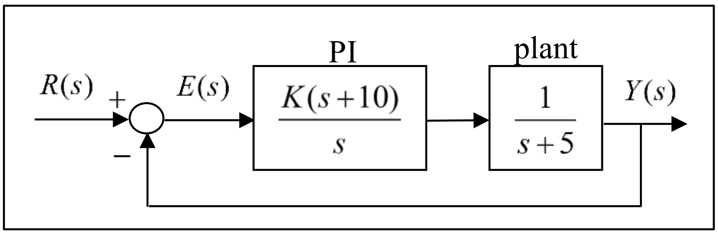
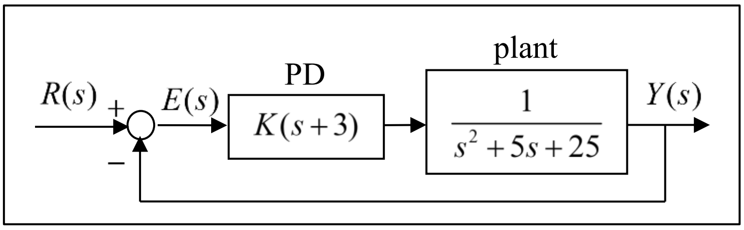
Introductory Control Systems
Exercises #10 – Steady-State Error

1. A ***proportional-integral*** ("PI") controller is used to control a 1st order plant as shown. The system has input $R(s)$, output $Y(s)$, and error $E(s)$. Find $\frac{E}{R}(s)$ the ***error transfer function***, and then find the range of values for the parameter K so the system has a ***steady-state error*** e_{ss} less than 0.01 for a ***unit ramp*** input $\left(R(s)=1/s^2\right)$. Assume $K>0$.



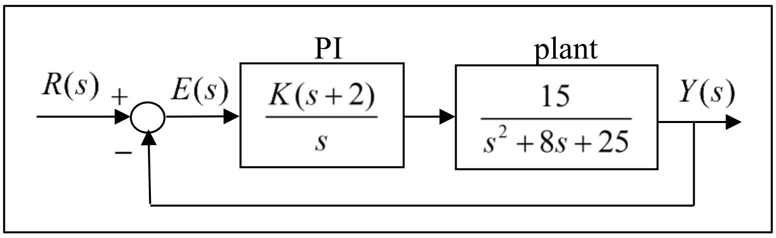
Answers: $\boxed{\frac{E}{R}(s)=\frac{s(s+5)}{s(s+5)+K(s+10)}}$; $K>50$

2. A ***proportional-derivative*** ("PD") controller is used to control a 2nd order plant as shown. The system has input $R(s)$, output $Y(s)$, and error $E(s)$. Find $\frac{E}{R}(s)$ the ***error transfer function***, and find e_{ss} the ***steady-state error*** associated with a ***unit step*** input in terms of the parameter K.



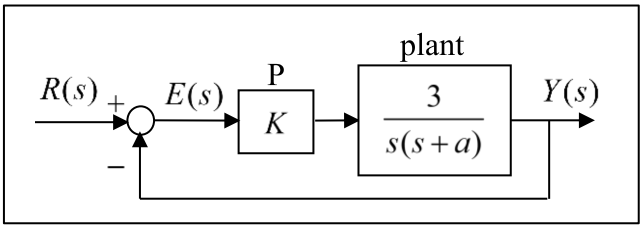
Answers: $\boxed{\frac{E}{R}(s)=\frac{s^2+5s+25}{s^2+5s+25+K(s+3)}}$; $\boxed{e_{ss}=\frac{25}{25+3K}}$

3. A ***proportional-integral*** ("PI") controller is used to control a 2nd order plant as shown. The system has input $R(s)$, output $Y(s)$, and error $E(s)$. Find $\frac{E}{R}(s)$ the ***error transfer function***, and then find the range of values for the parameter K so the system has a steady-state error less than 0.1 to a ***unit ramp*** input $\left(R(s)=1/s^2\right)$. Assume $K>0$.



