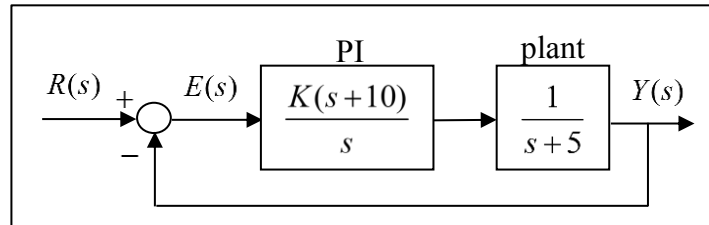


## Introductory Control Systems

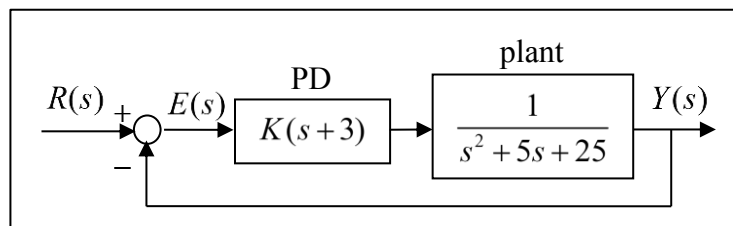
### Exercises #10 – Steady-State Error

1. A **proportional-integral** (“PI”) controller is used to control a 1<sup>st</sup> order plant as shown. The system has input  $R(s)$ , output  $Y(s)$ , and error  $E(s)$ . Find  $\frac{E}{R}(s)$  the **error transfer function**, and then find the range of values for the parameter  $K$  so the system has a **steady-state error**  $e_{ss}$  less than 0.01 for a **unit ramp** input ( $R(s) = 1/s^2$ ). Assume  $K > 0$ .



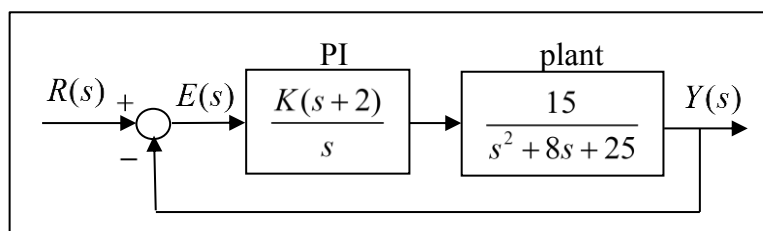
Answers:  $\frac{E}{R}(s) = \frac{s(s+5)}{s(s+5) + K(s+10)}$ ;  $K > 50$

2. A **proportional-derivative** (“PD”) controller is used to control a 2<sup>nd</sup> order plant as shown. The system has input  $R(s)$ , output  $Y(s)$ , and error  $E(s)$ . Find  $\frac{E}{R}(s)$  the **error transfer function**, and find  $e_{ss}$  the **steady-state error** associated with a **unit step** input in terms of the parameter  $K$ .



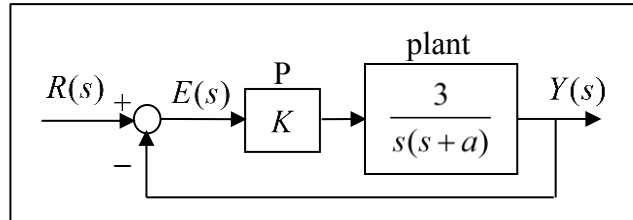
Answers:  $\frac{E}{R}(s) = \frac{s^2 + 5s + 25}{s^2 + 5s + 25 + K(s+3)}$ ;  $e_{ss} = \frac{25}{25 + 3K}$

3. A **proportional-integral** (“PI”) controller is used to control a 2<sup>nd</sup> order plant as shown. The system has input  $R(s)$ , output  $Y(s)$ , and error  $E(s)$ . Find  $\frac{E}{R}(s)$  the **error transfer function**, and then find the range of values for the parameter  $K$  so the system has a steady-state error less than 0.1 to a **unit ramp** input ( $R(s) = 1/s^2$ ). Assume  $K > 0$ .



