3. **Photon Condensate**

Consider a cavity that has two low-lying electromagnetic modes, with photon energies ϵ_0 and ϵ_1 (and energy difference $\Delta = \epsilon_1 - \epsilon_0 > 0$), that are separated by a large energy gap from all higher modes such that we can neglect higher mode contributions to any thermodynamic quantity. The quantum state of this cavity is specified by giving the numbers N_0 and N_1 of photons in the two modes.

First, suppose that these two modes exchange energy with a heat bath at temperature T in such a way that the total number $N = N_0 + N_1$ of photons in these two cavity modes (but not their individual numbers) is conserved. This is an idealization of an actual experimental setup involving laser optics.

(a) There are $N+1$ possible states of the cavity, labeled by the number N_1 of photons in the upper state ($N_1 = 0, 1, \ldots, N$). According to Boltzmann statistics, what are the occupation probabilities $p(N_1)$ of these states at bath temperature T?

(b) Your result from (a) simplifies in the limit of large N. Calculate the expectation value of the number of photons in the upper state in this limit, showing that it remains finite when $\Delta > 0$. The remaining photons go into the lower energy mode, which thus becomes a kind of Bose-Einstein condensate in this large N limit.

Now suppose that instead the system can exchange not only energy with the heat bath, but also photons, so that the total photon number N is not fixed. We must now use the grand canonical ensemble, an approach that involves a chemical potential parameter μ that we adjust to achieve the desired mean photon number. Note that even though these are photons, allow $\mu \neq 0$ in order to fix $\langle N \rangle$.

(c) Show that to have a large mean photon number $\langle N \rangle$, we must set $\mu = \epsilon_0 - k_B T / \langle N \rangle + ...$, where the neglected terms (....) are small compared to those shown explicitly. In the limit of large $\langle N \rangle$, does this ensemble have a different limiting value of $\langle N_1 \rangle$ from what you obtained in this limit in part (b)?

(d) Finally, in the large photon number limit, show that the fluctuations in the total photon number (and therefore in the photon number of the lower mode) are enormous in this grand canonical ensemble: $\langle (\delta N)^2 \rangle \cong \langle N \rangle^2$.