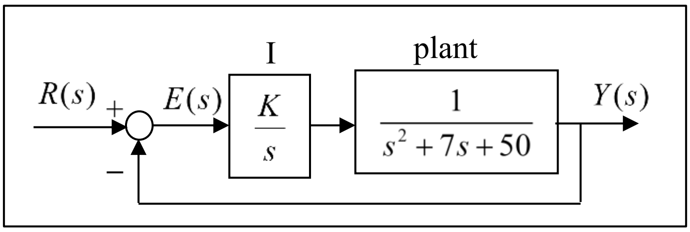
Answers: $\boxed{\dfrac{E}{R}(s)=\dfrac{s(s^2+8s+25)}{s(s^2+8s+25)+15K(s+2)}}$; $\boxed{K>8.33}$

4. A ***proportional*** ("P") controller is used to control a 2nd order plant as shown. The system has input $R(s)$, output $Y(s)$, and error $E(s)$. Find $\dfrac{E}{R}(s)$ the ***error transfer function***, and then find e_{ss} the ***steady-state error*** associated with a ***unit ramp*** input in terms of the parameters a and K.



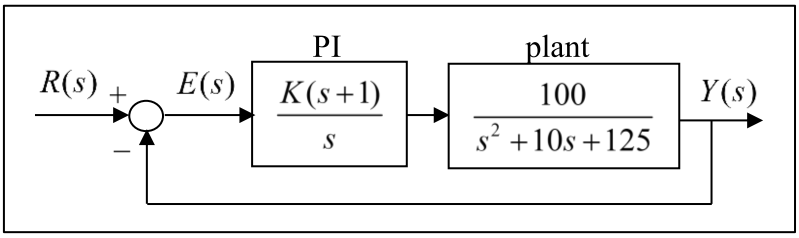
Answers: $\boxed{\dfrac{E}{R}(s)=\dfrac{s(s+a)}{s^2+as+3K}}$, $\boxed{e_{ss}=\dfrac{a}{3K}}$

5. An ***integral*** ("I") controller is used to control a 2nd order plant as shown. The system has input $R(s)$, output $Y(s)$, and error $E(s)$. Find $\dfrac{E}{R}(s)$ the ***error transfer function***, and then find the range of values for the gain K so e_{ss} the steady-state error due to a ***unit ramp*** input is less than 1.0. Assume $K>0$.



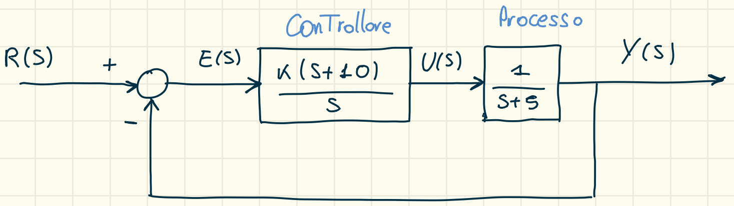
Answers: $\boxed{\dfrac{E}{R}(s)=\dfrac{s(s^2+7s+50)}{s^3+7s^2+50s+K}}$; $\boxed{K>50}$

6. A ***proportional-integral*** ("PI") controller is used to control a 2nd order plant as shown. The system has input $R(s)$, output $Y(s)$, and error $E(s)$. Find $\dfrac{E}{R}(s)$ the ***error transfer function***, and then find the range of values for the parameter K so the steady-state error to a ***unit ramp*** input is less than 0.1. Assume $K>0$.



Answers: $\boxed{\dfrac{E}{R}(s)=\dfrac{s(s^2+10s+125)}{s(s^2+10s+125)+100K(s+1)}}$; $\boxed{K>12.5}$

ES 1



1) $\frac{E(S)}{R(S)} = T_{r \to e} = S(S) = \frac{1}{1 + C(S)\,G(S)} = \frac{1}{1 + \frac{K(S+10)}{S(S+5)}} = \frac{S(S+5)}{S(S+5) + K(S+10)}$

2) e_{ss} per RAMPA UNITARIA $r(t) = t \cdot 1\!\!1(t) \rightleftharpoons R(S) = \frac{1}{S^2}$ $(K>0)$ $\Rightarrow e_r < 1\% = 0.01$