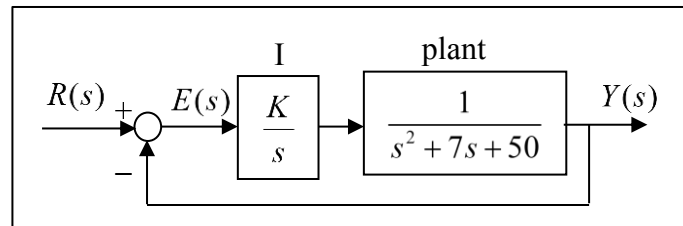
Answers:  $\boxed{\frac{E}{R}(s) = \frac{s(s^2 + 8s + 25)}{s(s^2 + 8s + 25) + 15K(s + 2)}}; \boxed{K > 8.33}$

4. A **proportional** (“P”) controller is used to control a 2<sup>nd</sup> order plant as shown. The system has input  $R(s)$ , output  $Y(s)$ , and error  $E(s)$ . Find  $\frac{E}{R}(s)$  the **error transfer function**, and then find  $e_{ss}$  the **steady-state error** associated with a **unit ramp** input in terms of the parameters  $a$  and  $K$ .



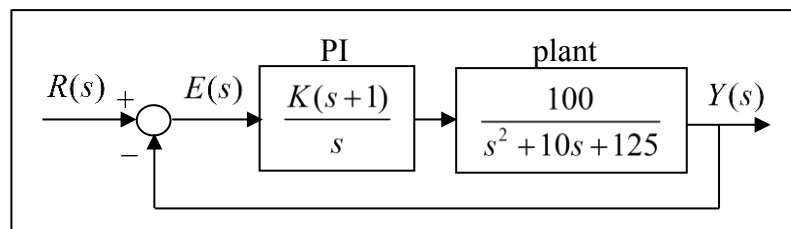
Answers:  $\boxed{\frac{E}{R}(s) = \frac{s(s+a)}{s^2 + as + 3K}}, \boxed{e_{ss} = \frac{a}{3K}}$

5. An **integral** (“I”) controller is used to control a 2<sup>nd</sup> order plant as shown. The system has input  $R(s)$ , output  $Y(s)$ , and error  $E(s)$ . Find  $\frac{E}{R}(s)$  the **error transfer function**, and then find the range of values for the gain  $K$  so  $e_{ss}$  the steady-state error due to a **unit ramp** input is less than 1.0. Assume  $K > 0$ .



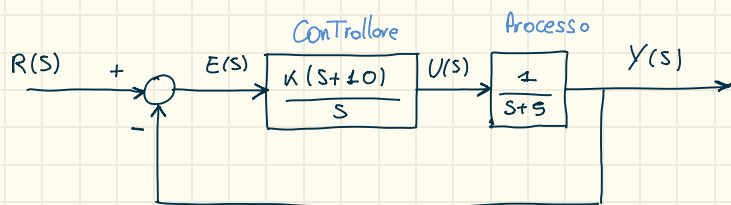
Answers:  $\boxed{\frac{E}{R}(s) = \frac{s(s^2 + 7s + 50)}{s^3 + 7s^2 + 50s + K}}; \boxed{K > 50}$

6. A **proportional-integral** (“PI”) controller is used to control a 2<sup>nd</sup> order plant as shown. The system has input  $R(s)$ , output  $Y(s)$ , and error  $E(s)$ . Find  $\frac{E}{R}(s)$  the **error transfer function**, and then find the range of values for the parameter  $K$  so the steady-state error to a **unit ramp** input is less than 0.1. Assume  $K > 0$ .



Answers:  $\boxed{\frac{E}{R}(s) = \frac{s(s^2 + 10s + 125)}{s(s^2 + 10s + 125) + 100K(s + 1)}}; \boxed{K > 12.5}$

ES 1



$$1) \frac{E(s)}{R(s)} = T_{e \rightarrow e} = S(s) = \frac{1}{1 + C(s)G(s)} = \frac{1}{1 + \frac{K(s+10)}{S(s+5)}} = \frac{S(s+5)}{S(s+5) + K(s+10)}$$

2)  $e_{ss}$  per RAHPA UNITARIA  $e(t) = t \cdot \mathcal{L}(t) \Leftrightarrow R(s) = \frac{1}{s^2}$  ( $K > 0$ )  $\Rightarrow e_v < 1\% \Rightarrow 0.01$

F. Anello  $C(s) \cdot G(s) = L(s) = \frac{K(s+10)}{\cancel{S}(s+5)}$   
 Tipo 1  $\Rightarrow$  1 azione integrale

$e_p = 0$ ,  $e_v = \frac{1}{\mu}$  con  $\mu =$  guadagno di  $L(s)$

$$\mu = \lim_{s \rightarrow 0} \frac{K(s+10)}{\cancel{S}(s+5)} \cdot \cancel{s} = \frac{10K}{5} = 2K \Rightarrow e_v = \frac{1}{2K} < 0.01 \Rightarrow 2K > 100 \Rightarrow K > 50$$

