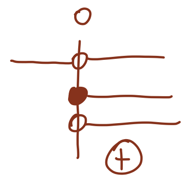
1. Studiare la seguente funzione e disegnarne il grafico: $y = \frac{\ln x}{\sqrt{x}}$.

$y = \ln x \cdot (x)^{-\frac{1}{2}}$

1) Dominio $\sqrt{x} \neq 0$, $x \geq 0$, $x > 0$

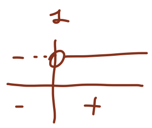
$\hookrightarrow x \neq 0$



$\Rightarrow \mathbb{D}: x > 0$

2) Segno: $f(x) > 0$ per $\ln x > 0 \rightarrow x > 1$

$\sqrt{x} > 0 \rightarrow \forall x \in \mathbb{R}$



$\Rightarrow f(x) > 0$ per $x > 1$

3) Simmetrie

$f(-x) = \frac{\ln(-x)}{\sqrt{-x}}$ $\neq f(x)$, $\neq -f(x)$ NO Simm

Tempo: 40' con ragionamenti

4) Intersez:

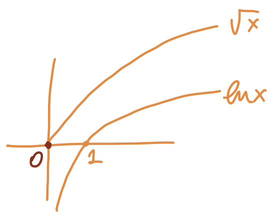
$\begin{cases} y = \frac{\ln x}{\sqrt{x}} \\ x = 0 \end{cases} \Rightarrow y = \frac{\ln(0)}{\sqrt{0}} \not\exists\, x \in \mathbb{D}$

$\begin{cases} y = \frac{\ln x}{\sqrt{x}} \\ y = 0 \end{cases} \rightarrow \frac{\ln x}{\sqrt{x}} = 0 \rightarrow \ln x = 0$ per $x = 1$

$\Rightarrow (1,0) \in f(x)$ Asse x

5) Asintoti

$\lim_{x \to 0^+} \frac{\ln x}{\sqrt{x}}$ $\ln x \rightarrow -\infty$, $\sqrt{x} \rightarrow 0$



$= \ln x \cdot \frac{1}{\sqrt{x}} \rightarrow -\infty$ ($\ln x \to -\infty$, $\frac{1}{\sqrt{x}} \to +\infty$) $\Rightarrow x = 0$ A.V.

$\lim_{x \to +\infty} \frac{\ln x}{\sqrt{x}}$ ($\ln x \to +\infty$, $\sqrt{x} \to +\infty$) $\sqrt{x} \gg \ln x \rightarrow 0 \Rightarrow y = 0$ A.O.

6) Max / min

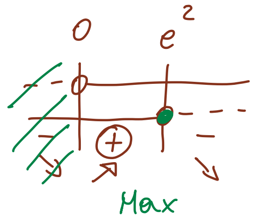
$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{\frac{2x - x\ln x}{2x\sqrt{x}}}{x} = \frac{2x^2 - x^2\ln x}{2x\sqrt{x}} = \frac{\cancel{x}(2x - x\ln x)}{2\cancel{x}\sqrt{x}}$

$\Rightarrow \frac{2x - x\ln x}{2\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{2x\sqrt{x} - x\sqrt{x}\ln x}{2x} = \frac{\cancel{x}(2\sqrt{x} - \sqrt{x}\ln x)}{2\cancel{x}} = \frac{2\sqrt{x} - \sqrt{x}\ln x}{2} > 0$ per

$\rightarrow 2\sqrt{x} - \sqrt{x}\ln x > 0 \rightarrow \sqrt{x}(2 - \ln x) > 0$

$\hookrightarrow \sqrt{x} > 0$ per $x > 0$

$\hookrightarrow \ln x < 2$ per $x < e^2$



$D[\sqrt{x}] = \frac{1}{2\sqrt{x}}$

$f(e^2) = \frac{\ln(e^2)}{\sqrt{2}} = \frac{2}{\sqrt{2}} \Rightarrow \left(e^2, \frac{2}{\sqrt{2}}\right) \in f(x)$ MAX

7 Flessi

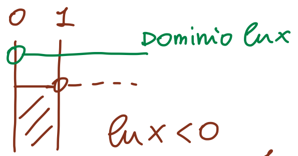
$f''(x) = \frac{\frac{1}{\sqrt{x}} - \left(\frac{\sqrt{x}\ln x}{2x} - \frac{\sqrt{x}}{x}\right)}{4} = \frac{\frac{\sqrt{x}}{x} - \left(\frac{\sqrt{x}\ln x - 2\sqrt{x}}{2x}\right)}{4} = \frac{\frac{2x\sqrt{x} - \sqrt{x}\ln x + 2\sqrt{x}}{2x}}{4}$

