

# SISTEMI DEL 1° ORDINE

Op. Preliminare: Piano complesso

ES:

$$G(s) = \frac{1}{sT+1}$$

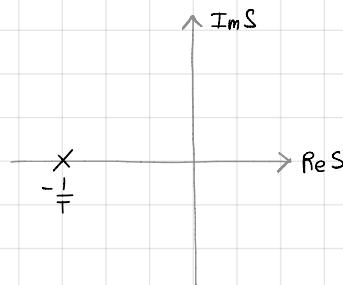
Con  $U_1(t) = \mathbb{1}(t)$

Svolgimento

1) Analisi INPUT

$$U_1(t) = \mathbb{1}(t) \Rightarrow U_1(s) = \frac{1}{s}$$

$$sT + 1 \rightarrow s = -\frac{1}{T}$$



2) Calcolo uscita in S

$$Y_1(s) = G(s) \cdot U(s) = \frac{1}{sT+1} \cdot \frac{1}{s} = \frac{1}{T(s+\frac{1}{T}) \cdot s} = \frac{\frac{1}{T}}{s(s+\frac{1}{T})} = \frac{E_1}{s} + \frac{E_2}{sT+1}$$

Forma Standard

I termini in s sono "soli"

$$\text{Polo: } s = -\frac{1}{T}$$

2.1) Calcolo residui

$$E_1 = \lim_{s \rightarrow 0} s \cdot \frac{\frac{1}{T}}{s(s+\frac{1}{T})} \rightarrow 1 E_1$$

$$E_2 = \lim_{s \rightarrow -\frac{1}{T}} (s + \frac{1}{T}) \frac{\frac{1}{T}}{s(s+\frac{1}{T})} = \frac{\frac{1}{T}}{-\frac{1}{T}} \rightarrow -1 E_2$$

Frazioni semplici:

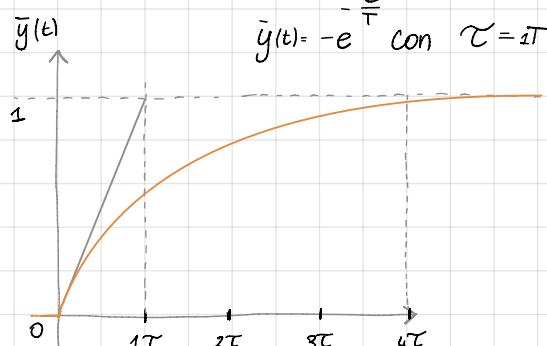
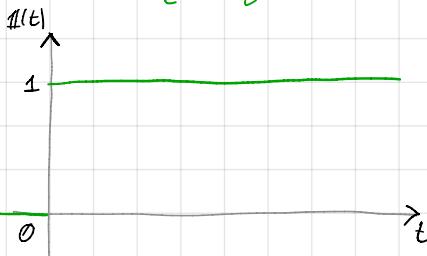
$$\Rightarrow Y_1(s) = G(s) \cdot U(s) = \frac{1}{s(sT+1)} = \frac{1}{s} - \frac{1}{sT+1}$$

3) Antitrasformo l'uscita

- $\frac{1}{s} \rightleftharpoons \mathbb{1}(t)$
- $\frac{1}{sT+1} \rightleftharpoons e^{-\frac{t}{T}}$

$$\Rightarrow y_1(t) = (1 - e^{-\frac{t}{T}}) \cdot \mathbb{1}(t) \quad \text{Ans 1}$$

4) Disegno gli andamenti



Stessa fdt ma input diverso

$$U_2(t) = t \cdot u(t) \Rightarrow U(s) = \frac{1}{s^2}$$

$$\Rightarrow Y_2(s) = \frac{1}{s^2} \cdot \frac{1}{sT+1} = \frac{\varepsilon_1}{s} + \frac{\varepsilon_2}{s^2} + \frac{\varepsilon_3}{sT+1}$$

$$\text{con } G(s) = \frac{1/T}{s+1/T} ; \bar{s} = -\frac{1}{T}$$

$$\varepsilon_2 = \lim_{s \rightarrow 0} s^2 \frac{1/T}{s(s+1/T)} = \textcircled{1} \varepsilon_2$$

$$\varepsilon_3 = \lim_{s \rightarrow -\frac{1}{T}} (s + \frac{1}{T}) \cdot \frac{1/T}{s^2(s+1/T)} = \frac{1/T}{1/T^2} = \textcircled{T} \varepsilon_3$$

$$= \frac{\varepsilon_1}{s} + \frac{\varepsilon_2}{s^2} + \frac{\varepsilon_3}{s+1/T} = \frac{s\varepsilon_1(s+1/T) + \varepsilon_2(s+1/T) + s^2\varepsilon_3}{s^2(s+1/T)}$$

usa il denominatore usato nel calcolo degli altri residui!

$$\begin{cases} s^2(\varepsilon_1 + \varepsilon_3) = 0 \\ s(\frac{1}{T}\varepsilon_1 + \varepsilon_2) = 0 \Rightarrow \varepsilon_1 = -\varepsilon_2 \cdot T \Rightarrow \varepsilon_1 = -T \varepsilon_2 \\ \varepsilon_2 \frac{1}{T} = \frac{1}{T} \Rightarrow \varepsilon_2 = 1 \end{cases}$$

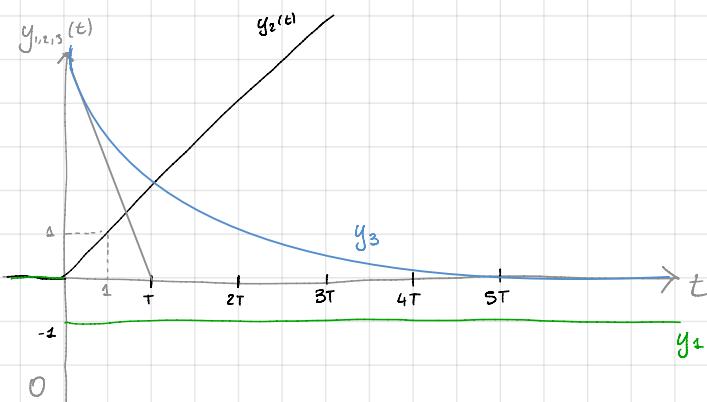
QED

$$\Rightarrow Y(s) = -\frac{T \cdot u(t)}{s} + \frac{1}{s^2} - \frac{T}{s+1/T} \quad \Leftrightarrow y(t) = (-T+t-Te^{-\frac{t}{T}}) \cdot u(t)$$

### • Andamento a regime e in zero

$$\text{TVI: } f(0) = \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} F(s) = \lim_{s \rightarrow \infty} -\frac{1}{s} + \frac{1}{s^2} + \frac{T}{sT+1} \Rightarrow \textcircled{0} f(0)$$

$$\text{TVF: } f(t \rightarrow \infty) = \lim_{s \rightarrow 0} -\frac{1}{s} + \frac{1}{s^2} + \frac{T}{sT+1} \Rightarrow \textcircled{+\infty} f(t \rightarrow \infty)$$



$$\lim_{s \rightarrow \infty} -\frac{1}{s} + \frac{1}{s^2} + \frac{T}{sT+1} \Rightarrow \textcircled{0} f(0)$$

$$\lim_{s \rightarrow 0} -\frac{1}{s} + \frac{1}{s^2} + \frac{T}{sT+1} \Rightarrow \textcircled{+\infty} f(t \rightarrow \infty)$$

$$\text{Per } t \rightarrow \infty, y(t) \rightarrow -1 + t + e^{-\frac{t}{T}} = \textcircled{t-1}$$