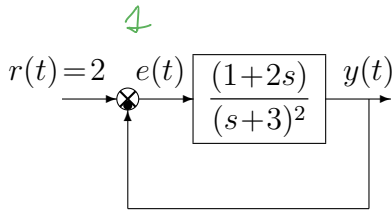
Procedimento alternativo

$$\frac{E(s)}{R(s)} = \frac{S(s+5)}{S(s+5) + K(s+10)} \Rightarrow E(s) = \frac{S(s+5)}{S(s+5) + K(s+10)} \cdot R(s) \Rightarrow e_{ss}(t) = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \cdot E(s)$$

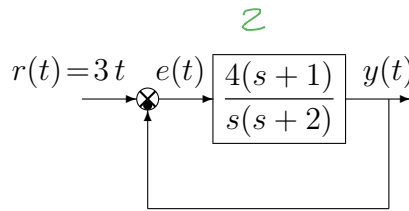
$$\Rightarrow e_{ss} = \lim_{s \rightarrow 0} \cancel{s} \cdot \frac{\cancel{S}(s+5)}{\cancel{S}(s+5) + K(s+10)} \cdot \frac{1}{\cancel{s^2}} = \frac{5}{[5 + K(s+10)]s} =$$

## Esempi

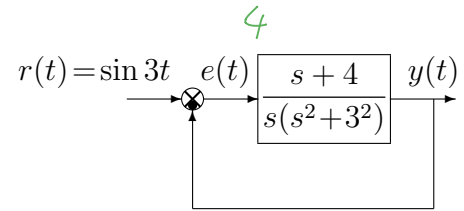
- Calcolare l'errore a regime  $e(\infty)$  per i seguenti sistemi retroazionati:



$$e(\infty) = \frac{2}{1 + \frac{1}{9}} = \frac{18}{10} = 1.8$$

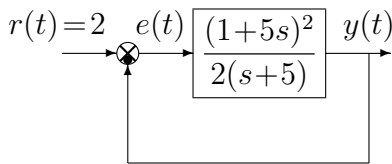


$$e(\infty) = \frac{3}{2} = 1.5$$

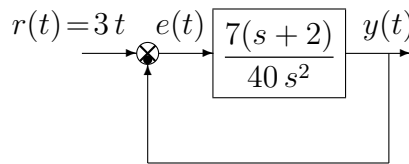


$$e(\infty) = 0$$

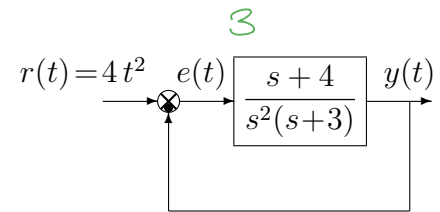
- Calcolare l'errore a regime  $e(\infty)$  per i seguenti sistemi retroazionati:



$$e(\infty) = \frac{2}{1 + \frac{1}{10}} = \frac{20}{11} = 1.8182$$

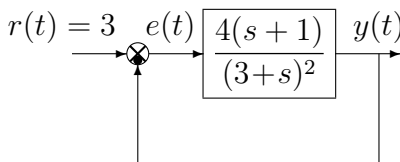


$$e(\infty) = 0$$

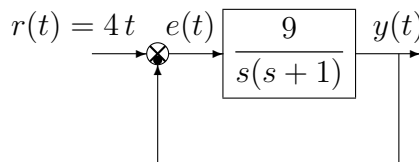


$$e(\infty) = 6$$

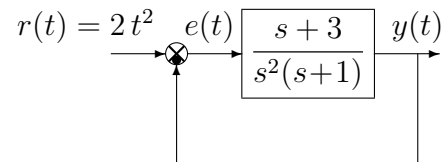
- Calcolare l'errore a regime  $e(\infty)$  per i seguenti sistemi retroazionati:



$$e(\infty) = \frac{3}{1 + \frac{4}{9}} = \frac{27}{13} = 2.08$$

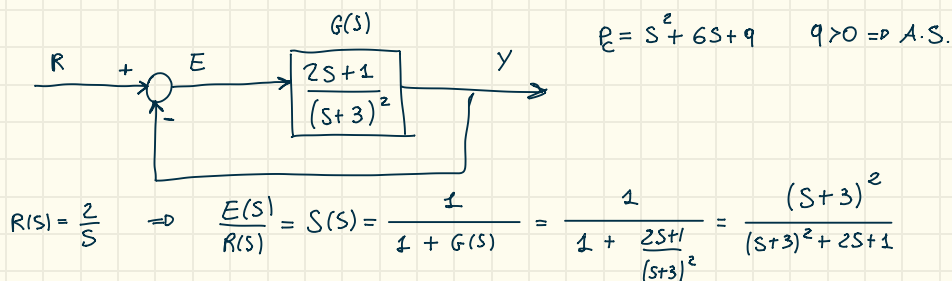


$$e(\infty) = \frac{4}{9} = 0.444$$



$$e(\infty) = \frac{4}{3} = 1.333$$

①

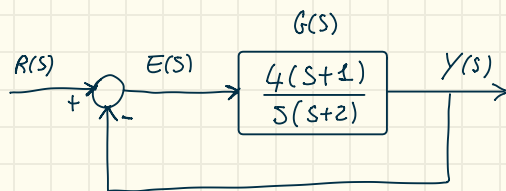


$$\Rightarrow E(s) = \frac{(s+3)^2}{(s+3)^2 + 2s+1} \cdot \frac{2}{s} \Rightarrow e_{ss} = \lim_{s \rightarrow 0} s \cdot \frac{(s+3)^2}{(s+3)^2 + 2s+1} \cdot \frac{2}{s} = \frac{9}{9+1} \cdot 2 = \frac{9}{5} = 1.8 \text{ Ans}$$

OPPURE Conoscendo le formule...

$$\mu_p = \lim_{s \rightarrow 0} \frac{2s+1}{(s+3)^2} = \frac{1}{9} \Rightarrow e_p = \frac{R_0}{1 + \mu_p} = \frac{2}{1 + 1/9} = 1.8$$

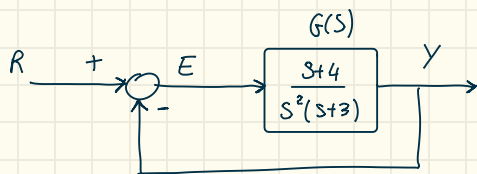
②



$$x(t) = 3t \Rightarrow R(s) = \frac{3}{s^2} \leftarrow R_0 \Rightarrow e(\infty) = e_{\text{velocità}}$$

$$\Rightarrow \mu_v = \lim_{s \rightarrow 0} s \cdot G(s) = \lim_{s \rightarrow 0} s \cdot \frac{4(s+1)}{s(s+2)} = \frac{4}{2} = 2 \mu_v \Rightarrow e_v = \frac{R_0}{\mu_v} = \frac{3}{2} = 1.5 \text{ Ans}$$

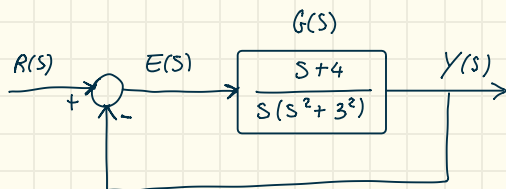
③



$$x(t) = 4t^2 \text{ siccome } \mathcal{L}[R_0 \cdot t^n \cdot 1(t)] = \frac{R_0 \cdot n!}{t^{n+1}} \Rightarrow R(s) = \frac{4 \cdot 2}{t^3} = \frac{8}{t^3} R_0$$

$$\Rightarrow \mu_a = \lim_{s \rightarrow 0} s^2 \cdot \frac{s+4}{s^2(s+3)} = \frac{4}{3} \mu_a \Rightarrow e_p = e_v = 0, e_a = \frac{R_0}{\mu_a} = \frac{8}{4/3} = 6 \text{ Ans}$$

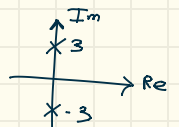
④

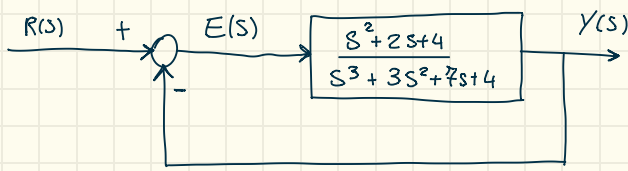


$$x(s) = \sin 3t \Rightarrow R(s) = \frac{3}{s^2 + 9} \Rightarrow s_{p,z} = \pm j3$$

$$\Rightarrow e_{ss} = 0 \Leftrightarrow s(s^2+3^2) \text{ ha due poli in } \pm j3$$

$$\bar{s}_1 = 0, s^2+9 \rightarrow s_{p,z} = \pm j3 \checkmark \Rightarrow e_{ss} = 0 \text{ Ans}$$





$$\begin{array}{c|cc} s^3 & 1 & 7 \\ s^2 & 3 & 4 \\ s^1 & 17/3 & 0 \\ s^0 & 4 & \end{array} \quad \frac{21-4}{3} = \frac{17}{3}$$