$= \frac{\frac{\cancel{\sqrt{x}}(2x - \ln x + 2)}{2\sqrt{x}}}{4} > 0$ per $2x - \ln x + 2 > 0$ D: $x > 0$

Pos

$\ln x > 0$ per $x > 1$

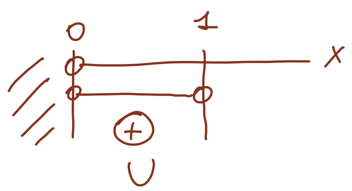
$-\ln x > 0$ per $x < 1$



$\ln x < 0$ per $0 < x < 1$

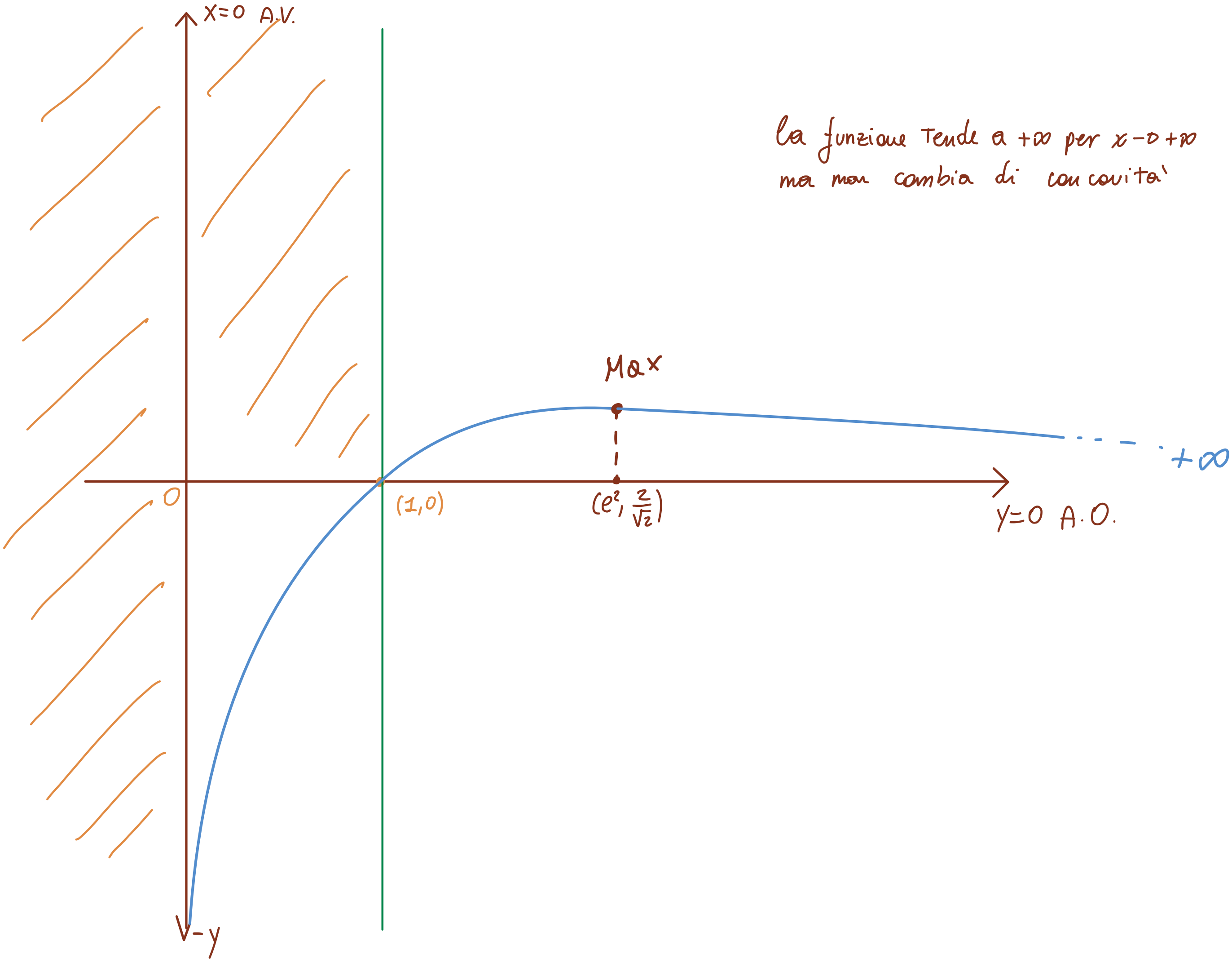
$2x > 0$ per $x > 0$

$\Rightarrow$



$\Rightarrow$ Flesso in $x = 1$

la funzione Tende a +∞ per $x \to +\infty$
ma non cambia di concavita'