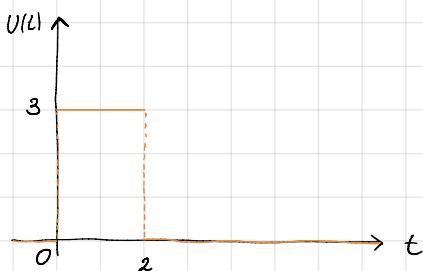
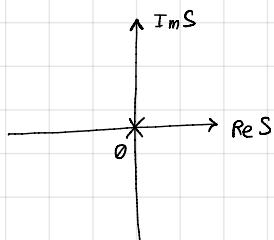
$$y \approx \textcircled{y} \text{ per } t \rightarrow \infty$$

Tolgo  $e^{-\frac{t}{T}}$  perché tende a zero mentre  $t \rightarrow \infty$

⇒ Morale: dopo  $4T$   $y(t)$  assomiglia molto a  $y(t) = t-1$

## ES: INTEGRATORE CON FINESTRA

$G(s) = \frac{1}{s}$   $\Rightarrow$  No zeri, 1 polo nell'origine  $\Rightarrow$  Mantiene l'uscita



$$U(t) = \begin{cases} 0 & t < 0 \\ 3 & 0 \leq t \leq 2 \\ 0 & t > 2 \end{cases}$$

(1) Scrivo il segnale di input con segnali notevoli:

$$\begin{cases} U_1(t) = 3\mathbb{1}(t) \\ U_2(t) = -3\mathbb{1}(t-2) \end{cases}$$

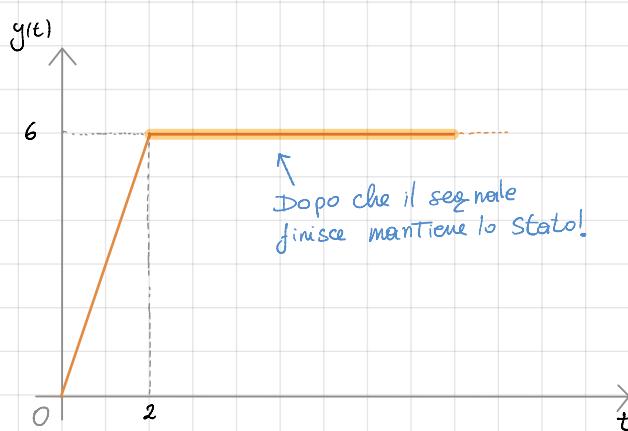
(2) Considero un segnale fittizio  $\hat{U}(t) = \mathbb{1}(t)$

$$\Rightarrow \hat{U}(s) = \frac{1}{s} \Rightarrow \hat{Y}(s) = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2} \Rightarrow \hat{y}(t) = t \cdot \mathbb{1}(t)$$

(3) Segnali reali

$$\begin{cases} y_1(t) = 3\hat{y}(t) = 3t \cdot \mathbb{1}(t) \\ y_2(t) = -3\hat{y}(t-2) = [-3(t-2)] \cdot \mathbb{1}(t) \end{cases} \Rightarrow y(t) = y_1 + y_2 = 3t \cdot \mathbb{1}(t) - 3(t-2) \cdot \mathbb{1}(t-2)$$

(4) Rappresento le uscite:



Per  $t \in (0, 2)$   $\Rightarrow y(t) = 3t$

Per  $t \in (2, +\infty)$   $\Rightarrow y(t) = 3t - 3t + 6 = 6$

(5) Considerazioni

Dobbiamo tenere conto che la funzione di trasferimento è un "integratore", quindi **integra il segnale di input**. Siccome il segnale in input, per l'intervallo da 0 a 2, è un segnale costante, sappiamo che l'integrale di un segnale costante è una retta, infatti l'osserviamo in uscita.

Quando il segnale in input si azzerà, l'area sottesa ad una costante di valore zero è proprio zero, quindi il contributo di quell'intervallo è zero, ma resta "l'area sottesa" calcolata fino a quel punto, quindi il segnale in uscita resta costante.

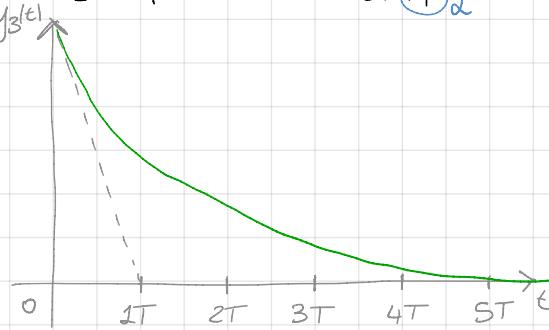
Stessa fdt ma input diverso

$$U_3(t) = \int \Leftrightarrow U(s) = \mathbb{1}(t)$$

$$\Rightarrow G(s) = \frac{1}{sT+1} = \frac{\frac{1}{T}}{s + \frac{1}{T}} \rightsquigarrow$$

$$\rightsquigarrow y_3(t) = \frac{1}{T} \cdot e^{-\frac{t}{T}} \quad \text{Ans}$$

$$Y_3(s) = \frac{\frac{1}{T}}{s + \frac{1}{T}} = \frac{\frac{1}{T}}{s + \frac{1}{T}} \quad \text{e sia' in fratti semplici}$$



### CONSIDERAZIONI

| INPUT                            | OUTPUT                                                                    |
|----------------------------------|---------------------------------------------------------------------------|
| $U_2(t) = t \cdot \mathbb{1}(t)$ | $y_2(t) = \left( -T + t - T e^{-\frac{t}{T}} \right) \cdot \mathbb{1}(t)$ |
| $U_1(t) = \mathbb{1}(t)$         | $y_1(t) = \left( 1 - e^{-\frac{t}{T}} \right) \cdot \mathbb{1}(t)$        |
| $U_3(t) = \int$                  | $y_3(t) = \frac{1}{T} \cdot e^{-\frac{t}{T}}$                             |

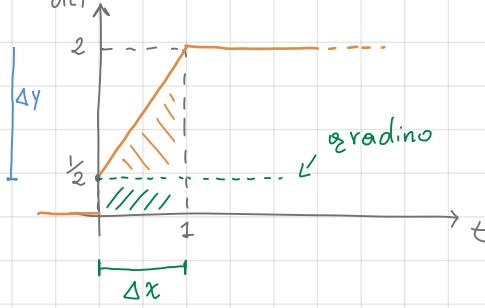
$$\text{Considerando } G(s) = \frac{1}{sT+1}$$

$$\frac{d}{dt} y_2(t) = \frac{d}{dt} \left[ -T + t - T e^{-\frac{t}{T}} \right] = \boxed{1 - e^{-\frac{t}{T}}} \quad \text{Risposta al gradino}$$

$$\frac{d}{dt} y_1(t) = \boxed{-\frac{1}{T} e^{-\frac{t}{T}}} \quad \text{Risposta all'impulso}$$

## ES - Lezione 17

$$G(S) = \frac{2S+3}{S+5}$$

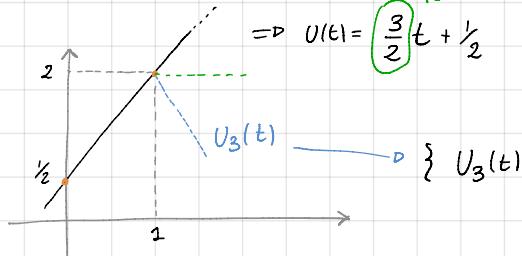


$$\Rightarrow \begin{cases} U_2(t) = \frac{3}{2}t \cdot \mathbb{1}(t) \\ U_1(t) = \mathbb{1}(t) \end{cases}$$

1) SCRIVO  $U(t)$  IN SEGNALE ELET.