F. Anello $C(S) \cdot G(S) = L(S) = \frac{K(S+10)}{S(S+5)}$

Tipo 1 $\Rightarrow$ 1 azione integrale

$e_p = 0$, $e_v = \frac{1}{\mu}$ con μ = guadagno di $L(S)$

$$\mu = \lim_{S \to 0} \frac{K(\cancel{S}+10)}{\cancel{S}(\cancel{S}+5)} \cdot \cancel{S} = \frac{10K}{5} = 2K \Rightarrow e_r = \frac{1}{2K} < 0.01 \to 2K > 100 \Rightarrow K > 50$$

Procedimento alternativo

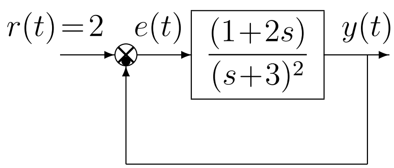
$$\frac{E(S)}{R(S)} = \frac{S(S+5)}{S(S+5)+K(S+10)} \Rightarrow E(S) = \frac{S(S+5)}{S(S+5)+K(S+10)} \cdot R(S) \Rightarrow e_{ss}(t) = \lim_{t \to \infty} e(t) = \lim_{S \to 0} S \cdot E(S)$$

$$\Rightarrow e_{ss} = \lim_{S \to 0} \cancel{S} \cdot \frac{\cancel{S}(S+5)}{\cancel{S}(S+5)+K(S+10)} \cdot \frac{1}{S^{\cancel{2}}} = \frac{5}{[5+K(S+10)]S} =$$

Esempi

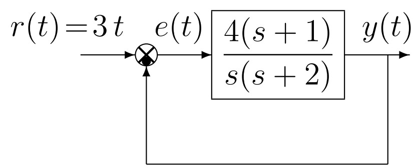
- Calcolare l'errore a regime $e(\infty)$ per i seguenti sistemi retroazionati:

1



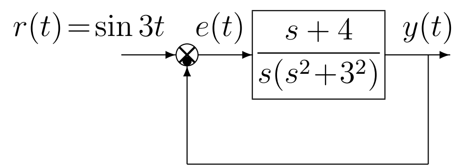
$$e(\infty) = \frac{2}{1+\frac{1}{9}} = \frac{18}{10} = 1.8$$

2



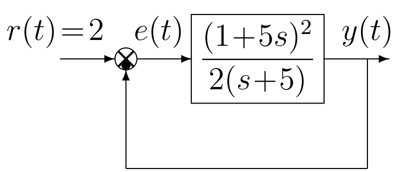
$$e(\infty) = \frac{3}{2} = 1.5$$

4

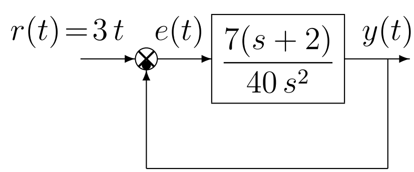


$$e(\infty) = 0$$

- Calcolare l'errore a regime $e(\infty)$ per i seguenti sistemi retroazionati:

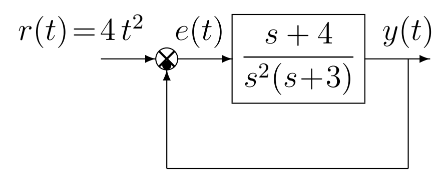


$$e(\infty) = \frac{2}{1+\frac{1}{10}} = \frac{20}{11} = 1.8182$$



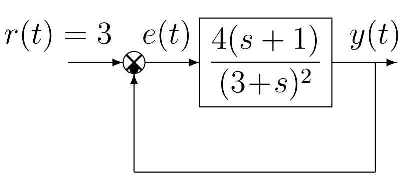
$$e(\infty) = 0$$

3

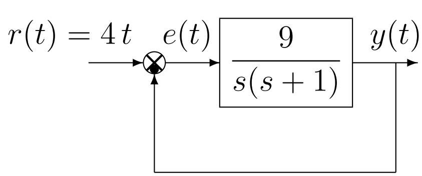


$$e(\infty) = 6$$

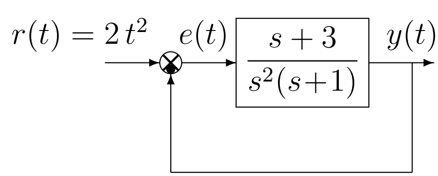
- Calcolare l'errore a regime $e(\infty)$ per i seguenti sistemi retroazionati:



