2. **Brownian Motion**

A solid spherical particle of radius b and mass M is suspended in a fluid, and is seen, using an optical microscope, to undergo Brownian motion with trajectory $\vec{r}(t)$. In this problem, you are asked to show that a measurement of the mean-square displacement, $\langle|\vec{r}(t_1) - \vec{r}(t_2)|^2\rangle$, can be used to determine Boltzmann's constant, k_B.

Assume the densities of the solid and fluid are identical, so buoyancy can be ignored. The cause of the Brownian motion is the rapidly fluctuating random force, $\vec{F}(t)$, due to collisions with the molecules of the fluid. The force has mean zero, $\langle\vec{F}(t)\rangle = 0$, and assume that its components (F_x, F_y, F_z) have two-time correlations of the form

$$\langle F_\alpha(t_1) F_\beta(t_2)\rangle = C\delta_{\alpha\beta}\ \delta(t_1 - t_2)$$

in terms of Kronecker and Dirac delta functions. The fluid has viscosity η and the system is isothermal at temperature T. Assume that the equation of motion of the particle is

$$M\frac{d^2\vec{r}(t)}{dt^2} + 6\pi\eta b\frac{d\vec{r}(t)}{dt} = \vec{F}(t)\ .$$

a) For a single realization of these random forces, express the velocity $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ as an integral involving the past forces, $\vec{F}(t')$ for $t' < t$.

b) Find the coefficient C as a function of the temperature T and the other constants mentioned above.

c) Calculate the mean square displacement $\langle|\vec{r}(t_1) - \vec{r}(t_2)|^2\rangle$ as a function of the time difference at large time difference $|t_2 - t_1|$, and explain how measuring this quantity can be used to determine Boltzmann's constant k_B, assuming that you also have measurements of all the constants above except for k_B and C.