2. Calcolare il limite: $\lim_{x\to 0} \dfrac{\sqrt[3]{1+x^2}-1}{(1-\cos x)\ln\left(1+\sin^2 x\right)}$.

$$= \frac{\cancel{\sqrt{1+0}-1}^{\,0}}{\underbrace{\cancel{1-\cos(0)}}_{0}\cdot \underbrace{\ln(\cancel{1+\sin^2(0)})}_{0}} = \frac{0}{0}$$

$$f(N) = D\left[(1+x^2)^{\frac{1}{3}}\right] = \frac{1}{3}(1+x^2)^{-\frac{2}{3}}\cdot 2x = \frac{2}{3}\,\frac{x}{\sqrt{(1+x^2)^2}}$$

$$f'(D) = \sin x \ln(1+\sin^2 x) + (1-\cos x)\cdot\frac{2\sin x\cos x}{1+\sin^2 x} = \sin x\ln(1+\sin^2 x) + \frac{2\sin x\cos x - 2\sin x\cos^2 x}{1+\sin^2 x}$$

$$= \frac{\sin x\ln(1+\sin^2 x)+\sin^3 x\ln(1+\sin^2 x)+2\sin x\cos x-2\sin x\cos^2 x}{1+\sin^2 x} = \frac{\sin x\ln(1+\sin^2 x)\left[1+\sin^2 x\right]+\sin x\cos x(2-2\cos x)}{1+\sin^2 x}$$

$$\lim_{x\to 0} \frac{\cancel{2x}^{\,0}}{3\underbrace{\sqrt{\cancel{(1+x^2)^2}}}_{1}}\cdot\frac{\cancel{1+\sin^2 x}^{\,0}}{\cancel{\sin x\ln(1+\sin^2 x)}\left[\cancel{1+\sin^2 x}\right]+\cancel{\sin x\cos x(2-2\cos x)}} = \frac{0}{3}\cdot\frac{1}{0} \to +\infty$$

$$= \frac{2x+2x\sin^2(x)}{3\sqrt{(1+x^2)^2}\sin x\ln(1+\sin^2 x)\left[1+\sin^2 x\right]+3\sqrt{(1+x^2)^2}\sin x\cos x(2-2\cos x)}$$

4. Calcolare il seguente integrale: $\int \arcsin^2 x dx$.

$$\int \arcsin x \, dx \underset{\text{Parti}}{=} \int \arcsin x \cdot D[x] \, dx$$

$$= x \operatorname{asin} x - \int \frac{1}{\sqrt{1-x^2}} \cdot x \, dx$$

pongo $t = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} \Rightarrow dx = -\frac{1}{\frac{1}{\cancel{2}} \frac{\cancel{2}x}{\sqrt{1-x^2}}} = -\frac{1}{\frac{x}{\sqrt{1-x^2}}}$

$$= -\frac{\sqrt{1-x^2}}{x} dt$$

$$\Rightarrow + \int \frac{1}{\cancel{\sqrt{1-x^2}}} \cancel{\cdot x} \cdot \frac{\cancel{\sqrt{1-x^2}}}{\cancel{x}} dt = \int dt = t$$

Tempo 20'

$$\Rightarrow \int \operatorname{asin} x \, dx = x \operatorname{asin} x + \sqrt{1-x^2} + C$$

$$\int \operatorname{asin}^2 x \, dx = \operatorname{asin} x \cdot \left(x \operatorname{asin} x + \sqrt{1-x^2} \right) - \int \frac{x \operatorname{asin} x}{\sqrt{1-x^2}} + \int dx$$

$$- \int \frac{x \operatorname{asin} x}{\sqrt{1-x^2}} dx$$ pongo $t = \sqrt{1-x^2} \Rightarrow dx = -\frac{\sqrt{1-x^2}}{x} dt \Rightarrow + \int \frac{\cancel{x} \operatorname{asin} x}{\cancel{\sqrt{1-x^2}}} \cdot \frac{\cancel{\sqrt{1-x^2}}}{\cancel{x}} dt = \int \operatorname{asin} x \, dx$

$$= x \operatorname{asin} x + \sqrt{1-x^2} + C$$

$$\Rightarrow \int \operatorname{asin}^2 x \, dx = \boxed{x \operatorname{asin}^2 x + \operatorname{asin} x \sqrt{1-x^2} + x \operatorname{asin} x + \sqrt{1-x^2} + C}$$

