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Time Dependent Noncommutative Geometry and Field Theory on the Nappi-Witten Spacetime

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Declaration

I hereby declare that the work presented in this thesis was carried out by myself at Heriot-Watt University, except where due acknowledgement is made, and not been submitted for any other degree.

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Signature of Supervisor:

Abstract

We characterise the worldvolume theories on symmetric and non-symmetric D-branes in the six-dimensional Nappi-Witten spacetime. We find classes of Euclidean and Lorentzian noncommutative D3-branes, the physical origins of which are described through the interplay between isometric embeddings of branes in the spacetime and the Penrose-Güven limit of $AdS_3 \times S^3$. A non-symmetric spacetime-filling D-brane is constructed to give a spatially varying noncommutativity, analogous to that of the Dolan-Nappi model.

We then describe an algebraic approach to the time-dependent noncommutative geometry of the Nappi-Witten spacetime and develop a formalism to construct and analyse field theories defined thereon. Various $\star$-products are derived in closed explicit form and the Hopf algebra of twisted isometries is constructed. Scalar field theories are defined using explicit forms of derivative operators, traces and noncommutative frame fields. Noncommutative worldvolume field theories of the aforementioned D-branes are also constructed.

All techniques throughout are presented in such a manner that they may be applied to generic homogeneous pp-waves supported by a constant Neveu-Schwarz flux.

Contents

Chapter 1

Introduction

1.1 Noncommutative Geometry

The formulation of quantum mechanics by Dirac [44] sets the commutativity of position operators by borrowing from classical mechanics, such that position operators commute with each other. However, this choice was merely convenience as there was no experimental evidence to question such a definition. It was not long before esteemed authors wrote about the effect of introducing noncommuting coordinates [94]. The motivation for studying noncommutative geometry was to hopefully find a fix for the infinities appearing in quantum field theory, but was sidelined as renormalisation became successful at achieving that goal.

Noncommutative geometry has seen recent interest since open string models have been shown to possess a noncommutative geometry [104, 90] where there is a non-zero B-field. Should string theory turn out to be a viable explanation of our universe, one would expect to be able to observe the noncommutativity on a quantum mechanical level. The coordinates would possess commutator brackets of the form

$$[\hat{x}^i, \hat{x}^j] = i\Theta^{ij}(\hat{x}) \tag{1.1}$$

where Θ is an antisymmetric matrix. However, no such noncommutativity has been experimentally observed [62]. The noncommutativity between position and momenta gives rise to the Heisenberg Uncertainty Principle and similarly (1.1) implies an uncertainty on the coordinates themselves

$$\Delta x^i \Delta x^j \geq \frac{1}{2}|\Theta^{ij}| \tag{1.2}$$

This means there is no longer an idea of "a point", but we instead have the notion of "Planck cells". The most widely studied type of noncommutative geometry is the canonical case where Θ is a constant antisymmetric matrix.

Unfortunately a noncommutative geometry will lead to a breaking of Lorentz invariance. Lorentz covariance of Θ means that different inertial observers see different noncommutativity of coordinates due to different projections of the noncommuting planes. This is perhaps one of the greatest hurdles for theories which rely upon a noncommutative geometry, but there is hope. Theories such as those on κ-Minkowski spacetime [43] preserve Lorentz symmetries as deformed quantum symmetries (a bonus from having a quantum algebra) and twisted symmetries even for constant Θ can realise Lorentz invariance by acting in a twisted way [25, 29].

A spacetime which is of particular interest is the Nappi-Witten background [76] which possesses a quantum algebra, is four dimensional with Minkowski signature and is an exactly solvable background for string theory. We shall look at the properties of this spacetime in Section 1.3 and proceed to investigate it in detail throughout this thesis.

Instead of calculating physics on a noncommutative space itself, we may equivalently stay in commutative space and replace pointwise multiplication by a deformed $\star$-product. A $\star$-commutator bracket between the coordinate functions produces the functional equivalent of (1.1)

$$[x^i, x^j]_\star = x^i \star x^j - x^j \star x^i = i\Theta^{ij}(x) \tag{1.3}$$

Although we shall only return to the complicated $\star$-product in detail for Chapter 4, a basic understanding shall be assumed in Chapter 3 whereby the reader recognises that in general $f \star g \neq g \star f$ for functions depending upon the coordinates and that the $\star$-product is not unique for a given algebra.

1.2 pp-Waves and the AdS/CFT Correspondence

A pp-wave spacetime is any Lorentzian manifold whose metric can be described in Brinkman coordinates [22] as

$$G = 2dx^+dx^- + |dz|^2 + H(x^+, x^-, z)\left(dx^+\right)^2 \tag{1.4}$$

where $H(x^+, x^-, z)$ is any smooth function. x^+ is a time-like coordinate[†], x^- is null and the z are euclidean coordinates. The term "pp" modestly stands for *plane-fronted waves with parallel propagation*. As we shall see in Chapter 2, Penrose [79] observed that near a null geodesic, every Lorentzian spacetime looks like a plane wave. pp-waves are an important family of exact solutions to Einstein's field equations and exhibit the characteristic effect of a gravitational wave on light.

The AdS/CFT correspondence [3] is the equivalence between a string or supergravity theory defined on anti de Sitter space and a conformal field theory defined on its conformal boundary, with dimension one lower. It is the most successfully tested realisation of the holographic principle [21].

The dynamics of strings in the backgrounds of pp-waves has been of interest recently for a variety of reasons. They provide explicit realisations of string theory in time-dependent backgrounds which is necessary for applications of string cosmology. They also provide scenarios in which the AdS/CFT correspondence may be tested beyond the supergravity approximation by taking the Penrose limit of an $\mathrm{AdS}_m \times \mathrm{S}^n$ background [13] and the BMN limit of the dual superconformal field theory [15]. The property of these backgrounds that make them appealing in these contexts is that string dynamics on them are solvable in some instances, even in the presence of non-trivial B-fields [15, 17, 75, 78, 83]. The spectrum of the theory can be studied in light-cone gauge wherein the two-dimensional σ-models become free, while scattering amplitudes can be analysed using light-cone string field theory.

D-branes (the D standing for Dirichlet) are membrane-like structures which are

[†]Please note that this convention has not been standardised in the literature; some authors label the time-like coordinate by x^-.

considered to be as physically fundamental to string theory as the strings themselves; they are the surfaces to which the strings are attached. If it were not for D-branes, energy could flow along a string, slip off the endpoint and vanish. Because the endpoints of open strings cannot detach from the D-branes to which they are affixed, the D-branes determine the boundary conditions for the string's equation of motion (either Neumann or Dirichlet), thereby ensuring conservation of energy. D-branes are typically classified by their spatial worldvolume dimension, which is indicated by a number written after the D. For example, A D0-brane is a "D-particle", a D1-brane is a line (sometimes called a "D-string") and a D2-brane is a plane. A "D-instanton" is a fixed point in all space-time coordinates,

When D-branes are added to such closed string backgrounds, in some cases decoupling limits exist in which one can freeze out massive open string modes and closed string excitations. The low-energy effective theory governing the dynamics of open strings living on the branes is non-gravitational and can be reformulated as a field theory. The typical result is a noncommutative gauge theory with a spacetime dependent noncommutativity parameter [26, 40, 38, 55, 56, 70]. The role of spacetime dependence in these worldvolume field theories leads to interesting violations of energy-momentum conservation [14, 82], and their potential time-dependence is especially important for cosmological applications. In some instances the decoupled open strings also have a dual description in terms of a gravitational theory via the AdS/CFT correspondence [55]. This suggests that the holographic description of cosmological spacetimes may be described by non-local field theories.

The general construction and analysis of noncommutative gauge theories on curved spacetimes is one of the most important outstanding problems in the applications of noncommutative geometry to string theory. These non-local field theories arise naturally as certain decoupling limits of open string dynamics on D-branes in curved superstring backgrounds in the presence of a non-constant background Neveu-Schwarz B-field. On a generic Poisson manifold M, they are formulated using the Kontesevich star-product [69] which is linked to a topological string theory known as the Poisson sigma-model [33]. Under suitable conditions, the quantisation of D-branes in the Poisson sigma-model which wrap coisotropic submanifolds of M, i.e. worldvolumes defined by first-class constraints, may be consistently carried out and related to the deformation quantisation in the induced Poisson bracket [34]. Branes defined by second-class constraints may also be treated by quantising Dirac brackets on the worldvolumes [24].

However, in other concrete string theory settings, most studies of noncommutative gauge theories on curved D-branes have been carried out only within the context of the AdS/CFT correspondence by constructing the branes as solutions in the dual supergravity description of the gauge theory (see for example [31, 26, 55, 60, 8]). It is important to understand how to describe the classical solutions and quantisation of these models directly at the field theoretic level in order to better understand to what extent the noncommutative field theories capture the non-local aspects of string theory and quantum gravity, and also to be able to extend the descriptions to more general situations which are not covered by the AdS/CFT correspondence.

Open string dynamics on the NW background are particularly interesting because it has the potential to display a time-dependent noncommutative geometry [38, 55], and hence the noncommutative field theories built on NW_6 can serve as interesting toy models for string cosmology which can be treated for the most part as ordinary field theories. However, this point is rather subtle for the present geometry [38, 56]. A particular gauge choice which leads to a time-dependent noncommutativity parameter breaks conformal invariance of the worldsheet sigma-model, i.e. it does not satisfy the Born-Infeld field equations, while a conformally invariant background yields a non-constant but time-independent noncommutativity. In this thesis we partially clarify this issue.

1.3 Nappi-Witten Spacetime

In this thesis, which is based on the publications [58, 59], we will study the noncommutative gauge theories that reside on some D-branes in the four-dimensional Nappi-Witten gravitational wave [76] and its six-dimensional generalisation [66]. We will refer to both of these pp-waves as Nappi-Witten spacetimes and denote them respectively by NW_4 and NW_6. In the full superstring setting the backgrounds we study are $\mathrm{NW}_4 \times \mathrm{T}^6$ and $\mathrm{NW}_6 \times \mathrm{T}^4$, although we shall not write the toroidal factors explicitly in what follows. We will only consider the noncommutative deformations of the bosonic parts of these string theories, and hence only a non-trivial NS background.

The interest in this particular class of pp-waves is that string theory in these backgrounds can be solved completely and in a fully covariant way [11, 23, 35, 47, 64, 65, 84]. They describe homogeneous gravitational waves (Hpp-waves) and represent the "minimal" deformation of flat spacetime by H-flux ($H = dB$, the flux associated to the NS field). They may be formulated as WZW models based on a twisted Heisenberg group, for which the wave is an exact solution of the worldsheet σ-model [66, 76].

The NW_6 spacetime already captures the generic features of higher-dimensional Hpp-waves. It can be regarded as the Penrose-Güven limit of the background $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ [13, 12] supported by an NS–NS three-form flux, which describes the near horizon geometry of an NS5/F1 bound state [50]. The dual superconformal field theory is believed to be the nonlinear σ-model with target space the symmetric product orbifold $\mathrm{Sym}^N(\mathrm{T}^4)$. Similarly, the plane wave metric of NW_4 arises from the Penrose limit of $\mathrm{AdS}_2 \times \mathrm{S}^2$ [13, 12].

Both NW_4 and the Penrose limit of $\mathrm{AdS}_3 \times \mathrm{S}^3$ are examples of non-dilatonic, pp-wave solutions of six-dimensional supergravity [88]. However, as we later discuss in detail, the Penrose-Güven limit of $\mathrm{AdS}_2 \times \mathrm{S}^2$ does *not* induce the full NS-supported geometry of the NW_4 spacetime. Therefore, contrary to some claims [41, 87], the four-dimensional Nappi-Witten spacetime cannot be studied as the Penrose-Güven limit of $\mathrm{AdS}_2 \times \mathrm{S}^2$. Instead, it arises as a Penrose-Güven limit of the near horizon geometry of NS5-branes [52], on which string theory is dual to little string theory. This feature can be understood by regarding $AdS_2 \times S^2$ as the worldvolume of a symmetric D-brane in the $AdS_3 \times S^3$ spacetime, while NW_4 may only be realised as the worldvolume of a non-symmetric D-brane in NW_6.

We shall find that the most natural plane wave limits of embedded $AdS_2 \times S^2$ submanifolds of $AdS_3 \times S^3$ correspond to two classes of symmetric D-branes in NW_6. The first one is a *flat* euclidean D3-brane in a constant magnetic field, which carries a noncommutative worldvolume field theory with constant noncommutativity parameter determined by the constant time slices of the plane wave background. The second one is a Lorentzian D3-brane isometric to NW_4 with vanishing NS fields but with a null worldvolume electric field, which is described in the decoupling limit by a non-gravitational theory of noncommutative open strings, rather than by a noncommutative field theory. It is tempting to speculate that the full NW_4 deformation of this noncommutative open string theory describes the dynamics of the dual little string theory.

This problem is not peculiar to the class of plane wave geometries that we study, and it leads us into a detailed investigation of how D-branes behave under the Penrose-Güven limit of a spacetime. Similar analyses in some specific contexts are considered in [36, 91, 87]. We formulate and solve this problem in some generality, and then apply it to our specific backgrounds of interest. With this motivation at hand, we then proceed to reanalyse the classification of the symmetric D-branes of NW_6, elaborating on the analysis initiated in [48, 87] and extending it to a detailed study of the worldvolume supergravity fields supported by each of these branes.

We also clarify some points which were missed in the analysis of [87]. In each instance we identify the $AdS_3 \times S^3$ origin of the brane in question, and quantise its worldvolume geometry using standard techniques and the representation theory of the twisted Heisenberg group [11, 23, 64, 98]. We will find that most of these branes support *local* worldvolume effective field theories, because on most of them the pertinent supergravity form fields are trivial. In fact, we find that all symmetric D-branes in NW_6 (both twisted and untwisted) have vanishing NS–NS three-form flux, and only the two classes of branes mentioned above support a non-vanishing gauge-invariant two-form field. The overall consistency of these results, along with their agreement with the exact boundary conformal field theory description of Cardy branes in NW_4 [36], provides an important check that the standard techniques for quantisation of worldvolume geometries in compact group manifolds (see [92] for a review) extend to these classes of non-compact (and non-semisimple) Lie groups.

Somewhat surprisingly, even the spacetime filling symmetric D5-brane in NW_6 has trivial supergravity form fields. Motivated by this fact, we systematically construct the noncommutative geometry underlying the non-local field theory living on a non-symmetric D5-brane wrapping NW_6. The resulting noncommutativity is non-constant, but independent of the plane wave time coordinate. This agrees with the recent analysis in [56] of the Dolan-Nappi model [38] which describes a time-dependent noncommutative geometry on the worldvolume of a D3-brane wrapping NW_4. However, the background used in [38] is not conformally invariant and hence not a closed string background. Correctly reinstating conformal invariance [56] gives a spatially dependent but time-independent noncommutativity parameter. We elaborate on this noncommutativity somewhat and show that it may be regarded as arising from a formal quantisation of the twisted Heisenberg algebra.

In Section 1.3.1 we review the definition and geometrical properties of the four-dimensional Nappi-Witten spacetime. We show that its natural pp-wave isometry group is isomorphic to the six-dimensional twisted Heisenberg group, paving the way to an analysis of the isometric embeddings $NW_4 \hookrightarrow NW_6$. We also emphasise the time-independent harmonic oscillator character of point-particle dynamics in these backgrounds, as it helps to clarify the nature of the noncommutative worldvolume field theories constructed later on. In Section 1.3.4 we study the interplay between isometric embeddings and Penrose-Güven limits of branes, first in generality and then to the particular instances of Nappi-Witten spacetimes. From this analysis it becomes clear that both the NW_4 and NW_6 gravitational waves are necessarily wrapped by non-symmetric D-branes. In Section 3.1 we begin our analysis of the symmetric branes in NW_6, beginning with those described by conjugacy classes of the twisted Heisenberg group.

We identify classes of null branes (with degenerate worldvolume metrics), and show that their quantised geometries are commutative but generically differ from those of the classical conjugacy classes due to a unitary rotational symmetry of the background. We also find a class of euclidean D3-branes and show, directly from the representation theory of the twisted Heisenberg group, that their worldvolumes carry a Moyal-type noncommutativity akin to that induced on branes in constant magnetic fields [39, 90, 99, 100]. This sort of noncommutativity is natural from the point of view of the time-independent harmonic oscillator dynamics. In Section 3.2 we analyse symmetric branes in NW_6 which are described by twisted conjugacy classes. We show, again through explicit quantisation via representation theory and analysis of the worldvolume supergravity fields, that the low-energy effective field theories on *all* twisted D-branes are local.

1.3.1 Definitions

In this section we will define and analyse the geometry of the Nappi-Witten spacetime NW_4 [76]. It is a four-dimensional homogeneous spacetime of Minkowski signature which defines a monochromatic plane wave. It is further equipped with a supergravity NS B-field of constant flux, which in the presence of D-branes is responsible for the spacetime noncommutativity of the pp-wave. We will emphasise the simple, time-independent harmonic oscillator form of the dynamics in this background, as it will play a crucial role in subsequent sections.

The spacetime NW_4 is defined as the group manifold of the Nappi-Witten group, the universal central extension of the two-dimensional euclidean group $\mathrm{ISO}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$. The corresponding simply connected group $\mathcal{N}_4$ is homeomorphic to four-dimensional Minkowski space $\mathbb{E}^{1,3}$. Its non-semisimple Lie algebra $\mathfrak{n}_4$ is generated by elements $\mathsf{P}^{\pm}$, J, T obeying the commutation relations

$$\begin{aligned}
\left[\mathsf{P}^{+},\mathsf{P}^{-}\right] &= 2i\mathsf{T} \\
\left[\mathsf{J},\mathsf{P}^{\pm}\right] &= \pm i\mathsf{P}^{\pm} \\
\left[\mathsf{T},\mathsf{J}\right] &= \left[\mathsf{T},\mathsf{P}^{\pm}\right] = 0
\end{aligned} \tag{1.5}$$

This is just the three-dimensional Heisenberg algebra extended by an outer automorphism which rotates the noncommuting coordinates. The twisted Heisenberg algebra may be regarded as defining the harmonic oscillator algebra of a particle moving in one-dimension, with the additional generator J playing the role of the number operator (or equivalently the oscillator hamiltonian). It is a solvable algebra whose properties are much more tractable than, for instance, those of the su(2) or sl$(2,\mathbb{R})$ Lie algebras which are at the opposite extreme.

The centre of the universal enveloping algebra $U(\mathfrak{n}_4)$ contains the central element T of the Lie algebra $\mathfrak{n}_4$ and also the quadratic Casimir element

$$\mathsf{C}_4 = 2\mathsf{J}\mathsf{T} + \frac{1}{2}\left(\mathsf{P}^{+}\mathsf{P}^{-} + \mathsf{P}^{-}\mathsf{P}^{+}\right) \tag{1.6}$$

The most general invariant, non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{n}_4 \times \mathfrak{n}_4 \to \mathbb{R}$ is defined by [76]

$$\begin{aligned}
\left\langle\mathsf{P}^{+},\mathsf{P}^{-}\right\rangle &= 2\left\langle\mathsf{J},\mathsf{T}\right\rangle = 2 \\
\left\langle\mathsf{J},\mathsf{J}\right\rangle &= b \\
\left\langle\mathsf{P}^{\pm},\mathsf{P}^{\pm}\right\rangle &= \left\langle\mathsf{T},\mathsf{T}\right\rangle = 0 \\
\left\langle\mathsf{J},\mathsf{P}^{\pm}\right\rangle &= \left\langle\mathsf{T},\mathsf{P}^{\pm}\right\rangle = 0
\end{aligned} \tag{1.7}$$

for any $b \in \mathbb{R}$. This inner product has Minkowski signature (when $b = 0$), so that the group manifold of $\mathcal{N}_4$ possesses a homogeneous, bi-invariant Lorentzian metric defined by the pairing of the Cartan-Maurer left-invariant, $\mathfrak{n}_4$-valued one-forms $g^{-1}dg$ for $g \in \mathcal{N}_4$ as

$$ds_4^2 = \left\langle g^{-1}dg, g^{-1}dg\right\rangle \tag{1.8}$$

A generic group element $g \in \mathcal{N}_4$ may be parametrised as

$$g(u,v,a,\overline{a}) = e^{a\mathsf{P}^+ + \overline{a}\mathsf{P}^-} e^{\theta u \mathsf{J}} e^{\theta^{-1} v \mathsf{T}} \tag{1.9}$$

where $u, v \in \mathbb{R}$, $a \in \mathbb{C}$, and the parameter $\theta \in \mathbb{R}, \theta > 0$ controls the strength of the NS B-field background. In these global coordinates, the Cartan-Maurer one-form is given by

$$g^{-1}dg = e^{-i\theta u} da \mathsf{P}^+ + e^{i\theta u} d\overline{a} \mathsf{P}^- + \theta du \mathsf{J} + \left(\theta^{-1} dv + i a d\overline{a} - i \overline{a} da\right) \mathsf{T} \tag{1.10}$$

so that the metric (1.8) reads

$$ds_4^2 = 2dudv + |da|^2 + 2i\theta \left(a d\overline{a} - \overline{a} da\right) du + b\theta^2 du^2 \tag{1.11}$$

The metric (1.11) assumes the standard form of the plane wave metric for a conformally flat, indecomposable Cahen-Wallach Lorentzian symmetric spacetime CW_4 in four dimensions [30] upon introduction of Brinkman harmonic coordinates (x^+, x^-, z) [22] defined by rotating the transverse plane at a Larmor frequency as $u = x^+$, $v = x^-$ and $a = e^{\frac{i\theta}{2} x^+} z$. In these coordinates the metric assumes the stationary form

$$ds_4^2 = 2dx^+ dx^- + |dz|^2 + \theta^2 \left(b - \frac{1}{4}|z|^2\right) \left(dx^+\right)^2 \tag{1.12}$$

revealing the pp-wave nature of the geometry for $b = 0$. The physical meaning of the arbitrary parameter b will be elucidated below. It may be set to zero by exploiting the translational symmetry of the geometry in x^- to shift $x^- \to x^- - \frac{\theta^2 b}{2} x^+$, which corresponds to a Lie algebra automorphism of $\mathfrak{n}_4$. Note that on the null planes of constant $u = x^+$, the geometry becomes that of flat two-dimensional euclidean space $\mathbb{E}^2$. This is the geometry appropriate to the Heisenberg subgroup of $\mathcal{N}_4$, where the effects of the twisting generator J are turned off.

Thus far, the Nappi-Witten spacetime has been described geometrically as a four-dimensional Cahen-Wallach space CW_4. The spacetime NW_4 is further supported by a Neveu-Schwarz two-form field B_4 of constant field strength

$$H_4 = -\frac{1}{3} \langle g^{-1}dg, \left[g^{-1}dg, g^{-1}dg\right] \rangle = 2i\theta dx^+ \wedge dz \wedge d\overline{z} = dB_4 \tag{1.13}$$

where

$$B_4 = -\frac{1}{2}\langle g^{-1}dg, \frac{\mathbb{1}+\mathrm{Ad}_g}{\mathbb{1}-\mathrm{Ad}_g} g^{-1}dg\rangle = 2i\theta x^+ dz \wedge d\bar{z} \tag{1.14}$$

is defined to be non-zero only on those vector fields lying in the range of the operator $\mathbb{1}-\mathrm{Ad}_g$ on $T_g\mathcal{N}_4$, i.e. on vectors tangent to the conjugacy class containing $g \in \mathcal{N}_4$. The corresponding contracted two-form H_4^2 compensates exactly the constant Riemann curvature of the metric (1.12), so that NW_4 provides a viable supergravity background. In fact, in this case the cancellation is exact at the level of the full string equations of motion, so that the plane wave is an exact background of string theory [76]. It is the presence of this B-field that induces noncommutativity of the string background in the presence of D-branes.