~~A. S. stabile~~

Routh Va  
Calcolato per il  
Polinomio a ciclo  
chiuso  
 $1 + L(s)$

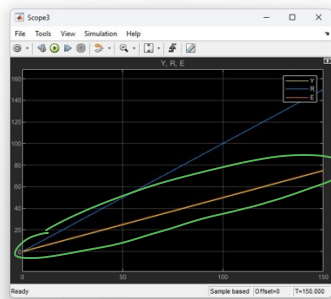
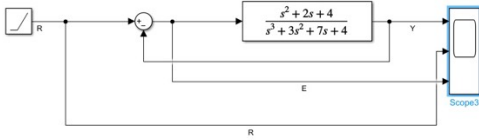
$$r(t) = t \cdot 1(t) \Rightarrow R(s) = \frac{1}{s^2} \quad R_0 = 1$$

$$\Rightarrow e_v = \frac{1}{\mu_v} \quad \text{con } \mu_v = \lim_{s \rightarrow 0} s \cdot \frac{s^2 + 2s + 4}{s^3 + 3s^2 + 7s + 4} = 0 \Rightarrow e_{ss} = \frac{1}{\mu_v} = \frac{1}{0} = \infty$$

$$q=0 \Rightarrow \mu = \lim_{s \rightarrow 0} s^0 \cdot L(s) = 1 \neq 0$$

$$e_{ss} = \infty \text{ perche' } q < 1$$

$$q=0 \Rightarrow e_v = \infty, e_v \propto \lim_{t \rightarrow \infty} t \cdot 1(t)$$

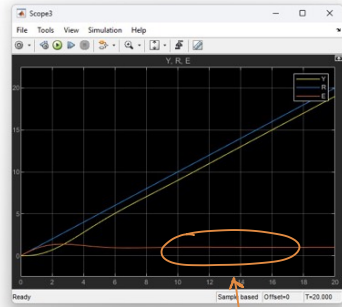
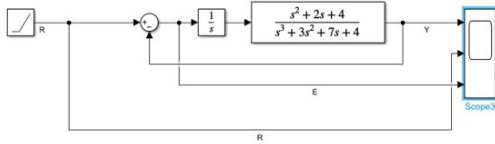


$e(t)$  e  $\dot{e}(t)$  hanno più o meno lo stesso andamento

$$\Rightarrow e(t) \propto \dot{e}(t) = t \cdot 1(t)$$

$$\Rightarrow e_{ss} = \lim_{t \rightarrow \infty} \dot{e}(t) = \infty$$

CON INTEGRATORE  $\Rightarrow q=1 \Rightarrow e_v \propto \frac{1}{\mu_v}$



$$e_v \text{ è finito e } \propto \frac{1}{\mu_v}$$

$\Rightarrow$

$e_{ss}$  Rimane Costante

Nel II caso  $C(s) = \frac{1}{s} \Rightarrow L(s) = \frac{s^2 + 2s + 4}{s(s^3 + 3s^2 + 7s + 4)}$

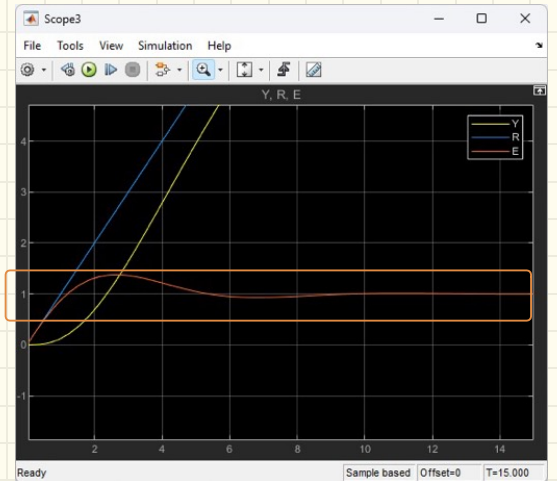
Controllare Nuovamente Polinomio a ciclo chiuso con Routh

ORA  $q=1$  Corretto

$$\Rightarrow \mu_v = \lim_{s \rightarrow 0} s \cdot \frac{s^2 + 2s + 4}{s(s^3 + 3s^2 + 7s + 4)} = 1 \Rightarrow e_v = \frac{1}{\mu_v} = 1$$

$$\Rightarrow e_v = \frac{1}{\mu_v} = 1$$

Infatti l'errore rimane costante ed 1 per  $t \rightarrow \infty$



Aggiungere un secondo integratore destabilizza il sistema