$$e(\infty) = \frac{3}{1+\frac{4}{9}} = \frac{27}{13} = 2.08$$

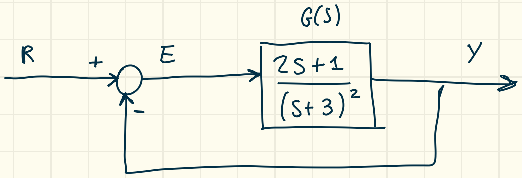


$$e(\infty) = \frac{4}{9} = 0.444$$



$$e(\infty) = \frac{4}{3} = 1.333$$

1



$P_c = S^2 + 6S + 9 \qquad 9 > 0 \Rightarrow A.S.$

$$R(S) = \frac{2}{S} \Rightarrow \frac{E(S)}{R(S)} = S(S) = \frac{1}{1+G(S)} = \frac{1}{1+\frac{2S+1}{(S+3)^2}} = \frac{(S+3)^2}{(S+3)^2+2S+1}$$

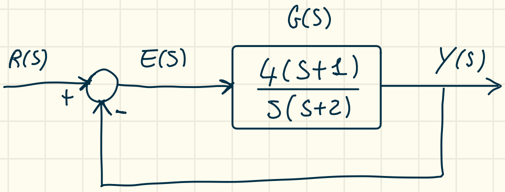
$$\Rightarrow E(S) = \frac{(S+3)^2}{(S+3)^2+2S+1} \cdot \frac{2}{S} \Rightarrow e_{ss} = \lim_{S\to 0} \cancel{S} \cdot \frac{(S+3)^2}{(S+3)^2+2S+1} \cdot \frac{2}{\cancel{S}} \to \frac{9}{9+1} \cdot 2 = \frac{9}{5} = 1.8$$

Ans

OPPURE Conoscendo le formule...

$$\mu_p = \lim_{S\to 0} \frac{2S+1}{(S+3)^2} = \frac{1}{9} \Rightarrow e_p = \frac{R_0}{1+\mu_p} = \frac{2}{1+1/9} = 1.8$$

2

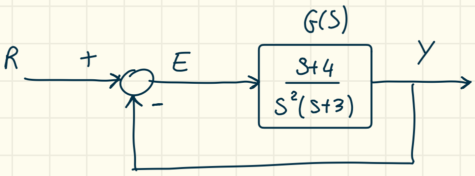


$$r(t) = 3t \rightleftharpoons R(S) = \frac{3 \leftarrow R_0}{S^2} \Rightarrow e(\infty) = e_{\text{Velocità}}$$

$$\Rightarrow \mu_v = \lim_{S\to 0} S \cdot G(S) = \lim_{S\to 0} \cancel{S}\, \frac{4(S+1)}{\cancel{S}(S+2)} = \frac{4}{2} = 2\ \mu_v \rightsquigarrow e_v = \frac{R_0}{\mu_v} = \frac{3}{2} = 1.5$$