Trovo pendenza netta  $[0; 1]$ :  $\frac{t-0}{1-0} = \frac{U(t) - \frac{1}{2}}{2 - \frac{1}{2}} \Rightarrow t = \frac{2}{3}U(t) - \frac{1}{3}$

What??  
dovrebbe essere  $\frac{1}{2}$



$$\Rightarrow U(t) = U_1(t) + U_2(t) + U_3(t) = \mathbb{1}(t) + \frac{3}{2}t \cdot \mathbb{1}(t) - \frac{3}{2}(t-1) \cdot \mathbb{1}(t-1)$$

2) TRASFORMO

$$U(s) = \mathcal{L}[u(t)] = \frac{U_1}{\frac{1}{s}} + \frac{U_2}{\frac{3}{2} \frac{1}{s^2}} - \frac{U_3}{\frac{3}{2} e^{-s} \frac{1}{s^2}}$$

3) CALCOLO USCITA IN S

Considero il segnale fittizio

$$\bar{U}(t) = t \cdot \mathbb{1}(t) \Rightarrow U(s) = \frac{1}{s^2}$$

$$\Rightarrow \bar{Y}(s) = G(s) \cdot U(s) = \frac{2s+3}{s+5} \cdot \frac{1}{s^2} = \frac{\xi_1}{s} + \frac{\xi_2}{s^2} + \frac{\xi_3}{s+5}$$

$$\Rightarrow \xi_2 = \lim_{s \rightarrow 0} s^2 \cdot \frac{2s+3}{s^2(s+5)} = \frac{3}{5} \xi_2$$

$$\xi_3 = \lim_{s \rightarrow -5} (s+5) \cdot \frac{2s+3}{s^2(s+5)} \Rightarrow \frac{-7}{s} = -\frac{7}{3} \xi_3$$

$$\text{e.d. } \frac{s\xi_1(s+5) + \xi_2(s+5) + s^2\xi_3}{s^2(s+5)} \Rightarrow \begin{cases} s^2(\xi_1 + \xi_3) = 0 \Rightarrow \xi_1 = -\xi_3 \Rightarrow \xi_1 = \frac{7}{5} \\ s(\xi_1 + \xi_2) = 2 \Rightarrow \xi_1 + \xi_2 = 2 \Rightarrow \xi_1 = 2 - \xi_2 = \frac{10-3}{5} = \frac{7}{5} \\ 5\xi_2 = 3 \Rightarrow \xi_2 = \frac{3}{5} \end{cases} \text{ QED}$$

$$\Rightarrow \bar{Y}(s) = \frac{7}{5} \left( \frac{1}{s} \right) + \frac{3}{5} \left( \frac{1}{s^2} \right) - \frac{7}{5} \left( \frac{1}{s+5} \right) \Rightarrow \bar{y}(t) = \left( \frac{7}{5} + \frac{3}{5}t - \frac{7}{5}e^{-5t} \right) \cdot \mathbb{1}(t)$$

3.2) Calcolo segnali reali

$$U_2(t) = \frac{3}{2}t \cdot \mathbb{1}(t) = \frac{3}{2}\bar{U}(t) \Rightarrow \bar{y}_2(t) = \frac{3}{2}\bar{y}(t) = \left( \frac{21}{10} + \frac{9}{10}t - \frac{21}{10}e^{-5t} \right) \cdot \mathbb{1}(t)$$

$$U_3(t) = -\frac{3}{2}(t-1) \cdot \mathbb{1}(t-1) = -\frac{3}{2}\bar{U}(t-1) \Rightarrow \bar{y}_3(t) = -\frac{3}{2}\bar{y}(t-1) = \left[ -\frac{21}{10} - \frac{9}{10}(t-1) + \frac{21}{10}e^{-5(t-1)} \right] \cdot \mathbb{1}(t-1)$$

$$U_1(t) = \frac{1}{2} u(t) = \frac{1}{2} \frac{d}{dt} [\bar{U}(t)] = \left( \frac{4}{10} + \frac{4}{2} e^{-st} \right) u(t)$$

\* Le Costanti potrebbero essere errate...

4) Disegno dei Singoli Termini

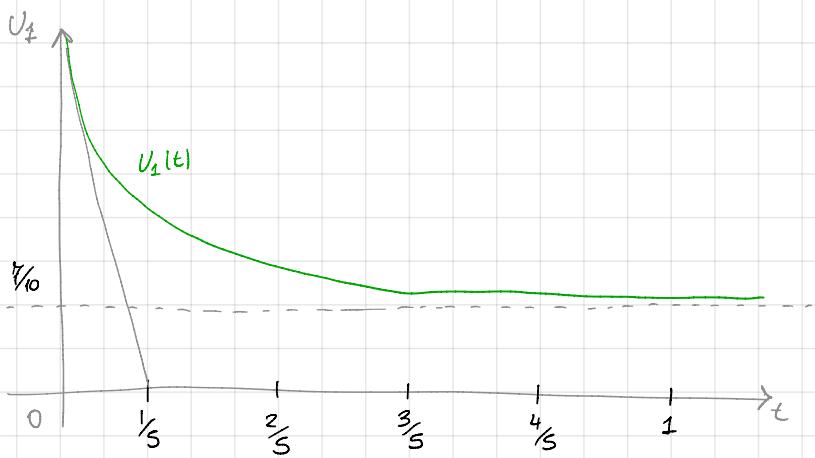
$$U_1(t) \approx \frac{4}{2} e^{-st} \Rightarrow C = \frac{1}{5}$$

4.1) Valori iniziali e finali

$$\text{Calcolo } Y(s) = \underset{\text{reale}}{\frac{2s+3}{s+5}} \left( \frac{1}{s} + \frac{3}{2} \frac{1}{s^2} - \frac{3}{2} e^{-s} \cdot \frac{1}{s^2} \right)$$

$$\text{TVI: } y(0^+) = \lim_{s \rightarrow \infty} s \left[ \frac{2s+3}{s+5} \left( \frac{1}{s} + \frac{3}{2} \cancel{\frac{1}{s^2}} - \frac{3}{2} e^{-s} \cancel{\frac{1}{s^2}} \right) \right] \rightarrow +\infty$$

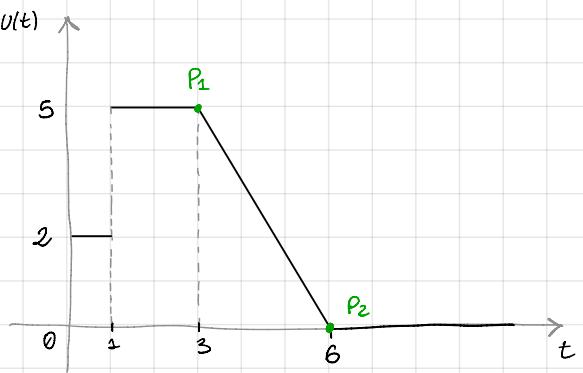
$$\text{TVF: } y(t \rightarrow \infty) = \lim_{s \rightarrow 0} s \left[ \frac{2s+3}{s+5} \left( \frac{1}{s} + \frac{3}{2} \cancel{\frac{1}{s^2}} - \frac{3}{2} e^{-s} \cancel{\frac{1}{s^2}} \right) \right] \text{ BOH}$$



## Esempio Suoluzione

$$G(S) = \frac{1}{S + \frac{1}{2}}$$

Determinare la risposta all'ingresso in figura



1) Costruiamo  $u(t)$  a partire dai segnali elem.

$$\begin{cases} U_1(t) = 2 \cdot \mathbb{1}(t) \\ U_2(t) = 3 \cdot \mathbb{1}(t-1) \end{cases} \quad U_1(t) + U_2(t) \text{ ottengo } 5 \mathbb{1}(t_0) \text{ in } t_0 = 3 ; \quad \text{devo trovare un segnale che ricrei la rampa in } [3; 6]$$

→ Il segnale sarà sicuramente una rampa, ma serve di una pendenza specifica...

$$\frac{t-3}{6-3} = \frac{U(t)-5}{0-5} \rightarrow U(t) = \left[ \left( -1 + \frac{1}{3}t \right) \cdot (-5) \right] + 5 = 5 - \frac{5}{3}t + 5 = -\frac{5}{3}t + 10$$

retta che descrive il segnale in  $[3; 6]$

Pendenza

Formula retta passante per 2 punti

$$\Rightarrow U_3(t) = -\frac{5}{3}((t-3)) \cdot \mathbb{1}(t-3)$$

Il segnale è  
shiftato ed  
inizia in  $t=3$

✓ pendenza opposta

$$U_4(t) = +\frac{5}{3}(t-6) \cdot \mathbb{1}(t-6)$$

A questo punto  $\bar{U}(t) = U_1 + U_2 + U_3$  è una rampa decrescente con pendenza  $-\frac{5}{3}$  (da  $t=3$  in poi); dobbiamo annullarla per valori da  $t=6$  in poi...

