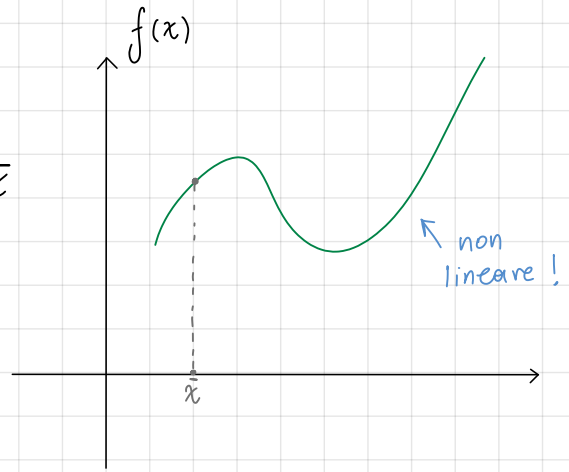


Linearizzazione : Come rendere lineare un sistema NON lineare

$f(x)$

Linearizziamo $f(x)$ intorno al punto a $x = \bar{x}$



Approx lin

$$f(x) = f(\bar{x}) + \left. \frac{df}{dx} \right|_{x=\bar{x}} (x-\bar{x}) + \frac{1}{2!} \frac{d^2f}{dx^2} (x-\bar{x})^2 + \dots + \frac{1}{n!} \frac{d^n f}{dx^n} (x-\bar{x})^n$$

-> Per approssimare linearmente ci basta approssimare al 1° Termine.

$$\rightarrow f(x) = \underbrace{f(\bar{x}) + \left. \frac{df}{dx} \right|_{x=\bar{x}} (x-\bar{x})}_{\beta} + \underbrace{\left. \frac{df}{dx} \right|_{x=\bar{x}}}_{\alpha} \epsilon$$

Al primo ordine
Deve essere sufficientemente PICCOLO

Esempi di funzioni linearizzate

$$f(x) = x^2$$

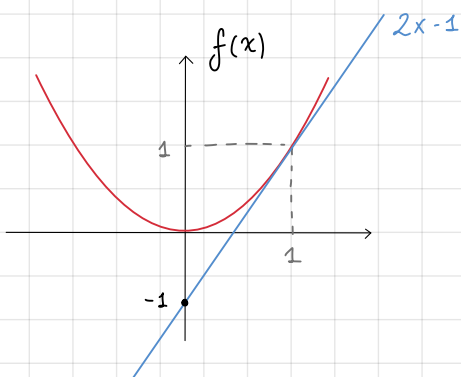
Vogliamo

$$\bar{x} = 1$$

$$\hat{f}(x) = f(\bar{x}) + \left. \frac{df}{dx} \right|_{x=\bar{x}} (x-\bar{x})$$

$$= 1 + 2x \Big|_{x=1} (x-1)$$

$$= 1 + 2x - 2 = 0 \rightarrow y = 2x - 1$$



Approssimazione per funzioni a 2 Variabili

$$f(x,y) = f(\bar{x}, \bar{y}) + \left. \frac{\partial f}{\partial x} \right|_{\bar{x}, \bar{y}} \cdot (x - \bar{x}) + \left. \frac{\partial f}{\partial y} \right|_{\bar{x}, \bar{y}} (y - \bar{y}) + \varepsilon$$

