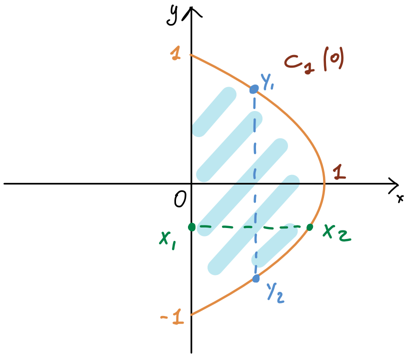
Continuo esercizi integr doppi:

ES:

$$\iint_D x \, dx \, dy$$




Asse y

$D_y = \{(x,y) \mid -1 \le y \le 1, \quad 0 < x < \sqrt{1-y^2}\}$

Eq circ: $x^2 + y^2 = 1 \xrightarrow{\text{in } f \text{ di } x} x^2 = 1 - y^2 \to x = \pm\sqrt{1-y}$

$$\Rightarrow \iint_D x \, dx = \int_{-1}^{1} dy \int_0^{\sqrt{1-y^2}} x \, dx \Rightarrow \int \left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy = \frac{1}{2}\int_{-1}^{1} 1 - y^2 \, dy = \frac{1}{2}\left[y - \frac{y^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$$

$$= \frac{1}{2}\left[\frac{4}{3}\right] = \frac{2}{3}$$

eq circ: $x^2 + y^2 = 1 \to y = \pm\sqrt{1-x^2}$

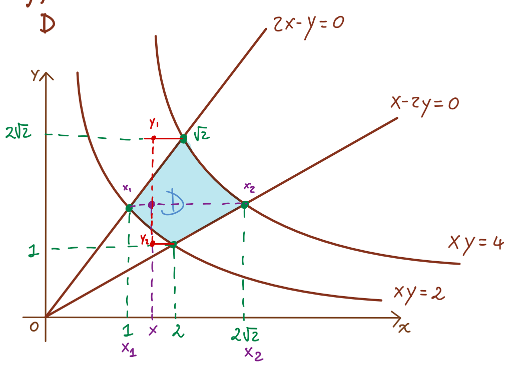
Asse x $\quad D_x = \{(x,y) \mid 0 \le x \le 1, \ -\sqrt{1-x^2} < y < \sqrt{1-x^2}\}$

$$\iint_D x \, dx \, dy = \int_0^1 x \, dy \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \int_0^1 x\left[\sqrt{1-x^2} + \sqrt{1-x^2}\right] = -\int_0^1 -2x\sqrt{1-x^2} \, dx = \left[\frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= -\frac{2}{3}(-1) = \frac{2}{3}$$

ES: "impegnativo"

$$\iint_D x^2 y^2 \, dx \, dy$$




1) Capire i punti di intersezione

$\begin{cases} x - 2y = 0 \\ xy = 2 \end{cases}$ $\begin{matrix} \to \frac{2}{y} - 2y = 0 \to \frac{2-2y^2}{y} = 0 \to +\frac{2y^2}{\cancel{y}} = +\frac{2}{y} \\ \to x = \frac{2}{y} \end{matrix}$

$\to 2y = \frac{2}{y} \to y = 1$

$\Rightarrow x = \frac{2}{y} = 2$

Facciamo questo per tutti i punti

2) Esprimere il dominio (rispetto ad x)

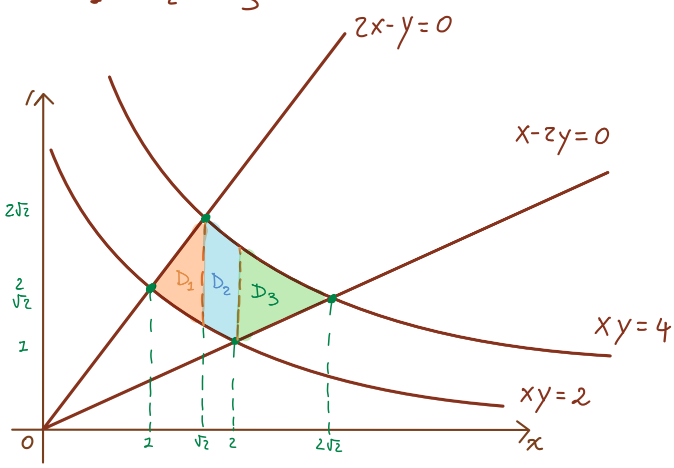
a) Troviamo l'intervallo di x: $I_x = (1, 2\sqrt{2})$

b) Troviamo $I_y$: quale f "sta sotto" e quale "sopra"?