⇒ Segnale in figura:  $U(t) = U_1(t) + U_2(t) + U_3(t) + U_4(t)$

2) Trasformate di Laplace

$$U_1(t) \Rightarrow U_1(S) = \frac{2}{S} \text{ gain}$$

$$U_3(t) \Rightarrow U_3(S) = -\frac{5}{3} \frac{e^{-3S}}{S^2}$$

$$U_2(t) \Rightarrow U_2(S) = \frac{3}{S} e^{-S}$$

$$U_4(t) \Rightarrow U_4(S) = \left( \frac{5}{3} \frac{e^{-6S}}{S^2} \right) \text{ Ritardo}$$

Rampa  
guadagno

3) Calcolo dell'uscita nel dominio S

→ Ci conviene usare il segnale fittizio  $\hat{U}(t) = t \cdot \mathbb{1}(t) \Rightarrow \hat{U}(S) = \frac{1}{S^2}$

$$\Rightarrow \hat{Y}(S) = G(S) \cdot \hat{U}(S) = \frac{1}{S + \frac{1}{2}} \cdot \frac{1}{S^2} = \frac{1}{S^2(S + \frac{1}{2})} = \frac{\varepsilon_1}{S} + \frac{\varepsilon_2}{S^2} + \frac{\varepsilon_3}{S + \frac{1}{2}}$$

$$\varepsilon_2 = \lim_{S \rightarrow 0} (S^2) \cdot \frac{1}{S^2(S + \frac{1}{2})} \rightarrow \frac{1}{\frac{1}{2}} = 2 \varepsilon_2$$

$$\frac{S(s+\frac{1}{2})\varepsilon_1 + (s+\frac{1}{2})\varepsilon_2 + s^2\varepsilon_3}{S^2(S+\frac{1}{2})} = \frac{S^2\varepsilon_1 + S\frac{1}{2}\varepsilon_1 + S\varepsilon_2 + \frac{1}{2}\varepsilon_2 + S^2\varepsilon_3}{S^2(S+\frac{1}{2})}$$

$$\varepsilon_3 = \lim_{S \rightarrow -\frac{1}{2}} (S + \frac{1}{2}) \cdot \frac{1}{S^2(S + \frac{1}{2})} \rightarrow 4 \varepsilon_1$$

$$S^2(\varepsilon_1 + \varepsilon_3) = 0 \Rightarrow \varepsilon_1 = -\varepsilon_3 = -4 \varepsilon_1$$

$$S(\frac{1}{2}\varepsilon_1 + \varepsilon_2) = 0 \Rightarrow \varepsilon_1 = -2\varepsilon_2 = -4 \varepsilon_2$$

$$\frac{1}{2}\varepsilon_2 = 1 \Rightarrow \varepsilon_2 = 2$$

$$\Rightarrow \hat{Y}(S) = -\frac{4}{S} + \frac{2}{S^2} + \frac{4}{S + \frac{1}{2}}$$

Uscita in S

#### 4.a) Antitrasformata

$$\hat{Y}(s) \Leftrightarrow \hat{y}(t) = -4 \cdot 1(t) + 2t \cdot 1(t) + 4 \cdot e^{-\frac{t}{2}} \cdot 1(t) = \boxed{\left[ -4 + 2t + 4e^{-\frac{t}{2}} \right] \cdot 1(t)}$$

#### 5) Valore iniziale e finale

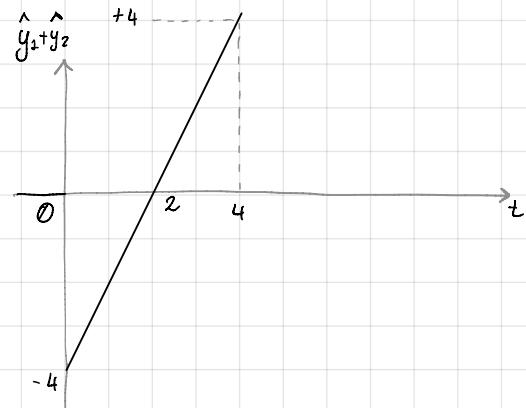
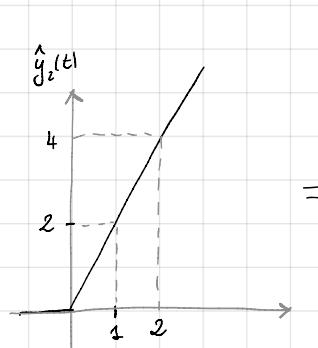
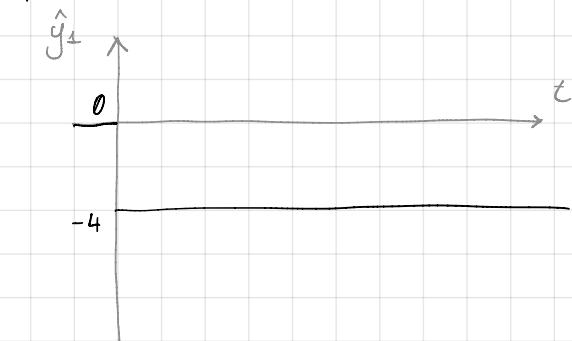
$$\hat{y}(0^+) = \lim_{t \rightarrow 0^+} \hat{y}(t) = \lim_{s \rightarrow \infty} s \cdot \hat{Y}(s) = \lim_{s \rightarrow \infty} s \cdot \left( -\frac{4}{s} + \frac{2}{s^2} + \frac{4}{s + \frac{1}{2}} \right) = \lim_{s \rightarrow \infty} -4 + \frac{2}{s} + \frac{8}{s(s + \frac{1}{2})}$$

$$\hat{y}(\infty) = \lim_{t \rightarrow \infty} \hat{y}(t) = \lim_{s \rightarrow 0} s \cdot \hat{Y}(s) = \lim_{s \rightarrow 0} s \cdot \left( -4 + \frac{2}{s} + \frac{8}{s(s + \frac{1}{2})} \right) = -4 + \infty$$

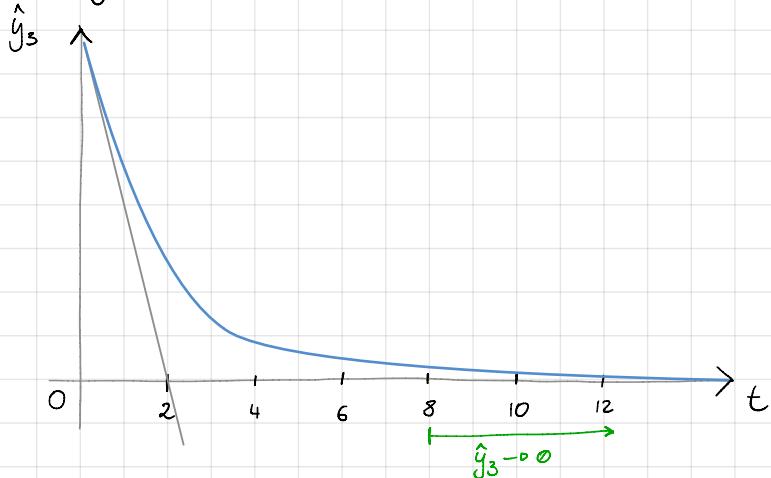
Valore finale

la rampa fa tendere l'uscita a  $\infty$

#### 4.b) Disegno gli Andamenti



$$\hat{y}_3(t) = 4e^{-\frac{t}{2}}$$



All'entrata  $\hat{U}(t) = t \cdot 1(t)$

#### 5. Trovo le uscite specifiche

Siccome le entrate reali sono:

$$\begin{cases} U_1(t) = 2 \cdot 1(t) \\ U_2(t) = 3 \cdot 1(t-1) \end{cases} \quad \begin{cases} U_3(t) = -\frac{5}{3} \cdot 1(t-3) \cdot 1(t-3) \\ U_4(t) = \frac{5}{3} \cdot 1(t-6) \cdot 1(t-6) \end{cases}$$