5. Calcolare l'integrale del seguente problema di Cauchy: $\begin{cases} y'' + 3y' - 4y = -5e^{-4x} \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$.

$y''+3y'-4y = 5e^{-4x}$

$\Rightarrow \lambda^2+3\lambda-4=0 \Rightarrow \Delta = 9-4\cdot(-4)=25 \Rightarrow \lambda_{1,2} = \frac{-3\pm 5}{2} \begin{cases} 1 \\ -4 \end{cases}$

$\Rightarrow y_0(x) = c_1 e^{x} + c_2 e^{-4x}$ $\quad f(x) = -5e^{-4x} \Rightarrow \gamma = -4 \Rightarrow h = 1$

$\Rightarrow y_p(x) = x^h \cdot e^{\gamma x} \cdot (P^n(x)) = x \cdot e^{-4x} \cdot (A) = Axe^{-4x} \Rightarrow y_p'(x) = Ae^{-4x} - Ax4e^{-4x} = e^{-4x}(A-4Ax)$

$y_p''(x) = -4Ae^{-4x} - [-A4e^{-4x} + Ax16e^{-4x}] = -\cancel{4Ae^{-4x}} + \cancel{A4e^{-4x}} - Ax16e^{-4x} = -e^{-4x}Ax16$

$\Rightarrow -e^{-4x}Ax16 + 3Ae^{-4x} - Ax12e^{-4x} - 4Axe^{-4x} = -5e^{-4x} \Rightarrow e^{-4x}(-\cancel{16Ax} + 3A - \cancel{12Ax} - \cancel{4Ax}) = -5e^{-4x}$

$\Rightarrow 3Ae^{-4x} = -5e^{-4x} \Rightarrow 3A = -5 \Rightarrow A = -\frac{5}{3}$

Tempo: 10'

$\Rightarrow y_p(x) = -\frac{5}{3}xe^{-4x} \Rightarrow y = -\frac{5}{3}xe^{-4x} + c_1e^{x} + c_2e^{-4x}$

Cauchy

$y(0) = \cancel{-\frac{5}{3}x}\underset{0}{\cancel{e^{-4x}}} + c_1\underset{1}{\cancel{e^{x}}} + c_2\underset{1}{\cancel{e^{-4x}}} = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow \begin{cases} c_1 = -c_2 \\ c_2 = -c_1 \end{cases}$

$y' = -\frac{5}{3}e^{-4x} + 4xe^{-4x} + c_1e^{x} - 4c_2e^{-4x} \Rightarrow y'(0) = -\frac{5}{3}\underset{1}{\cancel{e^{-4x}}} + \underset{0}{\cancel{4x}}e^{-4x} + c_1\underset{1}{\cancel{e^{x}}} - 4c_2\underset{1}{\cancel{e^{-4x}}} = 0$

$\Rightarrow c_1 - 4c_2 = \frac{5}{3} \underset{c_1=-c_2}{=} -c_2 - 4c_2 = \frac{5}{3} \Rightarrow -5c_2 = \frac{5}{3} \Rightarrow c_2 = -\frac{1}{3} \Rightarrow c_1 = \frac{1}{3}$

$\Rightarrow y = -\frac{5}{3}xe^{-4x} + \frac{1}{3}e^{x} - \frac{1}{3}e^{-4x}$

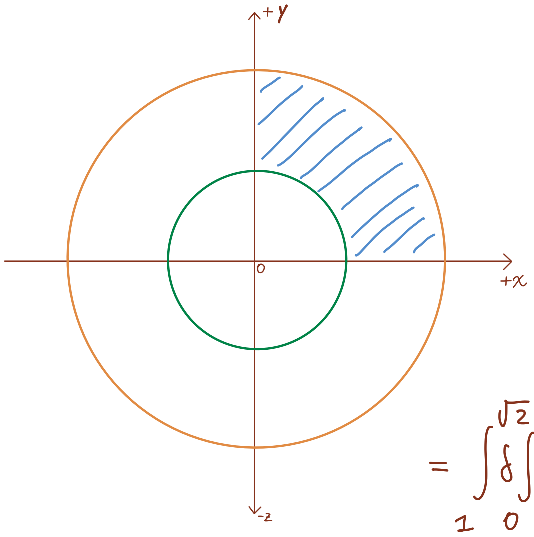
6. Calcolare l'integrale doppio $\int\int_D \frac{xy}{x^2+y^2}dxdy$, dove