Esempio

$$f(x,y) = xy$$

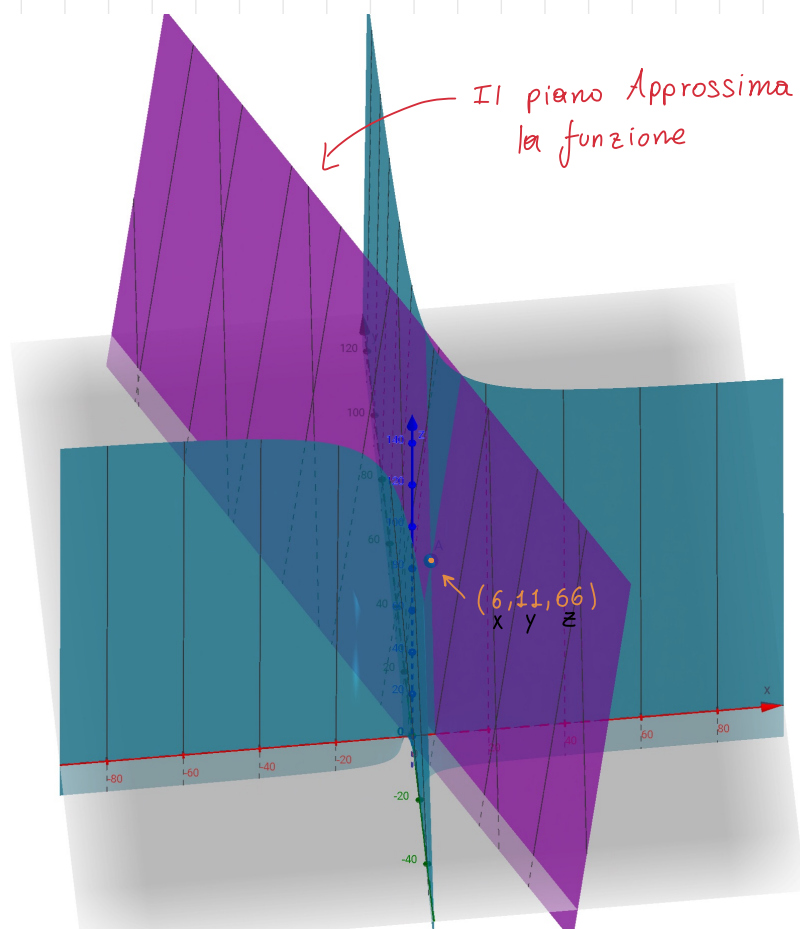
$$\text{con } x \in [5, 7] \text{ e } y \in [10, 12]$$

$$\Downarrow \\ \bar{x} = 6$$

$$\Downarrow \\ \bar{y} = 11$$

$$\Rightarrow \hat{f}(x,y) = 66 + 11 \cdot (x-6) + 6 \cdot (y-11) = \cancel{66} + 11x - \cancel{66} + 6y - 66$$

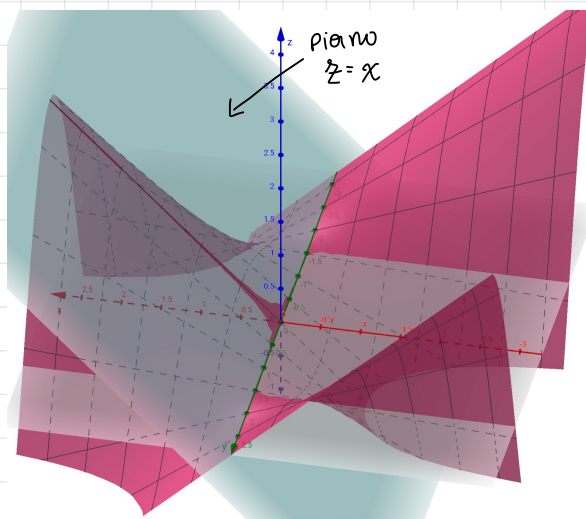
$$\Rightarrow \underline{\hat{f}(x,y) = 11x + 6y - 66}$$