Ricordando che  $\frac{d}{dt} t = 1(t)$  e che  $\frac{d}{dt} e^t = t$

$$\Rightarrow y_1(t) = 2 \cdot \frac{d}{dt} (\hat{y}(t)) = 2 \cdot \frac{d}{dt} \left[ -4 + 2t + 4e^{-\frac{t}{2}} \right] \cdot 1(t) = 2 \left( 2 - \frac{1}{2} 4e^{-\frac{t}{2}} \right) \cdot 1(t) = 2 \left( 1 - e^{-\frac{t}{2}} \right) \cdot 1(t)$$

$$y_2(t) = \frac{3}{2} \cdot y_1(t-1) = \frac{3}{2} \cdot \frac{3}{2} \left( 2 - 2e^{-\frac{t-1}{2}} \right) \cdot 1(t) = 6 \left( 1 - e^{-\frac{t-1}{2}} \right) \cdot 1(t-1)$$

mentre le uscite fittizie sono:

$$\begin{cases} \hat{y}_1(t) = -4 \cdot 1(t) \\ \hat{y}_2(t) = 2t \cdot 1(t) \end{cases} \quad \begin{cases} \hat{y}_3(t) = 4e^{-\frac{t}{2}} \cdot 1(t) \\ \hat{y}_4(t) = \frac{5}{3} \cdot 1(t-6) \cdot 1(t-6) \end{cases}$$

$$y_3(t) = -\frac{5}{3} \cdot \hat{y}(t-3) \cdot u(t) = -\frac{5}{3} \cdot \left( -4 + 2(t-3) + 4e^{-\frac{t-3}{2}} \right) = +\frac{20}{3} - \frac{10}{3}t + \frac{30}{3}e^{-\frac{t-3}{2}}$$

$$= \frac{50}{3} - \frac{10}{3}t - \frac{20}{3}e^{-\frac{t-3}{2}} = \frac{10}{3} \left( 5 - t - 2e^{-\frac{t-3}{2}} \right) \cdot u(t-3)$$

$$y_4(t) = \frac{5}{3} y(t-6) = \frac{5}{3} \left( -4 + 2(t-6) + 4e^{-\frac{t-6}{2}} \right) = -\frac{20}{3} + \frac{10}{3}t - 2 \cdot \frac{20}{3} + \frac{20}{3}e^{-\frac{t-6}{2}} \cdot e^6$$

$$= -\frac{80}{3} + \frac{10}{3}t + \frac{20}{3}e^{-\frac{t-6}{2}} = \frac{10}{3} \left( -8 + t + 2e^{-\frac{t-6}{2}} \right) \cdot u(t)$$

6) Possiamo scrivere la risposta considerando diversi intervalli di tempo, a seconda dei ritardi

- $T_1 : [-\infty, 0] \rightsquigarrow y(t) = 0$

- $T_2 : [0, 1] \rightsquigarrow y(t) = 4(1-e^{-\frac{t}{2}})$

- $T_3 : [1, 3] \rightsquigarrow y(t) = y_1(t) + y_2(t) = 4(1-e^{-\frac{t}{2}}) + 6(1-e^{-\frac{t-1}{2}}) = 4-4e^{\frac{t}{2}}+6-6e^{-\frac{t-1}{2}} = 10 + e^{-\frac{t}{2}}(-4-6e^{\frac{1}{2}}) = 10 - e^{-\frac{t}{2}}(4+6\sqrt{e})$

- $T_4 : [3, 6] \rightsquigarrow y(t) = y_1 + y_2 + y_3 = 10 - e^{\frac{t}{2}}(4+6\sqrt{e}) + \frac{10}{3}(5-t-2e^{-\frac{t-3}{2}})$

$$= 10 + e^{-\frac{t}{2}}(-4-6\sqrt{e}) + \frac{80}{3} - \frac{10}{3}t - \frac{20}{3}e^{-\frac{t}{2}} \cdot e^{\frac{3}{2}}$$

$$= \frac{80}{3} + e^{-\frac{t}{2}} \left( -4-6\sqrt{e} - \frac{20}{3}e^{\frac{3}{2}} \right) - \frac{10}{3}t \quad \text{OPPURE} \quad = \frac{80}{3} - e^{\frac{t}{2}} \left( 4+6\sqrt{e} - \frac{20}{3}\sqrt{e^3} \right) - \frac{10}{3}t$$

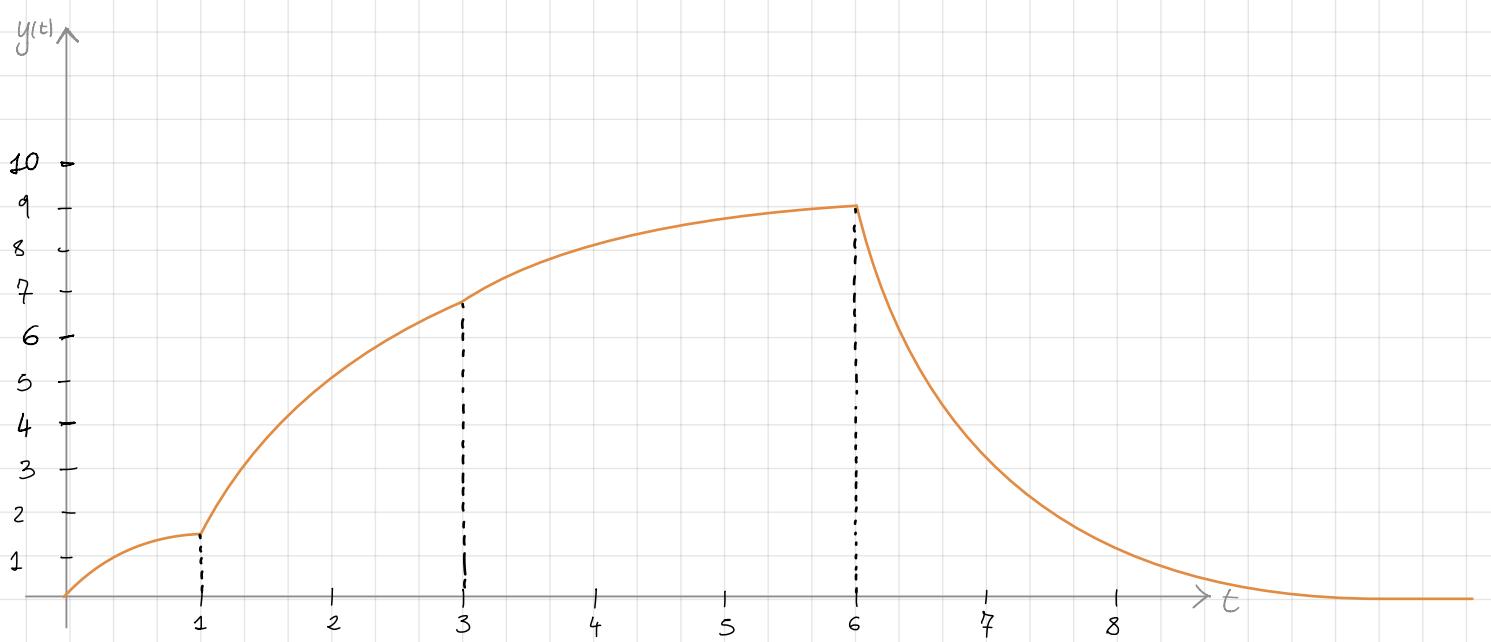
- $T_5 : [6, +\infty] = y_1 + y_2 + y_3 + y_4 = \frac{80}{3} - e^{\frac{t}{2}} \left( 4+6\sqrt{e} - \frac{20}{3}\sqrt{e^3} \right) - \frac{10}{3}t - \frac{80}{3} + \frac{10}{3}t + \frac{20}{3}e^{-\frac{t-6}{2}}$

$$\Rightarrow y(t) = +e^{\frac{t}{2}} \left( -4-6\sqrt{e} - \frac{20}{3}\sqrt{e^3} + \frac{20}{3}e^{\frac{3}{2}} \right) \approx 90e^{\frac{t}{2}}$$

