J09Q.1 - Dressed States and Rabi Oscillation

Problem

Rabi oscillation occurs in a quantum system when two states couple to an external field. Generally, the phenomenon can be understood from a semi-classical approach, which is quantum states + classical field. We are going to look at the problem by using a quantized field.

Considering a two-level spin system coupled to magnetic fields, the interaction Hamiltonian can be written as $H_1 = \gamma\vec{\sigma} \cdot \vec{B}$. Here γ is the coupling strength, $\vec{\sigma}$ represents Pauli matrices, and $\vec{B}$ is the magnetic field. The spin states are $|\uparrow\rangle$ and $|\downarrow\rangle$. Assuming $\vec{B}$ is composed of a DC longitudinal field $B - z\vec{z}$ and an AC transverse field $\vec{B}_T = B_x\vec{x} + B_y\vec{y}$, the interaction Hamiltonian is $H_1 = \gamma\sigma_z B_z + \gamma\vec{\sigma}_T \cdot \vec{B}_T$. To simplify the problem, we say that $\vec{B}_T$ is a rotating field, and its classical form is $\vec{B}_T = B_r \cos\omega t\,\vec{x} + B_r \sim \omega t\,\vec{y}$ with frequency $\omega = 2\gamma B_z/\hbar$.

Now we want to use a quantized magnetic field, so $\vec{B}_T$ is an operator, and $\vec{B}_T = B_r[(a + a^\dagger)\,\vec{x} + i(a - a^\dagger)\,\vec{y}]/2$, where a and $a^\dagger$ are the ladder operators of photon state $|n\rangle$ of the magnetic field. Here, the photon energy is $\hbar\omega = 2\gamma B_z$. If there is no coupling between the transverse field and the spin, we find state $|\uparrow\rangle|n\rangle$ and state $|\downarrow\rangle|n+1\rangle$ are degenerate. The degeneracy is lifted into orthogonal dressed states when the coupling of the transverse field is present.

a) Please use the full Hamiltonian (including the photon energy) to fin the energies of the dressed states as functions of B_r.

b) Assuming at $t = 0$ the probability of finding the system in the state $|\uparrow\rangle|n\rangle$ is 1 and in $|\downarrow\rangle|n+1\rangle$ is 0, how will the probabilities of the two states evolve (Rabi oscillation) for $t > 0$?