Lezione 4 del 04/10/23

TEMPO → TRASFORMATA → FREQUENZA → ANTI TRASFORMATA → TEMPO

Teorema del Valore finale

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)}$$

Se il limite tende a qualcosa

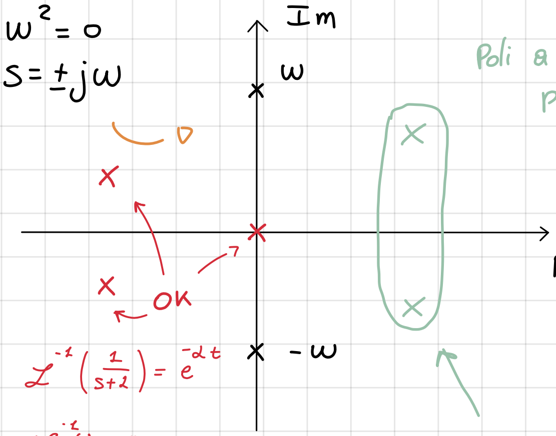
Questo teorema è utile perché potremmo voler trovare il valore finale di un sistema invece di calcolare come "ci arriva"

Non possiamo applicarlo per tutte quelle funzioni le quali hanno più di un polo sull'asse immaginario: ad esempio il la funzione seno.

Possiamo invece applicarlo alla funzione gradino

$s^2 + w^2 = 0$

per $s = \pm jw$



Poli a parte reale positiva!!

NO

$\frac{1}{s+\alpha} \to \mathcal{L}^{-1}\left(\frac{1}{s+\alpha}\right) = e^{-\alpha t}$

$\to \lim_{t \to \infty} \mathcal{L}^{-1}(\cdot) = \emptyset$

$\Rightarrow \lim_{s \to 0} s \cdot \frac{1}{s+\alpha} \to \emptyset$

$\frac{1}{s-\alpha} \Rightarrow s = \alpha$

$\to \mathcal{L}^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}$

$\lim_{t \to \infty} e^{\alpha t} \to \infty$

$\nexists$

DIMOSTRAZIONE

$$\lim_{s \to 0} s\,F(s) = \lim_{s \to 0}\left(f(0) + \mathcal{L}\left[\frac{df}{dt}\right]\right) = f(0) + \lim_{s \to 0} \mathcal{L}\left[\frac{df}{dt}\right]$$

$f(0)$

$$= f(0) + \lim_{s \to 0} \int_0^{\infty} \frac{df}{dt} e^{-st}\, dt = f(0) + \int_0^{\infty} \frac{df}{dt}\, dt = f(0) + \left[f(t)\right]_0^{\infty}$$

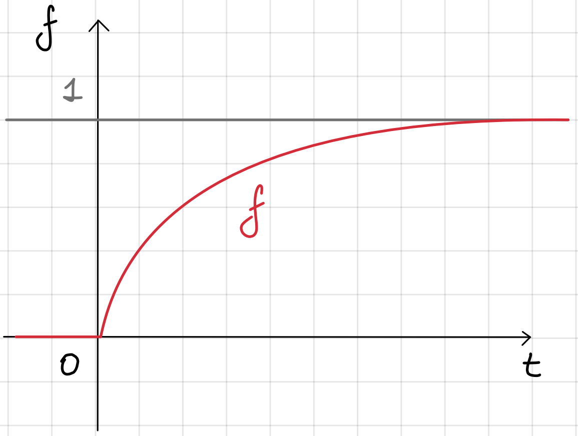
$\to 1$

$$= \lim_{t \to \infty} f(t)$$

QED

ESERCIZIO Esempio

$$f(t) = (1 - e^{-3t}) \cdot \mathbb{1}(t) \quad \rightarrow \quad \lim_{t \to \infty} f(t) \rightarrow \; ?$$



(1) TRASFORMATA

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{1}{s+3} = \frac{\cancel{s}+3-\cancel{s}}{s(s+3)} = \underbrace{\frac{3}{s(s+3)}}_{F(s)}$$

(2) Teorema

$$\lim_{s \to 0} s\,F(s) = \lim_{s \to 0} \frac{3}{\cancel{s}(s+3)}\cancel{s} \rightarrow 1$$