$\Rightarrow$ Non c'è una sola f! $\quad I = (1,2) \to f(xy=2)$, $\quad I = (2, 2\sqrt{2}) \to f(x-2y=0)$ (in basso)

$\to$ Come possiamo vedere le funzioni cambiano; come facciamo?

$D = D_1 \cup D_2 \cup D_3$




$\Rightarrow D_1 = \left\{(x,y) \mid 1 \le x \le \sqrt{2}, \quad \frac{2}{x} \le y \le 2x\right\}$

$D_2 = \left\{(x,y) \mid \sqrt{2} < x \le 2, \quad \frac{2}{x} \le y \le \frac{4}{x}\right\}$

$D_3 = \left\{(x,y) \mid 2 < x \le 2\sqrt{2}, \quad \frac{x}{2} < y < \frac{4}{x}\right\}$

3) Calcoliamo l'integrale:

$$\iint_D x^2 y^2 \, dx \, dy = \iint_{D_1} + \iint_{D_2} + \iint_{D_3}$$

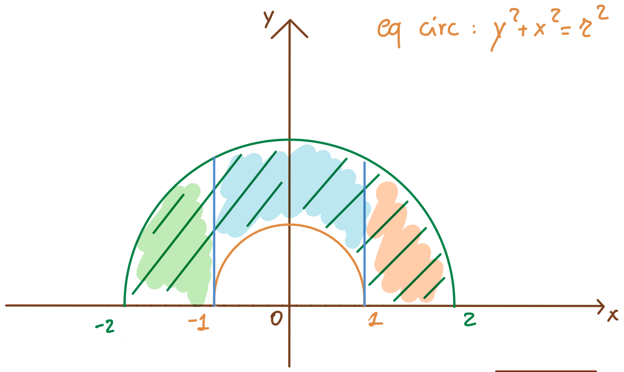
$$\iint_D x^2 y^2 \, dx\, dy = \int_1^{\sqrt{2}} x^2 dx \int_{2/x}^{2x} y^2 dy = \int_1^{\sqrt{2}} x^2 \left[\frac{y^3}{3}\right]_{2/x}^{2x} dx = \int_1^{\sqrt{2}} x^2 \cdot \left[\frac{8x^3}{3} - \frac{\frac{8}{x^3}}{3}\right] = \dots$$

ES:

$$\iint_D \frac{y}{x^2+y^2}\, dx\, dy$$

$\rightarrow$ Dove D è la corona circolare di raggi 1 e 2 contenta nel semipiano $y \geq 0$

eq circ : $y^2+x^2=2^2$




* La corona circ è fatta dallo spazio compreso tra 2 cerchi. In questo caso i cerchi hanno raggi 1 e 2.

I) Rispetto ad $x$ $\quad D_1 = \{(x,y) / -2 \leq x \leq 2 \ , \ ? < y < ?$

$D_1 = \{(x,y) / -2 \leq x \leq -1 \ , \quad 0 \leq y < \sqrt{4-x^2}$

$D_2 = \{(x,y) / -1 \leq x \leq 1 \ , \quad \sqrt{1-x^2} \leq y \leq \sqrt{4-x^2}$

Circ esterna : $x^2+y^2 = 4 \rightarrow y = \sqrt{4-x^2}$

Circ interna : $x^2+y^2 = 1 \rightarrow y = \sqrt{1-x^2}$

$D_3 = \{(x,y) / \ 1 \leq x \leq 2 \ , \quad 0 \leq y \leq \sqrt{4-x^2}\}$

$$\Rightarrow \iint_D \frac{y}{x^2+y}\, dx\, dy = \iint_{D_1} + \iint_{D_2} + \iint_{D_3} = \frac{1}{2}\int_{-2}^{-1} dx \int_0^{\sqrt{4-x^2}} \frac{2y}{\underbrace{x^2}_{\text{costante}}+y^2}\, dy = \frac{1}{2}\int_{-2}^{-1} dx \left[\ln|x^2+y^2|\right]_0^{\sqrt{4-x^2}} dx$$