$$D = \{(x,y) : 1 \le x^2+y^2 \le 2, x \ge 0, y \ge 0\}.$$

$z = x^2+y^2$ Indica un cerchio

$x^2+y^2=1$ cerchio di $r=1$

$x^2+y^2=2$ cerchio di $r=\sqrt{2}$



In coordinate polari

$$D = \{(\rho,\theta) / 1 < \rho < \sqrt{2},\ 0 < \theta < \frac{\pi}{2}\}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow \int_1^{\sqrt{2}}\int_0^{\pi/2} \frac{\rho\cos\theta\,\rho\sin\theta}{\rho^2\cos^2\theta + \rho^2\sin\theta} \cdot \rho\, d\rho\, d\theta$$

$$= \int_1^{\sqrt{2}}\rho\int_0^{\pi/2} \frac{\cancel{\rho^2}\cos\theta\sin\theta}{\cancel{\rho^2}\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}} d\rho\, d\theta = \int_1^{\sqrt{2}}\rho\int_0^{\pi/2}\cos\theta\sin\theta\, d\rho\, d\theta$$

$$\Rightarrow \int \underset{f}{\cos\theta}\,\underset{g}{\sin\theta}\, d\theta \overset{\text{PARTI}}{=} -\cos^2\theta - \int \sin\theta\cos\theta\, d\theta = \int\cos\theta\sin\theta\, d\theta$$

pongo $\int\cos\theta\sin\theta = I$

$$\Rightarrow 2I = -\cos^2\theta \Rightarrow \int\cos\theta\sin\theta\, d\theta = -\frac{\cos^2\theta}{2}$$

OPPURE $\int\cos\theta\sin\theta\, d\theta \Rightarrow$ pongo $t = \sin\theta \Rightarrow d\theta = \frac{1}{\cos\theta}dt \Rightarrow \int\cancel{\cos\theta}\sin\theta\cdot\frac{1}{\cancel{\cos\theta}}dt$

$$\Rightarrow \int t\,dt = \frac{t^2}{2} + c \Rightarrow \frac{\sin^2(\theta)}{2}$$

$$\Rightarrow \int_1^{\sqrt{2}}\rho\left[\frac{\sin^2(\theta)}{2}\right]_0^{\frac{\pi}{2}} d\rho = \int_1^{\sqrt{2}}\rho\left[\frac{1}{2}\right]d\rho = \frac{1}{2}\int_1^{\sqrt{2}}\rho\, d\rho = \frac{1}{2}\left[\frac{\rho^2}{2}\right]_1^{\sqrt{2}} = \frac{1}{2}\left[1-\frac{1}{2}\right] = \frac{1}{4}$$

Equivalente

uso $\int\cos\theta\sin\theta\, d\theta = -\frac{\cos^2\theta}{2} \Rightarrow \int_1^{\sqrt{2}}\rho\left[-\frac{\cos^2\theta}{2}\right]_0^{\frac{\pi}{2}}d\rho = \int_1\rho\left[0+\frac{1}{2}\right]d\rho = \frac{1}{2}\left[\frac{\rho^2}{2}\right] = \frac{1}{4}$

3. Studiare la seguente serie numerica: $\sum_{n=1}^{+\infty}\left(e^{\frac{n^2+n}{n^2+1}}-e\right)$.

$$\lim_{n\to+\infty} e^{\frac{n^2+n}{n^2+1}} - e \to \left[\frac{n^2+n}{n^2+1}\right] = \left[\frac{\cancel{n^2}(1+\cancel{\frac{1}{n}})}{\cancel{n^2}(1+\cancel{\frac{1}{n^2}})}\right] \to 1$$

$$\Rightarrow \lim_{n\to+\infty} a_n = e^{1} - e = \emptyset$$

Il criterio di divergenza non è conclusivo.

$\Rightarrow$ la serie NON converge