1.3.2 Isometries

The realisation of the geometry of NW_4 as a standard plane wave of Cahen-Wallach type enables us to study its isometry group using the standard classification [16]. Writing $\partial_\pm := \partial/\partial x^\pm$, the metric (1.12) has the obvious null Killing vector

$$T = \theta\partial_- \tag{1.15}$$

generating translations in x^- and characterising a pp-wave, and also the null Killing vector

$$J = \theta^{-1}\partial_+ \tag{1.16}$$

generating translations in x^+. An analysis of the Killing equations [16] shows that there are also four extra Killing vectors $P^{(k)}$, $P'^{(k)}$, $k = 1,2$ which generate twisted translations in the transverse plane $z \in \mathbb{C}$ to the motion of the plane wave. Denoting $\partial := \partial/\partial z$, they are given in the form

$$\begin{aligned} P^{(k)} &= c^{(k)}(x^+)\partial + \bar{c}^{(k)}(x^+)\bar{\partial} - \theta^{-1}\left(\dot{c}^{(k)}(x^+)\bar{z} + \dot{\bar{c}}^{(k)}(x^+)z\right)\partial_- \\ P'^{(k)} &= c'^{(k)}(x^+)\partial + \bar{c}'^{(k)}(x^+)\bar{\partial} - \theta^{-1}\left(\dot{c}'^{(k)}(x^+)\bar{z} + \dot{\bar{c}}'^{(k)}(x^+)z\right)\partial_- \end{aligned} \tag{1.17}$$

where the dots denote differentiation with respect to the light-cone time coordinate $u = x^+$, and the complex-valued coefficient functions in (1.17) solve the harmonic oscillator equation of motion

$$\dot{c}(x^+) = -\frac{\theta^2}{4}c(x^+) \tag{1.18}$$

The four linearly independent solutions of (1.18) are characterised by their initial conditions on the null surface $x^+ = 0$ as

$$c^{(k)}(0) = \delta_{k1} + i\delta_{k2} \qquad \dot{c}^{(k)}(0) = 0$$

$$c'^{(k)}(0) = 0 \qquad \dot{c}'^{(k)}(0) = \theta\,(\delta_{k1} + i\delta_{k2}) \tag{1.19}$$

The solutions of (1.18) and (1.19) are given by

$$c^{(1)}(x^+) = \cos\frac{\theta x^+}{2} \qquad c^{(2)}(x^+) = i\cos\frac{\theta x^+}{2}$$

$$c'^{(1)}(x^+) = 2\sin\frac{\theta x^+}{2} \qquad c'^{(2)}(x^+) = 2i\sin\frac{\theta x^+}{2} \tag{1.20}$$

An interesting feature of these functions is that they generate the Rosen form [86] of the plane wave metric (1.12). It is defined by the transformation to local coordinates (u, v, y^1, y^2) given by

$$\begin{aligned}
u &= x^+ \\
v &= x^- - \frac{\theta}{4}(z+\overline{z})^2\tan\frac{\theta x^+}{2} - \frac{\theta}{4}(z-\overline{z})^2\cot\frac{\theta x^+}{2} \\
y^1 &= \frac{1}{2}\left((z+\overline{z})\sec\frac{\theta x^+}{2} + i(z-\overline{z})\csc\frac{\theta x^+}{2}\right) \\
y^2 &= \frac{1}{2}\left((z+\overline{z})\sec\frac{\theta x^+}{2} - i(z-\overline{z})\csc\frac{\theta x^+}{2}\right)
\end{aligned} \tag{1.21}$$

under which the metric becomes

$$ds_4^2 = 2dudv + C_{ij}(u)dy^i dy^j + b\theta^2 du^2 \tag{1.22}$$

where

$$C(u) = \big(C_{ij}(u)\big) = \begin{pmatrix} 1 & \cos\theta u \\ \cos\theta u & 1 \end{pmatrix} \tag{1.23}$$

This form of the metric is degenerate at the conjugate points where $\cos\theta u = \pm 1$. The harmonic oscillator solutions (1.20) then generate an orthonormal frame for the transverse plane metric (1.23),

$$C(u) = E(u)E^{\top}(u) \tag{1.24}$$

with

$$E = \frac{1}{2}\begin{pmatrix} c^{(1)} + \frac{1}{2}c'^{(2)} + \bar{c}^{(1)} + \frac{1}{2}\bar{c}'^{(2)} & -\left(c^{(2)} - \frac{1}{2}c'^{(1)} + \bar{c}^{(2)} - \frac{1}{2}\bar{c}'^{(1)}\right) \\ c^{(1)} + \frac{1}{2}c'^{(2)} + \bar{c}^{(1)} + \frac{1}{2}\bar{c}'^{(2)} & c^{(2)} - \frac{1}{2}c'^{(1)} + \bar{c}^{(2)} - \frac{1}{2}\bar{c}'^{(1)} \end{pmatrix} \tag{1.25}$$

satisfying the symmetry condition

$$\dot{E}(u)E^{\top}(u) = E(u)\dot{E}^{\top}(u) \tag{1.26}$$

Note that in contrast to the Brinkman coordinate system, in the Rosen form (1.22) two extra commuting translational symmetries in the transverse plane (y^1, y^2) are manifest, while time translation symmetry is lost.

By defining $P^{\pm} := P'^{(1)} \pm iP^{(1)}$ and $Q^{\pm} := P'^{(2)} \pm iP^{(2)}$, the six Killing vectors generated by the basic Cahen-Wallach structure of the plane wave may be summarised as

$$\begin{aligned} T &= \theta\partial_- \\ J &= \theta^{-1}\partial_+ \\ P^{\pm} &= \left(\sin\frac{\theta x^+}{2} \pm ie^{\mp\frac{i\theta}{2}x^+}\right)\left(\partial + \bar{\partial}\right) - \theta e^{\mp\frac{i\theta}{2}x^+}\left(z + \bar{z}\right)\partial_- \\ Q^{\pm} &= \left(i\sin\frac{\theta x^+}{2} \mp e^{\mp\frac{i\theta}{2}x^+}\right)\left(\partial - \bar{\partial}\right) + i\theta e^{\mp\frac{i\theta}{2}x^+}\left(z - \bar{z}\right)\partial_- \end{aligned} \tag{1.27}$$

Together, they generate the harmonic oscillator algebra $\mathfrak{n}_6$ of a particle moving in *two* dimensions,

$$\begin{aligned}
\left[P^\alpha, Q^\beta\right] &= 0 \qquad\qquad \alpha,\beta=\pm \\
\left[T, P^\pm\right] &= \left[T, Q^\pm\right] = [T, J] = 0 \\
\left[P^+, P^-\right] &= \left[Q^+, Q^-\right] = 2iT \\
\left[J, P^\pm\right] &= \pm i P^\pm \\
\left[J, Q^\pm\right] &= \pm i Q^\pm
\end{aligned} \tag{1.28}$$

This isometry algebra acts transitively on the null planes of constant x^+ and it generates a central extension $\mathcal{N}_6$ of the subgroup

$$\mathcal{S}_5 = \mathrm{SO}(2) \ltimes \mathbb{R}^4 \tag{1.29}$$

of the four-dimensional euclidean group $\mathrm{ISO}(4) = \mathrm{SO}(4) \ltimes \mathbb{R}^4$, where SO(2) is the diagonal subgroup of $\mathrm{SO}(2) \times \mathrm{SO}(2) \subset \mathrm{SO}(4)$. It is defined by extending the commutation relations (1.5) by generators $\mathsf{Q}^\pm$ obeying relations as in (1.28). The quadratic Casimir element $\mathsf{C}_6 \in U(\mathfrak{n}_6)$ and inner product on $\mathfrak{n}_6$ are defined in the obvious way by extending (1.6) and (1.7) symmetrically under $\mathsf{P}^\pm \leftrightarrow \mathsf{Q}^\pm$.

Following the analysis of the previous section, one can show that the group manifold of $\mathcal{N}_6$ is a six-dimensional Cahen-Wallach space CW_6, with Brinkman metric

$$ds_6^2 = 2dx^+dx^- + |dz|^2 + \theta^2\left(b - \frac{1}{4}|z|^2\right)\left(dx^+\right)^2 \tag{1.30}$$

where $\boldsymbol{z}^\top = (z, w) \in \mathbb{C}^2$, which carries a constant Neveu-Schwarz three-form flux

$$\begin{aligned}
H_6 &= -2i\theta dx^+ \wedge d\overline{\boldsymbol{z}}^\top \wedge d\boldsymbol{z} = dB_6 \\
B_6 &= -2i\theta x^+ d\overline{\boldsymbol{z}}^\top \wedge d\boldsymbol{z}
\end{aligned} \tag{1.31}$$

It thereby defines a six-dimensional version NW_6 of the Nappi-Witten pp-wave [66]. This observation will be exploited in the ensuing sections to view the Nappi-Witten wave as an isometrically embedded D-submanifold $\mathrm{NW}_4 \hookrightarrow \mathrm{NW}_6$. In this setting, it corresponds to a symmetry-breaking D3-brane in a non-zero H-flux.

However, for the Nappi-Witten wave this is not the end of the story. Because of the bi-invariance of the metric (1.8), the actual isometry group is the direct product $\mathcal{N}_4 \times \overline{\mathcal{N}_4}$ acting by left and right multiplication on the group $\mathcal{N}_4$ itself. Since the left and right actions of the central generator T coincide, the isometry group is seven-dimensional. In the present basis, the missing generator from the list (1.27) is the left-moving copy $\overline{J}$ of the oscillator hamiltonian with

$$\left[\overline{J}, P^{\pm}\right] = \mp i P^{\pm} + Q^{\pm} \left[\overline{J}, Q^{\pm}\right] = \mp i Q^{\pm} - P^{\pm} \tag{1.32}$$

and it is straightforward to compute that it is given by

$$\overline{J} = -\theta^{-1}\partial_{+} - i\left(z\partial - \overline{z}\overline{\partial}\right) \tag{1.33}$$

The vector field $J + \overline{J}$ generates rigid rotations in the transverse plane.

1.3.3 Commutative Field Theory

Standard covariant quantisation of a massless relativistic scalar particle in NW_4 leads to the Klein-Gordon equation in the curved background,

$$\Box_4 \phi = 0 \tag{1.34}$$

where

$$\Box_4 = 2\partial_{+}\partial_{-} - \theta^2\left(b - \frac{1}{4}|z|^2\right)\partial_{-}^2 + |\partial|^2 \tag{1.35}$$

is the laplacian corresponding to the Brinkman metric (1.12). It coincides with the Casimir (1.6) expressed in terms of left or right isometry generators (1.27), (1.33). The dependence on the light-cone coordinates $x^{\pm}$ drops out of the Klein-Gordon equation because of the isometries generated by the Killing vectors (1.15) and (1.16).

By using a Fourier transformation of the covariant Klein-Gordon field ϕ along the x^{-} direction,

$$\phi\left(x^{+}, x^{-}, z, \overline{z}\right) = \int_{-\infty}^{\infty} dp^{+} \psi\left(x^{+}, z, \overline{z}; p^{+}\right) e^{ip^{+}x^{-}} \tag{1.36}$$

we may write (1.34) equivalently as

$$\left[|\partial|^2 + 2ip^+\partial_+ + \left(b - \frac{1}{4}|z|^2\right)\left(\theta p^+\right)^2\right]\psi\left(x^+, z, \overline{z}; p^+\right) = 0 \tag{1.37}$$

Introducing the time parameter τ through

$$u = x^+ = p^+\tau \tag{1.38}$$

the differential equation (1.37)) becomes the Schrödinger wave equation

$$i\frac{\partial\psi\left(\tau, z, \overline{z}; p^+\right)}{\partial\tau} = \left[-\frac{1}{2}|\partial|^2 + \frac{1}{2}\left(\frac{\theta p^+}{2}\right)^2|z|^2 - \frac{b}{2}\left(\theta p^+\right)^2\right]\psi\left(\tau, z, \overline{z}; p^+\right) \tag{1.39}$$

for the non-relativistic two-dimensional harmonic oscillator with a time independent frequency given by the light-cone momentum and H-flux as $\omega = |\theta p^+|/2$. The only role of the arbitrary parameter b is to shift the zero-point energy of the harmonic oscillator, and it thereby carries no physical significance.

Let us remark that the same hamiltonian that appears in the Schrödinger equation (1.39) could also have been derived in light-cone gauge in the plane wave metric (1.12) starting from the massless relativistic particle Lagrangian

$$L = \dot{x}^+\dot{x}^- + \frac{\theta^2}{2}\left(b - \frac{1}{4}|z|^2\right)\left(\dot{x}^+\right)^2 + \frac{1}{2}|\dot{z}|^2 \tag{1.40}$$

describing free geodesic motion in the Nappi-Witten spacetime. In the light-cone gauge, the light-cone momentum is $p^+ = p_- = \partial L/\partial\dot{x}^- = \dot{x}^+ = 1$, while the hamiltonian is $J = p_+ = \partial L/\partial\dot{x}^+$. Imposing the mass-shell constraint $L = 0$ at $\dot{x}^+ = 1$ gives the equation of motion for x^-, which when substituted into J yields exactly the hamiltonian appearing on the right-hand side of (1.39) with transverse momentum $p_\perp = \dot{z} = -i\partial$.

The time-independence of the effective dynamics here follows from homogeneity of the plane wave geometry, which prevents dispersion along the light-cone time direction. These calculations give the quantisation of a particle in NW_4 only in the commutative geometry limit, i.e. in the spacetime CW_4, because they do not incorporate the supergravity B-field supported by the Nappi-Witten spacetime. In the following we will describe how to incorporate the deformation of CW_4 caused by the non-trivial NS-sector. Henceforth we will drop the zero-point energy and set $b = 0$.

1.3.4 Isometric Embeddings of Branes

A remarkable feature of the Nappi-Witten spacetime is the extent to which it shares common features with many of the more "standard" curved spaces. It is formally similar to the spacetimes built on the $SL(2,\mathbb{R})$ and $SU(2)$ group manifolds, but in many ways is much simpler. As a twisted Heisenberg group, it lies somewhere in between these curved spaces and the flat space based on the usual Heisenberg algebra. One way to see this feature at a quantitative level is by examining Penrose-Güven limits involving the (universal covers of the) $SL(2,\mathbb{R})$ and $SU(2)$ group manifolds which produce the spacetime NW_4. This will provide an aid in understanding various physical properties which arise in later constructions.

In looking for D-submanifolds, we are primarily interested in D-embeddings which are NS-supported and thereby carry a noncommutative geometry. As we will discuss, this involves certain important subtleties that must be carefully taken into account. As the Nappi-Witten spacetime can be viewed as a Cahen-Wallach space, i.e. as a plane wave, its geometry will arise as Penrose limits of other metrics. This opens up the possibility of extracting features of NW_4 by mapping them directly from properties of simpler, better studied noncommutative spaces. In this section we will begin with a thorough general analysis of the interplay between Penrose-Güven limits and isometric embeddings of Lorentzian manifolds, and derive simple criteria for the limit and embedding to commute. Then we apply these results to derive the possible limits that can be used to describe the NS-supported D-embeddings of NW_4.

1.4 Outline of Thesis

The outline of this thesis is as follows. In Chapter 2 we introduce the concept of the Penrose-Güven limit and impose a constraint for isometric embedding diagrams. We then examine the Lie branes of NW_6 in Chapter 3, highlighting the implied noncommutative geometries. We close the Chapter with the discovery of a time dependent, noncommutative geometry on NW_6 which exhibits the same algebra as $\mathfrak{n}_6$.

The discovery of such a spacetime is motivation for studying three classes of Nappi-Witten $\star$-products in Chapter 4. We construct the necessary tools for investigating the associated free scalar field theories and Chapter 5 concludes by investigating the field theories of our three orderings and the regularly embedded D-branes in NW_6.

Chapter 2

Penrose-Güven Limits and Isometric Embeddings

2.1 The Penrose-Güven Plane Wave Limit

In [79], Penrose showed that any Lorentzian spacetime has a limiting spacetime which is a plane wave. This limit can be thought of as a "first order approximation" along a null geodesic. The limiting spacetime depends on the choice of null geodesic, and hence any spacetime can have more than one Penrose limit.

Let (M, G) be a d-dimensional lorentzian spacetime. We can always introduce local Penrose coordinates $(U, V, \boldsymbol{Y})$, $\boldsymbol{Y}^\top = (Y^i) \in \mathbb{R}^{d-2}$ in the neighbourhood of a segment of a null geodesic $\gamma \subset M$ which contains no conjugate points, whereby the metric assumes the form [71]

$$G = \left(2dU + \alpha dV + \beta_i dY^i\right) dV + C_{ij} dY^i dY^j \tag{2.1}$$

where α, β_i and C_{ij} are functions of the coordinates and C_{ij} is a symmetric positive-definite matrix. This coordinate system breaks down when $\det C = 0$, signalling the existence of a conjugate point. This coordinate system has the advantage that a null geodesic congruence is singled out by constant V and Y^i, with U being an affine parameter along these geodesics. The geodesic $\gamma(U)$ is the one at $V = Y^i = 0$.

In string backgrounds one also has to consider generic supergravity p-form gauge potentials A with $(p+1)$-form field strengths $F = dA$ in order to compensate non-trivial spacetime curvature effects. The Güven extension of the Penrose limit to general supergravity fields shows that any supergravity background has plane wave limits which are also supergravity backgrounds [54]. It requires the local temporal gauge choice

$$A_{Ui_1\cdots i_{p-1}} = 0 \tag{2.2}$$

in order to ensure well-defined potentials in the limit. With this gauge choice, which can always be achieved via a gauge transformation $A \mapsto A + d\Lambda$ leaving the flux F invariant, we can write general potentials and field strengths in the neighbourhood of a null geodesic γ on M as

$$\begin{aligned} A &= a_{i_1\ldots i_p} dV \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_{p-1}} \\ &\quad + b_{i_1\ldots i_p} dY^{i_1} \wedge \ldots \wedge dY^{i_p} \\ &\quad + c_{i_1\ldots i_p} dU \wedge dV \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_{p-2}} \end{aligned} \tag{2.3}$$

$$\begin{aligned} F &= \left(\frac{\partial b_{i_1\ldots i_{p+1}}}{\partial U}\right) dU \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_p} \\ &\quad + d_{i_1\ldots i_{p+1}} dY^{i_1} \wedge \ldots \wedge dY^{i_{p+1}} \\ &\quad + e_{i_1\ldots i_{p+1}} dU \wedge dV \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_{p-1}} \\ &\quad + f_{i_1\ldots i_{p+1}} dV \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_p} \end{aligned} \tag{2.4}$$

where a, b, c, d, e and f are functions of the coordinates. The Penrose-Güven limit starts with the one-parameter family of local diffeomorphisms $\psi_\lambda : M \to M$, $\lambda \in \mathbb{R}$ defined by a rescaling of the Penrose coordinates as

$$\psi_\lambda(U, V, \boldsymbol{Y}) = (u, \lambda^2 v, \lambda \boldsymbol{y}) \tag{2.5}$$

One then defines new fields which are related to the original ones by a diffeomorphism, a rescaling, and (in the case of potentials) possibly a gauge transformation, by the well-defined limits

$$\begin{aligned}
\widetilde{G} &= \lim_{\lambda\to 0} \lambda^{-2}\psi_\lambda^* G \\
\widetilde{A} &= \lim_{\lambda\to 0} \lambda^{-p}\psi_\lambda^* A \\
\widetilde{F} &= \lim_{\lambda\to 0} \lambda^{-p}\psi_\lambda^* F
\end{aligned} \tag{2.6}$$

Due to (2.5), the only functions in (2.1), (2.3) and (2.4) which survive this limit are $C_{ij}(u) = C_{ij}(U,0,\mathbf{0})$, $b_{i_1\cdots i_p}(u) = b_{i_1\cdots i_p}(U,0,\mathbf{0})$ and $c_{i_1\cdots i_{p-2}}(u) = c_{i_1\cdots i_{p-2}}(U,0,\mathbf{0})$, which are just the pull-backs of the tensor fields C, b and c to the null geodesic γ. Explicitly, we obtain a pp-wave metric and supergravity fields in Rosen coordinates $(u, v, \boldsymbol{y})$ [86] as

$$\widetilde{G} = 2dudv + C_{ij}(u)dy^i dy^j \tag{2.7}$$

$$\begin{aligned}
\widetilde{A} &= b_{i_1\ldots i_p}(u)dy^{i_1}\wedge\ldots\wedge dy^{i_p} \\
&\quad + c_{i_1\ldots i_p}(u)du\wedge dv\wedge dy^{i_1}\wedge\ldots\wedge dy^{i_{p-2}}
\end{aligned} \tag{2.8}$$

$$\widetilde{F} = \frac{\partial b_{i_1\ldots i_{p+1}}(u)}{\partial u} du\wedge dy^{i_1}\wedge\ldots\wedge dy^{i_p} \tag{2.9}$$

The physical effect of this limit is to blow up a neighbourhood of the null geodesic γ, giving the local background as seen by an observer moving at the speed of light in M. It can be thought of as an infinite volume limit. We may set $c_{i_1\cdots i_{p-2}}(u) = 0$ in (2.8) via the local gauge transformation

$$\begin{aligned}
\widetilde{A} &\longmapsto \widetilde{A} + d\widetilde{\Lambda} \\
\widetilde{\Lambda} &= -\left(\int^u du'\, c_{i_1\cdots i_{p-2}}(u')\right)\, dv\wedge dy^{i_1}\wedge\cdots\wedge dy^{i_{p-2}}
\end{aligned} \tag{2.10}$$

2.2 Isometric Embedding Diagrams

It is possible to generate a commutative isometric embedding diagram, whereby isometric embeddings are denoted by vertical arrows and Penrose-Güven limits

(PGL) by horizontal arrows:

$$
\begin{array}{ccc}
M & \xrightarrow{\text{PGL}} & \widetilde{M} \\
\iota \Big\uparrow & & \Big\uparrow \widetilde{\iota} \\
N & \xrightarrow{\text{PGL}} & \widetilde{N}
\end{array}
\tag{2.11}
$$

In order to ensure that the metric and fields commute in such a diagram (i.e. that $M \to \widetilde{M} \to \widetilde{N}$ yields the same result as $M \to N \to \widetilde{N}$), we must place some restrictions on the kind of embeddings $\iota, \widetilde{\iota}$ we can use.

We are interested in smooth, local isometric embeddings $\iota : W \subset N \hookrightarrow M$ of a Lorentzian manifold (N, g), also possibly supported by non-trivial p-form fields, from an open subset W of N onto a sub-manifold of M in codimension $m \geq 0$. Thus we require that $\iota : W \to \iota(W)$ be a diffeomorphism in the induced C^∞ structure, and that the derivative map $d\iota_x : T_x N \to T_{\iota(x)} M$ be one-to-one for all $x \in W$. The Lorentzian metric g of N is related to that of M through the pull-back

$$
g_x(\cdot, \cdot) = G_{\iota(x)}(d\iota(\cdot), d\iota(\cdot)) \tag{2.12}
$$

on $T_x W \otimes T_x W \to \mathbb{R}$, and p-form fields a on N are similarly related to those on M by

$$
a_x(\cdot, \ldots, \cdot) = A_{\iota(x)}(d\iota(\cdot), \ldots, d\iota(\cdot)) \tag{2.13}
$$

on $\otimes^p T_x W \to \mathbb{R}$. In what follows we will usually write $\iota(N)$ for the projection.

We shall examine situations in which the Penrose-Güven limit will simultaneously induce the Penrose-Güven limits of the ambient spacetime and of the embedded submanifold. This automatically restricts the types of possible embeddings. We will simplify matters somewhat by taking the Penrose-Güven limit of (N, g) along the same null geodesic γ as that used on (M, G). The embedded submanifold $\iota(N)$ is then the intersection of M with the hypersurface $Y^i = 0$, $\forall i \in I := \{j_1, \ldots, j_m\}$. With the additional requirement that the metric G restricts non-degenerately on $\iota(N)$, there is an orthogonal tangent space decomposition

$$
T_x M = T_x N \oplus T_x N^\perp \tag{2.14}
$$

With ∂_i the elements of a local basis of tangent vectors dual to the one-forms dY^i in an open neighbourhood of $x \in M$. The fibres of the normal bundle $TN^\perp \to \imath(N)$ over the embedding are given by $T_xN^\perp = \{\xi^\perp \in T_xM \mid G(\partial_i, \xi^\perp) = 0, \forall i \notin I\}$.

Suppose now that we are given another local isometric embedding $\widetilde{\imath} : \widetilde{W} \subset \widetilde{N} \hookrightarrow \widetilde{M}$ of lorentzian manifolds $(\widetilde{N}, \widetilde{g})$ and $(\widetilde{M}, \widetilde{G})$, again possibly in the presence of other supergravity fields. We are interested in the conditions under which the isometric embedding diagram (2.11) commutes. Such a commutative diagram can only be written down under very exact symmetry constraints on the geodesic restrictions of the transverse plane metric C_{ij} of (2.1) and supergravity tensor field $b_{i_1 \cdots i_p}$ of (2.3) in the directions normal to the embeddings $\imath(N) \subset M$ and $\widetilde{\imath}(\widetilde{N}) \subset \widetilde{M}$.

To formulate these symmetry requirements, let $\widetilde{\imath}(\widetilde{N})$ be realised as the intersection of $\widetilde{M}$ with the hypersurface $y^i = 0, \forall i \in \widetilde{I} := \{\tilde{j}_1, \ldots, \tilde{j}_m\}$. This realisation of the isometric embedding on the right-hand side of (2.11) is dictated by the embedding on the left-hand side and the Penrose limit.

Consider the submanifold, also denoted $\widetilde{N}$, defined by the intersection of M with the hypersurface $Y^i = 0, \forall i \in \widetilde{I}$. Denoting the normal bundle fibres as $T_{\tilde{x}}\widetilde{N}^\perp := \{\tilde{\xi}^\perp \in T_{\tilde{x}}M \mid G(\partial_i, \tilde{\xi}^\perp) = 0, \forall i \notin \widetilde{I}\}$ for $\tilde{x} \in \widetilde{W}$, there is an orthogonal tangent space decomposition

$$T_{\tilde{x}}M = T_{\tilde{x}}\widetilde{N} \oplus T_{\tilde{x}}\widetilde{N}^\perp \tag{2.15}$$

analogous to (2.14). Along the light-like null geodesic γ, where $x = \tilde{x} = (U, 0, \mathbf{0})$, we fix p tangent vectors $X, X_1, \ldots, X_{p-1} \in T_{(U,0,\mathbf{0})}M$, and use the Lorentzian metric and p-form gauge potentials to define the linear transformations

$$G_{(U,0,\mathbf{0})}(X, \cdot) : T_{(U,0,\mathbf{0})}M \longrightarrow T_{(U,0,\mathbf{0})}M \tag{2.16}$$

$$A_{(U,0,\mathbf{0})}(X_1, \ldots, X_{p-1}, \cdot) : T_{(U,0,\mathbf{0})}M \longrightarrow T_{(U,0,\mathbf{0})}M \tag{2.17}$$

The isometric embedding diagram (2.11) then commutes if, for every collection of tangent vectors $X, X_1, \ldots, X_{p-1} \in T_{(U,0,\mathbf{0})}M$, the restrictions of the linear maps (2.16) and (2.17) to the corresponding orthogonal projections in (2.14) and (2.15) agree

$$C_{(U,0,\mathbf{0})}(X,\cdot)|_{T_{(U,0,\mathbf{0})}N^{\perp}} = C_{(U,0,\mathbf{0})}(X,\cdot)|_{T_{(U,0,\mathbf{0})}\widetilde{N}^{\perp}} \tag{2.18}$$

$$b_{(U,0,\mathbf{0})}(X_1,\ldots,X_{p-1},\cdot)|_{T_{(U,0,\mathbf{0})}N^{\perp}} = b_{(U,0,\mathbf{0})}(X_1,\ldots,X_{p-1},\cdot)|_{T_{(U,0,\mathbf{0})}\widetilde{N}^{\perp}} \tag{2.19}$$

in the following sense. The normal subspaces $T_{(U,0,\mathbf{0})}N^{\perp}$ and $T_{(U,0,\mathbf{0})}\widetilde{N}^{\perp}$ of $T_{(U,0,\mathbf{0})}M$ are non-canonically isomorphic as vector spaces. Fixing one such isomorphism, there is then a one-to-one correspondence between normal vectors $\xi^{\perp} \in T_{(U,0,\mathbf{0})}N^{\perp}$ and $\tilde{\xi}^{\perp} \in T_{(U,0,\mathbf{0})}\widetilde{N}^{\perp}$, under which we require the transverse plane metric and p-form fields to coincide

$$C_{(U,0,\mathbf{0})}(X,\xi^{\perp}) = C_{(U,0,\mathbf{0})}(X,\tilde{\xi}^{\perp})$$

and

$$b_{(U,0,\mathbf{0})}(X_1,\ldots,X_{p-1},\xi^{\perp}) = b_{(U,0,\mathbf{0})}(X_1,\ldots,X_{p-1},\tilde{\xi}^{\perp})$$

These symmetry conditions together ensure that the same supergravity fields are induced on $\widetilde{\imath}(\widetilde{N})$ along the two different paths of the diagram (2.11), i.e. that the Penrose-Güven limit of M, along the null geodesic described above, induces simultaneously the Penrose-Güven limit of N. We stress that (2.18) and (2.19) are required to simultaneously hold under only a single isomorphism of m-dimensional vector spaces, and in all there are $\frac{1}{2}m(m+1)$ such commuting isometric embedding diagrams that can potentially be constructed for appropriate plane wave profiles.

