Lezione 4 del 04/10/23

TEMPO $\xrightarrow{\text{TRASFORMATA}}$ FREQUENZA $\xrightarrow{\text{ANTI TRASFORMATA}}$ TEMPO

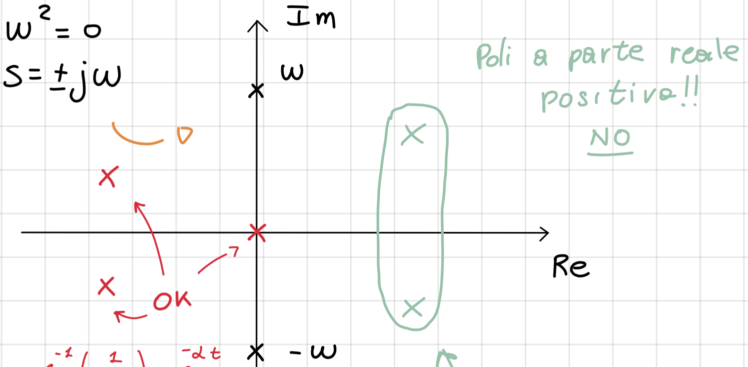
Teorema del Valore finale

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)}$$

Se il limite tende a qualcosa

> Questo teorema è utile perché potremmo voler trovare il valore finale di un sistema invece di calcolare come "ci arriva"
>
> Non possiamo applicarlo per tutte quelle funzioni le quali hanno più di un polo sull'asse immaginario: ad esempio il la funzione seno.
>
> Possiamo invece applicarlo alla funzione gradino

$s^2 + \omega^2 = 0$
per $s = \pm j\omega$



$\frac{1}{s+\alpha} \to \mathcal{L}^{-1}\left(\frac{1}{s+\alpha}\right) = e^{-\alpha t}$

$\to \lim_{t \to \infty} \mathcal{L}^{-1}(\cdot) = \emptyset$

$\Rightarrow \lim_{s \to \emptyset} s \cdot \frac{1}{s+\alpha} \to \emptyset$

$\frac{1}{s-\alpha} \Rightarrow s = \alpha$

$\to \mathcal{L}^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}$

$\lim_{t \to \infty} e^{\alpha t} \to \infty$

$\nexists$

DIMOSTRAZIONE

$$\lim_{s \to 0} s\,F(s) = \lim_{s \to \emptyset}\left(f(0) + \mathcal{L}\left[\frac{df}{dt}\right]\right) = f(0) + \lim_{s \to \emptyset} \mathcal{L}\left[\frac{df}{dt}\right]$$

$f(0)$

$$= f(0) + \lim_{s \to 0} \int_0^\infty \frac{df}{dt} e^{-st} dt = f(0) + \int_\emptyset^\infty \frac{df}{dt}\,dt = f(0) + \left[f(t)\right]_0^\infty$$

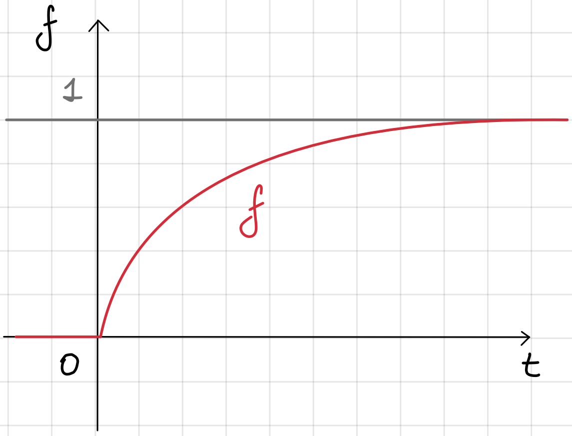
$e^{-st} \to 1$

$$= \lim_{t \to \infty} f(t)$$

QED

ESERCIZIO Esempio

$$f(t) = (1 - e^{-3t}) \cdot 1\!\!1(t) \longrightarrow \lim_{t \to \infty} f(t) \longrightarrow ?$$