7) Considerazioni

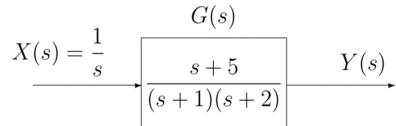
Per  $T_1, y \rightarrow 4$

Per  $T_2, y \rightarrow 10$



# SISTEMI DEL SECONDO ORDINE

Esercizio. Calcolare la risposta al gradino  $y(t)$  del seguente sistema:



$$G(s) = \frac{s+5}{(s+1)(s+2)}$$

$$\zeta_1 = -5; P_1 = -1; P_2 = -2$$

$$U(t) = 1(t) \Rightarrow U(s) = \frac{1}{s}$$

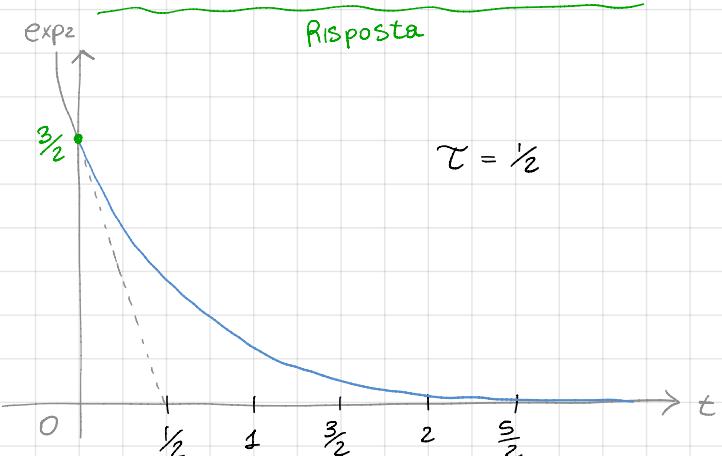
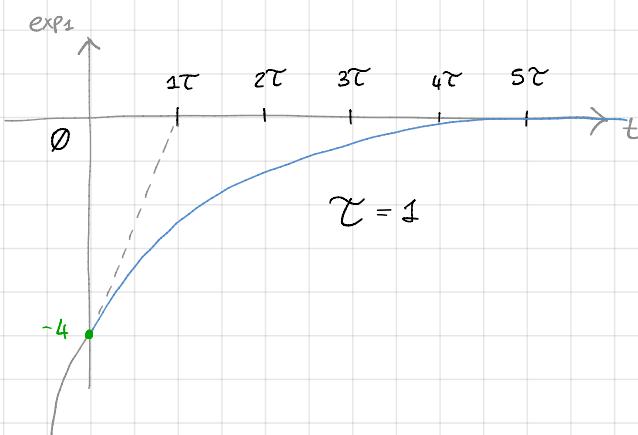
$$\Rightarrow Y(s) = \frac{s+5}{s(s+1)(s+2)} = \frac{\zeta_1}{s} + \frac{\zeta_2}{s+1} + \frac{\zeta_3}{s+2}$$

$$\zeta_1 = \lim_{s \rightarrow 0} s \cdot \frac{s+5}{s(s+1)(s+2)} = \frac{s}{1+2} = \frac{s}{2} \quad \text{circled } \zeta_1$$

$$\zeta_2 = \lim_{s \rightarrow -1} (s+1) \cdot \frac{s+5}{s(s+1)(s+2)} = \frac{+4}{-1} = 4 \quad \text{circled } \zeta_2$$

$$\zeta_3 = \lim_{s \rightarrow -2} (s+2) \cdot \frac{-2s+5}{s(s+1)(s+2)} = \frac{3}{2} \quad \text{circled } \zeta_3$$

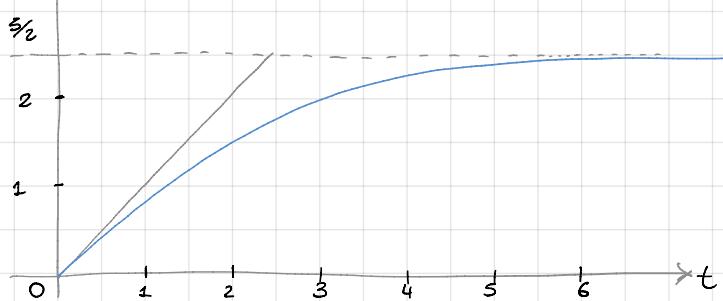
$$\Rightarrow Y(s) = \frac{5}{2} \frac{1}{s} - 4 \frac{1}{s+1} + \frac{3}{2} \frac{1}{s+2} \Rightarrow y(t) = \left( \frac{5}{2} - 4e^{-t} + \frac{3}{2}e^{-2t} \right) \cdot 1(t)$$



$$TVI: y(t=0) = \lim_{s \rightarrow \infty} s \cdot \left( \frac{5}{2s} - \frac{4}{s+1} + \frac{3}{2s+4} \right) \Rightarrow \frac{5}{2} - 4 + \frac{3}{2} = 0 \quad \text{Valore iniz.}$$

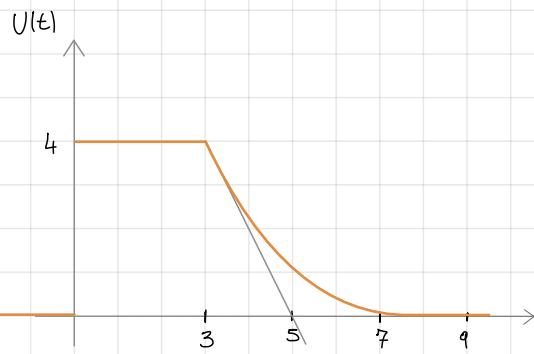
$$TVF: y(t \rightarrow \infty) = \lim_{s \rightarrow 0} s \left( \frac{5}{2s} - \frac{4}{s+1} + \frac{3}{2s+4} \right) = \frac{5}{2} - \frac{4s}{s+1} + \frac{3s}{2s+4} = \frac{5}{2} \quad \text{Valore fin}$$

$$\text{Pendenza in } t=0 \text{ mo } \dot{y}(t=0) = \lim_{s \rightarrow \infty} s \cdot G(s) = s \cdot \frac{s+5}{(s+1)(s+2)} = \frac{s \cdot \frac{s+5}{s}}{\frac{(s+1)s}{s} \frac{(s+2)s}{s}} = \frac{1}{1} \quad \text{Pendenza}$$



# POLI COMPLESSI E CONIUGATI

$$G(s) = \frac{s + 0.1}{(s^2 + 10s + 100)(s+2)}$$



ESERCIZIO LEZIONE 24

Con  $U(t) =$

$$\begin{cases} 0 & t < 0 \\ 4 & t \in (0, 3) \\ 4e^{-\frac{t-3}{2}} & t \geq 3 \end{cases}$$

$\hookrightarrow U'_3(t) = 4e^{-\frac{t-3}{2}} \cdot \mathbb{1}(t)$

(1) INDIVIDUARE I SEGNALI ELEMENTARI

$$\begin{cases} U_1(t) = 4 \cdot \mathbb{1}(t) & t \in [0, 3] \\ U_2(t) = -4 \cdot \mathbb{1}(t-3) & t \in [3, \infty) \\ U_3(t) = 4e^{-\frac{t-3}{2}} \cdot \mathbb{1}(t-3) & t \in [3, \infty) \end{cases}$$