These symmetry conditions are essentially just the simple statement that the restrictions of the embeddings $\imath$ and $\widetilde{\imath}$ to the null geodesic $\gamma(U)$ are equivalent. Nevertheless, there are many examples whereby the Penrose limit of the metric carries through in (2.11), but not the Güven extension to generic p-form supergravity fields, i.e. (2.18) can hold with (2.19) being violated. Conversely, there can be exotic isometric embeddings whereby the transverse metric violates the requirement (2.18), leading to target spacetimes with distinct pp-wave profiles induced by the Penrose limit of essentially the same lorentzian structure. An interesting example of this would be a situation wherein the metric is not preserved, but the other supergravity p-form fields are.

2.3 Hpp-Wave Limits

Let us now specialise the analysis of the previous section to a broad class of examples that are important to the analysis that follows. Consider a real, connected Lie group $\mathcal{G}$ possessing a bi-invariant metric. On the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$, this induces an invariant, non-degenerate inner product $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$. We will also write $\mathcal{G}$ for the group manifold.

Symmetric D-branes wrapping submanifolds $D\subset\mathcal{G}$ preserve the maximal (diagonal) symmetry group $\mathcal{G}\subset\mathcal{G}\times\overline{\mathcal{G}}$ allowed by conformal boundary conditions. They are described algebraically by twisted conjugacy classes of the group [7, 46, 48, 96]. Let $\Omega : \mathfrak{g}\to\mathfrak{g}$ be an outer automorphism of the Lie algebra of $\mathcal{G}$ preserving its inner product $\langle\cdot,\cdot\rangle$, and let $\omega : \mathcal{G}\to\mathcal{G}$ be the corresponding Lie group automorphism. The map ω is an isometry of the bi-invariant metric on $\mathcal{G}$, and so it generates an orbit of any point $g\in\mathcal{G}$ under the twisted adjoint action of the group as $g\mapsto \mathrm{Ad}^{\omega}_{h}(g) = hg\omega(h^{-1})$, $h\in\mathcal{G}$. We may thereby identify D with such an orbit as

$$D = \mathcal{C}^{\omega}_{g} = \{hg\omega(h^{-1})|\ h\in\mathcal{G}\} \tag{2.20}$$

To each such ω we can associate an equivalence class of D-branes foliating $\mathcal{G}$.

Since the metric on $\mathcal{G}$ is bi-invariant, the twisted conjugacy class (2.20) may be exhibited as a homogeneous space

$$\mathcal{C}^{\omega}_{g} = \mathcal{G}/\mathcal{Z}^{\omega}_{g} \tag{2.21}$$

where $\mathcal{Z}^{\omega}_{g}\subset\mathcal{G}$ is the stabiliser subgroup of the point g under the twisted adjoint action of $\mathcal{G}$ defined by

$$\mathcal{Z}^{\omega}_{g} = \{h\in\mathcal{G}|\ hg\omega(h^{-1}) = g\} \tag{2.22}$$

By this homogeneity, it will always suffice to determine the geometry (and any $\mathcal{G}$-invariant fluxes) at a single point, as all other points are related by the twisted adjoint action of the group, which is an isometry. A natural physical assumption is that the bi-invariant metric of $\mathcal{G}$ restricts non-degenerately to the twisted conjugacy classes (2.21). The normal bundle over the D-submanifold then has fibres given by

$$(T_g\mathcal{C}^{\omega}_{g})^{\perp} = T_g\mathcal{Z}^{\omega}_{g} \tag{2.23}$$

with $T_g\mathcal{C}_g^\omega \cap T_g\mathcal{Z}_g^\omega = \{0\}$, so that there is an orthogonal direct sum decomposition of the tangent bundle of $\mathcal{G}$ as

$$T_g\mathcal{G} = T_g\mathcal{C}_g^\omega \oplus T_g\mathcal{Z}_g^\omega \tag{2.24}$$

The space of normal vectors (2.23) may be identified with the Lie algebra of the stabiliser subgroup (2.22) given by

$$\mathfrak{z}_g^\omega = \{X \in \mathfrak{g} |\ g^{-1}Xg = \Omega(X)\} \tag{2.25}$$

These D-submanifolds are also NS-supported with B-field [95]

$$B_{g_0} = -\langle dhh^{-1}, \mathrm{Ad}_g \circ \Omega(dhh^{-1})\rangle \tag{2.26}$$

at $g_0 = hg\omega(h^{-1}) \in \mathcal{C}_g^\omega$, and they are stabilised against decay by the presence of a family of two-form Abelian gauge field strengths $F_{g_0}^{(\zeta)} = dA_{g_0}^{(\zeta)}$, $\zeta \in \mathfrak{z}_g^\omega$ given by [19]

$$F_{g_0}^{(\zeta)} = \langle \zeta, [dhh^{-1}, dhh^{-1}]\rangle \tag{2.27}$$

If ζ is a fractional symmetric weight of $\mathcal{G}$, then the integral of (2.27) over any two-sphere in the worldvolume $\mathcal{C}_g^\omega$ of the D-brane is an integer multiple of 2π. The two-forms

$$\mathcal{F}_{g_0}^{(\zeta)} := B_{g_0} + F_{g_0}^{(\zeta)} \tag{2.28}$$

are invariant under gauge transformations $B_{g_0} \mapsto B_{g_0} + d\Lambda$, $F_{g_0}^{(\zeta)} \mapsto F_{g_0}^{(\zeta)} - d\Lambda$ of the B-field. They are the gauge invariant combinations (in string units $\alpha' = 1$) that appear in the Dirac-Born-Infeld action governing the target space dynamics of the D-branes.

There are two ways in which the Penrose limit may be achieved within this setting [13]. In both instances it can be understood as an Inönü-Wigner group contraction of $\mathcal{G}$ whose limit $\widetilde{\mathcal{G}}$ is a non-compact, non-semisimple Lie group admitting a bi-invariant metric. In the first case we assume $\mathcal{G}$ is simple and consider a one-parameter subgroup $\mathcal{H} \subset \mathcal{G}$, which is necessarily geodesic relative to the bi-invariant metric. If it is also null, then it gives rise to a null geodesic and hence the Penrose limit can be taken as prescribed in the previous subsection. In the second case we consider a product group $\mathcal{G} = \mathcal{G}' \times \mathcal{H}$, where $\mathcal{G}'$ is a simple Lie group with a bi-invariant metric and $\mathcal{H} \subset \mathcal{G}'$ is a compact subgroup, with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}'$, which inherits a bi-invariant metric from $\mathcal{G}$ by pull-back. The product group $\mathcal{G}' \times \mathcal{H}$ then carries the bi-invariant product metric corresponding to the bilinear form $\langle \cdot, \cdot \rangle \oplus (-\langle \cdot, \cdot \rangle|_{\mathfrak{h}})$ on $\mathfrak{g}' \oplus \mathfrak{h}$. The submanifold $\mathcal{H} \subset \mathcal{H} \times \mathcal{H} \subset \mathcal{G}' \times \mathcal{H}$ given by the diagonal embedding is a Lie subgroup, and hence it is totally geodesic and maximally isotropic. The (generalised) Penrose limit of $\mathcal{G}' \times \mathcal{H}$ along $\mathcal{H}$ thereby yields a non-semisimple Lie group with a bi-invariant metric. The related group contraction can also be understood as an infinite volume limit along the compact directions of $\mathcal{G}' \times \mathcal{H}$, giving a semi-classical picture of the string dynamics in this background.

We may now attempt to specialise the isometric embedding diagram (2.11) to the diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\text{PGL}} & \widetilde{\mathcal{G}} \\ \imath \Big\uparrow & & \Big\uparrow \widetilde{\imath} \\ \mathcal{C}_g^\omega & \xrightarrow{\text{PGL}} & \mathcal{C}_{\tilde{g}}^{\tilde{\omega}} \end{array} \tag{2.29}$$

specifying the Penrose-Güven limit between twisted D-branes. We have used the fact that the Penrose limit of a maximally symmetric space is again a maximally symmetric space [13], so that the Penrose-Güven isometric embedding diagrams preserve the symmetry of the embedded spaces and symmetric D-branes map onto symmetric D-branes in the limit. To formulate the symmetry conditions (2.18) and (2.19) within this algebraic setting, we restrict to the stabiliser algebra (2.25). For any $h \in \mathcal{H}$ and any Lie algebra element $X \in \mathfrak{g}$, the isometric embedding diagram (2.29) then commutes if and only if

$$\langle X, \cdot \rangle|_{\mathfrak{z}_h^\omega} = \langle X, \cdot \rangle|_{\mathfrak{z}_h^{\tilde{\omega}}} \tag{2.30}$$

$$B_{h_0}(X, \cdot)|_{\mathfrak{z}_h^\omega} = B_{h_0}(X, \cdot)|_{\mathfrak{z}_h^{\tilde{\omega}}} \tag{2.31}$$

$$F_{h_0}^{(\zeta)}(X, \cdot)|_{\mathfrak{z}_h^\omega} = F_{h_0}^{(\zeta)}(X, \cdot)|_{\mathfrak{z}_h^{\tilde{\omega}}} \tag{2.32}$$

with the analogous restrictions on higher-degree p-form fields.

2.3.1 NW **Limits**

We will now apply these considerations to examine the possible D-embeddings of Nappi-Witten spacetimes. We start with the six-dimensional Lorentzian manifold $M = \mathrm{AdS}_3 \times \mathrm{S}^3$ describing the near horizon geometry of a bound state of fundamental strings and NS5-branes [50], equivalent to a two-dimensional superconformal field theory corresponding to the IR limit of the dynamics of parallel D1-branes and D5-branes. This identification requires that both factors share a common radius of curvature R. We can embed $\mathrm{AdS}_3 \times \mathrm{S}^3$ in the pseudo-euclidean space $\mathbb{E}^{2,6}$ as the intersection of the two quadrics

$$\left(x^0\right)^2 + \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 \;=\; R \tag{2.33}$$

$$\left(x^4\right)^2 + \left(x^5\right)^2 + \left(x^6\right)^2 + \left(x^7\right)^2 \;=\; R \tag{2.34}$$

with the induced metric. An explicit parametrisation is given by

$$\begin{aligned}
x^0 &= R\sqrt{1+r^2}\cos\tau\cosh\beta & \qquad x^4 &= R\cos\phi \\
x^1 &= R\sin\tau & \qquad x^5 &= R\chi\sin\phi\sin\psi \\
x^2 &= R\sqrt{1+r^2}\cos\tau\sinh\beta & \qquad x^6 &= R\sqrt{1-\chi^2}\sin\phi\sin\psi \\
x^3 &= Rr\cos\tau & \qquad x^7 &= R\sin\phi\cos\psi
\end{aligned} \tag{2.35}$$

In these coordinates the metric, NS-NS B-field and three-form flux on $\mathrm{AdS}_3 \times \mathrm{S}^3$ are given by

$$
\begin{aligned}
\frac{1}{R^2} G &= -d\tau^2 + \cos^2\tau \left(\frac{dr^2}{1+r^2} + \left(1+r^2\right) d\beta^2 \right) \qquad (2.36) \\
&\quad + d\phi^2 + \sin^2\phi \left(\frac{d\chi^2}{1-\chi^2} + \left(1-\chi^2\right) d\psi^2 \right) \\
-\frac{1}{2R^2} H/2 &= \cos^2\tau d\tau \wedge dr \wedge d\beta + \sin^2\phi d\phi \wedge d\chi \wedge d\psi \qquad (2.37) \\
-\frac{2}{R^2} 2B &= (\sin 2\tau + 2\tau)\, dr \wedge d\beta + (\sin 2\phi - 2\phi)\, d\chi \wedge d\psi \qquad (2.38)
\end{aligned}
$$

Viewing M as the group manifold of the Lie group $\mathrm{SU}(1,1) \times \mathrm{SU}(2)$ with its usual bi-invariant metric induced by the Cartan-Killing form, its embedded submanifolds wrapped by maximally symmetric D-branes are given by twisted conjugacy classes of the group. Here we will focus on the family of D3-branes which are isometric to $N = \mathrm{AdS}_2 \times \mathrm{S}^2$ and are given by the intersections of the hyperboloids (2.33) and (2.34) with the affine hyperplanes $x^3, x^5 =$ constant [18]. In this way we may exhibit a foliation of $\mathrm{AdS}_3 \times \mathrm{S}^3$ consisting of twisted D-branes, each of which is isometric to $\mathrm{AdS}_2 \times \mathrm{S}^2$. Within this family, the intersection of $\mathrm{AdS}_3 \times \mathrm{S}^3$ with the hyperplane defined by $x^3 = x^5 = 0$ is special for a variety of reasons. It corresponds to the fixed point set of the reflection isometry of $\mathbb{E}^{2,6}$ defined by $x^3 \mapsto -x^3$, $x^5 \mapsto -x^5$ while leaving fixed all other coordinates. This isometry preserves the embedding (2.33), (2.34) and hence induces an isometry of $\mathrm{AdS}_3 \times \mathrm{S}^3$. The metric (2.36) restricts nondegenerately to the corresponding $\mathrm{AdS}_2 \times \mathrm{S}^2$ submanifold, which is thereby totally geodesic and has equal radii of curvature R.

When we consider such embedded D-submanifolds, we should also add to the list of supergravity fields (2.36) through (2.38), the constant U(1) gauge field flux [18, 10]

$$
\frac{2}{R^2} F = \frac{2}{R^2} dA = 4\pi\kappa_q dr \wedge d\beta - 2\pi p d\chi \wedge d\psi \qquad (2.39)
$$

where $p \in \mathbb{Z}$ is the magnetic monopole number through S^2, while $\kappa_q \in \mathbb{R}$ is related to the quantum number $q \in \mathbb{Z}$ giving the Dirac-Born-Infeld electric flux through AdS_2. The presence of this two-form prevents the wrapped D3-branes from collapsing.

The quantity $\mathcal{F} = B + F$ is invariant under two-form gauge transformations of the B-field, but it is only the monopole flux and the dual Dirac-Born-Infeld electric displacement which are quantised. The addition of such worldvolume electric fields proportional to the volume form of AdS_2 along with worldvolume magnetic fields proportional to the volume form of S^2 still preserves supersymmetry. The sources of such fluxes are (p, q) strings connecting the D3-branes to the NS5/F1 black string background in the near horizon region [18]. The Dirac-Born-Infeld energy of the branes wrapping $AdS_2 \times S^2$ is locally minimised at $(\tau, \phi) = (2\pi\kappa_q, \pi p)$ [18, 10], and at those values the gauge-invariant two-form is given by

$$-\frac{2}{R^2}\mathcal{F} = \sin 2\tau dr \wedge d\beta + \sin 2\phi d\chi \wedge d\psi \tag{2.40}$$

Let us now consider the Penrose-Güven limit of M which produces the six-dimensional Nappi-Witten spacetime $\widetilde{M} = \mathrm{NW}_6$ [13]. As Lie groups, the Penrose limit can be interpreted as an Inönü-Wigner group contraction of $SU(1,1) \times SU(2)$ onto $\mathcal{N}_6$ [87]. Geometrically, it can be achieved along any null geodesic which has a non-vanishing velocity component tangent to the sphere S^3. For this, we change coordinates in the (τ, ϕ) plane

$$2\tau = U - V \qquad\qquad 2\phi = U + V \tag{2.41}$$

$$r = Y^1 \quad \beta = Y^2 \qquad\qquad \chi = Y^3 \quad \psi = Y^4$$

This enables us to represent the fields in (2.36) through (2.38) in the adapted coordinate forms (2.1), (2.3) and (2.4), which thereby exhibits $\frac{\partial}{\partial U} = \frac{1}{2}(\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \tau})$ as the null geodesic vector field with $G(\frac{\partial}{\partial U}, \frac{\partial}{\partial U}) = 0$. After the Penrose-Güven limit, the metric and Neveu-Schwarz fields along the geodesic $\gamma(U)$ are given by

$$\frac{1}{R^2}\widetilde{G} \;=\; 2dudv + \sin^2\frac{u}{2}d\boldsymbol{y}^2 \tag{2.42}$$

$$\frac{1}{R^2}\widetilde{H} \;=\; \cos^2\frac{u}{2}du \wedge dy^1 \wedge dy^2 - \sin^2\frac{u}{2}du \wedge dy^3 \wedge dy^4 \tag{2.43}$$

$$\frac{4}{R^2}\widetilde{B} \;=\; -(u + \sin u)dy^1 \wedge dy^2 + (u - \sin u)dy^3 \wedge dy^4 \tag{2.44}$$

with $\boldsymbol{y}^\top = (y^i) \in \mathbb{R}^4$. At this stage it is convenient to transform from Rosen coordinates to Brinkman coordinates

$$u = 2x^- \qquad v = x^+ + \frac{\cot x^-}{2}\vec{z}^2 \qquad y^i = \frac{z^i}{\sin x^-} \tag{2.45}$$

It is then straightforward to compute that one recovers the standard NS-supported geometry of NW_6, with supergravity fields $\frac{1}{R^2}\widetilde{G} = ds_6^2$, $\frac{1}{2R^2}\widetilde{H} = H_6$ and $\frac{2}{R^2}\widetilde{B} = B_6$ given by (1.30), (1.31).

The foliating hyperplanes $w = w_0 \in \mathbb{C}$ isometrically embed NW_4 in NW_6 with its standard geometry (1.12), (1.14) and zero-point energy $b = -\frac{\theta^2}{4}|w_0|^2$. The Penrose-Güven limit of the worldvolume flux (2.39) vanishes, $\widetilde{F} = 0$, because it is the field strength of a U(1) gauge field. On the other hand, the gauge invariant two-form (2.40) transforms as a potential under the Penrose-Güven limit and one finds

$$\mathcal{F}_6 := -\frac{2}{R^2}\widetilde{\mathcal{F}} = -i\cot\frac{\theta x^+}{2}\left[d\overline{\boldsymbol{z}}^\top \wedge d\boldsymbol{z} + \frac{\theta}{2}\cot\frac{\theta x^+}{2}dx^+ \wedge (\boldsymbol{z}^\top d\overline{\boldsymbol{z}} - \overline{\boldsymbol{z}}^\top d\boldsymbol{z})\right] \tag{2.46}$$

Strictly speaking, this field is only defined on the constant time slices $x^+ = \frac{\pi}{\theta}(2\kappa_q + p)$ of NW_6 induced by the energy minimising configurations described above. However, with the scaling transformations employed in the Penrose-Güven limit we can take the form (2.46) to be valid at all times. It induces a worldvolume two-form $\mathcal{F}_4 := \mathcal{F}_6|_{w=w_0}$ on NW_4.

Let us now examine the isometric embedding of the totally geodesic $AdS_2 \times S^2$ D-brane in $AdS_3 \times S^3$, which may be defined as the hyperplane $Y^1 = Y^3 = 0$. The geodetic property ensures that the Penrose limit of $AdS_3 \times S^3$ induces that of $AdS_2 \times S^2$, which yields the Cahen-Wallach symmetric space CW_4 [13]. However, the Güven extension breaks down. In this case the normal bundle fibre $T_{(U,0,\mathbf{0})}N^\perp$ is locally spanned by the vector fields ∂_2, ∂_4, while $T_{(U,0,\mathbf{0})}\widetilde{N}^\perp$ is spanned by ∂_3, ∂_4. There is thus no gauge transformation such that (2.19) is satisfied. On the other hand, since the transverse plane metric $C_{ij}(u) = \sin^2\frac{u}{2}\delta_{ij}$ in (2.42) is proportional to the identity, (2.18) is trivially satisfied and so the Penrose limit of the $AdS_2 \times S^2$ metric coincides with that of CW_4.

One cannot rectify this problem by choosing an alternative embedding of $AdS_2 \times S^2$ for which the B-field is non-vanishing, such as that with $\tau, \phi =$ constant, however, we may still find an interesting CW_4 spacetime by using such an alternative embedding. This case corresponds with the minimal energy symmetric D3-branes and allows the Penrose limit of $\mathrm{AdS}_2 \times \mathrm{S}^2$ to be taken in the adapted coordinates $U = \chi + r$, $V = \frac{1}{2}(\chi - r)$, $Y^1 = \beta$, $Y^2 = \psi$. After another suitable change to Brinkman coordinates it leads to the anticipated CW_4 geometry. However, now the CW_4 branes carry a non-vanishing null worldvolume flux which may be written in Brinkman coordinates as

$$\frac{2}{R^2}\widetilde{F} = \frac{\pi\theta}{2}\csc\frac{\theta x^+}{2}\left[(2\kappa_q - ip)dx^+ \wedge dw + (2\kappa_q + ip)dx^+ \wedge d\overline{w}\right] \tag{2.47}$$

2.3.2 Embedding Diagrams for NW Spacetimes

Let us now describe in more detail the two simple and obvious remedies to the problem which we raised in the previous Section. The first one modifies the embedding $\widetilde{\imath}$ on the right-hand side of the diagram (2.11) to be the intersection of $\widetilde{M} = \mathrm{NW}_6$ with the hyperplane $y^1 = y^3 = 0$, so that $T_{(U,0,0)}N^\perp = T_{(U,0,0)}\widetilde{N}^\perp$ and the conditions (2.18), (2.19) are always trivially satisfied. Now the pull-backs of the Neveu-Schwarz fields (2.43), (2.44) vanish, as does that of the two-form (2.46), while the pull-back of the metric (2.42) is still the standard metric on CW_4. This embedding thereby preserves the basic Cahen-Wallach structure $\mathrm{CW}_4 \subset \mathrm{NW}_6$, and the vanishing of the other supergravity form fields on CW_4 is indeed induced now by the Penrose-Güven limit from $\mathrm{AdS}_2 \times \mathrm{S}^2$. We may thereby write the commuting embedding diagram

$$\begin{array}{ccc} \mathrm{AdS}_3 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_6 \\ \imath \uparrow & & \uparrow \widetilde{\imath} \\ \mathrm{AdS}_2 \times \mathrm{S}^2 & \xrightarrow{\mathrm{PGL}} & \mathrm{CW}_4 \end{array} \tag{2.48}$$

which describes the Penrose-Güven limit between *commutative*, maximally symmetric Lorentzian D3-branes in $\mathrm{AdS}_3 \times \mathrm{S}^3$ and the six-dimensional Nappi-Witten spacetime NW_6. We may explicitly confirm that the path $\mathrm{AdS}_3 \times \mathrm{S}^3 \to \mathrm{AdS}_2 \times \mathrm{S}^2$

$\to \mathrm{CW}_4$ gives

$$\widetilde{G} = 2dx^- dx^+ + \frac{\bar{z}^2}{4}\left(dx^-\right)^2 + (d\bar{z})^2 \tag{2.49}$$

$$\widetilde{H} = d\widetilde{B} = 0 \tag{2.50}$$

Due to the vanishing of the worldvolume flux and the fact that $\pi_2(\mathrm{S}^3) = 0$, these D-branes can be unstable. They may shrink to zero sise completely corresponding to point-like D-instantons, to euclidean D-strings induced at a point in S^3 with worldvolume geometries $\mathrm{AdS}_2 \subset \mathrm{AdS}_3$, or to euclidean D-strings sitting at a point in AdS_3 and wrapping $\mathrm{S}^2 \subset \mathrm{S}^3$ on the left-hand side of the diagram (2.48). These symmetric decay products will be generically lost in the Penrose-Güven limit, as will become evident in the ensuing sections.

Alternatively, we may choose to modify the embedding $\imath$ on the left-hand side of the diagram (2.11) to be the intersection of $M = \mathrm{AdS}_3 \times \mathrm{S}^3$ with the hyperplane $Y^1 = Y^2 = 0$. Again $T_{(U,0,\mathbf{0})}N^\perp = T_{(U,0,\mathbf{0})}\widetilde{N}^\perp$ and so the conditions (2.18) and (2.19) are trivially satisfied. This hyperplane corresponds to the intersection of the hyperboloid (2.33) with $x^2 = x^3 = 0$, while (2.34) is left unchanged. It thereby defines a totally geodesic embedding of $\mathrm{S}^{1,0} \times \mathrm{S}^3$ in $\mathrm{AdS}_3 \times \mathrm{S}^3$. This does not define a twisted conjugacy class of the Lie group $\mathrm{SU}(1,1) \times \mathrm{SU}(2)$ and so does not correspond to a symmetric D-brane [18]. Instead, it arises in the near horizon geometry of a stack of NS5-branes [52]. The pull-backs of the NS–NS fields (2.37), (2.38) are non-vanishing, and the null geodesic defined by (2.41) spins along an equator of the sphere S^3. The Penrose-Güven limit thus induces the complete NS-supported geometry of NW_4 [35], and it can be thought of as a group contraction of $\mathrm{U}(1) \times \mathrm{SU}(2)$ along $\mathrm{U}(1)$ onto $\mathcal{N}_4$. We may thereby write the commuting embedding diagram

$$\begin{array}{ccc} \mathrm{AdS}_3 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_6 \\ \imath \Big\uparrow & & \Big\uparrow \widetilde{\imath} \\ \mathrm{S}^{1,0} \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_4 \end{array} \tag{2.51}$$

describing *noncommutative* branes. The Lorentzian NW_4 D3-brane, supported by non-trivial worldvolume fields, is stabilised against decay by its worldvolume two-form $\mathcal{F}_4 = \mathcal{F}_6|_{z=0}$. One may confirm explicitly that the path $\mathrm{AdS}_3 \times \mathrm{S}^3 \to \mathrm{NW}_6 \to \mathrm{NW}_4$ yields

$$\begin{aligned} \widetilde{G} &= 2dx^- dx^+ + \frac{\vec{z}^2}{4}\left(dx^-\right)^2 + (d\vec{z})^2 && (2.52) \\ \widetilde{H} &= -dx^- \wedge dz^1 \wedge dz^2 && (2.53) \\ \widetilde{B} &= -x^- dz^1 \wedge dz^2 && (2.54) \end{aligned}$$

which is the Nappi-Witten WZW model.