(1) TRASFORMATA

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{1}{s+3} = \frac{\cancel{s}+3-\cancel{s}}{s(s+3)} = \underbrace{\frac{3}{s(s+3)}}_{F(s)}$$

(2) Teorema

$$\lim_{s \to 0} s\,F(s) = \lim_{s \to 0} \frac{3}{\cancel{s}(s+3)}\cancel{s} \longrightarrow 1$$

Valore a cui Tende la funzione

TEOREMA DEL VALORE INIZIALE

$$\boxed{f(0^+) = \lim_{s \to \infty} s\,F(s)}$$

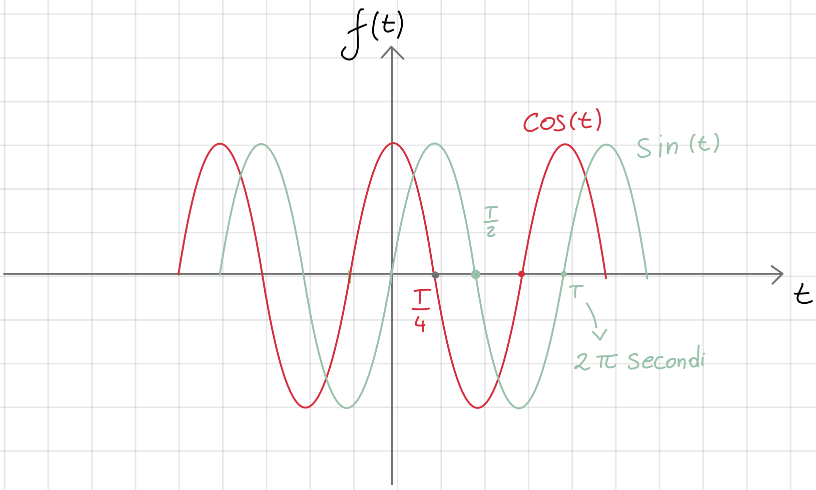
DIMOSTRAZIONE

$$f(0^+) = \lim_{s \to \infty} \left(\mathcal{L}\left[\frac{df}{dt}\right] + f(0^+) \right) = f(0^+) + \lim_{s \to \infty} \int_{0^+}^{\infty} \frac{df}{dt} e^{-st} dt = f(0^+)$$

$$f \cdot 0 = 0$$

QED

Esempio



$$\omega = \frac{2\pi}{T} = 2\pi f \quad ; \quad \omega t = \pi$$

$$\Rightarrow \cancel{\omega} = \frac{2\,\omega t}{T} \longrightarrow t = \frac{T}{2}$$

Si Annulla

periodico di $T = K\pi$

$$\Rightarrow t = k\frac{\pi}{2}$$

ES $f = 50\,Hz \longrightarrow T = \frac{1}{50}\,Sec = 20\,mS \longrightarrow$ Se $\omega = 1 \Rightarrow f = \frac{1}{2\pi} \Rightarrow T = 2\pi\ Sec$

E' un TEMPO

ESEMPIO

$$1(t)\,\sin \omega t\Big|_{t=0^-} = 0$$

$$1(t)\,\cos \omega t\Big|_{t=0^+} = 1$$

ESEMPIO

$$\cos(\omega t) \Rightarrow \lim_{s\to\infty} s\cdot F(s) = \lim_{s\to\infty} s\cdot \frac{s}{s^2+\omega^2} = \lim_{s\to\infty} \frac{\cancel{s^2}}{\cancel{s^2}+\omega^2} = 1$$

Valore iniziale del coseno

TEOREMA INTEGRALE REALE

$$\boxed{\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}}$$

$$f^{-1}(0) = \int f(t)\,dt\Big|_{t=0}$$

DIMOSTRAZIONE

$$\mathcal{L}\left[\int f(t)\,dt\right] = \int_0^\infty \Big(\underbrace{\int f(t)\,dt}_{g}\Big)\cdot e^{-st}\,dt = \int_0^\infty \Big(\underbrace{\int f(t)\,dt}_{g}\Big)\cdot \Big(\underbrace{\frac{d}{dt}-\frac{1}{s}e^{-st}}_{f}\Big)\,dt$$