Valore a cui Tende la funzione

TEOREMA DEL VALORE INIZIALE

$$\boxed{f(0^+) = \lim_{s \to \infty} s\,F(s)}$$

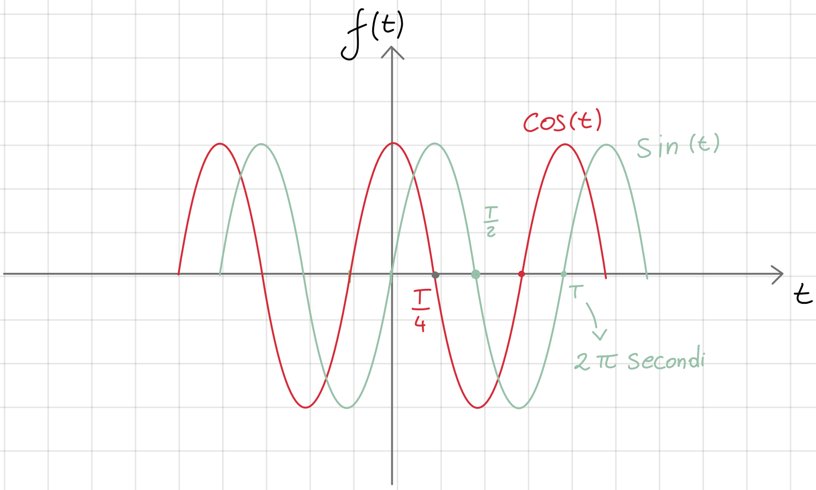
DIMOSTRAZIONE

$$f(0^+) = \lim_{s \to \infty} \left(\mathcal{L}\left[\frac{df}{dt}\right] + f(0^+) \right) = f(0^+) + \lim_{s \to \infty} \int_{0^+}^{\infty} \frac{df}{dt} e^{-st}\, dt = f(0^+)$$

$$f \cdot 0 = 0$$

QED

Esempio



$$\omega = \frac{2\pi}{T} = 2\pi f \quad ; \quad \omega t = \pi$$

$$\Rightarrow \cancel{\omega} = \frac{2\,\omega t}{T} \rightarrow t = \frac{T}{2}$$

Si Annulla

periodico di $T = K\pi$

$$\Rightarrow t = K\frac{\pi}{2}$$

ES $f = 50\,Hz \rightarrow T = \frac{1}{50}\,Sec = 20\,mS \rightarrow$ Se $\omega = 1 \Rightarrow f = \frac{1}{2\pi} \Rightarrow T = 2\pi\ Sec$

E' un TEMPO

ESEMPIO

$$1\!\!1(t)\,\sin \omega t\Big|_{t=0^-} = 0$$

$$1\!\!1(t)\,\cos \omega t\Big|_{t=0^+} = 1$$

ESEMPIO

$$\cos(\omega t) \Rightarrow \lim_{s\to\infty} s \cdot F(s) = \lim_{s\to\infty} s \cdot \frac{s}{s^2+\omega^2} = \lim_{s\to\infty} \frac{\cancel{s^2}}{\cancel{s^2}+\omega^2} = 1$$

Valore iniziale del coseno

TEOREMA INTEGRALE REALE

$$\boxed{\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}}$$

$$f^{-1}(0) = \int f(t)\,dt\Big|_{t=0}$$

DIMOSTRAZIONE

$$\mathcal{L}\left[\int f(t)\,dt\right] = \int_0^\infty \Big(\underbrace{\int f(t)\,dt}_{g}\Big) \cdot e^{-st}\,dt = \int_0^\infty \Big(\underbrace{\int f(t)\,dt}_{g}\Big) \cdot \Big(\underbrace{\frac{d}{dt} - \frac{1}{s}e^{-st}}_{f}\Big)\,dt$$

Applico la Def

PARTI

$$e^{-st} = \frac{d}{dt} -\frac{1}{s} e^{-st}$$

$$= \left[\int f(t)\,dt \cdot -\frac{1}{s}e^{-st}\right]_0^\infty + \frac{1}{s}\underbrace{\int_0^\infty f(t)\cdot e^{-st}\,dt}_{F(s)}$$