Ans

3



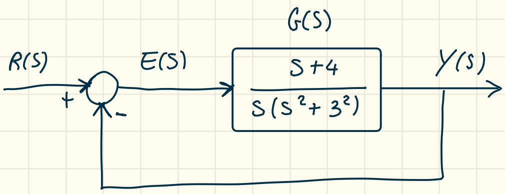
$$r(t) = 4t^2 \quad \text{Siccome } \mathcal{L}\left[R_0 \cdot t^n \cdot 1\!\!1(t)\right] = \frac{R_0 \cdot n!}{t^{n+1}}$$

$$\Rightarrow R(S) = \frac{4 \cdot 2}{t^3} = \frac{8}{t^3} \ R_0$$

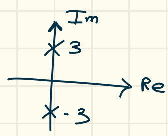
$$\Rightarrow \mu_a = \lim_{S\to 0} \cancel{S^2} \frac{S+4}{\cancel{S^2}(S+3)} \to \frac{4}{3} \mu_a \Rightarrow e_p = e_v = 0,\ e_a = \frac{R_0}{\mu_a} = \frac{\cancel{8}^{\,2}}{\cancel{4}/3} = 6 \ e_a$$

Ans

4

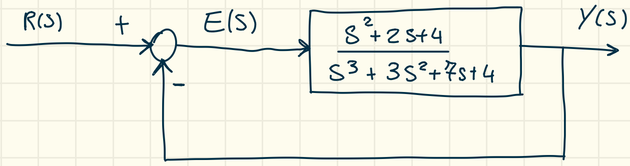


$$r(S) = \sin 3t \rightleftharpoons R(S) = \frac{3}{S^2+9} \Rightarrow S_{1,2} = \pm j3$$



$\rightsquigarrow e_{ss} = 0 \Leftrightarrow S(S^2+3^2)$ ha due poli in $\pm j3$

$\bar{S}_1 = 0,\quad S^2+9 \to S_{1,2} = \pm j3$ ✓ $\Rightarrow e_{ss} = \emptyset$ Ans



s^3	1	7
s^2	3	4
s^1	5	0
s^0	4	

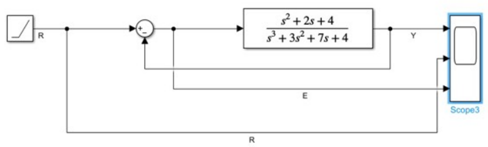
A. Stabile

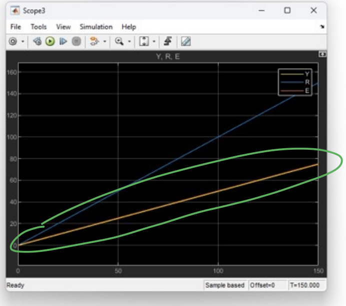
$$\frac{21-4}{3} = \frac{15}{3} = 5$$

$$r(t) = t \cdot 1(t) \rightleftharpoons R(s) = \frac{1}{s^2} \qquad R_0 = 1$$

$$\rightarrow e_v = \frac{1}{\mu_v} \quad \text{con} \quad \mu_v = \lim_{s \to 0} s \, \frac{s^2+2s+4}{s^3+3s^2+7s+4} = 0 \quad \Rightarrow \quad e_{ss} = \frac{1}{\mu_v} = \frac{1}{0} = \infty$$

$g = 0 \rightarrow e_v = \infty$, $e_v \propto \lim_{t \to \infty} t \cdot 1(t)$



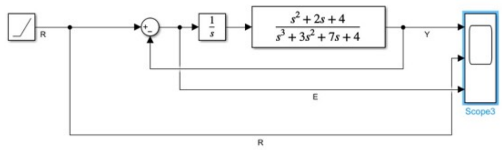


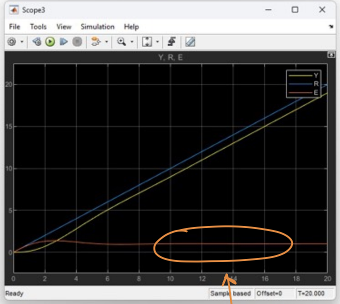
$e(t)$ e $r(t)$ hanno più o meno lo stesso andamento