These results are all consistent with the remark made just after (2.29). The brane $\mathrm{AdS}_2 \times \mathrm{S}^2$ is a symmetric D-submanifold of $\mathrm{AdS}_3 \times \mathrm{S}^3$. As we discuss in detail in Section 3.2, NW_4 is *not* a symmetric D-brane in NW_6, although CW_4 is. Consistently, $\mathrm{S}^{1,0} \times \mathrm{S}^3 \subset \mathrm{AdS}_3 \times \mathrm{S}^3$ is not a maximally symmetric D-embedding, but rather a member of the hierarchy of factorising symmetry-breaking D-branes in $\mathrm{SU}(1,1) \times \mathrm{SU}(2)$ which preserves the action of an $\mathbb{R} \times \mathrm{SU}(2)$ subgroup and is localised along a product of images of twisted conjugacy classes [81]. Similarly, both the $\mathrm{AdS}_3 \times \mathrm{S}^3$ and NW_6 branes are non-symmetric, while CW_6 is the worldvolume of a spacetime-filling twisted D5-brane in NW_6.

Chapter 3

Lie Branes of NW_6

3.1 Noncommutative Branes in NW_6

In this section and the next we will identify which *symmetric* D-branes of the six-dimensional Nappi-Witten spacetime NW_6 support a noncommutative worldvolume geometry. After identifying all supergravity fields living on the respective worldvolumes, we will proceed to quantise the classical geometries using standard techniques [92]. A potential obstruction to this procedure is that, as in any curved string background, a non-vanishing H-flux can lead to non-associative deformations of worldvolume algebras typically giving rise to variants of quantum group algebras that are deformations of standard noncommutative geometries for which there is no general notion of quantisation.

In certain semi-classical limits the NS-flux vanishes, $H = 0$, such as in the conformal field theory description based on a compact group $\mathcal{G}$ in the limit of infinite Kac-Moody level $k \to \infty$ [5]. Such limits correspond to field theory limits of the string theory whereby the Neveu-Schwarz B-field (or more precisely the gauge invariant two-form (2.28)) induces a symplectic structure on the brane worldvolume, which can be quantised in principle. In our situation we can identify $k = \theta^{-2}$, and the semi-classical limit corresponds to the limit in which the plane wave approaches flat spacetime. Such limits will always arise for the branes that we encounter in this thesis.

We will support our characterisations by comparison with known results from the boundary conformal field theory of the Nappi-Witten model [36, 61]. In this section we deal with branes described by conjugacy classes of the Nappi-Witten group $\mathcal{N}_6$. Generally, for the ordinary conjugacy classes ($\omega = \mathbb{1}$ in the notation of (2.20)), the worldvolumes inherit a natural symplectic form via the exponential map from the usual Kirillov-Kostant form on coadjoint orbits, which coincides with the symplectic structure induced by the B-field when $H = 0$. The conjugacy classes, given as orbits of the adjoint action of the Lie group $\mathcal{G}$ on itself, are then identified with representations of $\mathfrak{g}$, obtained from quantisation of coadjoint orbits, via the non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. If V_g is an irreducible module over the Lie algebra $\mathfrak{g}$ corresponding to the D-brane $D = \mathcal{C}_g := \mathcal{C}_g^{\mathbb{1}}$, then the noncommutative algebra of functions on the worldvolume is given by [5]

$$\mathcal{A}(\mathcal{C}_g) = \mathrm{End}(V_g) \tag{3.1}$$

The worldvolume algebra (3.1) carries a natural (adjoint) action of the group $\mathcal{G}$. In the present case we will find that these quantised conjugacy classes carry a noncommutative geometry which is completely analogous to that carried by D3-branes in flat space $\mathbb{E}^4$ with a uniform magnetic field on their worldvolume [39, 90, 99, 100], as expected from the harmonic oscillator character of dynamics in Nappi-Witten spacetime described at length in Section 1.3.1. We shall also find an interesting class of commutative null branes whose quantum geometry generically differs significantly from that of the classical conjugacy classes, due to a transverse U(2) rotational symmetry of the NW_6 background.

3.1.1 General Construction

It will prove convenient to introduce the doublet $(\underline{\mathsf{P}}^{\pm})^\top := (\mathsf{P}^{\pm}, \mathsf{Q}^{\pm})$ of generators and write Brinkman coordinates on $\mathcal{N}_6$ as

$$g(x^+, x^-, z, \overline{z}) = e^{\frac{\theta}{2}x^+\mathsf{J}} e^{z^\top \underline{\mathsf{P}}^+ + \overline{z}^\top \underline{\mathsf{P}}^-} e^{\frac{\theta}{2}x^+\mathsf{J}} e^{\theta^{-1}x^-\mathsf{T}} \tag{3.2}$$

In these coordinates one can work out the adjoint action

$$\mathrm{Ad}_{g(x^+,x^-,z,\overline{z})} g(x_0^+, x_0^-, z_0, \overline{z}_0)$$

corresponding to a fixed point $(x_0^+, x_0^-, z_0) \in \mathrm{NW}_6$ [48, 87], and the conjugacy classes can be written explicitly as the submanifolds

$$\begin{aligned} \mathcal{C}_{(x_0^+,x_0^-,z_0)} &= \left\{ \left(x_0^+, x_0^- - \frac{\theta}{2}|z|^2 \sin\theta x_0^+ + \theta \mathrm{Im}\left[z_0^\top \overline{z} e^{\frac{i\theta}{2}x^+} \cos\frac{\theta x_0^+}{2} \right], \right.\right. \\ &\qquad \left.\left. e^{i\theta x^+} z_0 - 2i \sin\frac{\theta x_0^+}{2} e^{\frac{i\theta}{2}x^+} z \right) \,\middle|\, \begin{matrix} x^+ \in \mathrm{S}^1 \\ z \in \mathbb{C}^2 \end{matrix} \right\} \end{aligned} \tag{3.3}$$

where we have used periodicity to restrict the light-cone time coordinate to $x^+, x_0^+ \in \mathrm{S}^1 = \mathbb{R}/2\pi\theta^{-1}\mathbb{Z}$. The null planes $x^+ = x_0^+$ are thus invariants of the conjugacy classes and can be used to distinguish the different D-submanifolds of NW_6. There are two types of euclidean branes generically associated to these conjugacy classes that we shall now proceed to describe in detail. In each case we first describe the classical geometry, and then proceed to quantise the orbits.

3.1.2 Null Branes

We begin with the "degenerate" cases where $x_0^+ = 0$.

When $\boldsymbol{z}_0 = \boldsymbol{0}$, the conjugacy class (3.3) corresponds to a D-instanton sitting at the point $(0, x_0^-, \boldsymbol{0}) \in \mathrm{NW}_6$. When $\boldsymbol{z}_0^\top := (z_0, w_0) \neq \boldsymbol{0}$ we denote the conjugacy classes by $\mathcal{C}_{|z_0|,|w_0|} := \mathcal{C}_{(0,x_0^-,\boldsymbol{z}_0)}$. Since $\mathcal{C}_{|z_0|,|w_0|} = \{(0, x^-, e^{i\theta x^+} \boldsymbol{z}_0 | x^- \in \mathbb{R}, x^+ \in \mathrm{S}^1\} \cong \mathbb{R} \times \mathrm{S}^1$, the resulting object may be thought of as a cylindrical brane extended along the null light-cone direction x^- with fixed radii $|z_0|, |w_0|$ in the two transverse planes $(z_0, w_0) \in \mathbb{C}^2$, and will therefore be refered to as a "null" brane. However, the quantisation of the classical worldvolume will generally turn out to depend crucially on the radii. We can understand this dependence heuristically as follows. Generally, the Cahen-Wallach metric (1.30) possesses an SO(4) rotational symmetry of its transverse space $\boldsymbol{z} \in \mathbb{C}^2 \cong \mathbb{R}^4$, which is broken to U(2) by the Neveu-Schwarz background (1.31). When $|z_0| = 0$ or $|w_0| = 0$, one has $\mathcal{C}_{|z_0|,0} \cong \mathcal{C}_{0,|w_0|} \cong \mathbb{R} \times \mathrm{S}^1$. However, when both $|z_0|$ and $|w_0|$ are non-zero, we may use this U(2) symmetry to rotate $w_0 \mapsto e^{i\phi} w_0$ at fixed z_0 by an arbitrary phase. In this case the cylindrical brane is parametrised effectively by two independent periodic coordinates, and one has

$$\begin{aligned} \mathcal{C}_{|z_0|,|w_0|} &= \left\{ (0, x^-, e^{i\theta x^+} z_0, e^{i\theta y^+} w_0) \mid x^- \in \mathbb{R}, x^+, y^+ \in \mathrm{S}^1 \right\} \\ &\cong \begin{cases} \text{point} & z_0 = w_0 = 0 \\ \mathbb{R} \times \mathrm{S}^1 \times \mathrm{S}^1 & z_0, w_0 \neq 0 \\ \mathbb{R} \times \mathrm{S}^1 & \text{otherwise} \end{cases} \end{aligned} \tag{3.4}$$

In each case it carries a degenerate metric

$$ds_6^2 \mid_{\mathcal{C}_{|z_0|,|w_0|}} = \theta^2 |z_0|^2 \left(dx^+\right)^2 + \theta^2 |w_0|^2 \left(dy^+\right)^2 \tag{3.5}$$

Such a brane generically has no straightforward interpretation as a D-brane, as the corresponding Dirac-Born-Infeld action is ill-defined. Nevertheless, it will play a role in our ensuing analysis and so we shall analyse it in some detail.

When $\boldsymbol{z}_0 \neq \mathbf{0}$, the centraliser of

$$g_0 := g(0, x_0^-, \boldsymbol{z}_0, \overline{\boldsymbol{z}}_0)$$

is the subgroup of $\mathcal{N}_6$ parametrised as the submanifold

$$\mathcal{Z}_{|z_0|,|w_0|} = \left\{ (0, x^-, \boldsymbol{z}) \mid \mathrm{Im}\, \boldsymbol{z}_0^\top \overline{\boldsymbol{z}} = 0 \right\} \cong \mathbb{R}^4 \tag{3.6}$$

with a degenerate metric, while $\mathcal{Z}_{0,0} \cong \mathcal{N}_6$. The Neveu-Schwarz fields vanish on the brane,

$$H_6 \mid_{\mathcal{C}_{|z_0|,|w_0|}} = B_6 \mid_{\mathcal{C}_{|z_0|,|w_0|}} = 0 \tag{3.7}$$

while an elementary calculation using (1.5), (1.7), (1.10) and (3.6) shows that the Abelian gauge field fluxes (2.27) also vanish on the null brane worldvolume,

$$F_6^{(\zeta)} \mid_{\mathcal{C}_{|z_0|,|w_0|}} = 0 \tag{3.8}$$

for any $\zeta = \theta^{-1} x^- \mathsf{T} + \boldsymbol{z}^\top \underline{\mathsf{P}}^+ + \overline{\boldsymbol{z}}^\top \underline{\mathsf{P}}^-$ in the tangent space to the centraliser (3.6). This suggests that these conjugacy classes should describe *commutative* branes. Since the H-field vanishes on the null branes, we can use the standard coadjoint orbit method as discussed earlier.

Quantisation

Let us now describe the algebra of functions (3.1) on the null brane worldvolume. The Lie algebra $\mathfrak{n}_6$ has three types of unitary irreducible representations $\mathcal{D}^{p^+,p^-}$: $U(\mathfrak{n}_6) \to \mathrm{End}(V^{p^+,p^-})$ labelled by light-cone momenta $p^\pm \in \mathbb{R}$ [11, 23, 64]. On each module V^{p^+,p^-}, elements of the centre of the universal enveloping algebra $U(\mathfrak{n}_6)$ are proportional to the identity operator $\mathbb{1}$. In particular, the central element T acts as

$$\mathcal{D}^{p^+,p^-}(\mathsf{T}) = i\theta p^+ \mathbb{1} \tag{3.9}$$

According to (1.38), the modules corresponding to the null branes live in the class $V^{0,p^-}_{\alpha,\beta}$, $\alpha, \beta \in [0, \infty)$ of continuous representations with $p^+ = 0$. In this case, in addition to T there are two other Casimir operators corresponding to the quadratic elements $\mathsf{P}^+\mathsf{P}^-$ and $\mathsf{Q}^+\mathsf{Q}^-$ of $U(\mathfrak{n}_6)$ with the eigenvalues

$$\mathcal{D}^{0,p^-}_{\alpha,\beta}\left(\mathsf{P}^+\mathsf{P}^-\right) = -\alpha^2 \mathbb{1}\ , \ \ \mathcal{D}^{0,p^-}_{\alpha,\beta}\left(\mathsf{Q}^+\mathsf{Q}^-\right) = -\beta^2 \mathbb{1}\ . \tag{3.10}$$

As a separable Hilbert space, the module $V^{0,p^-}_{\alpha,\beta}$ may be expressed as the linear span

$$V^{0,p^-}_{\alpha,\beta} = \bigoplus_{n,m\in\mathbb{Z}} \mathbb{C} \cdot \left|{}^{n\,m}_{\alpha\beta}; p^-\right\rangle \tag{3.11}$$

with the non-trivial actions of the generators of $\mathfrak{n}_6$ given by

$$\begin{aligned}
\mathcal{D}^{0,p^-}_{\alpha,\beta}\left(\mathsf{P}^\pm\right) \left|{}^{nm}_{\alpha\beta}; p^-\right\rangle &= i\alpha \left|{}^{n\mp1\,m}_{\alpha\quad\beta}; p^-\right\rangle \\
\mathcal{D}^{0,p^-}_{\alpha,\beta}\left(\mathsf{Q}^\pm\right) \left|{}^{nm}_{\alpha\beta}; p^-\right\rangle &= i\beta \left|{}^{n\,m\mp1}_{\alpha\beta}; p^-\right\rangle \\
\mathcal{D}^{0,p^-}_{\alpha,\beta}\left(\mathsf{J}\right) \left|{}^{nm}_{\alpha\beta}; p^-\right\rangle &= i\left(\theta^{-1}p^- - n - m\right) \left|{}^{n\,m}_{\alpha\beta}; p^-\right\rangle
\end{aligned} \tag{3.12}$$

The inner product is defined such that the basis (3.11) is orthonormal,

$$\left\langle{}^{n\,m}_{\alpha\beta}; p^- \,\middle|\, {}^{n'm'}_{\alpha\,\beta}; p^-\right\rangle = \delta_{nn'}\delta_{mm'} \tag{3.13}$$

These representations have no highest or lowest weight states. Note that shifting the labels n or m by 1 gives an equivalent representation

$$V^{0,p^-}_{\alpha,\beta} \cong V^{0,p^-+\theta}_{\alpha,\beta} \tag{3.14}$$

In other words, only the representations $V^{0,p^-}_{\alpha,\beta}$ with $p^- \in [0,\theta)$ are inequivalent. This is analogous to the periodicity constraint imposed on the light-cone coordinate x^+ in (3.3).

Among these representations is the trivial one-dimensional representation $V^{0,0}_{0,0}$ which corresponds to the D-instantons found above. To generically associate them with the null brane worldvolumes obtained before, we use the correspondence between Casimir elements and class functions on the group, which are respectively constant in irreducible representations and on conjugacy classes. Note that the geodesics in NW_6 obey $p^+ = \dot{x}^+$ and $p^- = \dot{x}^- - \frac{\theta^2}{4}|\boldsymbol{z}|^2 p^+$ (c.f. (1.40)). At $p^+ = 0$, we may thereby identify the parameters α and β with the radii $|z_0|$ and $|w_0|$ in the two transverse planes (z_0, w_0), while p^- is identified with a fixed value of the light-cone position x^-. The quantised algebra of functions on the conjugacy class (3.4) is thus given by

$$\mathcal{A}\left(\mathcal{C}_{|z_0|,|w_0|}\right) = \mathrm{End}\left(V^{0,p^-}_{|z_0|,|w_0|}\right) \tag{3.15}$$

where a generic element $\hat{f} \in \mathrm{End}(V^{0,p^-}_{\alpha,\beta})$ is a complex linear combination

$$\hat{f} = \sum_{n,m,n',m' \in \mathbb{Z}} f_{n,m;n',m'} \,\big|\, {}^{n\,m}_{\alpha\beta}; p^- \big\rangle \big\langle {}^{n'm'}_{\alpha\,\beta}; p^- \big| \tag{3.16}$$

In this context, we may appropriately regard the conjugacy class (3.4) as the worldvolume of a symmetric Dp-brane with $p = -1, 1, 0$ and volume element (3.5). Because of (3.14), in the quantum geometry the light-cone position is restricted to a finite interval $x^- \in [0, \theta)$.

This identification can be better understood by introducing the coherent states

$$\big|{}^{x^+ y^+}_{\alpha\ \beta}; p^-\big\rangle := \sum_{n,m \in \mathbb{Z}} e^{in\theta x^+ + im\theta y^+} \big|{}^{n\,m}_{\alpha\beta}; p^-\big\rangle \qquad x^+, y^+ \in \mathrm{S}^1 \tag{3.17}$$

on which the non-trivial symmetry generators are represented as differential operators

$$\begin{aligned}
\mathcal{D}^{0,p^-}_{\alpha,\beta}\left(\mathsf{P}^\pm\right) \big|{}^{x^+ y^+}_{\alpha\ \beta}; p^-\big\rangle &= i\alpha e^{\pm i\theta x^+} \big|{}^{x^+ y^+}_{\alpha\ \beta}; p^-\big\rangle \\
\mathcal{D}^{0,p^-}_{\alpha,\beta}\left(\mathsf{Q}^\pm\right) \big|{}^{x^+ y^+}_{\alpha\ \beta}; p^-\big\rangle &= i\beta\, e^{\pm i\theta y^+} \big|{}^{x^+ y^+}_{\alpha\ \beta}; p^-\big\rangle \\
\mathcal{D}^{0,p^-}_{\alpha,\beta}\left(\mathsf{J}\right) \big|{}^{x^+ y^+}_{\alpha\ \beta}; p^-\big\rangle &= i\theta^{-1}\left(p^- - \frac{\partial}{\partial x^+} - \frac{\partial}{\partial y^+}\right) \big|{}^{x^+ y^+}_{\alpha\ \beta}; p^-\big\rangle
\end{aligned} \tag{3.18}$$

The conjugate state to (3.17) is given by

$$\left\langle {}^{x^+\,y^+}_{\alpha\;\;\beta} \right| = \sum_{n,m\in\mathbb{Z}} e^{-in\theta x^+ - im\theta y^+} \; \left\langle {}^{n\,m}_{\alpha\beta} ; p^- \right| \tag{3.19}$$

so that the inner product is

$$\left\langle {}^{x_1^+\,y_1^+}_{\alpha\;\;\beta} ; p^- \,\middle|\, {}^{x_2^+\,y_2^+}_{\alpha\;\;\beta} ; p^- \right\rangle = \delta\left(x_1^+ - x_2^+\right)\delta\left(y_1^+ - y_2^+\right) \tag{3.20}$$

while the resolution of unity is

$$\mathbb{1} = \theta^2 \int_0^{2\pi\theta^{-1}} \frac{dx^+}{2\pi} \int_0^{2\pi\theta^{-1}} \frac{dy^+}{2\pi} \left| {}^{x^+\,y^+}_{\alpha\;\;\beta} ; p^- \right\rangle \left\langle {}^{x^+\,y^+}_{\alpha\;\;\beta} ; p^- \right| \tag{3.21}$$

Thus the states (3.17) form an over-complete basis for the Fock space $V^{0,p^-}_{\alpha,\beta}$. The metric and Kirillov-Kostant symplectic two-form on the orbit can be computed as the matrix elements (3.17) of the operators

$$\left|\mathcal{D}^{0,p^-}_{|z_0|,|w_0|}(g^{-1}dg)\right|^2$$

and

$$\mathcal{D}^{0,p^-}_{|z_0|,|w_0|}([g^{-1}dg, g^{-1}dg])$$

respectively. They coincide with those computed above as the pull-backs of the Nappi-Witten geometry to the conjugacy classes (3.4).

We may now attempt to view the worldvolume algebra (3.15) as a deformation of the classical algebra of functions on the conjugacy class (3.4). On the null hyperplanes $x^- =$ constant, we define an isomorphism of underlying vector spaces

$$\Delta : \mathrm{C}^\infty\left(\mathcal{C}_{|z_0|,|w_0|}\right) \longrightarrow \mathrm{End}\left(V^{0,p^-}_{|z_0|,|w_0|}\right) \tag{3.22}$$

in the following manner. Decomposing any smooth function $f \in \mathrm{C}^\infty(\mathcal{C}_{|z_0|,|w_0|})$ as a Fourier series over the torus $\mathrm{S}^1 \times \mathrm{S}^1$ given by

$$f\left(x^+, y^+\right) = \sum_{n,m\in\mathbb{Z}} f_{n,m}\, e^{in\theta x^+ + im\theta y^+} \tag{3.23}$$

we write

$$\hat{f} = \Delta(f) := \sum_{n,m\in\mathbb{Z}} f_{n,m} \left(\frac{1}{i|z_0|}\mathcal{D}^{0,p^-}_{|z_0|,|w_0|}\left(\mathsf{P}^{\varepsilon(n)}\right)\right)^{|n|} \left(\frac{1}{i|w_0|}\mathcal{D}^{0,p^-}_{|z_0|,|w_0|}\left(\mathsf{Q}^{\varepsilon(m)}\right)\right)^{|m|} \tag{3.24}$$

where the label $\varepsilon(n) = \pm$ corresponds to the sign of $n \in \mathbb{Z}$. The inverse map is given by

$$
\begin{aligned}
f\left(x^+, y^+\right) &:= \Delta^{-1}\big(\hat{f}\big)\left(x^+, y^+\right) \\
&= \theta^2 \int_0^{2\pi\theta^{-1}} \frac{d\tilde{x}^+}{2\pi} \int_0^{2\pi\theta^{-1}} \frac{d\tilde{y}^+}{2\pi} \, \langle{}^{\tilde{x}^+ \ \tilde{y}^+}_{|z_0|\,|w_0|}; p^- | \hat{f} |{}^{x^+ \ y^+}_{|z_0|\,|w_0|}; p^- \rangle
\end{aligned} \tag{3.25}
$$

As expected, from (3.18) one finds that the functions corresponding to the generators of $\mathfrak{n}_6$ on $V^{0,p^-}_{|z_0|,|w_0|}$ coincide with the coordinates of the conjugacy classes (3.4),

$$
\begin{aligned}
\Delta^{-1}\left(\mathcal{D}^{0,p^-}_{|z_0|,|w_0|}\left(\mathsf{P}^\pm\right)\right)\left(x^+, y^+\right) &= i|z_0| \, e^{\pm i\theta x^+} \\
\Delta^{-1}\left(\mathcal{D}^{0,p^-}_{|z_0|,|w_0|}\left(\mathsf{Q}^\pm\right)\right)\left(x^+, y^+\right) &= i|w_0| \, e^{\pm i\theta y^+} \\
\Delta^{-1}\left(\mathcal{D}^{0,p^-}_{|z_0|,|w_0|}(\mathsf{J})\right)\left(x^+, y^+\right) &= i\theta^{-1} p^-
\end{aligned} \tag{3.26}
$$

Generically, the classical conjugacy class (3.3) in this case corresponds to the diagonal subspace $x^+ = y^+$ of the space spanned by the coherent states (3.17).

Since the operator (3.24) is diagonal in the basis (3.17) with eigenvalue $f(x^+, y^+)$, we easily establish that in this case the map (3.22) is in fact an *algebra* isomorphism. Namely, the product of two operators $\hat{f}$ and $\hat{g}$ on $V^{0,p^-}_{|z_0|,|w_0|}$ corresponds to the pointwise multiplication of the associated functions on $\mathcal{C}_{|z_0|,|w_0|}$,

$$
\Delta^{-1}\big(\hat{f}\hat{g}\big)\left(x^+, y^+\right) = f\left(x^+, y^+\right) g\left(x^+, y^+\right) \tag{3.27}
$$

Thus the worldvolume algebra (3.15) in the present case describes a *commutative* geometry on the null branes, in agreement with the classical analysis. This is also in accord with the fact that the $p^+ = 0$ sector of the dynamics in Nappi-Witten spacetime does not feel the harmonic oscillator potential of Section 1.3.3, and thereby describes free motion in the transverse space.

3.1.3 D3-Branes

Let us now turn to the somewhat more interesting cases with $x_0^+ \neq 0$, whereby the classical worldvolume geometries are nondegenerate.

In this case the conjugacy class (3.3) can be written after a trivial coordinate redefinition as

$$\mathcal{C}_{(x_0^+,x_0^-,z_0)} = \{(x_0^+, x_0^- + \frac{\theta}{4}\left(|z_0|^2 - |z'|^2\right)\cot\frac{\theta x_0^+}{2}, z') \mid z' \in \mathbb{C}^2\} \cong \mathbb{E}^4 \tag{3.28}$$

It may be labelled as $\mathcal{C}_{x_0^+,\chi_0}$, with $\chi_0 := x_0^- + \frac{\theta}{4}|z_0|^2\cot\frac{\theta x_0^+}{2}$, so that the corresponding branes have four-dimensional worldvolumes located at

$$x^+ = x_0^+ \qquad\qquad x^- = \chi_0 - \frac{\theta}{4}|z|^2\cot\frac{\theta x_0^+}{2} \tag{3.29}$$

The worldvolume metric is non-degenerate and given by

$$ds_6^2 \mid_{\mathcal{C}_{x_0^+,\chi_0}} = |dz|^2 \tag{3.30}$$

so that in this case the conjugacy classes are wrapped by flat euclidean D3-branes. The NS fields on the worldvolume are given by flat space forms

$$H_6|_{\mathcal{C}_{x_0^+,\chi_0}} = 0 \tag{3.31}$$

$$B_6|_{\mathcal{C}_{x_0^+,\chi_0}} = -2i\theta x_0^+ d\overline{z}^\top \wedge dz \tag{3.32}$$

The vanishing of the H-flux again means that we can apply standard semi-classical quantisation techniques to these D-branes.