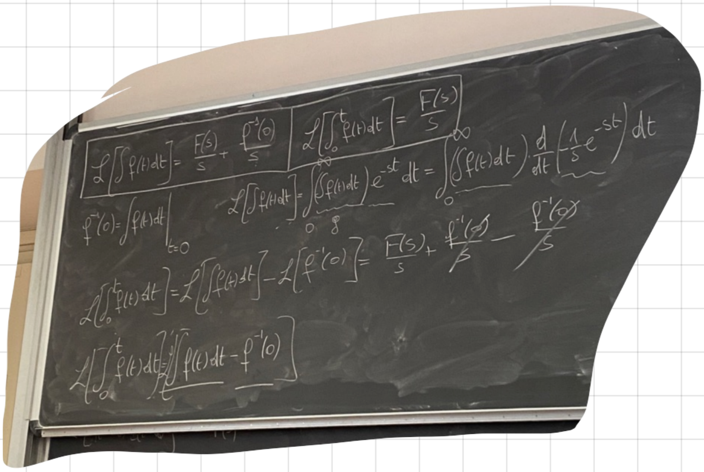
$$= \frac{F(s)}{s} + \frac{1}{s}e^{-st}\int f(t)\,dt\Big|_{t=0}$$

QED

CASO PARTICOLARE

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s}$$

perché $\displaystyle\int_0^t f(\tau)\,d\tau = \int f(t)\,dt - f^{-1}(0)$



Derivazioni nel dominio della variabile s (complessa)

$$\mathcal{L}[t \cdot f(t)] = -\frac{d}{ds} F(s)$$

DIMOSTRAZIONE

$$\int_0^{\infty} t\, f(t) \cdot e^{-st}\, dt = -\int_0^{\infty} f(t) \cdot \frac{d}{ds} e^{-st}\, dt = -\frac{d}{ds}\int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$\frac{d}{ds} e^{-st} = -te^{-st}$$

$$= -\frac{d}{ds} F(s) \quad \text{QED}$$

ESEMPI

- $\mathcal{L}[3t\, \mathbb{1}(t)] = 3\,\mathcal{L}[t \cdot \mathbb{1}(t)] = -3\frac{d}{ds}\frac{1}{s} = \frac{3}{s^2}$

- $\mathcal{L}[t \cdot \sin(\omega t)\, \mathbb{1}(t)] = -\frac{d}{dt}\mathcal{L}[\sin(\omega t)\, \mathbb{1}(t)] = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) =$

$$= +\frac{2\omega s}{(s^2+\omega^2)^2}$$

- ES 25

$$F(s) = \frac{2s+1}{s^2+s+1}$$

Q: Valore iniziale di F e F'
$f(0^+)$ $\quad f'(0^+)$

$$f(0^+) = \lim_{s\to\infty} s\,F(s) = \frac{2s^2+s}{s^2+s+1} \to 2$$

$$\hat{F}(s)$$

$$\mathcal{L}\left[\frac{df}{dt}\right] = s\,F(s) - f(0^+) = \frac{2s^2+s}{s^2+s+1} - 2 = \frac{\cancel{2s^2}+s-\cancel{2s^2}-2s-2}{s^2+s+1} = \frac{-s+2}{s^2+s+1}$$

$$\Rightarrow f'(0^+) = \lim_{s\to\infty} s\,\hat{F}(s) = -1 \quad \text{SOL}$$

ANTI TRASFORMATA

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

ESEMPIO : $F(s) = \dfrac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n}$

n POLI e m zeri

IMPO

SCOMPOSIZIONE IN FRATTI SEMPLICI

$$\boxed{F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n} = \sum_{i=1}^{n} \frac{r_i}{s+p_i}}$$

Residuo i-esimo Associato al polo

$\to \; f_1 \cdot \dfrac{1}{s+p_i} \Rightarrow f = r_i \cdot e^{-p_i t}$

$$\Rightarrow \quad f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\sum_{i=1}^{n} \frac{r_i}{s+p_i}\right] = \sum_{i=1}^{n} \mathcal{L}^{-1}\left[\frac{r_i}{s+p_i}\right] = \sum_{i=1}^{n} r_i \mathcal{L}^{-1}\left[\frac{1}{s+p_i}\right]$$

$$= \sum_{i=1}^{n} r_i \mathcal{L}^{-1}[F(s)] = \sum_{i=1}^{n} r_i e^{-p_i t} \text{, per } t \geq 0$$