(2) TRASFORMATA Dei segnali  $U_k(t)$

$$U_1(t) \Leftrightarrow U_1(s) = \frac{4}{s}$$

$$U_2(t) \Leftrightarrow U_2(s) = -\frac{4}{s} \cdot e^{-3s}$$

$$U_3 : U_3'(t) = 4e^{-\frac{t-3}{2}} \cdot \mathbb{1}(t) \Leftrightarrow U_3'(s) = 4 \cdot \frac{1}{s + \frac{1}{2}} \Rightarrow U_3'(s) = \mathcal{L}[U_3(t-3)]$$

*Ritardo*

$$\Rightarrow U_3(s) = \frac{4}{s + \frac{1}{2}} \cdot e^{-3s}$$

(3) Segnali fintizi

$$\text{Scelgo } \hat{U}_1(t) = \mathbb{1}(t) \Leftrightarrow \hat{U}_1(s) = \frac{1}{s}$$

$$\hat{U}_2(t) = e^{-\frac{t}{2}} \Leftrightarrow \hat{U}_2(s) = \frac{1}{s + \frac{1}{2}}$$

(3.a) Uscita a  $\hat{U}_1(s)$

$$\hat{Y}_1(s) = \frac{s + 0.1}{s(s^2 + 10s + 100)(s+2)}$$

⚠ Controllo che i poli non siano ulteriormente scomponibili...

$$s^2 + 10s + 100 \Rightarrow P_{1,2} = -5 \pm 8.7j \Rightarrow \text{Non scomponibili...}$$

$$\Rightarrow \hat{Y}_1(s) = \frac{\zeta_1}{s} + \frac{\zeta_2}{s+2} + \frac{\zeta_3 + \zeta_4 s}{s^2 + 10s + 100} = \frac{(s^2 + 10s + 100)(s+2)\zeta_1 + s(s+2)\zeta_2 + s(\zeta_3 + \zeta_4 s)(s+2)}{s(s^2 + 10s + 100)(s+2)}$$

\* A questo punto calcolo i residui e considero solo il num

$$\zeta_1 = \lim_{s \rightarrow 0} s \cdot \frac{s+0.1}{s(s^2+10s+100)(s+2)} \rightarrow \frac{0.1}{200} = 5 \times 10^{-4} \quad \zeta_1$$

$$\zeta_2 = \lim_{s \rightarrow -2} (s+2) \frac{s+0.1}{s(s^2+10s+100)(s+2)} \rightarrow \frac{-1.9}{-168} = 113 \times 10^{-4} \quad \zeta_2$$

$$(s^2+10s+100)(s+2)\zeta_1 + s\zeta_2(s^2+10s+100) + s(\zeta_3 + \zeta_4 s)(s+2) = s^3\zeta_1 + 2s^2\zeta_1 + 10s^2\zeta_1 + 20s\zeta_1 + 100s\zeta_1 + 200\zeta_1 \\ + s^3\zeta_2 + 10s^2\zeta_2 + 100s\zeta_2 + s^2\zeta_3 + 2s\zeta_3 + s^3\zeta_4 + 2s^2\zeta_4$$

$$\begin{cases} s^3(\zeta_1 + \zeta_2 + \zeta_4) = 0 \\ s^2(2\zeta_1 + 10\zeta_1 + 10\zeta_2 + \zeta_3 + 2\zeta_4) = 0 \\ s(20\zeta_1 + 100\zeta_1 + 100\zeta_2 + 2\zeta_3) = 1 \\ 200\zeta_1 = 0.1 \end{cases} \quad \text{QED}$$

$$\zeta_3 = -954 \times 10^{-4} \quad \zeta_4 = -118 \times 10^{-4}$$

(a)

$$\hat{\chi}_1(s) = \frac{\zeta_1}{s} + \frac{\zeta_2}{s+2} + \frac{\zeta_3 + \zeta_4 s}{s^2 + 10s + 100}$$

Termine  
doppio prodotto      Termine

! Non siamo ancora pronti!

→ Dobbiamo esprimere (a) in modo da ricavare sin e cos

Forma Standard:  $\frac{W_d}{(s+jW_n)^2 + (W_d)^2}$  → quindi dobbiamo riscrivere num e det

### • QUADRATO DI BINOMIO $-(s+jW_n)^2$

→ Per trarre  $jW_n$  basta guardare il termine doppio prodotto di (a)

$$(s+jW_n)^2 = s^2 + 2s jW_n + (jW_n)^2 = s^2 + 10s + 100$$

$$\Rightarrow 2jW_n = 10 \Rightarrow jW_n = \frac{10}{2} = \underline{jW_n = 5} \quad (1)$$

### • TERMINE AL QUADRATO $(W_d)^2$

Con questo termine dobbiamo "Apparire" quello che resta dal quadrato di binomio:

$$(s+5)^2 = s^2 + 10s + 25 + (W_d)^2 = s^2 + 10s + 100$$

$$\Rightarrow W_d^2 = 100 - 25 \Rightarrow W_d^2 = 75 \Rightarrow \underline{W_d = 5\sqrt{3}} \quad (2)$$

$$= \frac{\zeta_3 + \zeta_4 s}{s^2 + 10s + 100} = \frac{\zeta_3 + \zeta_4 s}{(s+5)^2 + (5\sqrt{3})^2} = \zeta_4 \frac{\frac{\zeta_3}{s+5} + s + 5 - 5}{(s+5)^2 (5\sqrt{3})^2} = \zeta_4 \frac{s+5}{(s+5)^2 (5\sqrt{3})^2} + \zeta_4 \frac{\frac{\zeta_3 - 5\zeta_4}{s+5} \cdot s}{(s+5)^2 (5\sqrt{3})^2}$$

Metto in evidenza per lasciare  $s$  da solo !

OK!

$\left| \begin{array}{l} \frac{\zeta_3}{s+5} \\ \zeta_4 \end{array} \right| \frac{s\sqrt{3}}{(s+5)^2 (5\sqrt{3})^2}$

→ Sostituisco i residui

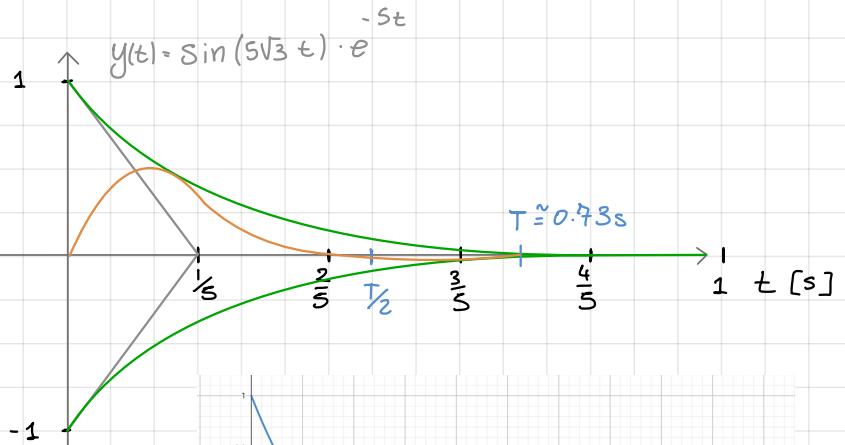
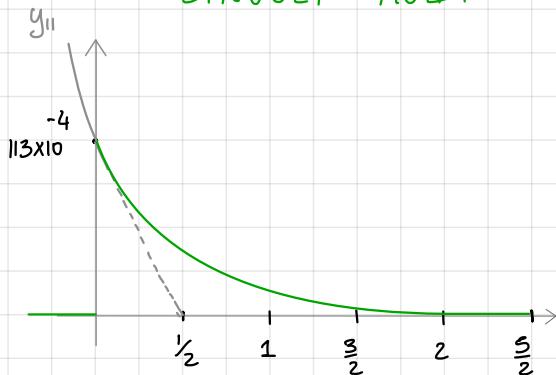
$$\hat{Y}_1(s) = \frac{\zeta_1}{s} + \frac{\zeta_2}{s+2} + \zeta_4 \frac{s+5}{(s+5)^2 + (5\sqrt{3})^2} + \frac{\zeta_3 - 5\zeta_4}{\zeta_4 \cdot 5\sqrt{3}} \frac{s\sqrt{3}}{(s+5)^2 + (5\sqrt{3})^2}$$

$$= \frac{5 \times 10^{-4}}{s} + \frac{113 \times 10^{-4}}{s+2} - 0.01 \cdot \frac{s}{(s+5)^2 + (5\sqrt{3})^2} - 0.36 \cdot \frac{s\sqrt{3}}{(s+5)^2 + (5\sqrt{3})^2}$$