If $x_0^+ = \frac{\pi}{\theta}$ then these branes, like the null branes, are associated with the conjugate points of the Rosen plane wave geometry defined by (1.21) through (1.23). In this case the conjugacy class $\mathcal{C}_{x_0^-} := \mathcal{C}_{(\frac{\pi}{\theta},x_0^-,z_0)}$ is a four-plane labelled by the fixed light-cone position $x^- = x_0^-$. When $x_0^+ \neq 0, \frac{\pi}{\theta}$, the brane worldvolume is a paraboloid corresponding to a point-like object travelling at the speed of light while simultaneously expanding or contracting in a three-sphere in the transverse space $z \in \mathbb{C}^2$ according to (3.29). Since these branes lie in the set of conjugate-free points, we may analyse them using the Penrose-Güven limit of Section 2.3.1, which yields a non-vanishing gauge invariant two-form field via pull-back of (2.46) to the conjugacy classes as

$$\mathcal{F}_{x_0^+} := \mathcal{F}_6|_{\mathcal{C}_{x_0^+,\chi_0}} = -i\cot\frac{\theta x_0^+}{2} d\overline{z}^\top \wedge dz \tag{3.33}$$

Thus these D-branes are expected to carry a noncommutative geometry for $x_0^+ \neq 0, \frac{\pi}{\theta}$. Extrapolating (3.33) to $x_0^+ = \frac{\pi}{\theta}$ shows that the conjugacy classes $\mathcal{C}_{x_0^-}$ are expected to support a commutative worldvolume geometry like the null branes, despite their non-vanishing B-field.

The Penrose-Güven limit can also be used to understand the physical origin of the euclidean D3-branes. They may be described through the commuting isometric embedding diagram

$$\begin{array}{ccc} \mathrm{AdS}_3 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_6 \\ {\scriptstyle \imath'}\big\uparrow & & \big\uparrow {\scriptstyle \widetilde{\imath}'} \\ \mathrm{AdS}_2 \times \mathrm{S}^2 & \xrightarrow{\mathrm{PGL}'} & \mathbb{E}^4 \end{array} \tag{3.34}$$

where the primes indicate that the limit is taken along a null geodesic which does *not* pass through the $\mathrm{AdS}_2 \times \mathrm{S}^2$ brane worldvolume, i.e. the embeddings $\imath'$ and $\widetilde{\imath}'$ are defined by the constant time slices $\tau, \phi =$ constant and $x^+ =$ constant, respectively. Thus the resulting geometry is not of plane wave type. The brane on the left-hand side of (3.34) originates as a flat D3-brane connected orthogonally to a distant NS5/F1 black string by a stretched (p, q) string [18], the origin of the worldvolume flux (3.33). As the (p, q) string pulls the flat D3-brane, it deforms its worldvolume geometry, leading to an $\mathrm{AdS}_2 \times \mathrm{S}^2$ brane in the near-horizon region of the black string. The Penrose-Güven limit in (3.34) pulls the deformed branes away into the asymptotically flat region of the black string, decompactifying it as $R \to \infty$ onto the flat euclidean D3-brane on the right-hand side of the isometric embedding diagram. However, since the diagram commutes, the standard fuzzy geometry of the $\mathrm{AdS}_2 \times \mathrm{S}^2$ brane induces, through the usual scaling limit [28], a Moyal type noncommutative geometry on the instantonic $\mathbb{E}^4$ brane. This argument is consistent with the fact that the (modified) Penrose-Güven limit is also a map between symmetric D-branes. We shall now substantiate this physical picture through explicit computation of the quantised worldvolume geometry.

Quantisation

To describe the algebra of functions (3.1) on the D3-brane worldvolume, we start with the irreducible representations $\mathcal{D}^{p^+,p^-} : U(\mathfrak{n}_6) \to \mathrm{End}(V^{p^+,p^-})$ having $p^+ > 0$. In addition to (3.9), the quadratic Casimir element $\mathsf{C}_6 = 2\mathsf{J}\mathsf{T} + \frac{1}{2}[(\underline{\mathsf{P}}^+)^\top \underline{\mathsf{P}}^- + (\underline{\mathsf{P}}^-)^\top \underline{\mathsf{P}}^+]$ acts as a scalar operator

$$\mathcal{D}^{p^+,p^-}(\mathbb{C}_6) = -2p^+ \left(p^- + \theta\right) \mathbb{1} \tag{3.35}$$

This operator is positive for all $p^- \in (-\infty, -\theta)$. As a separable Hilbert space, the module V^{p^+,p^-} may be exhibited as the linear span

$$V^{p^+,p^-} = \bigoplus_{n,m\in\mathbb{N}_0} \mathbb{C} \cdot |n,m;p^+,p^-\rangle \tag{3.36}$$

with the non-trivial actions of the Nappi-Witten generators given by

$$\begin{aligned}
\mathcal{D}^{p^+,p^-}\left(\mathsf{P}^+\right)|n,m;p^+,p^-\rangle &= 2i\theta p^+ n|n-1,m;p^+,p^-\rangle \qquad (3.37)\\
\mathcal{D}^{p^+,p^-}\left(\mathsf{P}^-\right)|n,m;p^+,p^-\rangle &= i|n+1,m;p^+,p^-\rangle\\
\mathcal{D}^{p^+,p^-}\left(\mathsf{Q}^+\right)|n,m;p^+,p^-\rangle &= 2i\theta p^+ m|n,m-1;p^+,p^-\rangle\\
\mathcal{D}^{p^+,p^-}\left(\mathsf{Q}^-\right)|n,m;p^+,p^-\rangle &= i|n,m+1;p^+,p^-\rangle\\
\mathcal{D}^{p^+,p^-}(\mathsf{J})|n,m;p^+,p^-\rangle &= i\left(\theta^{-1}p^- - n - m\right)|n,m;p^+,p^-\rangle
\end{aligned}$$

The inner product on the basis (3.36) is

$$\langle n,m;p^+,p^- \mid n',m';p^+,p^-\rangle = \left(2\theta p^+\right)^{n+m} n!m!\delta_{nn'}\delta_{mm'} \tag{3.38}$$

This representation admits a highest weight state $|0,0;p^+,p^-\rangle$ on which $-i\mathcal{D}^{p^+,p^-}(\mathsf{J})$ has weight $\theta^{-1}p^-$.

To associate these representations with euclidean D3-brane worldvolumes, we note that the constraint on the light-cone position in (3.29) can be written in the semi-classical limit $\theta \to 0$ as

$$2x_0^+ x^- + |z|^2 = 2x_0^+ \chi_0 \tag{3.39}$$

The relation (3.39) agrees with the Casimir eigenvalue constraint (3.35) under the identifications of the light-cone time x_0^+ with momentum p^+ as before and the class variable χ_0 with $p^- + \theta$. We will soon identify the worldvolume coordinates $\boldsymbol{z} \in \mathbb{C}^2$ with the operators $\mathcal{D}^{p^+,p^-}(\underline{\mathsf{P}}^+)$. Thus the quantised algebra of functions on the conjugacy class is given by

$$\mathcal{A}\big(\mathcal{C}_{x_0^+,\chi_0}\big) = \mathrm{End}\big(V^{p^+,p^-}\big) \tag{3.40}$$

where a generic element $\hat{f} \in \mathrm{End}(V^{p^+,p^-})$ is a complex linear combination

$$\hat{f} = \sum_{n,m,n',m' \in \mathbb{N}_0} f_{n,m;n',m'} |n,m;p^+,p^-\rangle\langle n',m';p^+,p^-| \tag{3.41}$$

As before, it is convenient to work in the conventional coherent state basis of the Fock module V^{p^+,p^-} defined for $\boldsymbol{z}^\top = (z,w) \in \mathbb{C}^2$ by

$$|\boldsymbol{z};p^+,p^-\rangle := e^{-\boldsymbol{z}^\top \underline{\mathsf{P}}^-}|0,0;p^+,p^-\rangle = \sum_{n,m\in\mathbb{N}_0} \frac{z^n w^m}{i^{n+m} n! m!} |n,m;p^+,p^-\rangle \tag{3.42}$$

on which the non-trivial symmetry generators are represented by differential operators

$$\begin{aligned}
\mathcal{D}^{p^+,p^-}\left(\underline{\mathsf{P}}^+\right)|\boldsymbol{z};p^+,p^-\rangle &= 2\theta p^+ \boldsymbol{z}|\boldsymbol{z};p^+,p^-\rangle \\
\mathcal{D}^{p^+,p^-}\left(\underline{\mathsf{P}}^-\right)|\boldsymbol{z};p^+,p^-\rangle &= -\boldsymbol{\partial}|\boldsymbol{z};p^+,p^-\rangle \\
\mathcal{D}^{p^+,p^-}\big(\mathsf{J}\big)|\boldsymbol{z};p^+,p^-\rangle &= i\left(\theta^{-1}p^- - \boldsymbol{z}^\top\boldsymbol{\partial}\right)|\boldsymbol{z};p^+,p^-\rangle
\end{aligned} \tag{3.43}$$

with $\boldsymbol{\partial}^\top := (\frac{\partial}{\partial z}, \frac{\partial}{\partial w})$. The conjugate state to (3.42) is given by

$$\langle \boldsymbol{z};p^+,p^-| = \langle 0,0;p^+,p^-|e^{\overline{\boldsymbol{z}}^\top \underline{\mathsf{P}}^+} \tag{3.44}$$

so that the inner product of coherent states is

$$\langle \boldsymbol{z};p^+,p^- \mid \boldsymbol{z}';p^+,p^-\rangle = e^{2\theta p^+ \overline{\boldsymbol{z}}^\top \boldsymbol{z}'} \tag{3.45}$$

while their completeness relation is

$$\mathbb{1} = \left(2\theta p^+\right)^2 \int\limits_{\mathbb{C}^2} d\varrho(\boldsymbol{z},\overline{\boldsymbol{z}})\, e^{-2\theta p^+|\boldsymbol{z}|^2} |\boldsymbol{z};p^+,p^-\rangle\langle \boldsymbol{z};p^+,p^-| \tag{3.46}$$

where $d\varrho(z,\overline{z}) = \frac{1}{\pi^2}|dz\wedge d\overline{z}\wedge dw\wedge d\overline{w}|$ is the standard flat measure on $\mathbb{E}^4\cong\mathbb{C}^2$. We will now use these states to describe the worldvolume algebra (3.40) as a deformation of the classical algebra of functions on the conjugacy class $\mathcal{C}_{x_0^+,\chi_0}$.

For this, we construct an isomorphism of underlying vector spaces

$$\Delta_* : \mathrm{C}^\infty\big(\mathcal{C}_{x_0^+,\chi_0}\big) \longrightarrow \mathrm{End}\big(V^{p^+,p^-}\big) \tag{3.47}$$

by expanding any smooth function $f\in\mathrm{C}^\infty(\mathcal{C}_{x_0^+,\chi_0})$ as a Taylor series over $\mathbb{C}^2$ given by

$$f\left(z,\overline{z}\right) = \sum_{n,m,n',m'\in\mathbb{N}_0} f_{n,m;n',m'}\, z^n w^m \overline{z}^{n'}\overline{w}^{m'} \tag{3.48}$$

and defining

$$\begin{aligned}\hat{f} = \Delta_*(f) \quad &:= \quad \sum_{n,m,n',m'\in\mathbb{N}_0} \frac{f_{n,m;n',m'}}{\left(2\theta p^+\right)^{n+m+n'+m'}}\,\mathcal{D}^{p^+,p^-}\left(\mathsf{P}^-\right)^{n'}\mathcal{D}^{p^+,p^-}\left(\mathsf{Q}^-\right)^{m'} \\ &\qquad \times\mathcal{D}^{p^+,p^-}\left(\mathsf{P}^+\right)^{n}\mathcal{D}^{p^+,p^-}\left(\mathsf{Q}^+\right)^{m}\end{aligned} \tag{3.49}$$

The inverse map is provided by the normalised matrix elements of operators as

$$f\left(z,\overline{z}\right) := \Delta_*^{-1}\big(\hat{f}\big)\left(z,\overline{z}\right) = e^{-2\theta p^+|z|^2}\langle z;p^+,p^-|\hat{f}|z;p^+,p^-\rangle \tag{3.50}$$

As before, the functions corresponding to the generators of the Nappi-Witten algebra $\mathfrak{n}_6$ coincide with the coordinates of the conjugacy classes $\mathcal{C}_{x_0^+,\chi_0}$,

$$\begin{aligned}\Delta_*^{-1}\Big(\mathcal{D}^{p^+,p^-}\left(\underline{\mathsf{P}}^+\right)\Big)\left(z,\overline{z}\right) &= 2\theta p^+ z \\ \Delta_*^{-1}\Big(\mathcal{D}^{p^+,p^-}\left(\underline{\mathsf{P}}^-\right)\Big)\left(z,\overline{z}\right) &= 2\theta p^+ \overline{z} \\ \Delta_*^{-1}\Big(\mathcal{D}^{p^+,p^-}\left(\mathsf{J}\right)\Big)\left(z,\overline{z}\right) &= i\left(\theta^{-1}p^- - 2\theta p^+|z|^2\right)\end{aligned} \tag{3.51}$$

The map (3.47) in this case is *not* an algebra homomorphism and it can be used to deform the pointwise multiplication of functions in the algebra $\mathrm{C}^\infty(\mathcal{C}_{x_0^+,\chi_0})$, giving an associative $*$-product defined by

$$\begin{aligned}\left(f*g\right)\left(z,\overline{z}\right) \quad &:= \quad \Delta_*^{-1}\big(\hat{f}\hat{g}\big)\left(z,\overline{z}\right) = e^{-2\theta p^+|z|^2}\langle z;p^+,p^-|\hat{f}\hat{g}|z;p^+,p^-\rangle \\ &= \quad \left(2\theta p^+\right)^2\int\limits_{\mathbb{C}^2} d\varrho\left(z',\overline{z}'\right)e^{-2\theta p^+(|z'|^2+|z|^2)} \\ &\qquad \times\langle z;p^+,p^-|\hat{f}|z';p^+,p^-\rangle\langle z';p^+,p^-|\hat{g}|z;p^+,p^-\rangle\end{aligned} \tag{3.52}$$

We can express this $*$-product more explicitly in terms of a bi-differential operator acting on $C^\infty(\mathcal{C}_{x_0^+,\chi_0}) \otimes C^\infty(\mathcal{C}_{x_0^+,\chi_0}) \to C^\infty(\mathcal{C}_{x_0^+,\chi_0})$ by writing the normalised matrix elements in (3.52) using translation operators as [6]

$$
\begin{aligned}
e^{-z^\top \partial'} e^{z'^\top \partial} f(z,\bar{z}) &= e^{-z^\top \partial'} \frac{\langle z; p^+, p^- | \hat{f} | z+z'; p^+, p^- \rangle}{\langle z; p^+, p^- \mid z+z'; p^+, p^- \rangle} \qquad (3.53)\\
&= \frac{\langle z; p^+, p^- | \hat{f} | z'; p^+, p^- \rangle}{\langle z; p^+, p^- \mid z'; p^+, p^- \rangle}
\end{aligned}
$$

The translation operator $e^{-z^\top \partial'} e^{z'^\top \partial}$, acting on z'-independent functions, can be expressed as a normal ordered exponential $\circ\!\!\circ \exp(z'-z)^\top \overrightarrow{\boldsymbol{\partial}} \circ\!\!\circ$ with derivatives ordered to the right in each monomial of the Taylor series expansion of the exponential function. In this way we may write the $*$-product (3.52) as

$$
\begin{aligned}
(f*g)(z,\bar{z}) &= \left(2\theta p^+\right)^2 \int_{\mathbb{C}^2} d\varrho\left(z',\bar{z}'\right) f(z,\bar{z}) \circ\!\!\circ \exp \overleftarrow{\boldsymbol{\partial}}^\top \left(z'-z\right) \circ\!\!\circ \\
&\quad \times e^{-2\theta p^+ |z'-z|^2} \circ\!\!\circ \exp\left(\bar{z}'-\bar{z}\right)^\top \overrightarrow{\bar{\boldsymbol{\partial}}} \circ\!\!\circ g(z,\bar{z}) \qquad (3.54)
\end{aligned}
$$

and performing the Gaussian integral in (3.54) leads to our final form

$$
(f*g)(z,\bar{z}) = f(z,\bar{z}) \exp\Big(\frac{1}{2\theta p^+} \overleftarrow{\boldsymbol{\partial}}^\top \overrightarrow{\bar{\boldsymbol{\partial}}}\Big) g(z,\bar{z}) \qquad (3.55)
$$

The $*$-product (3.55) is the Voros product [101] on four-dimensional noncommutative euclidean space $\mathbb{E}^4_\Theta$ corresponding to the Poisson bi-vector

$$
\Theta = -\frac{i}{2\theta p^+} \bar{\boldsymbol{\partial}}^\top \wedge \boldsymbol{\partial} \qquad (3.56)
$$

whose components are proportional to the inverse of the magnetic field (or equivalently harmonic oscillator frequency) $\omega = \frac{1}{2}\theta p^+$ in the effective particle dynamics in Nappi-Witten spacetime described in Section 1.3.3. This product is *not* the same as the standard Moyal product

$$
(f \star g)(z,\bar{z}) := f(z,\bar{z}) \exp\left[\frac{1}{4\theta p^+}\Big(\overleftarrow{\boldsymbol{\partial}}^\top \overrightarrow{\bar{\boldsymbol{\partial}}} - \overleftarrow{\bar{\boldsymbol{\partial}}}^\top \overrightarrow{\boldsymbol{\partial}}\Big)\right] g(z,\bar{z}) \qquad (3.57)
$$

which arises from Weyl operator ordering, rather than the normal ordering prescription employed in (3.49). Although different, these two products are cohomologically *equivalent* [101], because the invertible differential operator $\mathcal{T} := \exp \frac{1}{4\theta p^+} |\boldsymbol{\partial}|^2$

gives an algebra isomorphism $\mathcal{T} : (\mathrm{C}^\infty(\mathcal{C}_{x_0^+,\chi_0}), \star) \to (\mathrm{C}^\infty(\mathcal{C}_{x_0^+,\chi_0}), *)$, i.e. $\mathcal{T}(f \star g) = \mathcal{T}(f) * \mathcal{T}(g) \forall f, g \in \mathrm{C}^\infty(\mathcal{C}_{x_0^+,\chi_0})$.

Finally, to deal with the geometry in the case $p^+ < 0$, we construct a lowest-weight module $\widetilde{V}^{p^+,p^-}$ which defines the representation conjugate to $V^{p^+,-p^-}$ above by interchanging the roles of the generators $\underline{\mathsf{P}}^+ \leftrightarrow \underline{\mathsf{P}}^-$ and replacing the generators J, T with their reflections $-\mathsf{J}, -\mathsf{T}$. The two representations have the same quadratic Casimir eigenvalue (3.35) and are dual to each other as

$$\widetilde{V}^{p^+,p^-} \cong \left(V^{p^+,p^-}\right)^* \tag{3.58}$$

The $\star$-product is generically given as (3.55) or (3.57) with p^+ replaced by $|p^+|$, and the worldvolume algebra (3.40) is canonically isomorphic as a vector space to

$$\mathcal{A}\left(\mathcal{C}_{x_0^+,\chi_0}\right) = V^{p^+,p^-} \otimes \widetilde{V}^{p^+,p^-} \qquad\qquad p^+ > 0 \tag{3.59}$$

The $\mathcal{N}_6$-module structure is then determined by the Clebsch-Gordan decomposition of (3.59) into the irreducible continuous representations of the Nappi-Witten algebra as

$$\mathcal{A}\left(\mathcal{C}_{x_0^+,\chi_0}\right) = \bigoplus_0^\infty d\alpha\,\alpha \bigoplus_0^\infty d\beta\,\beta\, V^{0,0}_{\alpha,\beta} \tag{3.60}$$

The organisation of the quantised worldvolume algebra into irreducible representations associated with null branes owes to the fact that the isometry generators of the noncommutative D3-branes in NW_6 are given by the Killing vectors $P^\pm$, $Q^\pm$ and $J + \overline{J}$ in (1.27), 1.33 with $T = 0$, corresponding to translations in each transverse plane $z, w \in \mathbb{C}$ along with simultaneous rotations of the two planes. These isometries generate the subgroup (1.29) which coincides with the group $\mathrm{Inn}(\mathfrak{n}_6)$ of inner automorphisms of the Nappi-Witten Lie algebra. The symmetric untwisted D3-brane breaks the generic rotational symmetry $\mathrm{U}(2) = \mathrm{SU}(2) \times \mathrm{U}(1)$ to $\mathrm{U}(1) \cong \mathrm{SO}(2)$, leaving the overall isometry subgroup (1.29) which is precisely the symmetry group of noncommutative euclidean space in four dimensions with equal magnetic fields through each parallel plane [9, 27]. Thus the embedding of flat branes into NW_6 realises explicitly the usual breaking of $\mathrm{ISO}(4)$ invariance in passing to the noncommutative space $\mathbb{E}^4_\Theta$.

3.1.4 Open String Description

Let us now compare the semi-classical results obtained above with the predictions from the open string dynamics on the NS supported D-branes in the Seiberg-Witten decoupling limit. Generally, let G be the closed string metric on a D-brane and $\mathcal{F} = B + F$ the gauge-invariant two-form which we assume is non-degenerate. Whenever $d(\mathcal{F} - G\mathcal{F}^{-1}G) = 0$, the momentum of an open string attached to the D-brane is small in the low-energy limit [57]. This is just the requirement $H = 0$ of a vanishing NS flux in the limit of large B-field. The strings are then very short and see only a small portion of the worldvolume, which is approximately flat. The same expressions that apply to flat backgrounds can thereby be applied in these instances [5, 57]. In particular, the noncommutativity parameters and the open string metric may be computed from the usual Seiberg-Witten formulas [90]

$$\Theta = \frac{1}{\mathcal{F} - G\mathcal{F}^{-1}G} = -\frac{1}{G+\mathcal{F}}\mathcal{F}\frac{1}{G-\mathcal{F}} \tag{3.61}$$

$$G_{\rm o} = G - \mathcal{F}G^{-1}\mathcal{F} \tag{3.62}$$

For the flat euclidean D3-branes of the previous subsection, we substitute (3.30) and (3.33) in (3.61) to get the bi-vector

$$\Theta = -\frac{i}{(\mathcal{F}_{x_0^+})_{z\bar{z}} + (\mathcal{F}_{x_0^+})^{-1}_{z\bar{z}}}\overline{\boldsymbol{\partial}}^\top \wedge \boldsymbol{\partial} = -\frac{i}{2}\sin\theta x_0^+\overline{\boldsymbol{\partial}}^\top \wedge \boldsymbol{\partial} \tag{3.63}$$

This does not agree with (3.56) in the limit $\theta \to 0$, as one would have naively expected, but the reason for the discrepancy is simple. The semi-classical analysis of the previous subsection is strictly speaking valid only in the limit of large B-field, for which the formula (3.61) reduces to $\Theta = B^{-1}$ and coincides with (3.56) on the D3-branes. This is equivalent to the zero-slope field theory limit ($\alpha' \to 0$) of the open string dynamics, and it yields the Kirillov-Kostant Poisson bi-vector on the coadjoint orbits corresponding to the symplectic two-form B. This situation is characteristic of branes in curved backgrounds [5, 57], and we will regard the $\theta \to 0$ limit of (3.63), for which $\Theta = -\frac{i\theta p^+}{2}\overline{\boldsymbol{\partial}}^\top \wedge \boldsymbol{\partial}$, as "dual" to the field theoretic quantity (3.56) with respect to the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}_6$.

In [58] we find the intrinsic SU(2) symmetry of the NS background implies long strings, of light-cone momenta $p^+ > 1$, can move freely in the two transverse planes to the Nappi-Witten pp-wave and correspond to spectral-flowed null brane states [11]. The long strings in NW_6 thus correspond to fundamental string states. The spectral flow of long string states also implies that the strong NS-field limit gives a flat space theory [35], just like the semiclassical limit.

3.2 Twisted Noncommutative Branes in NW_6

The quantisation of twisted conjugacy classes is more intricate [4]. The ω-twisted D-branes in the Lie group $\mathcal{G}$ are labelled by representations of the invariant subgroup

$$\mathcal{G}^\omega := \{g \in \mathcal{G} \mid \omega(g) = g\} \tag{3.64}$$

In a neighbourhood of the identity of $\mathcal{G}$, a twisted conjugacy class $\mathcal{C}_g^\omega$ may be regarded as a fibration

$$\begin{array}{ccc} \breve{\mathcal{C}}_g & \longrightarrow & \mathcal{C}_g^\omega \\ & & \downarrow \\ & & \mathcal{G}/\mathcal{G}^\omega \end{array} \tag{3.65}$$

where $\breve{\mathcal{C}}_g$ is an ordinary conjugacy class of $\mathcal{G}^\omega$ and $\mathcal{G}^\omega$ acts on $\mathcal{G}$ by right multiplication. In particular, $\breve{\mathcal{C}}_g$ can be identified with a coadjoint orbit of the subgroup (3.64), with the standard linear Poisson structure coinciding with that induced by pull-back $B|_{\mathcal{G}^\omega}$ of the Neveu-Schwarz B-field when $H = 0$. Semiclassically, after quantisation $\mathcal{C}_g^\omega = \mathcal{G} \times_{\mathcal{G}^\omega} \breve{\mathcal{C}}_g$ becomes a trivial bundle with noncommutative fibers labelled by irreducible modules $\breve{V}_g^\omega$ over the group $\mathcal{G}^\omega$ but with a classical base space $\mathcal{G}/\mathcal{G}^\omega$ [4]. The associative noncommutative algebra of functions on the worldvolume in this case is thus given by

$$\mathcal{A}\left(\mathcal{C}_g^\omega\right) = \left(\mathrm{C}^\infty(\mathcal{G}) \otimes \mathrm{End}\big(\breve{V}_g^\omega\big)\right)^{\mathcal{G}^\omega} \tag{3.66}$$

where the superscript denotes the $\mathcal{G}^\omega$-invariant part and $\mathcal{G}^\omega$ acts on $\mathrm{C}^\infty(\mathcal{G})$ through the induced derivative action of right isometries of $\mathcal{G}$. Again, the worldvolume algebra (3.66) carries a natural action of the group $\mathcal{G}$, now through the induced action on $C^\infty(\mathcal{G})$ by right isometries.