Applico la Def

PARTI

$$\frac{d}{dt}-\frac{1}{s}e^{-st}$$

$$= \left[\int f(t)\,dt \cdot -\frac{1}{s}e^{-st}\right]_0^\infty + \frac{1}{s}\underbrace{\int_0^\infty f(t)\cdot e^{-st}\,dt}_{F(s)}$$

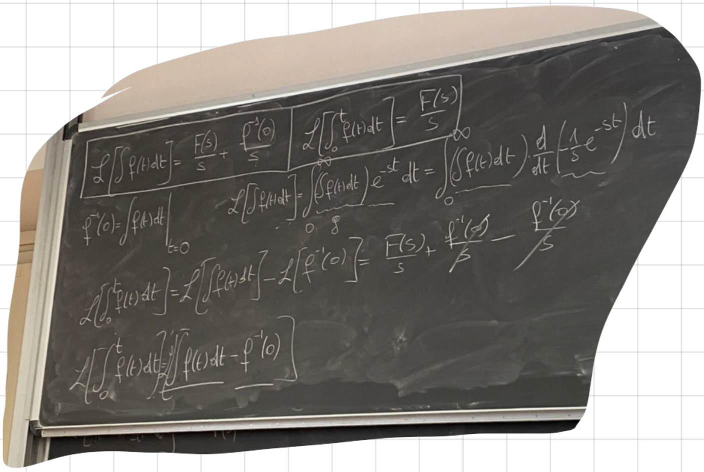
$$= \frac{F(s)}{s} + \frac{1}{s}e^{-st}\int f(t)\,dt\Big|_{t=0}$$

QED

CASO PARTICOLARE

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s}$$

perchè $$\int_0^t f(\tau)\,d\tau = \int f(t)\,dt - f^{-1}(0)$$



Derivazioni nel dominio della variabile s (complessa)

$$\mathcal{L}[t \cdot f(t)] = -\frac{d}{ds} F(s)$$

DIMOSTRAZIONE

$$\int_0^\infty t\, f(t) \cdot e^{-st}\, dt = -\int_0^\infty f(t) \cdot \frac{d}{ds} e^{-st}\, dt = -\frac{d}{ds} \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\frac{d}{ds} e^{-st} = -t e^{-st}$$

$$= -\frac{d}{ds} F(s) \quad \text{QED}$$

ESEMPI

- $$\mathcal{L}[3t\, \mathbb{1}(t)] = 3\,\mathcal{L}[t \cdot \mathbb{1}(t)] = -3 \frac{d}{ds} \frac{1}{s} = \frac{3}{s^2}$$
- $$\mathcal{L}[t \cdot \sin(\omega t)\, \mathbb{1}(t)] = -\frac{d}{dt} \mathcal{L}[\sin(\omega t)\, \mathbb{1}(t)] = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) =$$
  $$= +\frac{2\omega s}{(s^2+\omega^2)^2}$$
- ES 25

$$F(s) = \frac{2s+1}{s^2+s+1}$$

Q: Valore iniziale di F e F'
$f(0^+)$ $f'(0^+)$

$$f(0^+) = \lim_{s \to \infty} s\, F(s) = \frac{2s^2+s}{s^2+s+1} \to 2$$

$$\mathcal{L}\left[\frac{df}{dt}\right] = s\, F(s) - f(0^+) = \frac{2s^2+s}{s^2+s+1} - 2 = \frac{\cancel{2s^2}+s-\cancel{2s^2}-2s-2}{s^2+s+1} = \frac{-s+2}{s^2+s+1}$$

$\hat{F}(s)$

$$\Rightarrow f'(0^+) = \lim_{s \to \infty} s\, \hat{F}(s) = -1 \quad \text{SOL}$$