$$= \frac{1}{2}\int_{-2}^{-1} \ln|x^2+4-x^2| - \ln|x| \, dx = \frac{1}{2}\int_{-2}^{-1} \ln|4|\, dx - \cancel{\frac{1}{2}}\int_{-2}^{-1} \ln|x|^{\cancel{2}} dx = [x]_{-2}^{-1} \frac{1}{2}\ln|4| - \int_{-2}^{-1} \ln|x|\, dx = \frac{\ln 4}{2}[x]_{-2}^{-1} - [x\ln x - x]_{-2}^{-1}$$

$$= -\frac{\ln 4}{2}[-2+1] - [-\cancel{2}\ln 2 + \cancel{2}] - [-\cancel{1}\ln 1 + \cancel{1}] = +\frac{\ln 4}{2} - \ln 2 + \ln 1$$

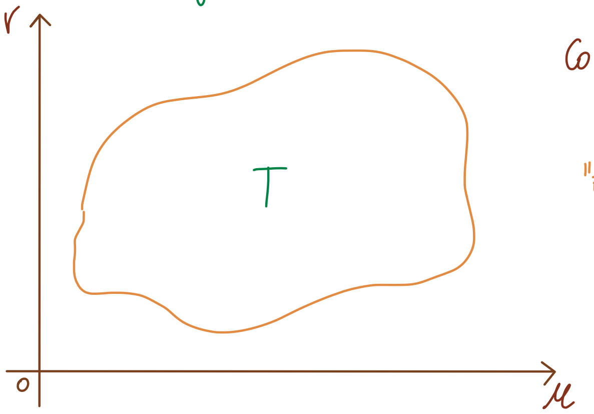
# Cambiamento di variabili

Cosa facevamo nel caso di 1 variabile?

$$\int_a^b f(x)\,dx \qquad x = g(t) \quad \to \quad \int f(g(t)) \cdot g'(t)\,dt \quad \text{con } g \text{ invertibile} \to t = g^{-1}(x)$$

$$\Rightarrow \int_{g^{-1}(a)}^{g^{-1}(b)} f[g(t)] \cdot g'(t)\,dt$$

$T$ dominio regolare in piano $uv$




Consideriamo le funz. derivabili con derivate continue $[C^1(T)]$

"Fi" maiuscolo $\to \Phi \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \quad (u,v) \in T$

$$\underset{g}{\Phi} : \underset{t}{(u,v) \in T} \to \underset{g(t)}{[x(u,v), y(u,v)]} \in D = \Phi(T)$$

Nel cambiamento di variabile in funz a 1 variabile, avevamo il termine $g'(t)$; l'equivalente è:

## Matrice Jacobiana

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

Derivata di $x$ rispetto ad $u$

## Determinante Jacobiano

– anche detto jacobiano
come "Hessiano"

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

## Teorema di cambiamento di variabili per int. doppi

$T, D$ domini regolari di $\mathbb{R}^2$

$\Phi : T \to D$ invertibile e di classe $C^1(T)$ e tale che il suo jacobiano sia $\neq 0$ in tutto $T$

ovvero derivabile e con tutte le deriv parz 1ª continue.

Sia $f : \underset{\Phi(T)}{D} \to \mathbb{R}$ continua in $D$.

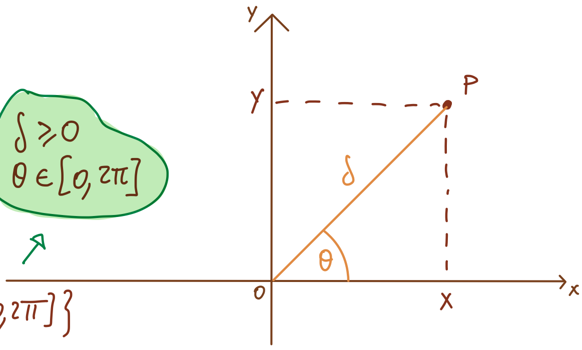
Allora

$$\iint_{D=\Phi(T)} f(x,y)\,dx\,dy = \iint_T f[x(u,v), y(u,v)] \cdot \underbrace{\left| \det \frac{\partial(x,y)}{\partial(u,v)} \right|}_{g'(t)} du\,dv$$