If $\omega = \mathbb{1}$, then $\mathcal{G}^\omega = \mathcal{G}$ and (3.66) reduces to the definition (3.1). If the module $\check{V}_g^\omega$ is one-dimensional (e.g. the trivial representation of $\mathcal{G}^\omega$), then $\mathrm{End}(\check{V}_g^\omega) \cong \mathbb{C}$ and the algebra (3.66) becomes the commutative algebra of functions $\mathcal{A}(\mathcal{C}_g^\omega) \cong C^\infty(\mathcal{G}/\mathcal{G}^\omega)$ on $\mathcal{G}$ invariant under the action of the subgroup $\mathcal{G}^\omega \subset \mathcal{G}$ by right isometries. It is this latter situation that we shall discover in the present case. All twisted D-branes in NW_6 support *commutative* worldvolume geometries, given by the algebra of functions on the classical twisted conjugacy class, consistent with the remarks made in Section 2.3.2. This result will be supported by the fact that all twisted branes have vanishing NS flux $H = 0$, so that again semi-classical quantisation applies. However, for certain lorentzian D-branes that we shall encounter, there are some important subtleties hidden in the structure of the worldvolume two-form fields that are invisible in the field theoretic analysis outlined here.

3.2.1 General Construction

The group of outer automorphisms of the Lie algebra $\mathfrak{n}_6$ is given by $\mathrm{Out}(\mathfrak{n}_6) = \mathbb{Z}_2 \ltimes \mathrm{SU}(2)/\mathbb{Z}_2$ [87], where the SU(2) factor is the transverse space rotational symmetry described in the previous section. There are thus two families of outer automorphisms $\omega_\pm^S$ parametrised by a matrix

$$S = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2) \tag{3.67}$$

with $a, b \in \mathbb{C}$ obeying $|a|^2 + |b|^2 = 1$ and with the identification $S \sim -S$. Corresponding to the identity element of $\mathbb{Z}_2$, the automorphism ω_+^S acts on a group element (3.2) via a rotation in the transverse space as

$$\omega_+^S(g) = g\left(x^+, x^-, Sz, \overline{S}\overline{z}\right) \tag{3.68}$$

with $S \neq \mathbb{1}$ (or else the automorphism is trivial). As a consequence, the invariant subgroup (3.64) in this case is given by

$$\mathcal{N}_6^{\omega_+^S} = \left\{ g\left(x^+, x^-, \mathbf{0}, \mathbf{0}\right) \in \mathcal{N}_6 \mid x^\pm \in \mathbb{R} \right\} \cong \mathbb{R}^2 \tag{3.69}$$

This is the two-dimensional abelian group of translations of the light-cone in NW_6, and its irreducible representations are all one-dimensional. Thus the quantised worldvolume algebra (3.66) in this case is $\mathcal{A}(\mathcal{C}_g^{\omega_+^S}) \cong \mathrm{C}^\infty(\mathbb{E}^4)$ corresponding to *commutative* euclidean D3-branes. We shall explicitly construct these branes in what follows, showing that the group (3.69) is isomorphic to the stabiliser of the twisted conjugacy class and hence its points (corresponding to its conjugacy classes) label the locations of the various D3-branes in NW_6. This characterisation is consistent with the fact that only the (trivial) one-dimensional subspace of V^{p^+,p^-} spanned by the ground state vector $|0,0;p^+,p^-\rangle$ is invariant under the action of the automorphism (3.68).

The other class of automorphisms ω_-^S combines the transverse space rotations with the non-trivial element of $\mathbb{Z}_2$ which acts by charge conjugation on the generators of $\mathfrak{n}_6$ to give

$$\omega_-^S(g) = g\left(-x^+, -x^-, S\overline{\boldsymbol{z}}, \overline{S}\boldsymbol{z}\right) \tag{3.70}$$

It defines an isomorphism $V^{p^+,p^-} \leftrightarrow \widetilde{V}^{p^+,p^-}$ on irreducible representations. The invariant subgroup (3.64) in this case is determined by the equation $S\overline{\boldsymbol{z}} = \boldsymbol{z}$ for $\boldsymbol{z} \in \mathbb{C}^2$. The only solution of this equation is $\boldsymbol{z} = \boldsymbol{0}$, unless the parameter b in (3.67) is purely imaginary in which case the subgroup reduces to the two-dimensional abelian group generated by the elements

$$\mathsf{P}_S = a\mathsf{P}^+ + \mathsf{P}^- + b\mathsf{Q}^+ \qquad\qquad \mathsf{Q}_S = \overline{a}\mathsf{Q}^+ + \mathsf{Q}^- + b\mathsf{P}^+ \tag{3.71}$$

Thus

$$\mathcal{N}_6^{\omega_-^S} = \{g\,(0,0,\boldsymbol{z},\overline{\boldsymbol{z}}) \in \mathcal{N}_6 \mid \boldsymbol{z} = S\overline{\boldsymbol{z}} \in \mathbb{C}^2\} \cong \begin{cases} \mathbb{R}^2 & b \in i\mathbb{R} \\ \{\mathbb{1}\} & \text{otherwise} \end{cases} \tag{3.72}$$

is again always an abelian group (of transverse space translations) with only one-dimensional irreducible representations, corresponding to commutative worldvolume geometries (3.66). As we show explicitly below, the first instance corresponds to a class of commutative, Lorentzian D3-branes wrapping $\mathrm{CW}_4 \subset \mathrm{NW}_6$, i.e. $\mathcal{A}(\mathcal{C}_g^{\omega_-^S}) \cong \mathrm{C}^\infty(\mathrm{CW}_4)$, with the points of (3.72) again parametrising the locations of the branes in NW_6. The second case represents a family of commutative, spacetime filling lorentzian D5-branes each isometric to CW_6, i.e. $\mathcal{A}(\mathcal{C}_g^{\omega_-^S}) \cong \mathrm{C}^\infty(\mathcal{N}_6)$. These results are consistent with the fact that only the self-dual null brane modules $V_{\alpha,\beta}^{0,0}$ and $V_{\alpha,\beta}^{0,\frac{\theta}{2}}$ are invariant under the action of the automorphism (3.70).

As in Section 3.1.1, one can now straightforwardly work out the twisted adjoint actions $\mathrm{Ad}^{\omega_\pm^S}_{g(x^+,x^-,z,\overline{z})} g(x_0^+, x_0^-, z_0, \overline{z}_0)$ corresponding to a fixed point $(x_0^+, x_0^-, z_0) \in \mathrm{NW}_6$ [48, 87]. For the automorphism (3.68) the twisted conjugacy classes can be written explicitly as the submanifolds

$$\begin{aligned} \mathcal{C}^{\omega_+^S}_{(x_0^+,x_0^-,z_0)} = \Big\{ \big(& x_0^+, x_0^- + \frac{\theta}{2} \mathrm{Im}\big[\overline{z}_0^\top (e^{-\frac{i\theta}{2} x_0^+} S + e^{\frac{i\theta}{2} x_0^+} \mathbb{1}) z e^{-\frac{i\theta}{2} x^+} \\ & + \frac{1}{2} z^\top (e^{-i\theta x_0^+} S - e^{i\theta x_0^+} \mathbb{1}) \overline{z} \big], \\ & e^{\frac{i\theta}{2}(x^+ - x_0^+)} (S - e^{i\theta x_0^+} \mathbb{1}) z + e^{i\theta x^+} z_0 \big) \ \Big| \ x^+ \in \mathrm{S}^1, z \in \mathbb{C}^2 \Big\} \end{aligned} \tag{3.73}$$

Again x_0^+ is an orbit invariant, and the corresponding branes are thus euclidean. In fact, except for the nature of their noncommutative worldvolume geometry, these branes are completely analogous to the branes described in the previous section. For the automorphism (3.70)) one finds instead the submanifolds

$$\begin{aligned} \mathcal{C}^{\omega_-^S}_{(x_0^+,x_0^-,z_0)} = \Big\{ \big(& x_0^+ + 2x^+, x_0^- + 2x^- - \theta \mathrm{Im}\big[\overline{z}_0^\top z e^{\frac{i\theta}{2}(x_0^+ + x^+)} \\ & - (\overline{z}_0^\top + e^{\frac{i\theta}{2}(x^+ + x_0^+)} \overline{z}^\top) S \overline{z} e^{\frac{i\theta}{2}(x^+ + x_0^+)} \big], \\ & z_0 - S\overline{z} e^{-\frac{i\theta}{2}(x^+ + x_0^+)} + z e^{\frac{i\theta}{2}(x^+ + x_0^+)} \big) \ \Big| \ x^+ \in \mathrm{S}^1, x^- \in \mathbb{R}, z \in \mathbb{C}^2 \Big\} \end{aligned} \tag{3.74}$$

wrapped by lorentzian branes. We will now briefly describe the supergravity fields supported by each of these classes of branes.

3.2.2 Euclidean D3-Branes

The geometry of the twisted conjugacy classes (3.73) is determined by the complex map on $\mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$z' := e^{\frac{i\theta}{2}(x^+ - x_0^+)} \Big(S - e^{i\theta x_0^+} \mathbb{1} \Big) z + e^{i\theta x^+} z_0 \tag{3.75}$$

Whenever the 2×2 matrix $S - e^{i\theta x_0^+}\mathbb{1}$ is invertible, this map defines an isomorphism of linear spaces and one has $\mathcal{C}^{\omega_+^S}_{(x_0^+, x_0^-, z_0)} \cong \mathbb{E}^4$ with the same worldvolume fields as in (3.30) and (3.31), the x^+-dependence of (3.73) cancelling out. However, although they are qualitatively the same as the euclidean D3-branes of Section 3.1.3, the worldvolume two-forms on these branes are different. The stabiliser of the twisted conjugacy class in this case is parametrised as the cylindrical submanifold

$$\mathcal{Z}^{\omega_+^S}_{(x_0^+, x_0^-, z_0)} = \left\{ \left(x^+, x^-, e^{\frac{i\theta}{2}(x_0^+ - x^+)}(1 - e^{i\theta x^+})(S - e^{i\theta x_0^+}\mathbb{1})^{-1} z_0 \right) \,\middle|\, \begin{array}{l} x^+ \in \mathrm{S}^1 \\ x^- \in \mathbb{R} \end{array} \right\} \tag{3.76}$$

from which we may compute the abelian gauge field fluxes (2.27) to be

$$F_6^{(\zeta)} \,\big|_{\mathcal{C}^{\omega_+^S}_{(x_0^+, x_0^-, z_0)}} = 2 i x_0^+ \, d\overline{z}^\top \wedge dz \tag{3.77}$$

Thus the gauge-invariant two-forms $\mathcal{F}_6^{(\zeta)}$ vanish on these twisted conjugacy classes, leading to the anticipated commutative worldvolume geometry.

We can also understand the origin of these branes through the Penrose-Güven limit of Section 2.3.1 by considering the commuting isometric embedding diagram

$$\begin{array}{ccc} \mathrm{AdS}_3 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_6 \\ {\scriptstyle \imath'}\big\uparrow & & \big\uparrow {\scriptstyle \widetilde{\imath}'} \\ \mathrm{H}^2 \times \mathrm{S}^2 & \xrightarrow{\mathrm{PGL}'} & \mathbb{E}^4 \end{array} \tag{3.78}$$

analogous to (3.34). The hyperbolic plane H^2 is wrapped by symmetric instantonic D-strings in AdS_3 [97], corresponding to conjugacy classes of the group $\mathrm{SU}(1,1)$, and it is obtained from the intersection of the three-dimensional hyperboloid (2.33) with the affine hyperplane $x^1 = \tau = 0$. Now the pull-back of the B-field (2.38) to the H^2 worldvolume vanishes, leading to a vanishing gauge-invariant two-form.

3.2.3 Spacetime Filling D5-Branes

Let us now examine the twisted conjugacy classes (3.74). By defining $y^+ := x_0^+ + 2x^+$ and $w := e^{\frac{i\theta}{4}(y^+ + x_0^+)} z$ we may express them as the submanifolds

$$\mathcal{C}^S_{z_0} = \left\{ (y^+, y^-, z_0 - S\overline{w} + w) \,\middle|\, y^+ \in \mathrm{S}^1, y^- \in \mathbb{R}, w \in \mathbb{C}^2 \right\} \tag{3.79}$$

The geometry of the twisted conjugacy class (3.79) is thereby determined by the *real* linear transformation on $\mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$\boldsymbol{w}' := S\overline{\boldsymbol{w}} - \boldsymbol{w} \tag{3.80}$$

whose determinant is straightforwardly worked out to be $4(\mathrm{Re}b)^2$. As long as the parameter b in (3.67) has a non-zero real part, the map (3.80) is a linear isomorphism and we can write the induced geometry in terms of the new transverse space coordinates $\boldsymbol{w}' \in \mathbb{C}^2$. After an irrelevant shift, the pull-back of the plane wave metric (1.30) to the twisted conjugacy class (3.79) is given by

$$ds_6^2 \,|_{\mathcal{C}^S_{z_0}} = 2dy^+dy^- + |d\boldsymbol{w}'|^2 - \frac{\theta^2}{4}\,|\boldsymbol{w}'|^2 \left(dy^+\right)^2 \tag{3.81}$$

and the corresponding worldvolumes are therefore wrapped by spacetime filling D-branes isometric to CW_6. From the identity

$$d\overline{\boldsymbol{w}}'^\top \wedge d\boldsymbol{w}' = d\overline{\boldsymbol{w}}^\top \wedge \left(\mathbb{1} - \overline{S}^\top S\right) d\boldsymbol{w} \tag{3.82}$$

and the unitarity of the matrix $S \in \mathrm{SU}(2)$ it follows that the pull-back of the NS background (1.31) is trivial,

$$H_6 \,|_{\mathcal{C}^S_{z_0}} = 0 = B_6 \,|_{\mathcal{C}^S_{z_0}} \tag{3.83}$$

Furthermore, since the corresponding stabiliser in this case is euclidean, having $x^+ = 0$, it is elementary to see that the pull-backs of the gauge-invariant two-forms also vanish,

$$\mathcal{F}_6^{(\zeta)} \,|_{\mathcal{C}^S_{z_0}} = 0 \tag{3.84}$$

In perfect agreement with the general analysis of Section 3.2.1, these D5-branes thus all carry commutative worldvolume geometries.

As described in Section 2.3.1, these D-branes originate through the Penrose limit of D5-branes isometric to $\mathrm{AdS}_3 \times \mathrm{S}^3$. The latter do *not* correspond to symmetric D-branes of the $\mathrm{SU}(1,1) \times \mathrm{SU}(2)$ group and, unlike the symmetric D5-branes obtained here which are spacetime filling branes of the six-dimensional plane wave, they do not fill the whole $\mathrm{AdS}_3 \times \mathrm{S}^3$ target space. They wrap only a three-dimensional submanifold of the S^3 component given by the complement in S^3 of the disjoint union of two circles [97]. While the Penrose limit de-compactifies the branes to spacetime filling ones, the Neveu-Schwarz background vanishes due to the non-trivial embedding $\imath$ represented through the isometric embedding diagram

$$\begin{array}{ccc} \mathrm{AdS}_3 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_6 \\ \imath \Big\uparrow & & \Big\uparrow \widetilde{\imath} \\ \mathrm{AdS}_3 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{CW}_6 \end{array} \tag{3.85}$$

with $\widetilde{\imath}$ constructed as above.

3.2.4 Lorentzian D3-Branes

We now describe the situations wherein the linear maps defined above are not of maximal rank, beginning with (3.80). In this case the parameter $b \in i\mathbb{R}$ is purely imaginary, and the map has real rank 2 so that the variable $\boldsymbol{w}'^\top = (w_1', w_2')$ lives in a one-dimensional complex subspace of $\mathbb{C}^2$. When $\mathrm{Re}a = a = 1$, one has $b = 0$ and $\boldsymbol{w}' \in i\mathbb{R}^2$. Defining $z \in \mathbb{C}$ by $z := 2\mathrm{Im}w_1 + 2i\mathrm{Im}w_2$, it follows that $|\boldsymbol{w}'| = |z|$ and the metric (3.81) coincides with the usual Cahen-Wallach geometry (1.12) of CW_4. When $\mathrm{Re}a < 1$, we may coordinatise the twisted conjugacy class (3.79) by $\boldsymbol{w} \in \mathbb{R}^2$. Then by defining

$$z := \sqrt{2(1-\mathrm{Re}a)}\left(w_1 - \frac{b}{2(1-\mathrm{Re}a)}w_2\right) + i\sqrt{2(1-\mathrm{Re}a) - \frac{b^2}{2(1-\mathrm{Re}a)}}\,w_2 \tag{3.86}$$

one finds $|\boldsymbol{w}'| = |z|$ and (3.81) again reduces to the form (1.12). The identity (3.82) once more implies the vanishing (3.83) of the NS background, so that the twisted conjugacy classes (3.79) are wrapped by curved lorentzian D3-branes isometric to CW_4. All of this is in perfect harmony with the general findings of Section 3.2.1.

The $\mathrm{AdS}_3 \times \mathrm{S}^3$ origin of these D3-branes is described by (2.48). However, now an apparent discrepency arises. Using (3.82) and the fact that the matrix (3.67) is symmetric whenever $b \in i\mathbb{R}$, one finds a non-vanishing pull-back of the worldvolume flux (2.46) given by

$$\mathcal{F}_6 \,|_{C^S_{z_0}} = -\frac{i\theta}{2}\cot^2\frac{\theta y^+}{2} dy^+ \wedge \left(\boldsymbol{z}_0^\top d\overline{\boldsymbol{w}}' - \overline{\boldsymbol{z}}_0^\top d\boldsymbol{w}'\right) \tag{3.87}$$

This worldvolume two-form depends on the locations of the branes in NW_6, and it should induce a spacetime noncommutative geometry on the D3-brane worldvolume. However, this flux corresponds to a worldvolume *electric* field, rather than a magnetic field, and D-branes in electric backgrounds do not exhibit a well-defined decoupling of the massive string states. Instead of field theories, such backgrounds lead to noncommutative open string theories [51, 89] which lie beyond the scope of the semi-classical analysis at the beginning of this section. Such a noncommutativity agrees with the non-trivial boundary three-point couplings [36] on the symmetric lorentzian D-membranes [48] wrapping the pp-waves $\mathrm{CW}_3 \hookrightarrow \mathrm{NW}_4$ defined by the volumes $\mathrm{Im}z =$ constant in the above. Given the origin of the null worldvolume flux (3.87) from Section 2.3.1 and the mechanism described after (3.34), this noncommutative open string theory could provide a curved space analog of the expected flat space S-duality with (p, q) strings.

3.2.5 D-Strings

Let us now turn to the degenerate cases of the map (3.75). The vanishing of the determinant of the 2×2 matrix $S - e^{i\theta x_0^+}\mathbb{1}$ shows that these branes arise when the fixed light-cone time coordinate obeys the equation

$$\cos\theta x_0^+ = \mathrm{Re}a \tag{3.88}$$

Since $|a| < 1$ in this case, these worldvolumes lie in the set of conjugate free points $x_0^+ \neq 0, \frac{\pi}{\theta}$ of the Rosen plane wave geometry and $S - e^{i\theta x_0^+}\mathbb{1}$ is of complex rank 1. Explicitly, with $\boldsymbol{z}^\top = (z, w)$ and $\xi \in \mathbb{C}$ defined by $\xi := bw - i(\sin\theta x_0^+ - \mathrm{Im}a)z$, one finds using (3.88) that

$$\left(S - e^{i\theta x_0^+}\mathbb{1}\right)\boldsymbol{z} = \begin{pmatrix} \xi \\ \frac{i\bar{b}}{\sin\theta x_0^+ - \mathrm{Im}a}\,\xi \end{pmatrix} \tag{3.89}$$

If $z_0 = (S - e^{i\theta x_0^+}\mathbb{1})w_0$ for some fixed $w_0 \in \mathbb{C}^2$, then a simple coordinate redefinition in (3.75) enables us to parametrise the twisted conjugacy class by a single complex variable. The pull-back of the NW_6 metric is non-degenerate and the classes in this case are wrapped by euclidean D1-branes. As in Section 3.2.2, all worldvolume fields in this case are easily found to be trivial. These branes do not explicitly appear in the general analysis of Section 3.2.1, and like the null branes of Section 3.1.2 their quantised worldvolume geometry, although commutative, differs from the classical one. They can nevertheless be regarded as subclasses of the twisted euclidean D3-branes constructed in Section 3.2.2. In particular, they originate from symmetric euclidean D-strings in $AdS_3 \times S^3$, either wrapping $H^2 \subset AdS_3$ and sitting at a point in S^3 or wrapping $S^2 \subset S^3$ and sitting at a point in AdS_3.

3.2.6 D-Membranes

Finally, if $z_0 \notin \mathrm{im}(S - e^{i\theta x_0^+}\mathbb{1})$ on $\mathbb{C}^2$, then from (3.75) it follows that the twisted conjugacy class (3.73) is isometric to $S^1 \times \mathbb{E}^2$. The metric ds_6^2 restricts non-degenerately, and once again all worldvolume form fields are trivial. The orbits are thus wrapped by euclidean D2-branes, which can likewise be regarded as subclasses of the D3-branes in Section 3.2.2. Like the spacetime filling D-branes of Section 3.2.3, these branes do not originate from symmetric D-branes in $AdS_3 \times S^3$, but rather from either of the trivially embedded AdS_3 or S^3 submanifolds.

3.3 The Dolan-Nappi Model

Introducing to NW_6 the one-form

$$\Lambda := -i\left(\theta^{-1}x_0^- + \theta x^+\right)\left(z^\top d\overline{z} - \overline{z}^\top dz\right) \tag{3.90}$$

on the null hypersurfaces of constant $x^- = x_0^-$, we may compute the corresponding two-form gauge transformation of the B-field in (1.31) to get

$$B_6^\Lambda := B_6 + d\Lambda = -i\theta dx^+ \wedge \left(z^\top d\overline{z} - \overline{z}^\top dz\right) + 2i\theta^{-1}x_0^- d\overline{z}^\top \wedge dz \tag{3.91}$$

With $x_0^+ = 0$ and restricted to the four-dimensional hypersurface defined by $\boldsymbol{z}^\top = (z, 0)$, the metric (1.12) and NS potential (3.91) coincide with those of the Dolan-Nappi model [38] describing a (non-symmetric) D3-brane with the complete NS-supported geometry of NW_4.

In [56] this geometry is realised as a null Melvin twist of a flat commutative D3-brane with twist parameter $\frac{\theta}{2}$ (in string units $\alpha' = 1$), leading to the Melvin universe with a boost. Despite the non-vanishing null NS three-form flux of NW_4, it can be argued from this realisation that the usual flat space Seiberg-Witten formulae (3.61), (3.62) hold in this closed string background with $F = 0$ a consistent solution to the corresponding Dirac-Born-Infeld equations of motion. The isometry with respect to which the background is twisted corresponds to the R-symmetry of the D3-brane worldvolume field theory, which thereby becomes a non-local theory of dipoles whose length is proportional to the R-charge. The open string metric (3.62) correctly captures the non-local dipole-like open string dynamics on the D3-brane.

Extrapolating this argument to $x_0^- \neq 0$ and to the full six-dimensional spacetime NW_6, a straightforward calculation gives the Seiberg-Witten bi-vector (3.61) with $F = 0$ for the background (1.30), (3.91) as

$$\Theta^\Lambda = -\frac{2i\theta}{\theta^2 + \left(x_0^-\right)^2} \left[\theta^2 \partial_- \wedge \left(\boldsymbol{z}^\top \boldsymbol{\partial} - \overline{\boldsymbol{z}}^\top \overline{\boldsymbol{\partial}}\right) + 4x_0^- \boldsymbol{\partial}^\top \wedge \overline{\boldsymbol{\partial}}\right] \tag{3.92}$$

while the corresponding open string metric (3.62) is given by

$$G_{\mathrm{o}}^\Lambda = 2dx^+dx^- + \frac{\theta^2 + \left(x_0^-\right)^2}{\theta^2} |d\boldsymbol{z}|^2 + 2ix_0^- \left(\boldsymbol{z}^\top d\overline{\boldsymbol{z}} - \overline{\boldsymbol{z}}^\top d\boldsymbol{z}\right) dx^+ \tag{3.93}$$

Since (3.92) is degenerate on the whole NW_6 spacetime, it does not define a symplectic structure. Generally, if the components of a bi-vector $\Theta := \Theta^{ij} \partial_i \wedge \partial_j$ obey

$$\Theta^{il} \partial_l \Theta^{jk} + \Theta^{jl} \partial_l \Theta^{ki} + \Theta^{kl} \partial_l \Theta^{ij} = 0 \tag{3.94}$$

for all i, j, k, then Θ defines a Poisson structure, i.e. it is a Poisson bi-vector and (3.94) is equivalent to the Jacobi identity for the corresponding Poisson brackets. If in addition Θ is invertible, then (3.94) is equivalent to the required closure condition $d(\Theta^{-1}) = 0$ for a symplectic two-form. It is easily checked that (3.92) satisfies

(3.94) and hence that it defines a Poisson bi-vector. In the flat space limit $x_0^- \to 0$ of (3.93), the corresponding quantisation of NW_6 is given by the associative Kontsevich $\star$-product [69] in this case.

