Esercizio 1. Calcolare

$$\lim_{x\to 0}\frac{\ln(1-\ln(1-x))\sin(x)}{1-cos(x)}.$$

$$\lim_{x\to 0}\frac{\ln(1-\ln(1-x))\sin x}{1-\cos(x)} = \left[\frac{0}{0}\right]$$

$\to$ Hôpital : $D_n' = \dfrac{\frac{\sin x}{1-x}}{1-\ln(1-x)} + \ln(1-\ln(1-x))\cos(x) =$

$$= \frac{\sin(x) + \ln(1-\ln(1-x))\cos(x)(1-x)(1-\ln(1-x))}{(1-x)(1-\ln(1-x))} \qquad D_d' = \sin(x)$$

$$\frac{\overbrace{\sin(x)}^{0} + \overbrace{\ln(1-\ln(1-x))}^{0}\overbrace{\cos(x)}^{1}\overbrace{(1-x)}^{1}\overbrace{(1-\ln(1-x))}^{1}}{\underbrace{(1-x)}_{1}\underbrace{(1-\ln(1-x))}_{1}\underbrace{\sin(x)}_{0}} = \overbrace{\frac{\cancel{\sin(x)}}{(1-x)(1-\ln(1-x))\cancel{\sin(x)}}}^{1} +$$

$$+ \frac{\ln(1-\ln(1-x))}{\sin(x)} \cdot \overbrace{\frac{\cos(x)(1-x)(1-\ln(1-x))}{(1-x)(1-\ln(1-x))}}^{1}$$

$$\Rightarrow \lim_{x\to 0} 1 + 1\cdot\frac{\ln(1-\ln(1-x))}{\sin(x)} = \left[\frac{0}{0}\right] \Rightarrow \text{Hôpital} \quad D_n' = \frac{\frac{1}{1-x}}{1-\ln(1-x)} = \frac{1}{(1-\ln(1-x))(1-x)}$$

$$D_d' = \cos(x) \Rightarrow \frac{1}{\underbrace{(1-\underbrace{\ln(1-x)}_{0})}_{1}\underbrace{(1-x)}_{1}\cdot\underbrace{\cos(x)}_{1}} = \lim_{x\to 0} 1+1\cdot 1 = 2$$

Tempo ~ 9'

Esercizio 2. Stabire se la serie numerica

$$\sum_{n=1}^{\infty} \frac{2^n(n^3+\sin(n))}{5^n}$$

è convergente.

$$\sum_{n=1}^{\infty} \frac{2^n(n^3+\sin(n))}{5^n}$$

1) $\lim_{n\to\infty} \frac{2^n(n^3+\sin(n))}{5^n} = \underbrace{\left(\frac{2}{5}\right)^n}_{0} \cdot \underbrace{(n^3+\sin(n))}_{+\infty}$

$= \sin(+\infty)$ è oscillante $\Rightarrow \lim_{n\to\infty} \left(\frac{2}{5}\right)^n \cdot n^3 = 0$?

2) Criterio del rapp

$$\frac{\dfrac{2^{(n+1)}((n+1)^3+\sin(n+1))}{5^{(n+1)}}}{\dfrac{2^n(n^3+\sin(n))}{5^n}} = \frac{2}{5} \cdot \frac{\cancel{2^n}[(n+1)^3+\sin(n+1)]}{\cancel{5^n}} \cdot \frac{\cancel{5^n}}{\cancel{2^n}} \frac{1}{(n^3+\sin(n))}$$

$$= \frac{2}{5} \lim \frac{(n+1)^3+\sin(n+1)}{n^3+\sin(n)}$$

stesso grado

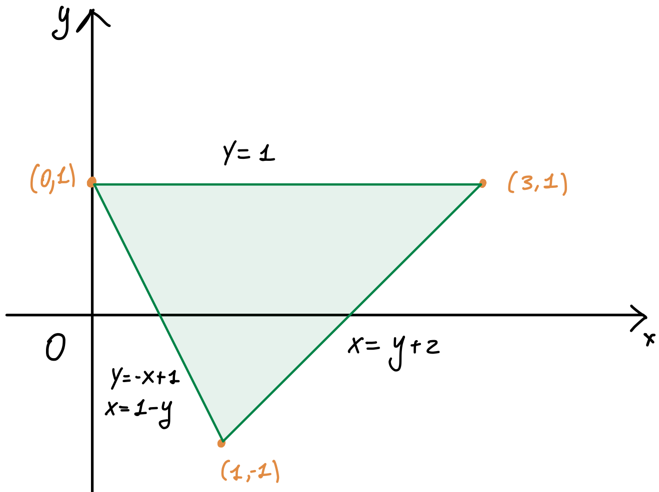
$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \to \frac{2}{5}$ converge

Tempo $\sim 10'$

Esercizio 3. Calcolare

$$\iint_D \sin(x)y\,dxdy$$

Dove D è il triangolo di vertici $A=(0,1)$ $B=(1,-1)$ $C=(3,1)$.



