M00T.2—Brownian Motion

Problem

A solid spherical particle of radius b and mass M is suspended in a fluid, and is seen, using an optical microscope, to undergo Brownian motion. You are asked to show that a measurement of the mean-square displacement $\langle[\vec{r}(t_1) - \vec{r}(t_2)]^2\rangle$ can be used to determine Boltzmann's constant.

Assume the densities of the solid and fluid are identical, so buoyancy can be ignored. The cause of the Brownian motion is a rapidly fluctuating force due to collisions with the molecules of the fluid. The force has mean zero, $\langle\vec{F}(t)\rangle = 0$, and two-time correlation of the form

$$\langle\vec{F}(t) \cdot \vec{F}(t')\rangle = C\delta(t - t').$$

The fluid has viscosity η and the system is isothermal at temperature T. The equation of motion of the particle is

$$M\frac{d^2\vec{r}}{dt^2} + 6\pi\eta b\frac{d\vec{r}}{dt} = \vec{F}(t).$$

a) Express the velocity at time t as an integral involving the past forces, $\{\vec{F}(t')\}_{t'<t}$.

b) Find the coefficient C of the force-correlation at temperature T, as a function of T and the other constants mentioned above.

c) Describe the rate of growth of the mean square displacement $\langle|\vec{r}(t_1) - \vec{r}(t_2)|^2\rangle$, and explain how its measurement can be used to determine Boltzmann's constant k_B.