$$f(x,y) = x \cdot \cos(y)$$

$$\text{in } \bar{x}=0, \bar{y}=0$$

$$\hat{f}(x,y) = 0 + 1(x-0) + -\sin(0) \cdot 0(y-0) = x$$

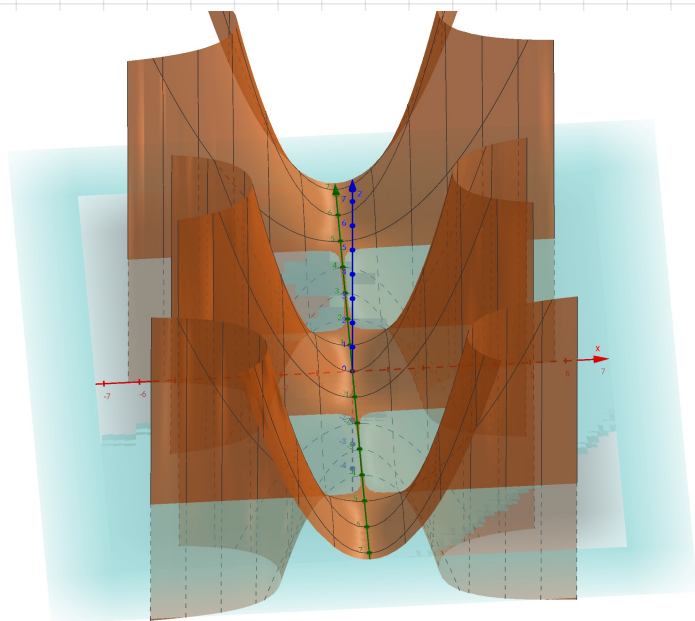


$$f(x,y) = x \sin y$$

$$\text{in } \bar{x}=0, \bar{y}=0$$

$$\hat{f}(x,y) = 0 + 0 + 0 = 0 \quad \Rightarrow \quad \text{Piano } z=0 \quad \text{Approssima } f(x,y) \text{ in } (0,0,0)$$

$$f(x,y) = x^2 \cos y \simeq z=0$$



Sistema non lineare

Scriviamo le eq del sys

$$\dot{x} = f(x, u) \quad \rightarrow \text{Linearizzazione} \quad \text{ma intorno a quale punto?}$$



PUNTO DI EQUILIBRIO del Sys

$$u = \bar{u}, \quad \bar{x}: f(\bar{x}, \bar{u}) = 0$$

$$\dot{x} \Big|_{x=\bar{x}} = \dot{\bar{x}} = f(\bar{x}, \bar{u}) = 0 \quad \leftarrow \text{La derivata NON cambia}$$

Linearizzazione attorno al punto di equilibrio

$$x = \bar{x} + \hat{x}$$

↑
punti di equilibrio

distanza
rispetto a dove
linearizzo

$$\dot{x} = \dot{\bar{x}} + \dot{\hat{x}}$$

la derivata
di un P.E. è
Zero

$$\dot{\bar{x}} = \dot{\hat{x}}$$

$$\dot{x} = \dot{\hat{x}} = f(\bar{x}, \bar{u}) + \underbrace{\nabla_x f}_{\substack{\text{GRADIENTE} \\ \text{ovvero la derivata} \\ \text{multidimensionale}}} \Big|_{\bar{x}, \bar{u}} \underbrace{(\hat{x})}_{\substack{\text{A} \\ \text{B}}} + \underbrace{\nabla_u f}_{\substack{\text{B} \\ \text{C}}} \Big|_{\bar{x}, \bar{u}} \underbrace{(\hat{u})}_{\substack{\text{D} \\ \text{E}}}$$

Derivate
parziali

$$f(x) = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{pmatrix} \quad \nabla f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