## Cambiamento in Coordinate Polari ← unici esercizi di cambiamento coordinate.

$$\Phi : (\rho,\theta) \in T \to (x(\rho,\theta),\ y(\rho,\theta)) \in D$$

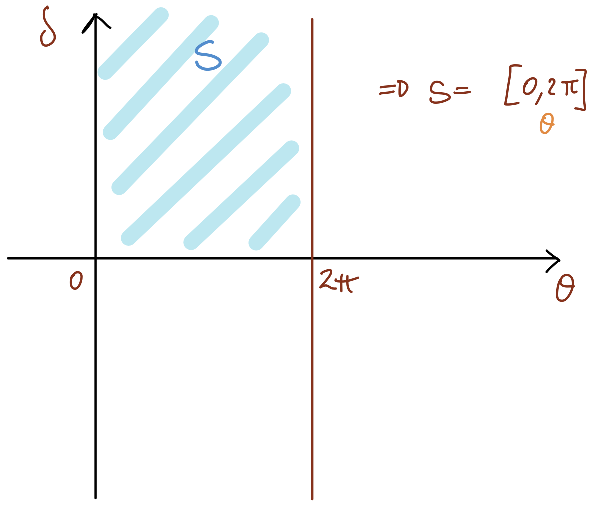
Tale che: $\begin{cases} x = x(\rho,\theta) = \rho\cos\theta \\ y = y(\rho,\theta) = \rho\sin\theta \end{cases}$



Quindi la funz. $\Phi : (\rho,\theta) \in S = \{(\rho,\theta) / \rho \geq 0,\ \theta \in [0,2\pi]\}$

$\rho \geq 0$
$\theta \in [0,2\pi]$

Se sono in un piano $\rho - \theta$:



$$\Rightarrow S = \underset{\theta}{[0,2\pi]} \cdot \underset{\rho}{[0,+\infty)}$$

### Jacobiano della Trasformazione

$$\det \frac{\partial(x,y)}{\partial(\rho,\theta)} = \begin{vmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} =$$

$$= \cos\theta \cdot \rho\cos\theta - [-\rho\sin\theta \cdot \sin\theta] = \rho\cos^2\theta + \rho\sin^2\theta$$

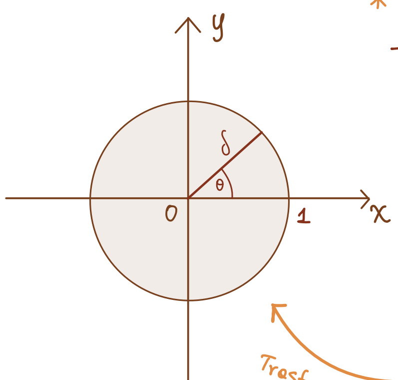
$$= \rho(\underbrace{\cos^2\theta + \sin^2\theta}_{\text{rel fond}}) = \rho$$

## Teorema cambiamento variabile in coordinate polari

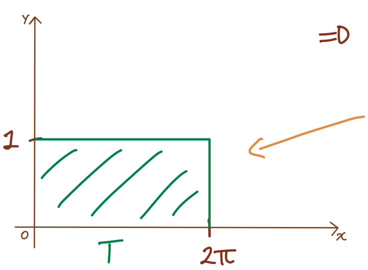
$D$ dominio regolare, $f$ è una $f$ continua in $D$, allora:

$$\iint_D f(x,y)\, dx\, dy = \int_{\Phi^{-1}(D)} f(\rho\cos\theta, \rho\sin\theta) \cdot \underbrace{\rho}_{\text{Determinante}}\, d\rho\, d\theta$$

ES: $\iint_D (x^2+y^2)(1-\sqrt{x^2+y^2})\, dx\, dy$ dove $D$ è la circonf: $C_1(0)$



Trasformato



* Potremmo svolgerlo con le formule di riduzione viste prima.
→ diventa più semplice se passiamo a coord polari

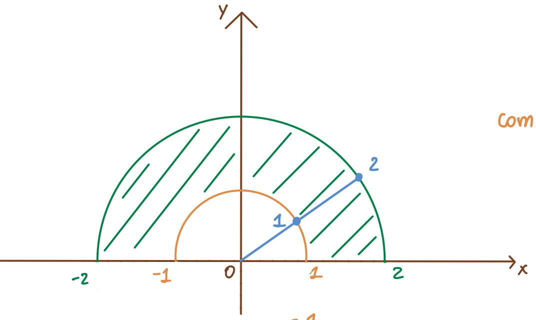
$D = \{(x,y) / x^2+y^2 \leq 1\}$ $\quad T = \{(\rho,\theta) / \underbrace{0 \leq \rho \leq 1}_{\text{Distanza}},\ \underbrace{0 \leq \theta \leq 2\pi}_{\text{Angolo}}\}$