$\Rightarrow e(t) \propto r(t) = t \cdot 1(t)$

$\Rightarrow e_{ss} = \lim_{t \to \infty} r(t) = \infty$

CON INTEGRATORE $\rightarrow g = 1 \rightarrow e_v \propto \frac{1}{\mu_v}$





e_v è finito e $\propto \frac{1}{\mu_v}$

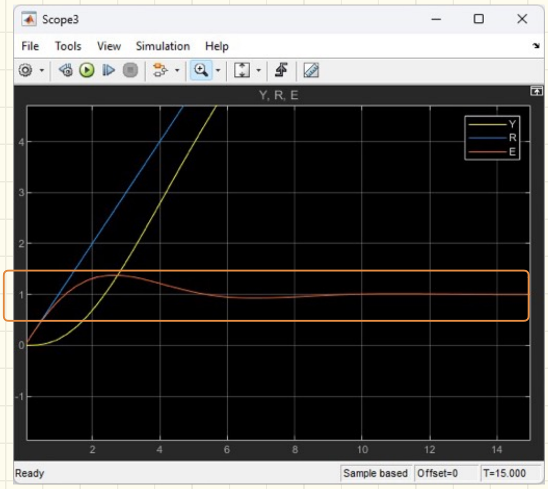
$\Rightarrow$

e_{ss} Rimane Costante

Nel II caso $\quad C(s) = \frac{1}{s} \quad \Rightarrow L(s) = \frac{s^2+2s+4}{s(s^3+3s^2+7s+4)}$

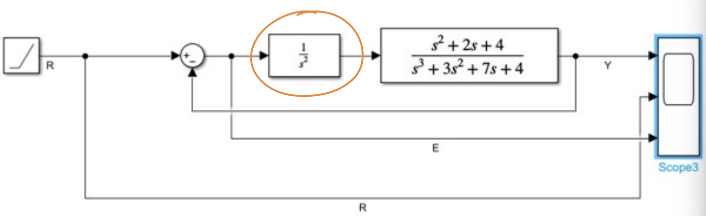
$$\Rightarrow \mu_v = \lim_{s \to 0} \cancel{s} \cdot \frac{s^2+2s+4}{\cancel{s}(s^3+3s^2+7s+4)} \rightarrow 1 \; \mu_v \quad \Rightarrow \quad e_v = \frac{1}{\mu_v} = 1$$

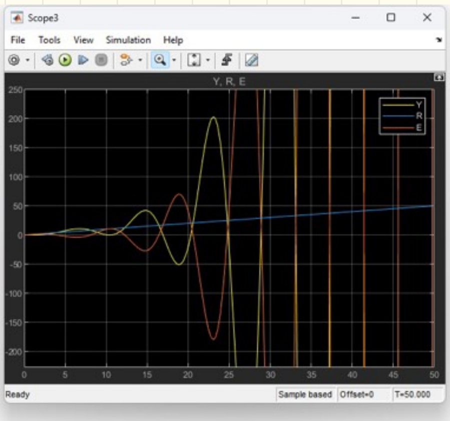
e_{ss}

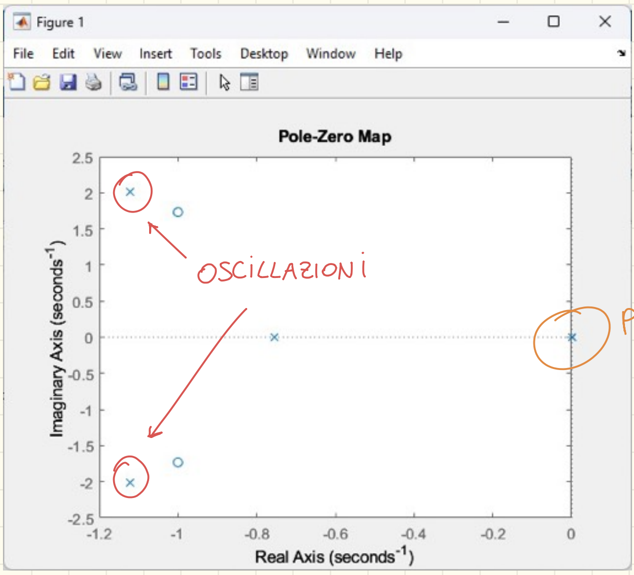


Infatti l'errore rimane costante ad 1 per $t \to \infty$

Aggiungere un secondo integratore destabilizza il sistema







Polo che da fastidio?