$$\frac{x-1}{3-1} = \frac{y+1}{1+1} \Rightarrow \frac{x-1}{2} = \frac{y+1}{2}$$

$$\Rightarrow x-1 = y+1 \Rightarrow \begin{array}{l} y = x-2 \\ x = y+2 \end{array}$$

$$D_y = \{(x,y) \,/\, -1 < y < 1 \,,\, 1-y < x < y+2\}$$

$$\Rightarrow \int_{-1}^{1}\int_{1-y}^{y+2} \sin(x)\, y\, dx\, dy = \int_{-1}^{1} y \int_{1-y}^{y+2} \sin x\, dx\, dy$$

$$= \int_{-1}^{1} y\, \Big[-\cos x\Big]_{1-y}^{y+2} dy = \int_{-1}^{1} y \cdot \Big[-\cos(y+2)+\cos(1-y)\Big] dy$$

$$= -\underbrace{\int_{-1}^{1} y\cos(y+2)\, dy}_{a} + \int_{-1}^{1} y\cos(1-y)\, dy$$

a) $-\int y\cos(y+2)\,dy \quad t = y+2 \Rightarrow dy = dt \Rightarrow -\int (t-2)(\cos(t))\,dt = -\int t\cos t + 2\int \cos t\, dt$

$\Rightarrow y = t-2$

$$\overset{\text{parti}}{=} -\Big[t\sin t - \int \sin t\, dt\Big] + 2\sin t = -t\sin t - \cos t + 2\sin t \underset{t=y+2}{=} -(y+2)\sin(y+2) - \cos(y+2) + 2\sin(y+2)$$

$$= -y\sin(y+2) - \cos(y+2)$$

b) $\int y\cos(1-y)\,dy \quad t = 1-y \Rightarrow dy = -dt$

$\Rightarrow y = 1-t$

$$\Rightarrow -\int (1-t)\cos(t)\, dt = -\int \cos(t) + \int t\cos t\, dt = -\sin t + \Big[t\sin t - \int \sin t\, dt\Big]$$

$$= -\sin t + t\sin t + \cos t \underset{t=1-y}{=} -\sin(1-y) + (1-y)\sin(1-y) + \cos(1-y)$$

$$\cos(1-y) - y\sin(1-y)$$

$$\Rightarrow \Big[-y\sin(y+2) - \cos(y+2)\Big]_{-1}^{1} = -\sin(3) - \cos(3) - \sin(1) + \cos(1)$$

$$\Rightarrow \Big[\cos(1-y) - y\sin(1-y)\Big]_{-1}^{1} = \underbrace{\cos(0)}_{1} - \underbrace{\sin(0)}_{0} - \cos(2) - \sin(2)$$

$\Rightarrow$ SUM $= -\sin(3) - \cos(3) - \sin(1) + \cos(1) - \cos(2) - \sin(2) + 1$

Tempo $\sim 15'$

Esercizio 4. Sia α un parametro **reale** e **positivo**. Discutere, al variare di α la differenziabilità della funzione $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = |\sin(xy)|^{\alpha}.$$

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$D: \mathbb{R}^2$

$\Rightarrow$ f è Diff quando $f(x_0,y_0) = \lim_{(x,y)\to P} \dfrac{f(x,y) - f(P) - f'_x(P)(x-x_0) - f'_y(P)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$

Ma in questo caso non abbiamo un punto $P=(x_0,y_0)$ ma un intero dominio.

$f(x,y)$ è Differenziabile se esistono le derivate parziali ed esse sono continue in $P_0=(x_0,y_0)$

$$\Rightarrow \left.\begin{array}{l} f'_x = \alpha|\sin(xy)|^{\alpha-1}\cdot\cos(xy)\cdot y \\ f'_y = \alpha|\sin(xy)|^{\alpha-1}\cdot\cos(xy)\cdot x \end{array}\right\}$$

Prendo in esame f'_x visto che hanno la stessa struttura

$\Rightarrow$ $\alpha|\sin(xy)|^{\alpha-1}\cos(xy)\cdot y \sim \alpha|\sin(xy)|^{\alpha-1}$

$\to \alpha=1 \to |\sin(xy)|^0 = 1$ no prob

$\to \alpha>1 \to \alpha|\sin(xy)|^{\alpha(-1)}$ $\to$ no prob

omesso

$\to \alpha<1 \to -a|\sin(xy)|^{-a} = \dfrac{-a}{(\sin(xy))^a}$

$\Rightarrow -\alpha = a$

$\Rightarrow$ Con $\alpha<1$ abbiamo una discontinuità: $\sin(xy)\neq 0$ per $xy\neq 0$ $\to x=0$ OR $\to y=0$

$\Rightarrow$ Per $\alpha<1$ $f(x,y)$ <u>NON</u> è differenziabile (in $D:\mathbb{R}^2$)

$$\frac{\alpha|\sin(xy)|^{\alpha}}{|\sin(xy)|}$$

$\alpha=1 \to z=1$

$\alpha>1 \to f$

$0 \to \dfrac{\alpha}{|\sin(xy)|}$

Tempo 9'42''

Esercizio 5. Si consideri la seguente forma differenziale

$$\omega = \frac{1}{1+y^2}\,dx + \left(y + \frac{2xy}{(1+y^2)^2}\right)\,dy.$$

Si dica se essa è esatta e, in caso positivo, si calcoli una primitiva.