$$\Rightarrow \iint_{\Phi^{-1}(D)} (\rho^2\cos^2\theta + \rho^2\sin^2\theta)(1-\sqrt{\rho^2(\cos^2\theta+\sin^2\theta)}) \cdot \underbrace{\rho}_{\text{Jacobiano}}\, d\rho\, d\theta$$

$$= \int_0^1 d\rho \int_0^{2\pi} \rho^2(1-\rho)\cdot\rho\, d\rho\, d\theta = \int_0^1 \rho^3 - \rho^4\, d\rho \cdot \int_0^{2\pi} d\theta$$

$$= \left[\frac{\rho^4}{4}\right]_0^1 - \left[\frac{\rho^5}{5}\right]_0^1 \cdot [\theta]_0^{2\pi} = 2\pi\left(\frac{1^4}{4} - \frac{1^5}{5}\right) = 2\pi \cdot \frac{5-4}{20} = \frac{\pi}{10}$$

ES: $\iint_D \frac{y}{x^2+y^2}\,dx\,dy$




Corona circolare vista prima:

$$= \int_1^2 d\rho \int_0^\pi \frac{\rho\sin\theta}{\rho^2\cos^2\theta+\rho^2\sin^2\theta}\cdot \rho\, d\theta$$

Jacobiano

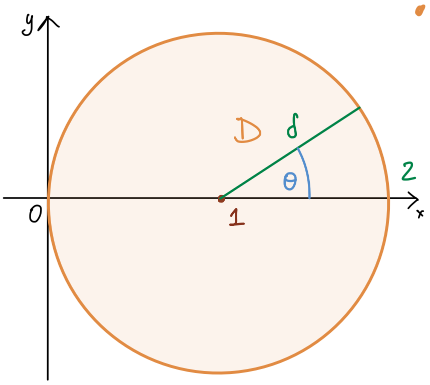
come varia $\rho$?   come varia $\theta$?

$$= \int_1^2 d\rho \int_0^\pi \frac{\cancel{\rho}\sin\theta}{\cancel{\rho^2}(\underbrace{\cos^2\theta+\sin^2\theta}_{1})}\cdot \cancel{\rho}\, d\rho\, d\theta = \int_1^2 [-\cos\theta]_0^\pi\, d\rho$$

$$= -\int_1^2 \underbrace{\cos\pi}_{-1} - \underbrace{\cos(0)}_{1}\, d\rho = 2\int_1^2 d\rho = 2[\rho]_1^2 = 2[2-1]=2$$

* Questo esercizio è molto più semplice se svolto usando le coordinate polari!

ES: $\iint_D \sqrt{x^2+y^2}\,dx\,dy$   Il centro della circ non è l'origine!




• Applichiamo le formule nel caso il polo è $(1,0)$

$$\Rightarrow \begin{cases} x = 1+\rho\cos\theta \\ y=\rho\sin\theta\end{cases}$$

Coord. pol. nel pto $(1,0)$

$D=\{(\rho,\theta) / 0\le\rho\le 1,\ 0\le\theta\le 2\pi\}$

$$\Rightarrow \int_0^1 d\rho\int_0^{2\pi}\sqrt{x^2+y^2}\cdot\rho\,d\theta\,d\rho = \int_0^1 d\rho\int_0^{2\pi}\sqrt{(1+\rho\cos\theta)^2+\rho^2\sin^2\theta}\cdot\rho\,d\theta\,d\rho$$

$$=\int_0^1 d\rho\int_0^{2\pi}\sqrt{1+2\rho\cos\theta+\underbrace{\rho^2\cos^2\theta+\rho^2\sin^2\theta}_{\text{nel fond}}}\cdot\rho\,d\theta\,d\rho = \int_0^1 d\rho\int_0^{2\pi}\sqrt{1+2\rho\cos\theta+\rho^2}\cdot\rho\,d\rho\,d\theta$$