$$\dot{\hat{x}} = A \hat{x} + B \hat{u}$$

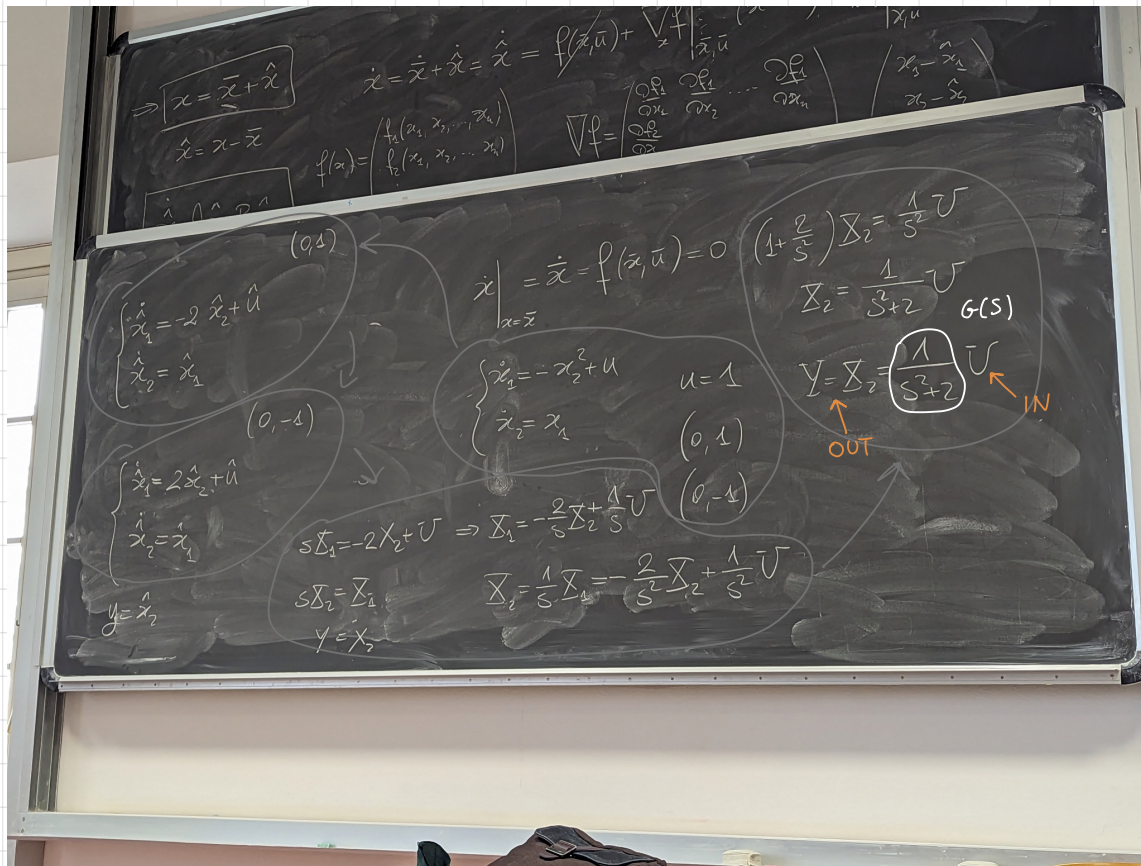
Sistema che sappiamo risolvere!

↑
Sistema dinamico
LINEARE Tempo invariante

SOLO nell'intorno del punto di equilibrio

Cambiando il punto di equilibrio le derivate rimangono le stesse, ma cambiano i punti in cui esse vengono calcolate; di conseguenza cambiando il punto di equilibrio cambieranno le matrici **A** e **B**

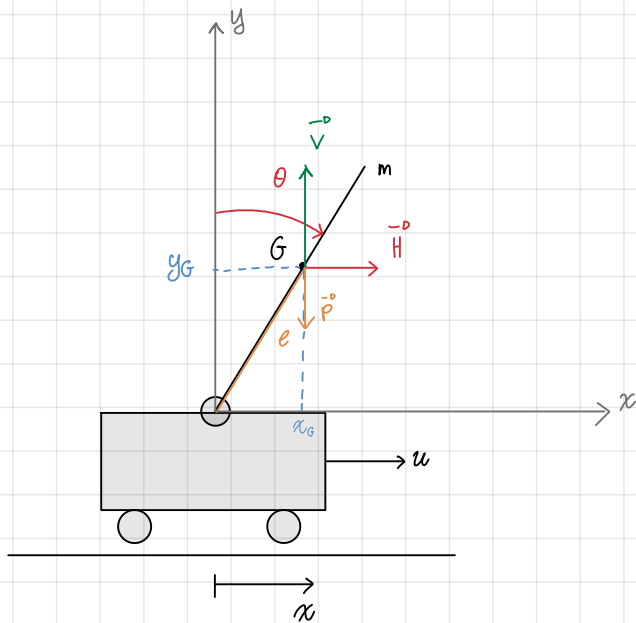
FOTO



PENDOLO INVERSO

Su un carrello

NON CHIESTO ALL'ORALE!

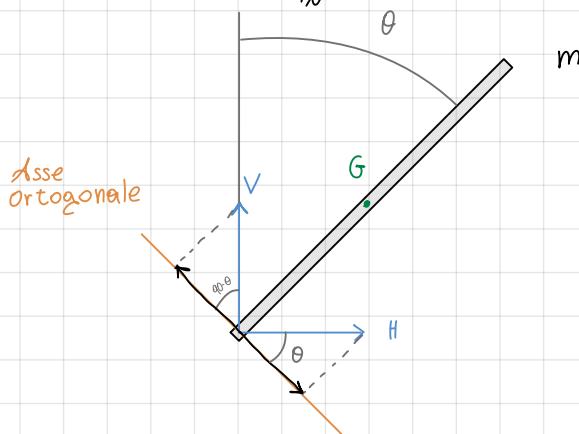


$$x_G = x + l \sin \theta$$

$$y_G = l \cos \theta$$

$$\begin{cases} m \cdot \frac{d^2 x_G}{dt^2} = H \\ m \cdot \frac{d^2 y_G}{dt^2} = V - mg \\ I \ddot{\theta} = \underbrace{V \cdot l \cos \theta}_{\text{COPPIA}} - \underbrace{H \cdot l \sin \theta}_{\text{COPPIA}} \\ M \ddot{x} = u - H \end{cases}$$

