THEORY Comparable
TYPE PARAMETERS S
DATA TYPES
Comparable(S)
constructors
Cons_Comp($SetDef : \mathbb{P}(S)$, $CompRel : \mathbb{P}(S \times S)$)
OPERATORS
ComparableWellCons ***predicate*** ($comp : Comparable(S)$)
direct definition
$CompRel(comp) \in SetDef(comp) \leftrightarrow SetDef(comp) \land SetDef(comp) \neq \emptyset$
reflexive ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x \cdot x \in SetDef(comp) \Rightarrow ((x \mapsto x) \in CompRel(comp))$
antireflexive ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x \cdot x \in SetDef(comp) \Rightarrow (x \mapsto x \notin CompRel(comp))$
symetrical ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x, y \cdot x \in SetDef(comp) \land y \in SetDef(comp) \Rightarrow ($
$(x \mapsto y \in CompRel(comp)) \Rightarrow (y \mapsto x \in CompRel(comp))$
$)$
asymetrical ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x, y \cdot x \in SetDef(comp) \land y \in SetDef(comp) \Rightarrow ($
$(x \mapsto y \in CompRel(comp)) \Rightarrow (y \mapsto x \notin CompRel(comp))$
$)$
antisymetrical ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x, y \cdot x \in SetDef(comp) \land y \in SetDef(comp) \Rightarrow ($
$(x \mapsto y \in CompRel(comp)) \land (y \mapsto x \in CompRel(comp)) \Rightarrow x = y$
$)$
transitive ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x, y, z \cdot x \in SetDef(comp) \land y \in SetDef(comp) \land z \in SetDef(comp) \Rightarrow ($
$(x \mapsto y \in CompRel(comp)) \land (y \mapsto z \in CompRel(comp)) \Rightarrow (x \mapsto z \in CompRel(comp))$
$)$
total ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x, y \cdot x \in SetDef(comp) \land y \in SetDef(comp) \Rightarrow ((x \mapsto y \in CompRel(comp)) \lor (y \mapsto x \in CompRel(comp)))$
equivalence ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$reflexive(comp) \land symetrical(comp) \land transitive(comp)$
preorder ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$reflexive(comp) \land transitive(comp)$
order ***predicate*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$preorder(comp) \land antisymetrical(comp)$
strict ***expression*** ($comp : Comparable(S)$)
well−definedness condition $ComparableWellCons(comp)$
direct definition
$Cons\_Comp(SetDef(comp), \{x \mapsto y \mid x \mapsto y \in CompRel(comp) \land x \neq y\})$
wellFounded ***predicate*** ($comp : Comparable(S)$)

