2. **Squeezing the Harmonic Oscillator**

The quantum harmonic oscillator, described by the Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$, with $[p, q] = \frac{\hbar}{i}$, is used to describe a remarkably diverse set of physical systems. Consider the class of harmonic oscillator 'squeezed state' wave functions of the form

$$\psi(q) = C \exp(-\alpha q^2/2)$$

where C is a normalization constant and α is an arbitrary *complex* number (with positive real part to ensure that the wave function is normalizable). States of this type can be prepared from the ground state by turning on a suitable interaction hamiltonian for the right length of time. It is not in general an energy eigenstate, but it has interesting and useful properties:

(a) The means of p and q in this state vanish. Calculate their variances and show that for *real* α this is a minimum uncertainty state, *i.e.* a state with $(\delta p)(\delta q) = \hbar/2$ (define $(\delta q)^2 = \langle (q - \langle q \rangle)^2 \rangle$, and similarly for δp).

(b) This state satisfies the equation

$$(\alpha q + ip/\hbar)\psi(q) = (\alpha q + \frac{\partial}{\partial q})\psi(q) = 0$$

Show that the time-evolved state $\psi_t(q) = \exp(-iH_0 t/\hbar)\psi(q)$ satisfies the same equation with a time-dependent (and in general complex) $\alpha(t)$. One way to do this is to go to the Heisenberg picture and time-evolve the operators p and q, rather than $\psi(q)$.

(c) You should find that $\alpha(t)$ becomes complex with time, even if $\alpha(0) = \alpha$ is real. So, in general, at $t > 0$ the state is no longer minimum uncertainty. Show that there are later times in the harmonic oscillator period where $\alpha(t)$ becomes real again (and the state recovers its minimum uncertainty character). Identify these real values of $\alpha(t)$ and comment on the nature of the minimum uncertainty states that are visited (assume that α is large, so that the initial minimum uncertainty state has small variance in q (and large variance in p).