se $p_i = 0$ abbiamo solo r_i

DIMOSTRAZIONE RESIDUO

$$\boxed{\lim_{s \to p_i} (s+p_i) F(s) = r_i}$$

perchè $\displaystyle\lim_{s \to p_k} (s+p_k) F(s) =$

$$= \lim_{s \to p_k} (s+p_k) \cdot \sum_{i=1}^{n} \frac{r_i}{s+p_i}$$

La sommatoria scompare perché tutti i termini sono zero tranne s+pk

Se ad esempio volessi calcolare il residuo del Polo due, k = 2 ma la sommatoria scorre sempre su tutti i poli

$$\Rightarrow \lim_{s \to p_k} (\cancel{s+p_k}) \cdot \frac{r_k}{\cancel{s+p_k}} = r_k$$

QED

Dim migliore giù

Es $n = 3 \to 3$ poli

$$F(S) = \frac{r_1}{S+P_1} + \frac{r_2}{S+P_2} + \frac{r_3}{S+P_3}$$

$$\lim_{S \to P_K} (S+P_K) \cdot F(S) = \lim_{S \to P_K} (S+P_K) \cdot \sum_{i=1}^{n} \frac{r_i}{S+P_i}$$

Voglio $r_1 \to K = 1 \Rightarrow r_1 = \lim_{S \to P_1} (S+P_1) \cdot \left[\frac{r_1}{S+P_1} + \frac{r_2}{S+P_2} + \frac{r_3}{S+P_3} \right]$

$$\to r_1 = \lim_{S \to -P_1} = r_1 + \frac{r_2 (S+P_1)}{(S+P_2)} + r_3 \frac{S+P_1}{S+P_2} \to r_1 + r_2 \cancelto{0}{\frac{-P_1+P_1}{S+P_2}} + r_3 \cancelto{0}{\frac{-P_1+P_1}{S+P_3}}$$

$$\Rightarrow \lim_{S \to -P_K} (S+P_K) \cdot F(S) = r_K$$

QED

ESEMPIO

$$F(s) = \frac{s+3}{s^2+3s+2}$$

Dimostrazione sopra

METODO 1. CON I LIMITI

(1) Radici Denom

$$s = \frac{-3 \pm \sqrt{9-8}}{2} \quad \begin{cases} \frac{-3+1}{2} = -1 \\ \frac{-3-1}{2} = -2 \end{cases}$$

$$\Rightarrow \quad F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{r_1}{s+1} + \frac{r_2}{s+2}$$

(2) Trovo r_1 e r_2

(a) trovo r_1

$$\lim_{s \to -1} \frac{\cancel{(s+1)}(s+3)}{\cancel{(s+1)}(s+2)} = \frac{s+3}{s+2} = \frac{-1+3}{-1+2} = 2$$

P_1

$$\left\{ \text{FORMA: } \sum_{i=1}^{n} r_i \, \mathcal{L}\left[\frac{1}{s+p_i}\right] \right\}$$

(b) trovo r_2

$$\lim_{s \to -2} \frac{\cancel{(s+2)}(s+3)}{(s+1)\cancel{(s+2)}} = \frac{s+3}{s+1} = \frac{-2+3}{-2+1} = -1$$

P_2

(3) Scrivo la TRASFORMATA

$$\left. \begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned} \right\} \quad \boxed{\frac{2}{s+1} - \frac{1}{s+2}}$$

Metodo 2

mcm $\to$ $(s+1)(s+2)$ $\to$ Uguagliamo

$$\frac{r_1(s+2) + r_2(s+1)}{(s+1)(s+2)} = \frac{(r_1+r_2)s + 2r_1 + r_2}{(s+1)(s+2)} \overset{\text{DEVE}}{=} \frac{s+3}{(s+1)(s+2)}$$

$$\Rightarrow \begin{cases} r_1 + r_2 = 1 \\ 2r_1 + r_2 = 3 \end{cases} \Rightarrow \begin{cases} r_2 = 1 - r_1 \\ \to 2r_1 + 1 - r_1 = 3 \end{cases} \to \Rightarrow \begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

Spiegazione FINE CLIP 12

$$\Rightarrow \quad f(t) = (2e^{-t} - e^{-2t})\, \mathbb{1}(t)$$