well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall X \cdot X \subseteq SetDef(comp) \land X \neq \emptyset \Rightarrow ($
$\exists m \cdot m \in X \land (\forall x \cdot x \in X \Rightarrow (x \mapsto m \notin CompRel(comp)))$
$)$
wellPartialOrder ***predicate*** $(comp : Comparable(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$order(comp) \land wellFounded(strict(comp))$
wellOrder ***predicate*** $(comp : Comparable(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$order(comp) \land total(comp) \land wellFounded(strict(comp))$
covers ***predicate*** $(comp : Comparable(S), a : S, b : S)$
well−definedness condition $a \in SetDef(comp), b \in SetDef(comp), ComparableWellCons(comp) \land order(comp)$
direct definition
$(a \mapsto b \in CompRel(comp)) \land a \neq b \land (\forall c \cdot c \in SetDef(comp) \land (a \mapsto c \in CompRel(comp)) \land (c \mapsto b \in CompRel(comp))$
$\Rightarrow ((c = a) \lor (b = a)))$
compose ***expression*** $(comp1 : Comparable(S), comp2 : Comparable(S))$
well−definedness condition $ComparableWellCons(comp1), ComparableWellCons(comp2), \top$
direct definition
$Cons\_Comp(SetDef(comp1) \cup SetDef(comp2), \{x, z \cdot x \in SetDef(comp1) \land z \in SetDef(comp2) \land$
$(\exists y \cdot y \in SetDef(comp1) \cap SetDef(comp2) \land x \mapsto y \in CompRel(comp1) \land y \mapsto z \in CompRel(comp2)) \mid x \mapsto z\})$
converse ***expression*** $(comp : Comparable(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$Cons\_Comp(SetDef(comp), \{x, y \cdot x \in SetDef(comp) \land y \in SetDef(comp) \land y \mapsto x \in CompRel(comp) \mid x \mapsto y\})$
complement ***expression*** $(comp : Comparable(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$Cons\_Comp(SetDef(comp), \{x, y \cdot x \in SetDef(comp) \land y \in SetDef(comp) \land x \mapsto y \notin CompRel(comp) \mid x \mapsto y\})$
equality ***expression*** $(comp : Comparable(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\{x \mapsto y \mid x \in S \land y \in S \land x = y\}$
equivalenceClass ***expression*** $(comp : Comparable(S), x : S)$
well−definedness condition $ComparableWellCons(comp) \land equivalence(comp)$
direct definition
$\{y \cdot y \in S \land (x \mapsto y) \in CompRel(comp) \mid y\}$
leftGeneralized ***expression*** $(comp : Comparable(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\{x, P \cdot x \in S \land P \in \mathbb{P}(S) \land P \neq \emptyset \land (\forall y \cdot y \in P \Rightarrow (x \mapsto y \in CompRel(comp))) \mid x \mapsto P\}$
rightGeneralized ***expression*** $(comp : Comparable(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\{P, x \cdot P \in \mathbb{P}(S) \land P \neq \emptyset \land x \in S \land (\forall y \cdot y \in P \Rightarrow (y \mapsto x \in CompRel(comp))) \mid P \mapsto x\}$
upperBound ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), B : S)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall t \cdot t \in T \Rightarrow t \mapsto B \in CompRel(comp)$
lowerBound ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), B : S)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall t \cdot t \in T \Rightarrow B \mapsto t \in CompRel(comp)$
bounds ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), m : S, M : S)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$lowerBound(comp, T, m) \land upperBound(comp, T, M)$
upperBounded ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\exists M \cdot M \in S \land upperBound(comp, T, M)$
lowerBounded ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S))$

well−definedness condition $ComparableWellCons(comp)$
direct definition
$\exists m \cdot m \in S \land lowerBound(comp, T, m)$
bounded ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\exists m, M \cdot m \in SetDef(comp) \land M \in SetDef(comp) \land bounds(comp, T, m, M)$
supremum ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), M : S)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$upperBound(comp, T, M) \land$
$(\forall m \cdot m \in S \land upperBound(comp, T, m) \Rightarrow M \mapsto m \in CompRel(comp))$
infimum ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), m : S)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$lowerBound(comp, T, m) \land$
$(\forall M \cdot M \in S \land lowerBound(comp, T, M) \Rightarrow M \mapsto m \in CompRel(comp))$
maximal ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), M : T)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x \cdot x \in T \land M \mapsto x \in CompRel(comp) \Rightarrow x = M$
minimal ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), M : T)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\forall x \cdot x \in T \land x \mapsto M \in CompRel(comp) \Rightarrow x = M$
maximum ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), M : S)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$(M \in T) \land (\forall x \cdot x \in T \Rightarrow x \mapsto M \in CompRel(comp))$
minimum ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S), M : S)$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$(M \in T) \land (\forall x \cdot x \in T \Rightarrow M \mapsto x \in CompRel(comp))$
hasMaximum ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\exists M \cdot M \in T \land maximum(comp, T, M)$
hasMinimum ***predicate*** $(comp : Comparable(S), T : \mathbb{P}(S))$
well−definedness condition $ComparableWellCons(comp)$
direct definition
$\exists m \cdot m \in T \land minimum(comp, T, m)$

AXIOMATIC DEFINITIONS

operations:

OPERATORS

Gmax ***expression*** $(comp : Comparable(S), T : \mathbb{P}(S)) \; : \; S$
well−definedness $ComparableWellCons(comp)$
Gmin ***expression*** $(comp : Comparable(S), T : \mathbb{P}(S)) \; : \; S$
well−definedness $ComparableWellCons(comp)$
Gsup ***expression*** $(comp : Comparable(S), T : \mathbb{P}(S)) \; : \; S$
well−definedness $ComparableWellCons(comp)$
Ginf ***expression*** $(comp : Comparable(S), T : \mathbb{P}(S)) \; : \; S$
well−definedness $ComparableWellCons(comp)$