Porto al primo integ

$$=\int_0^1\rho\,d\rho\int_0^{2\pi}\sqrt{1+2\rho\cos\theta+\rho^2}\,d\rho$$

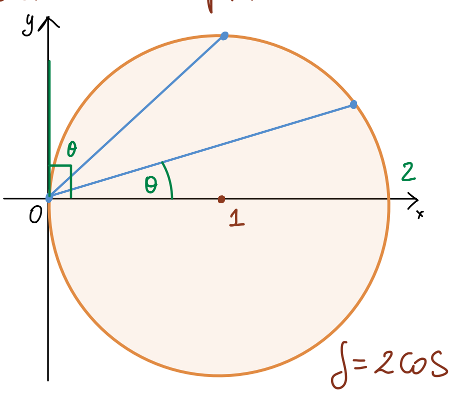
← con la radice è difficile da svolgere; facciamo finta che $f(x,y)$ era $x^2+y^2$:

Avremmo avuto: $\int_0^1\rho\,[\theta+2\rho\sin\theta+\rho^2\theta]_0^{2\pi}\,d\rho = \int_0^1\rho\,[2\pi+2\rho\sin(2\pi)+\rho^2(2\pi)]\,d\rho=\int_0^1\rho\,[2\pi+\rho^2 2\pi]\,d\rho$

$$=2\pi\int_0^1\rho+\rho^3\, d\rho = 2\pi\left[\frac{\rho^2}{2}+\frac{\rho^4}{4}\right]_0^1 = 2\pi\left[\frac12+\frac14\right] = \frac{3\pi}{2}$$

Senza la radice la risoluzione è semplice

Cosa ci insegna questo? In questo caso ci conveniva fare il cambiamento nelle classiche coordinate polari:




$\rho = 2\cos\theta$

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta\end{cases}$$

$\rho$ VARIA tra 0 e l'eq della circonferenza $C_1(1)$

$\to C: (x-1)^2+y^2=1$

Sviluppiamo l'eq: $y^2= 1-(x-1)^2$ ; $y^2=\cancel{1}-x^2+2x-\cancel{1}$

$\to y^2=2x-x^2 \to x^2+y^2-2x=0$   Eq cartesiana di C

Passiamo in coordinate polari

$$\delta^2 \underbrace{\cos^2\theta + \delta^2 \sin^2\theta}_{1} - 2\delta\cos\theta = 0 \to \delta^2 - 2\delta\cos\theta = 0$$

$$\delta(\delta - 2\cos\theta) = 0$$

$\hookrightarrow$ ~~$\delta = 0$~~ $\to$ NON ACC.

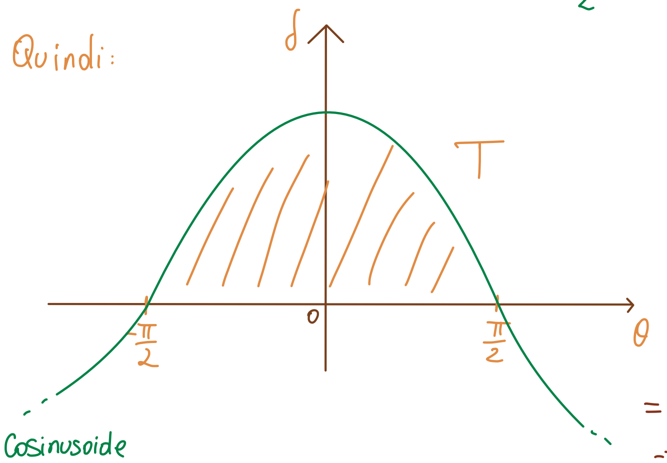
$\hookrightarrow$ $\delta - 2\cos\theta = 0$

Eq polare della circ

Quindi: Tra chi variano $\delta$ e $\theta$ ?

$$D = \{(\delta,\theta) \;/\; 0 \leq \delta \leq 2\cos\theta \;,\; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Quindi:




$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \sqrt{\underbrace{\delta^2\cos^2\theta + \delta^2\sin^2\theta}_{x^2+y^2}} \cdot \delta \, d\theta \, d\delta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \sqrt{\delta^2} \cdot \delta \, d\theta \, d\delta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int \delta^2 \, d\theta \, d\delta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\delta^3}{3}\right]_0^{2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^3\cos^3\theta}{3} \, d\theta = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \, d\theta$$

$$= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \cdot \cos\theta \, d\theta = \int \cos\theta - \int \sin^2\theta\cos\theta \, d\theta = \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{32}{9}$$

$\cos^2\theta \to 1 - \sin^2\theta$