$A: \ 1+y^2 \neq 0 \quad \text{per } y^2 \neq -1$

$A: \ \forall x,y \in \mathbb{R}^2 = \mathbb{R}^2 \to \text{C.C.}$

$$W = \frac{1}{1+y^2}\,dx + \left(y + \frac{2xy}{(1+y^2)^2}\right)dy$$

$$\Rightarrow X = \frac{1}{1+y^2} \qquad X'_y = D\left[(1+y^2)^{-1}\right] = -\frac{2y}{(1+y^2)^2}$$

Le derivate sono diverse!

$$Y = y + \frac{2xy}{(1+y^2)^2} \quad \Rightarrow \quad Y'_x = D\left[\frac{2y}{(1+y^2)^2}\cdot x\right] = \frac{2y}{(1+y^2)^2}$$

Siccome le derivate parziali non coincidono, la F.D. non è chiusa. Possiamo quindi affermare che non è esatta.

Se fosse stata esatta:

$$\int \frac{1}{1+y^2}\,dx = \frac{1}{1+y^2}\int dx = \frac{x}{1+y^2} + c(y)$$

$$\rightarrow D'_y = D\left[x(1+y^2)^{-1} + c(y)\right] = \cancel{\frac{2yx}{(1+y^2)^2}} + c'(y) = y + \cancel{\frac{2xy}{(1+y^2)^2}}$$

$$\rightarrow \quad c'(y) = y \quad \Rightarrow \quad c(y) = \int y\,dy \quad \Rightarrow \quad c(y) = \frac{1}{2}y^2 + K$$

$$\Rightarrow F(x,y) = \frac{2xy}{(1+y^2)^2} + \frac{1}{2}y^2 + K$$

Tempo ~ 8'

Esercizio 6. Si risolva il seguente problema di Cauchy

$$\begin{cases} y' = \frac{y\ln(y)}{x^2+x} \\ y(1) = e \end{cases}$$

$$y' = \frac{y\ln(y)}{x^2+x}$$

$$\Rightarrow \frac{y'}{y\ln(y)} = \frac{1}{x^2+x} \Rightarrow \frac{dy}{y\ln y} = \frac{dx}{x^2+x}$$

$$\Rightarrow \underbrace{\int \frac{dy}{y\ln y}}_{b} = \underbrace{\int \frac{dx}{x^2+x}}_{a}$$

a) $\frac{1}{x^2+x} = \frac{1}{x(x+1)} \Rightarrow \frac{A}{x} + \frac{B}{x+1} = \frac{Ax+A+Bx}{x(x+1)} = 1$

$$\Rightarrow \begin{cases} A+B=0 \Rightarrow \underline{B=-1} \\ \underline{A=1} \end{cases} \Rightarrow \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \Rightarrow \int \frac{1}{x}dx - \int\frac{1}{x+1}dx = \ln\left|\frac{x}{x+1}\right| + c$$

b) $t = \ln y \Rightarrow dy = y\,dt \Rightarrow \int \frac{dy}{\cancel{y}\ln y}\cancel{y} = \int \frac{1}{\ln y}dt \underset{t=\ln y}{=} \int \frac{1}{t}dt = \ln|t| \underset{t=\ln y}{=} \ln|\ln y|$

$$\Rightarrow \ln(\ln y) = \ln\left|\frac{x}{x+1}\right| + c \Rightarrow \ln(\ln y) - \ln\left(\frac{x}{x+1}\right) = c \Rightarrow$$

$$\ln\left(\frac{\ln y}{\frac{x}{x+1}}\right) = c \Rightarrow \frac{\ln y}{\frac{x}{x+1}} = \bar{c} \Rightarrow \ln y = \bar{c}\left(\frac{x}{x+1}\right) \Rightarrow y = e^{\left(\frac{x}{x+1}\right)\bar{c}}$$

Cauchy

$$y(1) = e^{\left(\frac{1}{1+1}\right)\bar{c}} = e \Rightarrow e^{\frac{1}{2}\bar{c}} = e \Rightarrow \frac{1}{2}\bar{c} = 1 \Rightarrow \bar{c} = 2$$

$$\Rightarrow \text{SOL}: \quad y = e^{\left(\frac{x}{x+1}\right)2}$$

Tempo: poco (<6')

Tempo totale ~ 58'