

Dipolo elettrico

Partiamo dalla eq di Maxwell per il campo elettrico:

$$L = \int \vec{F} \cdot d\vec{e} = U_A - U_B \quad \text{ma} \quad E = \frac{F}{q} \rightarrow \frac{U}{q} = V \Rightarrow \int \vec{E} \cdot d\vec{e} = V_A - V_B$$

$$\text{Se } L = \int \vec{F} \cdot d\vec{e} = -U \Rightarrow \frac{L}{q} = \int \vec{E} \cdot d\vec{e} = -V \Rightarrow \vec{E} \cdot d\vec{e} = -dV$$

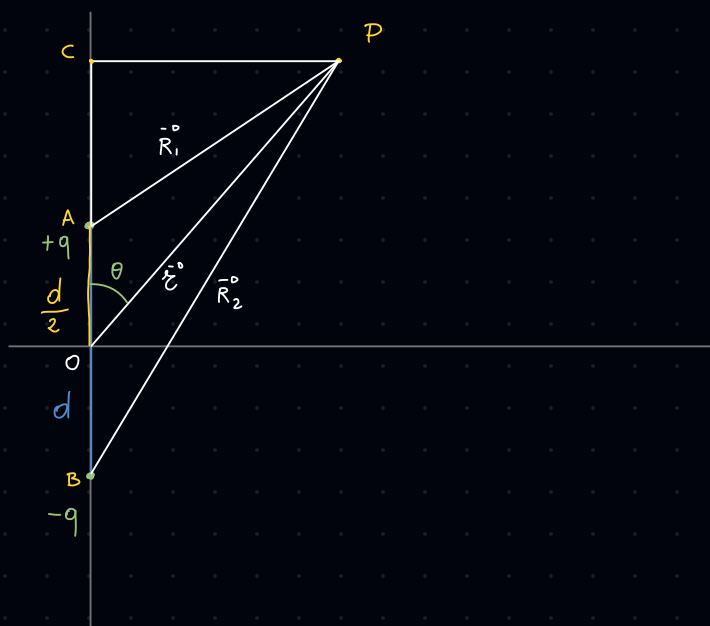
ma dV è del tipo $\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$

$$\begin{cases} d\vec{e} = \hat{i} dx + \hat{j} dy + \hat{k} dz \\ \vec{\nabla} V = \hat{i} \frac{\partial V}{\partial x} + \hat{j} \frac{\partial V}{\partial y} + \hat{k} \frac{\partial V}{\partial z} \end{cases}$$

$$\Rightarrow dV = \vec{\nabla} V \cdot d\vec{e} \Rightarrow \vec{E} \cdot d\vec{e} = -\vec{\nabla} V \cdot d\vec{e} \Rightarrow \vec{E} = -\vec{\nabla} V$$

Possiamo trovare il campo elettrico tramite il potenziale

Potenziale elettrico $V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r_A} - \frac{1}{r_B} \right]$



Approx di dipolo: $r \gg d$

conosco $\begin{cases} \vec{r} & \text{Posizione di P} \\ \theta & \text{Angolo tra } \vec{r} \text{ e } d \\ d & \text{distanza tra i poli} \end{cases}$

$$\begin{aligned} R_1^2 &= \overline{CP}^2 + \overline{CA}^2 = (r \sin \theta)^2 + \left(\overline{CO} - \frac{d}{2} \right)^2 \\ &= r^2 \sin^2 \theta + \left(r \cos \theta - \frac{d}{2} \right)^2 \\ &= r^2 \sin^2 \theta + r^2 \cos^2 \theta + \frac{d^2}{4} - d r \cos \theta \\ &= r^2 (\sin^2 \theta + \cos^2 \theta) + \frac{d^2}{4} - d r \cos \theta \\ &= r^2 + \frac{d^2}{4} - d r \cos \theta \\ &\stackrel{r \gg d}{\approx} r^2 - d r \cos \theta \end{aligned}$$

$$R_2^2 = \overline{CP}^2 + \overline{CB}^2 = (r \sin \theta)^2 + \left(\overline{CO} + \frac{d}{2} \right)^2 = r^2 + d r \cos \theta$$

$$\Rightarrow \begin{cases} R_1 = \sqrt{r^2 - d r \cos \theta} \\ R_2 = \sqrt{r^2 + d r \cos \theta} \end{cases} \Rightarrow \begin{cases} R_1 = \sqrt{r} \cdot \sqrt{r - d \cos \theta} \\ R_2 = \sqrt{r} \cdot \sqrt{r + d \cos \theta} \end{cases}$$