AXIOMS

$GmaxDef:$
$\forall comp, T \cdot comp \in Comparable(S) \land ComparableWellCons(comp) \land T \subseteq S \land hasMaximum(comp, T)$
$\Rightarrow (maximum(comp, T, Gmax(comp, T)))$
$GminDef:$
$\forall comp, T \cdot comp \in Comparable(S) \land ComparableWellCons(comp) \land T \subseteq S \land hasMinimum(comp, T)$
$\Rightarrow (minimum(comp, T, Gmin(comp, T)))$
$GsupDef:$
$\forall comp, T \cdot comp \in Comparable(S) \land ComparableWellCons(comp) \land T \subseteq S \land upperBounded(comp, T)$
$\Rightarrow (supremum(comp, T, Gsup(comp, T)))$
$GinfDef:$
$\forall comp, T \cdot comp \in Comparable(S) \land ComparableWellCons(comp) \land T \subseteq S \land lowerBounded(comp, T)$

$\Rightarrow (infimum(comp, T, Ginf(comp, T)))$

THEOREMS

converseDomain :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp)$
$\Rightarrow (dom(CompRel(converse(comp))) = ran(CompRel(comp)))$

converseRange :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp)$
$\Rightarrow (ran(CompRel(converse(comp))) = dom(CompRel(comp)))$

converseInvolutive :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp)$
$\Rightarrow (converse(converse(comp)) = comp)$

converseSymetry :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land symetrical(comp)$
$\Rightarrow symetrical(converse(comp))$

converseReflexivity :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land reflexive(comp)$
$\Rightarrow reflexive(converse(comp))$

converseAntireflexivity :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land antireflexive(comp)$
$\Rightarrow antireflexive(converse(comp))$

complementInvolutive :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp)$
$\Rightarrow (complement(complement(comp)) = comp)$

complementConverse :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp)$
$\Rightarrow (complement(converse(comp)) = converse(complement(comp)))$

complementSymetry :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land symetrical(comp)$
$\Rightarrow symetrical(complement(comp))$

complementReflexivity :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land ComparableWellCons(comp) \land reflexive(comp)$
$\Rightarrow antireflexive(complement(comp))$

complementAntireflexivity :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land antireflexive(comp)$
$\Rightarrow reflexive(complement(comp))$

totalRelationsReflexivity :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land total(comp)$
$\Rightarrow reflexive(comp)$

equivalenceClassEquity :
$\forall comp, x, y \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land equivalence(comp) \land$
$x \in SetDef(comp) \land y \in SetDef(comp) \Rightarrow ($
$((x \mapsto y) \in CompRel(comp)) \Leftrightarrow (equivalenceClass(comp, x) = equivalenceClass(comp, y))$
$)$

equivalenceClassNotEmpty :
$\forall comp, x \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land equivalence(comp) \land x \in SetDef(comp) \Rightarrow$
$(equivalenceClass(comp, x) \neq \emptyset)$

equivalenceClassCover :
$\forall comp \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land equivalence(comp) \Rightarrow ($
$(\bigcup equivalenceClass(comp1, x) \mid comp1 = comp \land x \in SetDef(comp)) = SetDef(comp)$
$)$

equivalenceClassDisjoint :
$\forall comp, x, y \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land equivalence(comp) \land$
$x \in SetDef(comp) \land y \in SetDef(comp) \land (x \mapsto y \notin CompRel(comp)) \Rightarrow ($
$(equivalenceClass(comp, x) \cap equivalenceClass(comp, y)) = \emptyset$
$)$

supremumMaximal :
$\forall comp, T, M \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land order(comp) \land T \subseteq S \land M \in S \Rightarrow$
$(supremum(comp, T, M) \land M \in T \Rightarrow maximal(comp, T, M))$

infimumMinimal :
$\forall comp, T, M \cdot comp \in (Comparable(S)) \land ComparableWellCons(comp) \land order(comp) \land T \subseteq S \land M \in S \Rightarrow$
$(infimum(comp, T, M) \land M \in T \Rightarrow minimal(comp, T, M))$

END