The important feature of the noncommutativity parameter (3.92) is that it is time independent, though non-constant. If we think of the light-cone position x^- as being dual to the Nappi-Witten generator J, then the form of (3.92) agrees with its representation in (3.26) and (3.51). On the other hand, the calculation of [38] provides evidence for a time-dependent Poisson bi-vector in the original closed string background (1.31). To make this precise, however, one would require a detailed understanding of the worldvolume stabilising flux on the NW_6 brane, which is difficult to determine for non-symmetric D-branes. The noncommutativity parameter and open string metric in the decoupling limit of D5-branes in Nappi-Witten spacetime NW_6 are thereby presently given by (3.92) and (3.93). In particular, at the special value $x_0^- = \theta$ and with the rescaling $\boldsymbol{z} \to \sqrt{2/\theta\tau}\,\boldsymbol{z}$, the metric (3.93) becomes that of CW_6 in global coordinates analogous to (1.11), while the non-vanishing Poisson brackets corresponding to (3.92) read

$$
\begin{aligned}
\{z_a, \overline{z}_b\} &= 2i\theta\tau\delta_{ab} \\
\{x^-, z_a\} &= -i\theta z_a \\
\{x^-, \overline{z}_a\} &= i\theta\overline{z}_a
\end{aligned}
\tag{3.95}
$$

for $a, b = 1, 2$. The Poisson algebra thereby coincides with the Nappi-Witten Lie algebra $\mathfrak{n}_6$ in this case and the metric on the branes with the standard curved geometry of the pp-wave. In the semi-classical flat space limit $\theta \to 0$, the quantisation of the brackets (3.95) thereby yields a noncommutative worldvolume geometry on D5-branes wrapping NW_6 which can be associated with a quantisation of $\mathfrak{n}_6$ (or more precisely of its dual $\mathfrak{n}_6^*$).

With a slight abuse of notation, we will denote the central coordinate τ as the plane wave time coordinate x^+. Our semi-classical quantisation will then be valid in the small time limit $x^+ \to 0$.

Chapter 4

$\star$-Products, Derivatives and Integrals

4.1 Quantisation

Our starting point in describing the noncommutative geometry of NW_6 will be at the algebraic level. We will consider the deformation quantisation of the dual $\mathfrak{n}^\vee$ to the Lie algebra $\mathfrak{n}$. Naively, one may think that the easiest way to carry this out is to compute $\star$-products on the pp-wave by taking the Penrose limits of the standard ones on S^3 and AdS_3 (or equivalently by contracting the standard quantisations of the Lie algebras su(2) and sl$(2,\mathbb{R})$). However, some quick calculations show that the induced $\star$-products obtained in this way are divergent in the infinite volume limit, and the reason why is simple. While the standard Inönü-Wigner contractions hold at the level of the Lie algebras [87], they need not necessarily map the corresponding universal enveloping algebras, on which the quantisations are performed. This is connected to the phenomenon that twisted conjugacy classes of branes are not necessarily related by the Penrose-Güven limit [58]. We must therefore resort to a more direct approach to quantising the spacetime NW_6.

For notational ease, we will write the algebra $\mathfrak{n}$ in the generic form

$$[\mathsf{X}_a, \mathsf{X}_b] = i\theta C^c_{ab} \mathsf{X}_c \tag{4.1}$$

where $(\mathsf{X}_a) := \theta(\mathsf{J}, \mathsf{T}, \mathsf{P}^i_\pm)$ are the generators of $\mathfrak{n}$ and the structure constants C^c_{ab} can be gleamed off from (1.5). The algebra (4.1) can be regarded as a formal deformation quantisation of the Kirillov-Kostant Poisson bracket on $\mathfrak{n}^\vee$ in the standard coadjoint orbit method. Let us identify $\mathfrak{n}^\vee$ as the vector space $\mathbb{R}^6$ with basis $\mathsf{X}_a^\vee := \langle \mathsf{X}_a, - \rangle : \mathfrak{n} \to \mathbb{R}$ dual to the X_a. In the algebra of polynomial functions $\mathbb{C}(\mathfrak{n}^\vee) = \mathbb{C}(\mathbb{R}^6)$, we may then identify the generators X_a themselves with the coordinate functions

$$\begin{aligned}
\mathsf{X}_{\mathsf{J}}(\boldsymbol{x}) &= x_{\mathsf{T}} = x^- \\
\mathsf{X}_{\mathsf{T}}(\boldsymbol{x}) &= x_{\mathsf{J}} = x^+ \\
\mathsf{X}_{\mathsf{P}^i_+}(\boldsymbol{x}) &= 2x_{\mathsf{P}^i_-} = 2\bar{z}_i \\
\mathsf{X}_{\mathsf{P}^i_-}(\boldsymbol{x}) &= 2x_{\mathsf{P}^i_+} = 2z_i
\end{aligned} \tag{4.2}$$

for any $\boldsymbol{x} \in \mathfrak{n}^\vee$ with component x_a in the $\mathsf{X}_a^\vee$ direction. These functions generate the whole coordinate algebra and their Poisson bracket Θ is defined by

$$\Theta(\mathsf{X}_a, \mathsf{X}_b)(\boldsymbol{x}) = \boldsymbol{x}\big([\mathsf{X}_a, \mathsf{X}_b]\big) \qquad \forall \boldsymbol{x} \in \mathfrak{n}^\vee \tag{4.3}$$

Therefore, when viewed as functions on $\mathbb{R}^6$ the Lie algebra generators have a Poisson bracket given by the Lie bracket, and their quantisation is provided by (4.1) with deformation parameter θ. We will explore various aspects of this quantisation and derive several (equivalent) star products on $\mathfrak{n}^\vee$.

4.2 Gutt Products

The formal completion of the space of polynomials $\mathbb{C}(\mathfrak{n}^\vee)$ is the algebra $C^\infty(\mathfrak{n}^\vee)$ of smooth functions on $\mathfrak{n}^\vee$. There is a natural way to construct a $\star$-product on the cotangent bundle $T^\vee\mathcal{N} \cong \mathcal{N} \times \mathfrak{n}^\vee$, which naturally induces an associative product on $C^\infty(\mathfrak{n}^\vee)$. This induced product is called the Gutt product. The Poisson bracket defined by (4.3) naturally extends to a Poisson structure $\Theta : C^\infty(\mathfrak{n}^\vee) \times C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$ defined by the Kirillov-Kostant bi-vector

$$\Theta = \frac{1}{2} C_{ab}^{c} x_c \partial^a \wedge \partial^b \tag{4.4}$$

where $\partial^a := \frac{\partial}{\partial x_a}$. The Gutt product constructs a quantisation of this Poisson structure. It is equivalent to the Kontsevich $\star$-product in this case [45], and by construction it keeps that part of the Kontsevich formula which is associative [93]. In general, within the present context, the Gutt and Kontsevich deformation quantisations are only identical for nilpotent Lie algebras.

The algebra $\mathbb{C}(\mathfrak{n}^\vee)$ of polynomial functions on the dual to the Lie algebra is naturally isomorphic to the symmetric tensor algebra $S(\mathfrak{n})$ of $\mathfrak{n}$. By the Poincaré-Birkhoff-Witt theorem, there is a natural isomorphism $\Omega : S(\mathfrak{n}) \to U(\mathfrak{n})$ with the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$. Using the above identifications, this extends to a canonical isomorphism

$$\Omega : \mathrm{C}^\infty\left(\mathbb{R}^6\right) \longrightarrow \overline{U(\mathfrak{n})^{\mathbb{C}}} \tag{4.5}$$

defined by specifying an ordering for the elements of the basis of monomials for $S(\mathfrak{n})$, where $\overline{U(\mathfrak{n})^{\mathbb{C}}}$ denotes a formal completion of the complexified universal enveloping algebra $U(\mathfrak{n})^{\mathbb{C}} := U(\mathfrak{n}) \otimes \mathbb{C}$. Denoting this ordering by $\circ\!\!\circ - \circ\!\!\circ$, we may write this isomorphism symbolically as

$$\Omega(x_{a_1} \cdots x_{a_n}) = \circ\!\!\circ \mathsf{X}_{a_1} \cdots \mathsf{X}_{a_n} \circ\!\!\circ \tag{4.6}$$

The original Gutt construction [53] takes the isomorphism Ω on $S(\mathfrak{n})$ to be symmetrisation of monomials. In this case $\Omega(f)$ is usually called the Weyl symbol of $f \in \mathrm{C}^\infty(\mathbb{R}^6)$ and the symmetric ordering $\circ\!\!\circ - \circ\!\!\circ$ of symbols $\Omega(f)$ is called Weyl ordering. In the following we shall work with three natural orderings appropriate to the algebra $\mathfrak{n}$.

The isomorphism (4.5) can be used to transport the algebraic structure on the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$ to the algebra of smooth functions on $\mathfrak{n}^\vee \cong \mathbb{R}^6$ and give the $\star$-product defined by

$$f \star g := \Omega^{-1}\left(\circ\!\!\circ \Omega(f) \cdot \Omega(g) \circ\!\!\circ\right) \qquad f, g \in \mathrm{C}^\infty\left(\mathbb{R}^6\right) \tag{4.7}$$

The product on the right-hand side of the formula (4.7) is taken in $U(\mathfrak{n})$, and it follows that $\star$ defines an associative, noncommutative product. Moreover, it represents a deformation quantisation of the Kirillov-Kostant Poisson structure on $\mathfrak{n}^\vee$, in the sense that

$$[x,y]_\star := x\star y - y\star x = i\theta\Theta(x,y) \qquad\qquad x,y\in\mathbb{C}_{(1)}\left(\mathfrak{n}^\vee\right) \tag{4.8}$$

where $\mathbb{C}_{(1)}(\mathfrak{n}^\vee)$ is the subspace of homogeneous polynomials of degree 1 on $\mathfrak{n}^\vee$. In particular, the Lie algebra relations (4.1) are reproduced by $\star$-commutators of the coordinate functions as

$$[x_a,x_b]_\star = i\theta C^c_{ab}x_c \tag{4.9}$$

in accordance with the Poisson brackets (3.95) and the definition (4.3).

Let us now describe how to write the $\star$-product (4.7) explicitly in terms of a bi-differential operator $\hat{\mathcal{D}} : C^\infty(\mathfrak{n}^\vee)\times C^\infty(\mathfrak{n}^\vee)\to C^\infty(\mathfrak{n}^\vee)$ [68]. Using the Kirillov-Kostant Poisson structure as before, we identify the generators of $\mathfrak{n}$ as coordinates on $\mathfrak{n}^\vee$. This establishes, for small $s\in\mathbb{R}$, a one-to-one correspondence between group elements $e^{s\mathsf{X}}$, $\mathsf{X}\in\mathfrak{n}$ and functions e^{sx} on $\mathfrak{n}^\vee$. Pulling back the group multiplication of elements $e^{s\mathsf{X}}\in\mathcal{N}$ via this correspondence induces a bi-differential operator $\hat{\mathcal{D}}$ acting on the functions e^{sx}. Since these functions separate the points on $\mathfrak{n}^\vee$, this extends to an operator on the whole of $C^\infty(\mathfrak{n}^\vee)$.

To apply this construction explicitly, we use the following trick [73, 20] which will also be useful for later considerations. By restricting to an appropriate Schwartz subspace of functions $f\in C^\infty(\mathbb{R}^6)$, we may use a Fourier representation

$$f(\boldsymbol{x}) = \int\limits_{\mathbb{R}^6}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^6}\tilde{f}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{4.10}$$

This establishes a correspondence between (Schwartz) functions on $\mathfrak{n}^\vee$ and elements of the complexified group $\mathcal{N}^{\mathbb{C}} := \mathcal{N}\otimes\mathbb{C}$. The products of symbols $\Omega(f)$ may be computed using (4.6), and the $\star$-product (4.7) can be represented in terms of a product of group elements in $\mathcal{N}^{\mathbb{C}}$ as

$$f\star g = \int\limits_{\mathbb{R}^6}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^6}\int\limits_{\mathbb{R}^6}\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^6}\tilde{f}(\boldsymbol{k})\tilde{g}(\boldsymbol{q})\Omega^{-1}\left({}^{\circ\circ}_{\circ\circ}e^{ik^a\mathsf{X}_a}{}^{\circ}_{\circ}\cdot{}^{\circ}_{\circ}e^{iq^a\mathsf{X}_a}{}^{\circ\circ}_{\circ\circ}\right) \tag{4.11}$$

Using the Baker-Campbell-Hausdorff formula, to be discussed below, we may write

$${}^{\circ\circ}_{\circ\circ}e^{ik^a\mathsf{X}_a}{}^{\circ}_{\circ}\cdot{}^{\circ}_{\circ}e^{iq^a\mathsf{X}_a}{}^{\circ\circ}_{\circ\circ} = {}^{\circ}_{\circ}e^{iD^a(\boldsymbol{k},\boldsymbol{q})\mathsf{X}_a}{}^{\circ}_{\circ} \tag{4.12}$$

for some function $\boldsymbol{D} = (D^a) : \mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}^6$. This enables us to rewrite the $\star$-product (4.11) in terms of a bi-differential operator $f \star g := \hat{\mathcal{D}}(f, g)$ given explicitly by

$$f \star g = f e^{ix \cdot [\boldsymbol{D}(-i \overleftarrow{\boldsymbol{\partial}}, -i \overrightarrow{\boldsymbol{\partial}}) + i \overleftarrow{\boldsymbol{\partial}} + i \overrightarrow{\boldsymbol{\partial}}]} g \tag{4.13}$$

with $\boldsymbol{\partial} := (\partial^a)$. In particular, the $\star$-products of the coordinate functions themselves may be computed from the formula

$$x_a \star x_b = - \frac{\partial}{\partial k^a} \frac{\partial}{\partial q^b} e^{i \boldsymbol{D}(\boldsymbol{k}, \boldsymbol{q}) \cdot x} \Big|_{\boldsymbol{k} = \boldsymbol{q} = \boldsymbol{0}} \tag{4.14}$$

Finally, let us describe how to explicitly compute the functions $D^a(\boldsymbol{k}, \boldsymbol{q})$ in (4.12). For this, we consider the Dynkin form of the Baker-Campbell-Hausdorff formula which is given for $\mathsf{X}, \mathsf{Y} \in \mathfrak{n}$ by

$$e^{\mathsf{X}} e^{\mathsf{Y}} = e^{\mathrm{H}(\mathsf{X}:\mathsf{Y})} \tag{4.15}$$

where $\mathrm{H}(\mathsf{X} : \mathsf{Y}) = \sum_{n \geq 1} \mathrm{H}_n(\mathsf{X} : \mathsf{Y}) \in \mathfrak{n}$ is generically an infinite series whose terms may be calculated through the recurrence relation

$$\begin{aligned} (n+1) \mathrm{H}_{n+1}(\mathsf{X} : \mathsf{Y}) &= \frac{1}{2} \big[\mathsf{X} - \mathsf{Y}, \mathrm{H}_n(\mathsf{X} : \mathsf{Y}) \big] \\ &\quad + \sum_{p=1}^{\lfloor n/2 \rfloor} \frac{B_{2p}}{(2p)!} \sum_{\substack{k_1, \dots, k_{2p} > 0 \\ k_1 + \dots + k_{2p} = n}} \big[\mathrm{H}_{k_1}(\mathsf{X} : \mathsf{Y}), \big[\dots, \big[\mathrm{H}_{k_{2p}}(\mathsf{X} : \mathsf{Y}), \mathsf{X} + \mathsf{Y} \big] \dots \big] \big] \end{aligned} \tag{4.16}$$

with $\mathrm{H}_1(\mathsf{X} : \mathsf{Y}) := \mathsf{X} + \mathsf{Y}$. The coefficients B_{2p} are the Bernoulli numbers which are defined by the generating function

$$\frac{s}{1 - e^{-s}} - \frac{s}{2} - 1 = \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} s^{2p} \tag{4.17}$$

The first few terms of the formula (4.15) may be written explicitly as

$$\begin{aligned} \mathrm{H}_1(\mathsf{X} : \mathsf{Y}) &= \mathsf{X} + \mathsf{Y} \\ \mathrm{H}_2(\mathsf{X} : \mathsf{Y}) &= \frac{1}{2} [\mathsf{X}, \mathsf{Y}] \\ \mathrm{H}_3(\mathsf{X} : \mathsf{Y}) &= \frac{1}{12} \big[\mathsf{X}, [\mathsf{X}, \mathsf{Y}] \big] - \frac{1}{12} \big[\mathsf{Y}, [\mathsf{X}, \mathsf{Y}] \big] \\ \mathrm{H}_4(\mathsf{X} : \mathsf{Y}) &= - \frac{1}{24} \big[\mathsf{Y}, \big[\mathsf{X}, [\mathsf{X}, \mathsf{Y}] \big] \big] \end{aligned} \tag{4.18}$$

Terms in the series grow increasingly complicated due to the sum over partitions in (4.16), and in general there is no closed symbolic form, as in the case of the Moyal product based on the ordinary Heisenberg algebra, for the functions $D^a(\boldsymbol{k}, \boldsymbol{q})$ appearing in (4.12). However, at least for certain ordering prescriptions, the solvability of the Lie algebra $\mathfrak{n}$ enables one to find explicit expressions for the $\star$-product (4.13) in this fashion. We will now proceed to construct three such products.

4.2.1 Time Ordering

The simplest Gutt product is obtained by choosing a "time ordering" prescription in (4.6) whereby all factors of the time translation generator J occur to the far right in any monomial in $U(\mathfrak{n})$. It coincides precisely with the global coordinatisation (1.9) of the Cahen-Wallach spacetime, and written on elements of the complexified group $\mathcal{N}^{\mathbb{C}}$ it is defined by

$$\Omega_* \left(e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right) = {}^*_*e^{ik^a\mathsf{X}_a}{}^*_* := e^{i(p_i^+\mathsf{P}^i_+ + p_i^-\mathsf{P}^i_-)}e^{ij\mathsf{J}}e^{it\mathsf{T}} \tag{4.19}$$

where we have denoted $\boldsymbol{k} := (j, t, \boldsymbol{p}^{\pm})$ with $j, t \in \mathbb{R}$ and $\boldsymbol{p}^{\pm} = \overline{\boldsymbol{p}^{\mp}} = (p_1^{\pm}, p_2^{\pm}) \in \mathbb{C}^2$. To calculate the corresponding $\star$-product $*$, we have to compute the group products

$$^{**}_{**}e^{ik^a\mathsf{X}_a}{}^*_* \cdot {}^*_*e^{ik'^a\mathsf{X}_a}{}^{**}_{**} \tag{4.20}$$

The simplest way to compute these products is to realise the six-dimensional Lie algebra $\mathfrak{n}$ as a central extension of the subalgebra $\mathfrak{s} = \mathrm{so}(2) \ltimes \mathbb{R}^4$ of the four-dimensional Euclidean algebra $\mathrm{iso}(4) = \mathrm{so}(4) \ltimes \mathbb{R}^4$ [87, 48]. Regarding $\mathbb{R}^4$ as $\mathbb{C}^2$ (with respect to a chosen complex structure), for generic $\theta \neq 0$ the generators of $\mathfrak{n}$ act on $\boldsymbol{w} \in \mathbb{C}^2$ according to the affine transformations $e^{ij\mathsf{J}} \cdot \boldsymbol{w} = e^{-\theta j}\boldsymbol{w}$ and $e^{i(p_i^+\mathsf{P}^i_+ + p_i^-\mathsf{P}^i_-)} \cdot \boldsymbol{w} = \boldsymbol{w} + i\theta\boldsymbol{p}^-$, corresponding to a combined rotation in the (12), (34) planes and translations in $\mathbb{R}^4 \cong \mathbb{C}^2$. The central element generates an abstract one-parameter subgroup acting as $e^{it\mathsf{T}} \cdot \boldsymbol{w} = e^{-\theta t}\boldsymbol{w}$ in this representation. From this action we can read off the group multiplication laws

$$
\begin{aligned}
e^{ij\mathsf{J}} e^{ij'\mathsf{J}} &= e^{i(j+j')\mathsf{J}} && (4.21) \\
e^{ij\mathsf{J}} e^{i(p_i^+ \mathsf{P}_+^i + p_i^- \mathsf{P}_-^i)} &= e^{i(e^{-\theta j} p_i^+ \mathsf{P}_+^i + e^{\theta j} p_i^- \mathsf{P}_-^i)} e^{ij\mathsf{J}} && (4.22) \\
e^{i(p_i^+ \mathsf{P}_+^i + p_i^- \mathsf{P}_-^i)} e^{i(p_i'^+ \mathsf{P}_+^i + p_i'^- \mathsf{P}_-^i)} &= e^{i[(p_i^+ + p_i)'^+ \mathsf{P}_+^i + (p_i^- + p_i)'^- \mathsf{P}_-^i]} e^{2\theta \mathrm{Im}(\boldsymbol{p}^+ \cdot \boldsymbol{p}'^-)\mathsf{T}} && (4.23)
\end{aligned}
$$

where the formula (4.22) displays the semi-direct product nature of the euclidean group, while (4.23) displays the group cocycle of the projective representation of the subgroup $\mathcal{S}$ of ISO(4), arising from the central extension, which makes the translation algebra noncommutative and is computed from the Baker-Campbell-Hausdorff formula.

Using (4.21) through (4.23) we may now compute the products (4.20)

$$
\begin{aligned}
{}^{**}_{**} e^{ik^a \mathsf{X}_a} {}^{*}_{*} \cdot {}^{*}_{*} e^{ik'^a \mathsf{X}_a} {}^{**}_{**} &= e^{i\mathbf{F}^* \cdot \mathsf{X}} && (4.24) \\
F_i^{*+} &= p_i^+ + e^{-\theta j} p_i'^+ \\
F_i^{*-} &= p_i^- + e^{\theta j} p_i'^- \\
F_j^* &= j + j' \\
F_t^* &= t + t' - \theta(e^{\theta j} \boldsymbol{p}^+ \cdot \boldsymbol{p}'^- - e^{-\theta j} \boldsymbol{p}^- \cdot \boldsymbol{p}'^+)
\end{aligned}
$$

From (4.14) we may compute the $\star$-products between the coordinate functions on

$\mathfrak{n}^\vee$ and easily verify the commutation relations of the algebra $\mathfrak{n}$,

$$
\begin{aligned}
x_a * x_a &= (x_a)^2 \qquad (4.25)\\
x_a * x^+ &= x^+ * x_a = x_a x^+ \\
z_1 * z_2 &= z_2 * z_1 = z_1 z_2 \\
\overline{z}_1 * \overline{z}_2 &= \overline{z}_2 * \overline{z}_1 = \overline{z}_1 \overline{z}_2 \\
x^- * z_i &= x^- z_i - i\theta z_i \\
z_i * x^- &= x^- z_i \\
x^- * \overline{z}_i &= x^- \overline{z}_i + i\theta \overline{z}_i \\
\overline{z}_i * x^- &= x^- \overline{z}_i \\
z_i * \overline{z}_i &= z_i \overline{z}_i - i\theta x^+ \\
\overline{z}_i * z_i &= z_i \overline{z}_i + i\theta x^+
\end{aligned}
$$

with $a = 1, \ldots, 6$ and $i = 1, 2$.