Approssimo $\sqrt{r - d \cos \theta}$ con Taylor in $x_0 = 0$

$$g_1(d) = \sqrt{r - d \cos \theta} = (r - d \cos \theta)^{\frac{1}{2}} \Rightarrow g_1'(d) = \frac{1}{2} (r - d \cos \theta)^{-\frac{1}{2}} \cdot (-\cos \theta) = -\frac{\cos \theta}{2(r - d \cos \theta)}$$

$$g_1(0) = (r - 0 \cos \theta)^{\frac{1}{2}} = \sqrt{r} \quad g_1'(0) = -\frac{\cos \theta}{2(r - 0)} = -\frac{1}{2\sqrt{r}} \cos \theta$$

$$\Rightarrow \text{Taylor} \Rightarrow \sqrt{z - d \cos \theta} \approx \sqrt{z} - \frac{\cos \theta}{2\sqrt{z}} \cdot d = \sqrt{z} - \frac{d \cos \theta}{2\sqrt{z}} = \bar{f}(d)$$

$$\Rightarrow R_1 \approx \sqrt{z} \cdot \bar{f}(d) = \sqrt{z} \cdot \left(\sqrt{z} - \frac{d \cos \theta}{2\sqrt{z}} \right) = z - \frac{d \cos \theta}{2} \quad (1)$$

$$\Rightarrow R_2 \approx z + \frac{d \cos \theta}{2} \quad (2)$$

Trovo il potenziale

$$V = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{z - \frac{d \cos \theta}{2}} - \frac{1}{z + \frac{d \cos \theta}{2}} \right) = k \left(\frac{z + \frac{d}{2} \cos \theta - z + \frac{d}{2} \cos \theta}{z^2 - \left(\frac{d}{2} \cos \theta \right)^2} \right)$$

$$= \frac{q}{4\pi\epsilon_0} \frac{d \cos \theta}{z^2 - \left(\frac{d}{2} \cos \theta \right)^2} \quad \text{ma } z \gg d \Rightarrow V = \frac{q}{4\pi\epsilon_0} \frac{d \cos \theta}{z^2} \quad (3)$$

Trovo il campo elettrico

$\vec{E} = -\vec{\nabla} V \Rightarrow$ Derivate parziali \Rightarrow Servono le coordinate cartesiane!

$$CO = z \cos \theta \Rightarrow \text{Battezzo } CO = z \Rightarrow \cos \theta = \frac{z}{r}$$

$$\text{inoltre eq sfera: } r^2 = x^2 + y^2 + z^2$$

$$\Rightarrow \frac{\cos \theta}{r^2} = \frac{z}{r} \cdot \frac{1}{r^2} = \frac{z}{(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$\Rightarrow V = \frac{q}{4\pi\epsilon_0} \frac{d \cdot z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \quad \text{ora siamo pronti} \quad -\frac{3}{2} (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot z x$$

$$\vec{E}_x = -\hat{i} \frac{\partial V}{\partial x} = \frac{q}{4\pi\epsilon_0} d \cdot z \cdot \frac{d}{dx} (x^2 + y^2 + z^2)^{-\frac{3}{2}} = + \frac{q}{4\pi\epsilon_0} \cdot \frac{3 \cdot d \cdot z x}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

$$x^2 + y^2 + z^2 = r^2 \Rightarrow \vec{E}_z = +\hat{i} \frac{3qd}{4\pi\epsilon_0} \cdot \frac{z \cdot x}{\sqrt{(r^2)^5}} = \hat{i} \frac{3qd}{4\pi\epsilon_0} \frac{z \cdot x}{z^5} \quad (A)$$

$$\vec{E}_y = \hat{j} \frac{3qd}{4\pi\epsilon_0} \cdot \frac{zy}{z^5} \quad (B)$$

$$\vec{E}_z = -\hat{k} \frac{qd}{4\pi\epsilon_0} \cdot \left(\frac{3z^2}{z^5} - \frac{1}{z^3} \right) = \frac{1}{4\pi\epsilon_0 z^3} \left(\frac{3z^2}{z^2} - 1 \right) = \frac{1}{4\pi\epsilon_0 z^3} (3 \cos^2 \theta - 1) \quad (C)$$