#### (4.a) ANTITRASFORMO

$$\hat{Y}_1(t) = \left[ 5 \times 10^{-4} + 113 \times 10^{-4} e^{-4} e^{-2t} \underbrace{y_{11}}_{y_{11}} + \underbrace{-118 \times 10^{-4} \cdot \cos(5\sqrt{3}t) \cdot e^{-5t}}_{y_{12}} - 0.36 \cdot \sin(5\sqrt{3}t) \cdot e^{-5t} \underbrace{y_{13}}_{y_{13}} \right] u(t)$$

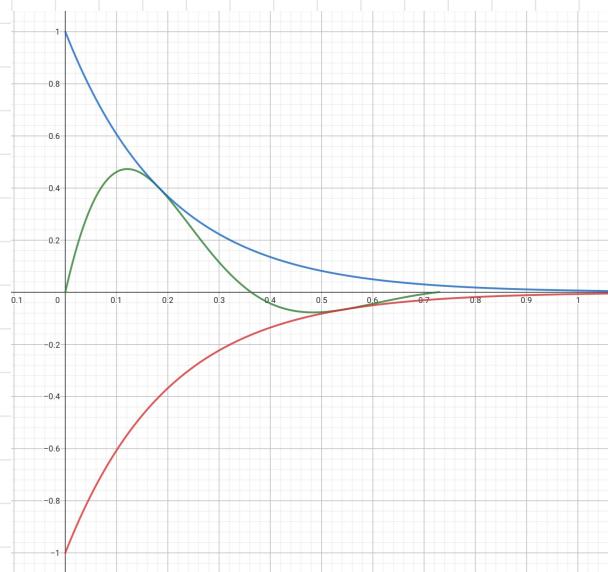
→ SINGOLI MODI



$$\sin(5\sqrt{3}t) \rightarrow T = \frac{2\pi}{5\sqrt{3}} \approx 0.23\pi \approx 0.73$$

• Valore iniziale

$$TVI: \lim_{s \rightarrow \infty} s \hat{Y}_1(s) \rightarrow 0$$



(3.b) Uscita a  $\hat{U}_2(s)$

$$\hat{U}_2(s) = \frac{1}{s + \frac{1}{2}}$$

$$\Rightarrow \hat{y}_2(s) = \frac{1}{(s + \frac{1}{2})} \cdot \frac{s + 0.1}{(s^2 + 10s + 100)(s+2)} = \frac{\varepsilon_1}{s + \frac{1}{2}} + \frac{\varepsilon_2}{s+2} + \frac{\varepsilon_3 + \varepsilon_4 s}{s^2 + 10s + 100}$$

Usa direttamente il sistema

$$(\varepsilon_3 + \varepsilon_4 s)(s+2)(s+\frac{1}{2})$$

$$\text{Num} \left[ \hat{y}_2(s) \right] = \varepsilon_1 (s^2 + 10s + 100)(s+2) + \varepsilon_2 (s + \frac{1}{2})(s^2 + 10s + 100) +$$

$$\Rightarrow \begin{cases} s^3 (\varepsilon_1 + \varepsilon_2) = 0 \\ s^2 (2\varepsilon_1 + 10\varepsilon_1 + 10\varepsilon_2 + \frac{1}{2}\varepsilon_2 + \varepsilon_3 + \varepsilon_3 + \varepsilon_4 + 2\varepsilon_4) = 0 \\ s (20\varepsilon_1 + 100\varepsilon_1 + 100\varepsilon_2 + 5\varepsilon_2 + 2\varepsilon_3 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_3 + \varepsilon_4) = 1 \\ 200\varepsilon_1 + 50\varepsilon_2 + \varepsilon_3 + \varepsilon_3 = 0.1 \end{cases}$$

$$\Rightarrow \begin{cases} \varepsilon_1 = -5.5 \times 10^{-3} \\ \varepsilon_2 = 5.5 \times 10^{-3} \\ \varepsilon_3 = 462.8 \times 10^{-3} \\ \varepsilon_4 = -305.8 \times 10^{-3} \end{cases}$$

$$\Rightarrow \hat{y}_2(s) = \frac{\varepsilon_1}{s + \frac{1}{2}} + \frac{\varepsilon_2}{s+2} + \frac{\varepsilon_3 + \varepsilon_4 s}{s^2 + 10s + 100} = \frac{\varepsilon_1}{s + \frac{1}{2}} + \frac{\varepsilon_2}{s+2} + \frac{\varepsilon_3 + \varepsilon_4 s}{(s+5)^2 + (5\sqrt{3})^2}$$

$$= \frac{\varepsilon_1}{s + \frac{1}{2}} + \frac{\varepsilon_2}{s+2} + \varepsilon_4 \frac{\frac{\varepsilon_3}{\varepsilon_4} + s + 5.5}{(s+5)^2 + (5\sqrt{3})^2} = \frac{\varepsilon_1}{s + \frac{1}{2}} + \frac{\varepsilon_2}{s+2} + \varepsilon_4 \frac{s+5}{(s+5)^2 + (5\sqrt{3})^2} + \varepsilon_4 \frac{\left(\frac{\varepsilon_3}{\varepsilon_4} - 5\right) \cdot \frac{5\sqrt{3}}{5\sqrt{3}}}{(s+5)^2 + (5\sqrt{3})^2}$$

$$= \frac{\varepsilon_1}{s + \frac{1}{2}} + \frac{\varepsilon_2}{s+2} + \varepsilon_4 \frac{s+5}{(s+5)^2 + (5\sqrt{3})^2} + \varepsilon_4 \frac{\frac{5\sqrt{3}}{\varepsilon_4 \cdot 5\sqrt{3}} \cdot \varepsilon_4}{(s+5)^2 + (5\sqrt{3})^2}$$

OK!

Non lo dimenticare!

(4.b) ANTITRASFORMO

$$\Rightarrow \hat{y}_2(t) = \left[ \varepsilon_1 e^{-\frac{t}{2}} + \varepsilon_2 e^{-2t} + \varepsilon_4 \cos(5\sqrt{3}t) e^{-5t} + \varepsilon_4 \left( \frac{\varepsilon_3 - 5\varepsilon_4}{\varepsilon_4 \cdot 5\sqrt{3}} \right) \sin(5\sqrt{3}t) \cdot e^{-5t} \right] \cdot u(t)$$

## (5) Segnali reali

Ricordiamo che

$$\begin{cases} u_1(t) = 4 \cdot \mathbb{1}(t) \\ u_2(t) = -4 \cdot \mathbb{1}(t-3) \\ u_3(t) = 4 \exp [(-t-3)/2] \end{cases}$$

Abbiamo scelto i segnali fintizi:

$$\begin{cases} \hat{u}_1(t) = \mathbb{1}(t) \\ \hat{u}_2(t) = e^{-\frac{t}{2}} \end{cases}$$

$$\Rightarrow \begin{cases} y_1(t) = 4 \hat{y}_1(t) \cdot \mathbb{1}(t) \\ y_2(t) = -4 \hat{y}_2(t-3) \cdot \mathbb{1}(t-3) \\ y_3(t) = 4 \hat{y}_3(t-3) \cdot \mathbb{1}(t-3) \end{cases} \Rightarrow y(t) = y_1(t) + y_2(t) + y_3(t)$$

-> Molto complesso da disegnare