J05T.2 - Bose Einstein Condensation

Problem

Consider $N \gg 1$ spinless noninteracting bosons contained in a n isotropic three-dimensional harmonic well. In terms of the position $\vec{r}$ and the momentum $\vec{p}$, the single-particle Hamiltonian is

$$H = \frac{1}{2m}|\vec{p}|^2 + \frac{1}{2}m\omega_0|\vec{r}|^2,$$

where the particles have mass m and the oscillations in the potential have natural frequency ω_0. The resulting energy levels depend on the three quantum numbers

$$E = \hbar\omega_0(n_x + n_y + n_z + (3/2)),$$

where each $n_i = 0, 1, 2, \ldots$. This can also be represented as energy levels that depend on a single quantum number $n = 0, 1, 2 \ldots; \epsilon_n = \epsilon_0 + n\hbar\omega_0$, but with a degeneracy $g_n = (n+1)(n+2)/2$, and $\epsilon_0 = \frac{3}{2}\hbar\omega_0$.

a) What is the specific heat $c_N(T)$ per particle, at fixed particle number N, in the "classical limit" where $k_BT/\hbar w_0$ is so large that $N_0 \ll 1$, where N_0 is the mean number of particles in the $n = 0$ state.

b) Find $c_N(T)$ at low temperatures $k_BT \ll \hbar\omega_0$, including the leading behavior for nonzero temperatures.

c) Find the chemical potential, $\mu(T, N)$ in the "classical limit". Above what temperature scale is the "classical limit" reached?

d) Find $\mu(T, N)$ for low temperatures $k_BT \ll \hbar\omega_0$, including the leading behavior for non-zero temperatures. (Hint: finad an exact expression for $\mu(T, N_0)$ at all temperatures and substitute the value of $N_0(T, N)$ in the temperature range of interest.)

e) Since the particles are bosons, $N_0(T, N)$ may be macroscopic (i.e. of order N) in a finite temperature range $T < T_{BEC}(N, \hbar\omega_0)$. Obtain an expression for T_{BEC} in the large-N limit. You can express any numerical constants as dimensionless integrals which you must define, but need not evaluate. (Hint: for $N \gg 1, k_BT_{BEC} \gg \hbar\omega_0$.)

Note: For large but finite N, there is no true phase transition at $T = T_{BEC}$, but a qualitative change in the system takes place over a small temperature range ΔT around T_{BEC} where $\Delta T/T_{BEC} \ll 1$.