3. **Particles on a Line**

Consider a system of N classical particles on a line restricted to positions

$$0 < x_1 < x_2 < ... < x_{N-1} < x_N \ .$$

The Hamiltonian is

$$H = f x_N + \sum_{n=1}^{N} (\frac{p_n^2}{2m} + U(x_n - x_{n-1})) \ ,$$

so there is a compressive force $f > 0$. We define $x_0 = 0$. The potential between neighboring particles vanishes for $x_n \geq (x_{n-1} + a)$, so $U(y) = 0$ for $y \geq a > 0$, where $y = x_n - x_{n-1}$. At shorter distances the potential is attractive and constant, so $U(y) = -U < 0$ for $0 < y < a$. Note that the compressive force may equivalently be viewed as an additional linear term $+fy$ in the potential.

The two parts of this problem may be done independently, although of course their correct answers are consistent with each other:

(a) Compute the mean length, $\langle x_N \rangle$, of this system at thermal equilibrium, as a function of the temperature T and the given parameters.

(b) There are three limiting regimes where this result simplifies:

(i) where the interparticle distances are mostly all such that $(x_n - x_{n-1}) \gg a$,

(ii) where the interparticle distances are mostly all such that $(x_n - x_{n-1}) \ll a$,

(iii) where the interparticle distances are mostly all such that $(x_n - x_{n-1}) < a$ but they are fairly uniformly distributed in $0 < (x_n - x_{n-1}) < a$.

State what the values of the parameters must be to be in each of these three simplifying limits, and what is $\langle x_N \rangle$ in each of these regimes.