ANTI TRASFORMATA

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

ESEMPIO :

$$F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

n POLI e m zeri

IMPO

SCOMPOSIZIONE IN FRATTI SEMPLICI

$$\boxed{F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n} = \sum_{i=1}^{n} \frac{r_i}{s + p_i}}$$

Residuo i-esimo Associato al polo

$$\to f_1 \cdot \frac{1}{s + p_i} \Rightarrow f = r_i \cdot e^{-p_i t}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\sum_{i=1}^{n} \frac{r_i}{s + p_i}\right] = \sum_{i=1}^{n} \mathcal{L}^{-1}\left[\frac{r_i}{s + p_i}\right] = \sum_{i=1}^{n} r_i \mathcal{L}^{-1}\left[\frac{1}{s + p_i}\right]$$

$$\Rightarrow \mathcal{L}^{-1}[F(s)] = \sum_{i=1}^{n}$$

DIMOSTRAZIONE RESIDUO

$$\boxed{\lim_{s \to p_i} (s + p_i) F(s) = r_i}$$

perché $\lim_{s \to p_k} (s + p_k) F(s) =$

$$= \lim_{s \to p_k} (s + p_k) \cdot \sum_{i=1}^{n} \frac{r_i}{s + p_i}$$

La sommatoria scompare perché tutti i termini sono zero tranne s+pk

Se ad esempio volessi calcolare il residuo del Polo due, k = 2 ma la sommatoria scorre sempre su tutti i poli

$$\Rightarrow \lim_{s \to p_k} \cancel{(s + p_k)} \cdot \frac{r_k}{\cancel{s + p_k}} = r_k$$

QED

ESEMPIO

$$F(s) = \frac{s+3}{s^2+3s+2}$$

↑ Dimostrazione sopra

METODO 1. CON I LIMITI

(1) Radici Denom

$$s = \frac{-3 \pm \sqrt{9-8}}{2} \quad \begin{cases} \frac{-3+1}{2} = -1 \\ \frac{-3-1}{2} = -2 \end{cases}$$

$$\rightarrow \quad F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{r_1}{s+1} + \frac{r_2}{s+2}$$

(2) Trovo r_1 e r_2

(a) trovo r_1

$$\lim_{s \to -1} \frac{\cancel{(s+1)}\,(s+3)}{\cancel{(s+1)}(s+2)} = \frac{s+3}{s+2} = \frac{-1+3}{-1+2} = 2$$

P_1

$$\left\{ \text{FORMA: } \sum_{i=1}^{n} r_i \, \mathcal{L}\left[\frac{1}{s+p_i}\right] \right\}$$

(b) trovo r_2

$$\lim_{s \to -2} \frac{\cancel{(s+2)}\,(s+3)}{(s+1)\cancel{(s+2)}} = \frac{s+3}{s+1} = \frac{-2+3}{-2+1} = -1$$

P_2

(3) Scrivo la TRASFORMATA

$$\left.\begin{array}{l} r_1 = 2 \\ r_2 = -1 \end{array}\right\} \quad \boxed{\frac{2}{s+1} - \frac{1}{s+2}}$$

Metodo 2

mcm $\rightarrow$ $(s+1)(s+2)$ $\rightarrow$ Uguagliamo

$$\frac{r_1(s+2) + r_2(s+1)}{(s+1)(s+2)} = \frac{(r_1+r_2)s + 2r_1 + r_2}{(s+1)(s+2)} \overset{\text{DEVE}}{=} \frac{s+3}{(s+1)(s+2)}$$

$$\Rightarrow \begin{cases} r_1 + r_2 = 1 \\ 2r_1 + r_2 = 3 \end{cases} \Rightarrow \begin{cases} r_2 = 1 - r_1 \\ \rightarrow 2r_1 + 1 - r_1 = 3 \rightarrow \end{cases} \begin{array}{l} r_1 = 2 \\ \Rightarrow r_2 = -1 \end{array}$$

Spiegazione FINE CLIP 12

$$\Rightarrow \quad f(t) = (2e^{-t} - e^{-2t})\,\mathbb{1}(t)$$