FORZE ORIZZ E VERT
BRACCIO



SOLUZIONE CLIP 12

$$(1) \quad m \frac{d^2 x_G}{dt^2} = m \frac{d^2 x}{dt^2} + m l \frac{d^2}{dt^2} \sin \theta = m \ddot{x} + m l \frac{d}{dt} \left(\frac{d}{dt} \sin \theta \right) = m \ddot{x} + m l \frac{d}{dt} (\dot{\theta} \cos \theta)$$

$$= m \ddot{x} + m l [\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta]$$

$$(2) \quad m \frac{d^2 y_G}{dt^2} = m \frac{d^2}{dt^2} (l \cos \theta) = m l \frac{d}{dt} (-\dot{\theta} \sin \theta) = m l (-\ddot{\theta} \sin \theta - \dot{\theta}^2 \cos \theta)$$

Sostituisco nel sys

$$\begin{cases} m \ddot{x} = H - m l (\ddot{\theta} \cos \theta + \dot{\theta}^2 \sin \theta) \\ 0 = m l \ddot{\theta} \sin \theta + m l \dot{\theta}^2 \cos \theta + V - m g \\ I \ddot{\theta} = V l \sin \theta - H l \cos \theta \\ M \ddot{x} = u - H \end{cases} \quad \rightarrow \text{Linearizziamo}$$

$$\begin{cases} m \ddot{x} = H - m l \ddot{\theta} & (1) \\ V = m g & (2) \\ \pm \ddot{\theta} = \frac{V}{l} \theta - \frac{H}{l} & (3) \\ M \ddot{x} = u - H & (4) \end{cases}$$

Risolvere il sistema

dalla (4)

$$H = m\ddot{x} + m\ell\ddot{\theta}$$

$$M\ddot{x} = u - m\ddot{x} - m\ell\ddot{\theta}$$

$$\Rightarrow (M+m)\ddot{x} + m\ell\ddot{\theta} = u$$

$$I\ddot{\theta} = m\ell\ddot{x} - m\ell^2\ddot{\theta}$$

se diamo le var di stato

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \theta \\ x_4 = \dot{\theta} \end{cases}$$

possiamo risolvere

$$\begin{cases} (M+m)\ddot{x} + m\ell\ddot{\theta} = u \\ (I+m\ell^2)\ddot{\theta} + m\ell\ddot{x} = m\ell\ddot{x} \end{cases}$$

modello semplificato
 $\Rightarrow I=0$

Se prendiamo $I=0$

$$\begin{cases} (M+m)\ddot{x} + m\ell\ddot{\theta} = u \\ \ddot{\theta}m\ell^2 + m\ell\ddot{x} = m\ell\ddot{x} \end{cases} \quad \text{L.T.} \Rightarrow \begin{cases} (M+m)s^2\bar{X} + m\ell s^2\bar{\Theta} = U \\ m\ell s^2\bar{\Theta} + m s^2\bar{X} = m\ell\bar{\Theta} \quad (2) \end{cases}$$

dalla (2) $s^2\bar{X} = m\ell\bar{\Theta} - m\ell s^2\bar{\Theta}$

nella (1) $(M+m)(m\ell\bar{\Theta} - \ell s^2\bar{\Theta}) + m\ell s^2\bar{\Theta} = U$

$$\begin{cases} (M+m)\ell\bar{\Theta} - (M+m)\ell s^2\bar{\Theta} + m\ell s^2\bar{\Theta} = U \\ \Rightarrow (M+m)\ell\bar{\Theta} - M\ell s^2\bar{\Theta} = U \quad \Rightarrow \end{cases}$$

$$G(s) = \frac{\bar{\Theta}(s)}{-U(s)} = \frac{1}{M\ell s^2 - (M+m)\ell}$$