2x)$

$\rightarrow \begin{cases} 4b = 0 \rightarrow b = 0 \\ -4a = 3 \rightarrow a = -\frac{3}{4} \end{cases}$ $\Rightarrow$ $y_p(x) = -\frac{3}{4}\cos(2x)$

$\Rightarrow y(x) = \dfrac{c_1\cos(\sqrt{3}x)}{e^x} + \dfrac{c_2\sin(\sqrt{3}x)}{e^x} - \dfrac{3}{4}\cos(2x)$

$\Rightarrow y(0) = \dfrac{c_1}{\cancel{e^x}_{\,1}} - \dfrac{3}{4} = 0 \rightarrow c_1 = \dfrac{3}{4}$

$y'(x) = \dfrac{-\sqrt{3}c_1\sin(\sqrt{3}x)e^x - e^x c_1\cos(\sqrt{3}x)}{e^{2x}} + \dfrac{\sqrt{3}c_2\cos(\sqrt{3}x)e^x + e^x c_2\sin(\sqrt{3}x)}{e^{2x}}$

$y'(0) = -c_1 + \sqrt{3}c_2 = \frac{1}{4} \xrightarrow{c_1 = \frac{3}{4}}$ $\qquad c_2 = \left(\frac{1}{4} + \frac{3}{4}\right)\sqrt{3} \Rightarrow c_2 = (3)^{-\frac{1}{2}}$

$\Rightarrow y(x) = y(x) = \dfrac{\frac{3}{4}\cos(\sqrt{3}x)}{e^x} + \dfrac{3^{-\frac{1}{2}}\sin(\sqrt{3}x)}{e^x} - \dfrac{3}{4}\cos(2x)$

1.101. Risolvere il problema di Cauchy:

$$\begin{cases} y'' - 2y' + 3y = 2\cos(3x) \\ y(0) = \frac{1}{3} \\ y'(0) = 0 \end{cases}$$

$\lambda^2 - 2\lambda + 3 = 0 \Rightarrow \Delta = 4 - 12 = -8 < 0$

$\Rightarrow \lambda_{1,2} = \frac{2 \pm 2i\sqrt{2}}{2}$ $1+i\sqrt{2}$, $1-i\sqrt{2}$

$\Rightarrow y_0(x) = e^x \left[c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)\right]$

$f(x) = 2\cos(3x) \Rightarrow \quad y_p(x) = a\cos(3x) + b\sin(3x)$

$\Rightarrow y_p'(x) = -3a\sin(3x) + 3b\cos(3x) \Rightarrow y_p''(x) = -9a\cos(3x) - 9b\sin(3x)$

$\Rightarrow \cos(3x)\left[-9a - 6b + 3a\right) + \sin(3x)\left[-9b + 6a + 3b\right] = 2\cos(3x)$

$\Rightarrow \begin{cases} -9a - 6b + 3a = 2 \\ -9b + 6a + 3b = 0 \end{cases} \Rightarrow \quad -6b + 6a = 0 \Rightarrow \quad a = b \Rightarrow a = -\frac{1}{6}$

$\Rightarrow -9b - 6b + 3b = 2 \Rightarrow -12b = 2 \Rightarrow b = -\frac{1}{6}$

$y_p(x) = -\frac{1}{6}\cos(3x) - \frac{1}{6}\sin(3x) \Rightarrow y(x) = e^x\left[c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)\right] - \frac{1}{6}\left(\cos(3x) + \sin(3x)\right.$

Cauchy: $y'(x) = e^x\left[c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)\right] + e^x\left[-c_1\sqrt{2}\sin(\sqrt{2}x) + c_2\sqrt{2}\cos(\sqrt{2}x)\right] +$

$+ \frac{1}{2}\sin(3x) - \frac{1}{2}\cos(3x)$

$y(0) = \underbrace{e^x}_{1}\left[c_1 \underbrace{\cos(\sqrt{2}x)}_{1} + c_2 \underbrace{\sin(\sqrt{2}x)}_{0}\right] - \frac{1}{6}\left(\underbrace{\cos(3x)}_{1} + \underbrace{\sin(3x)}_{0}\right) = c_1 - \frac{1}{6} = \frac{1}{3}$

$\Rightarrow c_1 = \frac{1}{2} \Rightarrow \quad y'(0) = c_1 + c_2\sqrt{2} - \frac{1}{2} = 0 \xrightarrow{c_1 = \frac{1}{2}} \frac{1}{2} + c_2\sqrt{2} - \frac{1}{2} = 0$

$\Rightarrow c_2 = 0$

$\Rightarrow y(x) = y(x) = e^x\left[\frac{1}{2}\cos(\sqrt{2}x)\right] - \frac{1}{6}\left(\cos(3x) + \sin(3x)\right.$

1.103. Risolvere il problema di Cauchy:

$$\begin{cases} y'' - y' - 2y = x^2 + 1 \\ y(0) = 2 \\ y'(0) = 0 \end{cases}$$

$\lambda^2 - \lambda - 2 = 0 \qquad \Delta = 1 + 8 = 9 > 0$

$\lambda_{1,2} = \frac{1 \pm 3}{2} \begin{cases} 2 \\ -1 \end{cases}$

$\Rightarrow y_0(x) = c_1 e^{2x} + c_2 e^{-x} \qquad f(x) = x^2 + 1 \qquad \gamma = 0$ NO RAD $\Rightarrow y_p(x) = ax^2 + bx + c$

$\rightarrow y'(x) = 2ax + b \qquad y''(x) = 2a$

$\rightarrow 2a - 2ax - b - 2ax^2 - 2bx - 2c = x^2 + 1$

$\rightarrow x^2(-2a) + x(-2a - 2b) + 2a - b - 2c = x^2 + 1$

$\Rightarrow \begin{cases} -2a = 1 \Rightarrow a = -\frac{1}{2} \\ -2a - 2b = 0 \rightarrow b = -\frac{1}{2} \\ 2a - b - 2c = 1 \end{cases} \rightarrow -1 + \frac{1}{2} - 2c = 1 \rightarrow c = -\frac{5}{4}$

$\Rightarrow y_p(x) = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{5}{4} \qquad y(x) = c_1 e^{2x} + c_2 e^{-x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{5}{4}$

$y'(x) = 2c_1 e^{2x} - c_2 e^{-x} - x - \frac{1}{2}$

$y(0) = c_1 + c_2 - \frac{5}{4} = 2 \rightarrow c_1 = \frac{13}{4} - c_2$

$y'(0) = 2c_1 - c_2 - \frac{1}{2} = 0 \rightarrow \qquad -2c_2 - c_2 = \frac{1}{2} - \frac{13}{2}$

$\rightarrow \quad c_2 = -2 \qquad \Rightarrow c_1 = \frac{13}{4} + 2 = \frac{13+8}{4} = \frac{21}{4}$

$\Rightarrow y(x) = \frac{21}{4}e^{2x} - 2e^{-x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{5}{4}$