From (4.12), (4.13) we find the $*$-product of generic functions $f, g \in C^\infty(\mathfrak{n}^*)$ given by

$$
\begin{aligned}
f * g \;=\; \mu \circ \exp\Big[& i\theta x^+ \left(e^{-i\theta\partial_-} \boldsymbol{\partial}^\top \otimes \overline{\boldsymbol{\partial}} - e^{i\theta\partial_-} \overline{\boldsymbol{\partial}}^\top \otimes \boldsymbol{\partial} \right) \qquad (4.26)\\
& + \overline{z}_i \left(e^{i\theta\partial_-} - 1 \right) \otimes \partial^i + z_i \left(e^{-i\theta\partial_-} - 1 \right) \otimes \overline{\partial}^i \Big] f \otimes g
\end{aligned}
$$

where $\mu(f \otimes g) = fg$ is the pointwise product. To second order in the deformation parameter θ we obtain

$$
\begin{aligned}
f * g \;=\;& fg - i\theta\left[x^+\left(\overline{\boldsymbol{\partial}} f\cdot\boldsymbol{\partial} g - \boldsymbol{\partial} f\cdot\overline{\boldsymbol{\partial}} g\right) - \overline{\boldsymbol{z}}\cdot\partial_- f\boldsymbol{\partial} g + \boldsymbol{z}\cdot\partial_- f\overline{\boldsymbol{\partial}} g\right] \qquad (4.27)\\
& -\theta^2\sum_{i=1,2}\Bigg[\frac{1}{2}\left(x^+\right)^2\left((\partial^i)^2 f(\overline{\partial}^i)^2 g - 2\overline{\partial}^i\partial^i f\overline{\partial}^i\partial^i g + (\overline{\partial}^i)^2 f(\partial^i)^2 g\right)\\
&\qquad -x^+\left(\partial^i\partial_- f\overline{\partial}^i g - \overline{\partial}^i\partial_- f\partial^i g\right)\\
&\qquad -x^+\overline{z}_i\left(\overline{\partial}^i\partial_- f(\partial^i)^2 g - \partial^i\partial_- f\overline{\partial}^i\partial^i g\right)\\
&\qquad +x^+ z_i\left(\overline{\partial}^i\partial_- f\overline{\partial}^i\partial^i g - \partial^i\partial_- f(\overline{\partial}^i)^2 g\right) - \overline{z}_i z_i\partial_-^2 f\overline{\partial}^i\partial^i g\\
&\qquad +\frac{1}{2}\left(\overline{z}_i^2\partial_-^2 f(\partial^i)^2 g + \overline{z}_i\partial_-^2 f\partial^i g + z_i\partial_-^2 f\overline{\partial}^i g + z_i^2\partial_-^2 f(\overline{\partial}^i)^2 g\right)\Bigg]\\
& +O\left(\theta^3\right)
\end{aligned}
$$

4.2.2 Symmetric Time Ordering

Our next Gutt product is obtained by taking a "symmetric time ordering" whereby any monomial in $U(\mathfrak{n})$ is the symmetric sum over the two time orderings obtained by placing J to the far right and to the far left. This ordering is induced by the group contraction of U(1) $\times$ SU(2) onto the Nappi-Witten group $\mathcal{N}_0$ [35], and it thereby induces the coordinatisation of NW_4 that is obtained from the Penrose-Güven limit of the spacetime $\mathrm{S}^{1,0}\times\mathrm{S}^3$, i.e. it coincides with the Brinkman coordinatisation of the Cahen-Wallach spacetime. On elements of $\mathcal{N}^{\mathbb{C}}$ it is defined by

$$
\Omega_\bullet\left(e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right) = {}^\bullet_\bullet e^{ik^a\mathsf{X}_a}{}^\bullet_\bullet := e^{\frac{i}{2}j\mathsf{J}}e^{i(p_i^+\mathsf{P}_+^i + p_i^-\mathsf{P}_-^i)}e^{\frac{i}{2}j\mathsf{J}}e^{it\mathsf{T}} \qquad (4.28)
$$

From (4.21) through (4.23) we can again easily compute the required group products to get

$$
\begin{aligned}
\vdots\,\vdots e^{ik^a \mathsf{X}_a} \vdots \cdot \vdots e^{ik'^a \mathsf{X}_a} \vdots\,\vdots &= e^{i\mathbf{F}^\bullet \cdot \mathsf{X}} \\
F_i^{\bullet +} &= e^{\frac{\theta}{2}j'} p_i^+ + e^{-\frac{\theta}{2}j} p_i'^+ \\
F_i^{\bullet -} &= e^{-\frac{\theta}{2}j'} p_i^- + e^{\frac{\theta}{2}j} p_i'^- \\
F_j^\bullet &= j + j' \\
F_t^\bullet &= t + t' - \theta e^{\frac{\theta}{2}(j+j')} \boldsymbol{p}^+ \cdot \boldsymbol{p}'^- - \theta e^{-\frac{\theta}{2}(j+j')} \boldsymbol{p}^- \cdot \boldsymbol{p}'^+
\end{aligned} \tag{4.29}
$$

With the same conventions as above, from (4.14) we may now compute the $\star$-products $\bullet$ between the coordinate functions on $\mathfrak{n}^\vee$ and again verify the commutation relations of the algebra $\mathfrak{n}$,

$$
\begin{aligned}
x_a \bullet x_a &= (x_a)^2 \\
x_a \bullet x^+ &= x^+ \bullet x_a = x_a x^+ \\
z_1 \bullet z_2 &= z_2 \bullet z_1 = z_1 z_2 \\
\overline{z}_1 \bullet \overline{z}_2 &= \overline{z}_2 \bullet \overline{z}_1 = \overline{z}_1 \overline{z}_2 \\
x^- \bullet z_i &= x^- z_i - \frac{i}{2}\theta z_i \\
z_i \bullet x^- &= x^- z_i + \frac{i}{2}\theta z_i \\
x^- \bullet \overline{z}_i &= x^- \overline{z}_i + \frac{i}{2}\theta \overline{z}_i \\
\overline{z}_i \bullet x^- &= x^- \overline{z}_i - \frac{i}{2}\theta \overline{z}_i \\
z_i \bullet \overline{z}_i &= z_i \overline{z}_i - i\theta x^+ \\
\overline{z}_i \bullet z_i &= z_i \overline{z}_i + i\theta x^+
\end{aligned} \tag{4.30}
$$

From (4.12) through (4.13) we find for generic functions the formula

$$
\begin{aligned}
f \bullet g \;&=\; \mu \circ \exp\Big\{ i\theta x^+ \left(e^{-\frac{i\theta}{2}\partial_-} \boldsymbol{\partial}^\top \otimes e^{-\frac{i\theta}{2}\partial_-} \overline{\boldsymbol{\partial}} - e^{\frac{i\theta}{2}\partial_-} \overline{\boldsymbol{\partial}}^\top \otimes e^{\frac{i\theta}{2}\partial_-} \boldsymbol{\partial} \right) \\
&\qquad\qquad + \bar{z}_i \left[\partial^i \otimes \left(e^{-\frac{i\theta}{2}\partial_-} - 1 \right) + \left(e^{\frac{i\theta}{2}\partial_-} - 1 \right) \otimes \partial^i \right] \\
&\qquad\qquad + z_i \left[\overline{\partial}^i \otimes \left(e^{\frac{i\theta}{2}\partial_-} - 1 \right) + \left(e^{-\frac{i\theta}{2}\partial_-} - 1 \right) \otimes \overline{\partial}^i \right] \Big\} f \otimes g
\end{aligned}
\tag{4.31}
$$

To second order in θ we obtain

$$
\begin{aligned}
f \bullet g \;&=\; fg - \frac{i}{2}\theta \Big[2x^+ \left(\overline{\boldsymbol{\partial}} f \cdot \boldsymbol{\partial} g - \boldsymbol{\partial} f \cdot \overline{\boldsymbol{\partial}} g \right) \\
&\qquad\qquad + \bar{\boldsymbol{z}} \cdot \left(\boldsymbol{\partial} f \partial_- g - \partial_- f \boldsymbol{\partial} g \right) + \boldsymbol{z} \cdot \left(\partial_- f \overline{\boldsymbol{\partial}} g - \overline{\boldsymbol{\partial}} f \partial_- g \right) \Big] \\
&\quad - \frac{1}{2}\theta^2 \sum_{i=1,2} \Big[\left(x^+\right)^2 \left((\overline{\partial}^i)^2 f (\partial^i)^2 g + (\partial^i)^2 f (\overline{\partial}^i)^2 g - 2 \overline{\partial}^i \partial^i f \overline{\partial}^i \partial^i g \right) \\
&\qquad - x^+ \left(\partial^i f \overline{\partial}^i \partial_- g + \overline{\partial}^i f \partial^i \partial_- g + \overline{\partial}^i \partial_- f \partial^i g + \partial^i \partial_- f \overline{\partial}^i g \right) \\
&\qquad + x^+ \bar{z}_i \left(\overline{\partial}^i \partial^i f \partial^i \partial_- g - \overline{\partial}^i \partial_- f (\partial^i)^2 g + \partial^i \partial_- f \overline{\partial}^i \partial^i g - (\partial^i)^2 f \overline{\partial}^i \partial_- g \right) \\
&\qquad + x^+ z_i \left(\overline{\partial}^i \partial^i f \overline{\partial}^i \partial_- g - \partial^i \partial_- f (\overline{\partial}^i)^2 g + \overline{\partial}^i \partial_- f \overline{\partial}^i \partial^i g - (\overline{\partial}^i)^2 f \partial^i \partial_- g \right) \\
&\qquad + \frac{1}{2} \bar{z}_i z_i \left(\overline{\partial}^i \partial_- f \partial^i \partial_- g + \partial^i \partial_- f \overline{\partial}^i \partial_- g - \partial_-^2 f \overline{\partial}^i \partial^i g - \overline{\partial}^i \partial^i f \partial_-^2 g \right) \\
&\qquad + \frac{1}{4} \bar{z}_i^2 \left((\partial^i)^2 f \partial_-^2 g - 2 \partial^i \partial_- f \partial^i \partial_- g + \partial_-^2 f (\partial^i)^2 g \right) \\
&\qquad + \frac{1}{4} z_i^2 \left((\overline{\partial}^i)^2 f \partial_-^2 g - 2 \overline{\partial}^i \partial_- f \overline{\partial}^i \partial_- g + \partial_-^2 f (\overline{\partial}^i)^2 g \right) \\
&\qquad + \frac{1}{4} \bar{z}_i \left(\partial^i f \partial_-^2 g + \partial_-^2 f \partial^i g \right) + \frac{1}{4} z_i \left(\partial_-^2 f \overline{\partial}^i g + \overline{\partial}^i f \partial_-^2 g \right) \Big] \\
&\quad + O\left(\theta^3\right)
\end{aligned}
$$

4.2.3 Weyl Ordering

The original Gutt product [53] is based on the "Weyl ordering" prescription by which all monomials in $U(\mathfrak{n})$ are completely symmetrised over all elements of $\mathfrak{n}$.

On $\mathcal{N}^{\mathbb{C}}$ it is defined by

$$\Omega_\star\left(e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right) = {}^\circ_\circ e^{ik^a\mathsf{X}_a}{}^\circ_\circ := e^{ik^a\mathsf{X}_a} \tag{4.32}$$

While this ordering is usually thought of as the "canonical" ordering for the construction of $\star$-products, in our case it turns out to be drastically more complicated than the other orderings. Nevertheless, we shall present here its explicit construction for the sake of completeness and for later comparisons.

It is an extremely arduous task to compute products of the group elements (4.32) directly from the Baker-Campbell-Hausdorff formula (4.16). Instead, we shall construct an isomorphism $\mathcal{G} : \overline{U(\mathfrak{n})^{\mathbb{C}}} \to \overline{U(\mathfrak{n})^{\mathbb{C}}}$ which sends the time-ordered product defined by (4.20) into the Weyl-ordered product defined by (4.32), i.e.

$$\mathcal{G} \circ \Omega_* = \Omega_\star \tag{4.33}$$

Then by defining $\mathcal{G}_\Omega := \Omega_*^{-1} \circ \mathcal{G} \circ \Omega_\star$, the $\star$-product $\star$ associated with the Weyl ordering prescription (4.32) may be computed as

$$f \star g = \mathcal{G}_\Omega\big(\mathcal{G}_\Omega^{-1}(f) * \mathcal{G}_\Omega^{-1}(g)\big) \qquad\qquad f, g \in C^\infty(\mathfrak{n}^\vee) \tag{4.34}$$

Explicitly, if

$${}^*_* e^{ik^a\mathsf{X}_a}{}^*_* = e^{iG^a(\boldsymbol{k})\mathsf{X}_a} \tag{4.35}$$

for some function $\boldsymbol{G} = (G^a) : \mathbb{R}^6 \to \mathbb{R}^6$, then the isomorphism $\mathcal{G}_\Omega : C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$ may be represented as the invertible differential operator

$$\mathcal{G}_\Omega = e^{i\boldsymbol{x}\cdot[\boldsymbol{G}(-i\boldsymbol{\partial})+i\boldsymbol{\partial}]} \tag{4.36}$$

This relation just reflects the fact that the time-ordered and Weyl-ordered $\star$-products simply represent different ordering prescriptions for the same algebra and are therefore (cohomologically) *equivalent*. We will elucidate this property more thoroughly in section 4.3. Thus once the map (4.35) is known, the Weyl ordered $\star$-product $\star$ can be computed in terms of the time-ordered $\star$-product $*$ of Section 4.2.1.

The functions $G^a(\boldsymbol{k})$ appearing in (4.35) are readily calculable through the Baker-Campbell-Hausdorff formula. It is clear from (4.20) that the coefficient of the time translation generator $\mathsf{J} \in \mathfrak{n}$ is simply

$$G^{j}(j,t,\boldsymbol{p}^{\pm}) = j \tag{4.37}$$

From (4.16) it is also clear that the only terms proportional to P^i_+ come from commutators of the form $[\mathsf{J},[\ldots,[\mathsf{J},\mathsf{P}^i_+]]\ldots]$, and gathering all terms we find

$$\begin{aligned}\sum_{i=1,2} G^{p_i^+}(j,t,\boldsymbol{p}^{\pm})\mathsf{P}^i_+ &= -i\sum_{n=0}^{\infty}\frac{B_n}{n!}\big[\underbrace{ij\mathsf{J},\big[\ldots,\big[ij\mathsf{J}}_{n},ip_i^+\mathsf{P}^i_+\big]\big]\ldots\big] \\ &= p_i^+\sum_{n=0}^{\infty}\frac{B_n}{n!}(-\theta j)^n\mathsf{P}^i_+\end{aligned} \tag{4.38}$$

Since $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0\ \forall k \geq 1$, from (4.17) we thereby find

$$G^{\boldsymbol{p}^+}(j,t,\boldsymbol{p}^{\pm}) = \frac{\boldsymbol{p}^+}{\phi_\theta(j)} \tag{4.39}$$

where we have introduced the function

$$\phi_\theta(j) = \frac{1-e^{-\theta j}}{\theta j} \tag{4.40}$$

obeying the identities

$$\phi_\theta(j)e^{\theta j} = \phi_{-\theta}(j) \qquad \phi_\theta(j)\phi_{-\theta}(j) = -\frac{2}{(\theta j)^2}\big(1-\cos(\theta j)\big) \tag{4.41}$$

In a completely analogous way one finds the coefficient of the P^i_- term to be given by

$$G^{\boldsymbol{p}^-}(j,t,\boldsymbol{p}^{\pm}) = \frac{\boldsymbol{p}^-}{\phi_{-\theta}(j)} \tag{4.42}$$

Finally, the non-vanishing contributions to the central element $\mathsf{T} \in \mathfrak{n}$ are given by

$$\begin{aligned}G^{t}(j,t,\boldsymbol{p}^{\pm})\mathsf{T} &= t\mathsf{T} - i\sum_{n=1}^{\infty}\frac{B_{n+1}}{n!}\Bigg(\big[ip_i^+\mathsf{P}^i_+,\big[\underbrace{ij\mathsf{J},\ldots\big[ij\mathsf{J}}_{n},ip_i^-\mathsf{P}^i_-\big]\ldots\big]\big] \\ &\qquad + \big[ip_i^-\mathsf{P}^i_-,\big[\underbrace{ij\mathsf{J},\ldots\big[ij\mathsf{J}}_{n},ip_i^+\mathsf{P}^i_+\big]\ldots\big]\big]\Bigg) \\ &= t\mathsf{T} + 4\theta\boldsymbol{p}^+\cdot\boldsymbol{p}^-\sum_{n=1}^{\infty}\frac{B_{n+1}}{n!}(-\theta j)^n\mathsf{T}\end{aligned} \tag{4.43}$$

By differentiating (4.38) and (4.40) with respect to $s = -\theta j$ we arrive finally at

$$G^t(j, t, \boldsymbol{p}^{\pm}) = t + 4\theta \boldsymbol{p}^+ \cdot \boldsymbol{p}^- \gamma_\theta(j) \tag{4.44}$$

where we have introduced the function

$$\gamma_\theta(j) = \frac{1}{2} + \frac{(1+\theta j)e^{-\theta j} - 1}{\left(e^{-\theta j} - 1\right)^2} \tag{4.45}$$

From (4.36) we may now write down the explicit form of the differential operator implementing the equivalence between the $\star$-products $*$ and $\star$ as

$$\begin{aligned} \mathcal{G}_\Omega \quad = \quad \exp &\left[-2i\theta x^+ \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial} \left(1 + \frac{2(1 - i\theta\partial_-)e^{i\theta\partial_-} - 1}{\left(e^{i\theta\partial_-} - 1\right)^2} \right) \right. \\ &\left. + \, \overline{\boldsymbol{z}} \cdot \boldsymbol{\partial} \left(\frac{i\theta\partial_-}{e^{i\theta\partial_-} - 1} - 1 \right) - \boldsymbol{z} \cdot \overline{\boldsymbol{\partial}} \left(\frac{i\theta\partial_-}{e^{-i\theta\partial_-} - 1} + 1 \right) \right] \end{aligned} \tag{4.46}$$

From (4.24) and (4.35) we may readily compute the products of Weyl symbols with the result

$$\begin{aligned} {}^{\circ\circ}_{\circ\circ} e^{ik^a \mathsf{X}_a} {}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ} e^{ik'^a \mathsf{X}_a} {}^{\circ\circ}_{\circ\circ} \quad &= \quad e^{i\mathbf{F}^\star \cdot \mathsf{X}} \\ F_i^{\star +} \quad &= \quad \frac{\phi_\theta(j) p_i^+ + e^{-\theta j} \phi_\theta(j') p_i'^+}{\phi_\theta(j + j')} \\ F_i^{\star -} \quad &= \quad \frac{\phi_{-\theta}(j) p_i^- + e^{\theta j} \phi_{-\theta}(j') p_i'^-}{\phi_{-\theta}(j + j')} \\ F_j^\star \quad &= \quad j + j' \\ F_t^\star \quad &= \quad t + t' + \theta\big(\phi_{-\theta}(j)\phi_{-\theta}(j')\boldsymbol{p}^+ \cdot \boldsymbol{p}'^- - \phi_\theta(j)\phi_\theta(j')\boldsymbol{p}^- \cdot \boldsymbol{p}'^+\big) \\ &\quad -4\theta \Big(\gamma_\theta(j + j')\big(\phi_{-\theta}(j)\boldsymbol{p}^+ + e^{\theta j}\phi_{-\theta}(j')\boldsymbol{p}'^+\big) \\ &\qquad \times \big(\phi_\theta(j)\boldsymbol{p}^- + e^{-\theta j}\phi_\theta(j')\boldsymbol{p}'^-\big) \\ &\qquad -\gamma_\theta(j)\phi_\theta(j)\phi_{-\theta}(j)\boldsymbol{p}^+ \cdot \boldsymbol{p}^- - \gamma_\theta(j')\phi_\theta(j')\phi_{-\theta}(j')\boldsymbol{p}'^+ \cdot \boldsymbol{p}'^- \Big) \end{aligned} \tag{4.47}$$

The $x_a \star x_b$ products are identical to those of the symmetric time ordering prescription (4.30). After some computation, from (4.12), (4.13) we find for generic functions $f, g \in C^\infty(\mathfrak{n}^\vee)$ the formula

$$
\begin{aligned}
f \star g \;&=\; \mu \circ \exp\Bigg\{ \theta x^+ \Bigg[\frac{1\otimes 1 + \big(i\theta(\partial_-\otimes 1 + 1\otimes\partial_-) - 1\otimes 1\big) e^{i\theta\partial_-}\otimes e^{i\theta\partial_-}}{\left(e^{i\theta\partial_-}\otimes e^{i\theta\partial_-} - 1\otimes 1\right)^2} \\
&\times \Bigg(\frac{4\boldsymbol{\partial}^\top\left(e^{-i\theta\partial_-}-1\right)}{\theta\partial_-} \otimes \frac{\overline{\boldsymbol{\partial}}\left(e^{-i\theta\partial_-}-1\right)}{\theta\partial_-} - \frac{3\overline{\boldsymbol{\partial}}^\top\left(e^{i\theta\partial_-}-1\right)}{\theta\partial_-} \otimes \frac{\boldsymbol{\partial}\left(e^{i\theta\partial_-}-1\right)}{\theta\partial_-} \\
&+ \frac{4\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}\sin^2\left(\frac{\theta}{2}\partial_-\right)}{\theta^2\partial_-^2} \otimes 1 - 1 \otimes \frac{4\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}\sin^2\left(\frac{\theta}{2}\partial_-\right)}{\theta^2\partial_-^2} \Bigg) \\
&+ \frac{4i\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}}{\theta\partial_-}\left(\frac{i\sin^2\left(\frac{\theta}{2}\partial_-\right)}{\theta\partial_-\left(e^{i\theta\partial_-}-1\right)} - 1\right) \otimes 1 + 1 \otimes \frac{4i\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}}{\theta\partial_-}\left(\frac{i\sin^2\left(\frac{\theta}{2}\partial_-\right)}{\theta\partial_-\left(e^{i\theta\partial_-}-1\right)} - 1\right) \\
&+ \frac{3\overline{\boldsymbol{\partial}}^\top\left(e^{i\theta\partial_-}-1\right)}{\theta\partial_-} \otimes \frac{\boldsymbol{\partial}\left(e^{i\theta\partial_-}-1\right)}{\theta\partial_-} + \frac{\boldsymbol{\partial}^\top\left(e^{-i\theta\partial_-}-1\right)}{\theta\partial_-} \otimes \frac{\overline{\boldsymbol{\partial}}\left(e^{-i\theta\partial_-}-1\right)}{\theta\partial_-} \Bigg] \\
&+ \frac{\overline{z}_i}{1\otimes e^{-i\theta\partial_-} - e^{i\theta\partial_-}\otimes 1} \Bigg[\frac{\partial^i}{\partial_-}\left(1-e^{i\theta\partial_-}\right)\otimes\partial_- - \partial_-\otimes\frac{\partial^i}{\partial_-}\left(1-e^{-i\theta\partial_-}\right) \\
&+ 1\otimes\partial^i e^{-i\theta\partial_-} - \partial^i e^{i\theta\partial_-}\otimes 1 - 1\otimes 2\partial^i \Bigg] \\
&+ \frac{z_i}{1\otimes e^{i\theta\partial_-} - e^{-i\theta\partial_-}\otimes 1} \Bigg[\frac{\overline{\partial}^i}{\partial_-}\left(1-e^{-i\theta\partial_-}\right)\otimes\partial_- - \partial_-\otimes\frac{\overline{\partial}^i}{\partial_-}\left(1-e^{i\theta\partial_-}\right) \\
&+ 1\otimes\overline{\partial}^i e^{i\theta\partial_-} - \overline{\partial}^i e^{-i\theta\partial_-}\otimes 1 - 1\otimes 2\overline{\partial}^i \Bigg] \Bigg\} f\otimes g \qquad (4.48)
\end{aligned}
$$

To second order in the deformation parameter θ we obtain

$$\begin{aligned} f \star g &= fg - \frac{i}{2}\theta \Big[2x^+ \left(\overline{\boldsymbol{\partial}} f \cdot \boldsymbol{\partial} g - \boldsymbol{\partial} f \cdot \overline{\boldsymbol{\partial}} g\right) \\ &\qquad + \overline{\boldsymbol{z}} \cdot (\boldsymbol{\partial} f \partial_- g - \partial_- f \boldsymbol{\partial} g) + \boldsymbol{z} \cdot \left(\partial_- f \overline{\boldsymbol{\partial}} g - \overline{\boldsymbol{\partial}} f \partial_- g\right)\Big] \end{aligned} \tag{4.49}$$

$$\begin{aligned} &-\frac{1}{2}\theta^2 \sum_{i=1,2} \Big[\left(x^+\right)^2 \left((\overline{\partial}^i)^2 f (\partial^i)^2 g + (\partial^i)^2 f (\overline{\partial}^i)^2 g - 2\overline{\partial}^i \partial^i f \overline{\partial}^i \partial^i g\right) \\ &\qquad -\frac{1}{3}x^+ \Big(\partial^i f \overline{\partial}^i \partial_- g + \overline{\partial}^i f \partial^i \partial_- g + \overline{\partial}^i \partial_- f \partial^i g + \partial^i \partial_- f \overline{\partial}^i g \\ &\qquad\qquad - 2\partial_- f \overline{\partial}^i \partial^i g - 2\overline{\partial}^i \partial^i f \partial_- g\Big) \\ &\qquad +x^+ \overline{z}_i \left(\overline{\partial}^i \partial^i f \partial^i \partial_- g - \overline{\partial}^i \partial_- f (\partial^i)^2 g + \partial^i \partial_- f \overline{\partial}^i \partial^i g - (\partial^i)^2 f \overline{\partial}^i \partial_- g\right) \\ &\qquad +x^+ z_i \left(\overline{\partial}^i \partial^i f \overline{\partial}^i \partial_- g - \partial^i \partial_- f (\overline{\partial}^i)^2 g + \overline{\partial}^i \partial_- f \overline{\partial}^i \partial^i g - (\overline{\partial}^i)^2 f \partial^i \partial_- g\right) \\ &\qquad +\frac{1}{2}\overline{z}_i z_i \left(\overline{\partial}^i \partial_- f \partial^i \partial_- g + \partial^i \partial_- f \overline{\partial}^i \partial_- g - \partial_-^2 f \overline{\partial}^i \partial^i g - \overline{\partial}^i \partial^i f \partial_-^2 g\right) \\ &\qquad +\frac{1}{4}\overline{z}_i^2 \left((\partial^i)^2 f \partial_-^2 g - 2\partial^i \partial_- f \partial^i \partial_- g + \partial_-^2 f (\partial^i)^2 g\right) \\ &\qquad +\frac{1}{4}z_i^2 \left((\overline{\partial}^i)^2 f \partial_-^2 g - 2\overline{\partial}^i \partial - f \overline{\partial}^i \partial_- g + \partial_-^2 f (\overline{\partial}^i)^2 g\right) \\ &\qquad +\frac{1}{6}\overline{z}_i \left(\partial^i f \partial_-^2 g + \partial_-^2 f \partial^i g - \partial_- f \partial^i \partial_- g - \partial^i \partial_- f \partial_- g\right) \\ &\qquad +\frac{1}{6}z_i \left(\partial_-^2 f \overline{\partial}^i g + \overline{\partial}^i f \partial_-^2 g - \partial_- f \overline{\partial}^i \partial_- g - \overline{\partial}^i \partial_- f \partial_- g\right)\Big] \\ &+O\left(\theta^3\right) \end{aligned} \tag{4.50}$$

Although extremely cumbersome in form, the Weyl-ordered product has several desirable features over the simpler time-ordered products. For instance, it is Hermitean owing to the property

$$\overline{f \star g} = \overline{g} \star \overline{f} \tag{4.51}$$

Moreover, while the $\mathfrak{n}$-covariance condition (4.8) holds for all of our $\star$-products, the Weyl product is in fact $\mathfrak{n}$-invariant, because for any $x \in \mathbb{C}_{(1)}(\mathfrak{n}^\vee)$ one has the stronger compatibility condition

$$[x, f]_\star = i\theta\Theta(x, f) \qquad \forall f \in C^\infty(\mathfrak{n}^\vee) \tag{4.52}$$

with the action of the Lie algebra $\mathfrak{n}$. In the next section we shall see that the Weyl-ordered $\star$-product is, in a certain sense, the generator of all other $\star$-products making it the "universal" product for the quantisation of the spacetime NW_6.

4.3 Weyl Systems

In this section we will use the notion of a generalised Weyl system introduced in [1] to describe some more formal aspects of the $\star$-products that we have constructed and to analyse the interplay between them. This generalises the standard Weyl systems [99] which may be used to provide a purely operator theoretic characterisation of the Moyal product, associated to the (untwisted) Heisenberg algebra. In that case, it can be regarded as a projective representation of the translation group in an even-dimensional real vector space. However, for the twisted Heisenberg algebra such a representation is not possible, since by definition the appropriate arena should be a central extension of the non-Abelian subgroup $\mathcal{S}_5$ (1.29) of the full euclidean group ISO(4). This requires a generalisation of the standard notion which we will now describe and use it to obtain a very useful characterisation of the noncommutative geometry induced by the algebra $\mathfrak{n}$.

Let $\mathbb{V}$ be a five-dimensional real vector space. In a suitable (canonical) basis, vectors $\boldsymbol{k} \in \mathbb{V} \cong \mathbb{R} \times \mathbb{C}^2$ will be denoted (with respect to a chosen complex structure) as

$$\boldsymbol{k} = \begin{pmatrix} j \\ \boldsymbol{p}^+ \\ \boldsymbol{p}^- \end{pmatrix} \tag{4.53}$$

with $j \in \mathbb{R}$ and $\boldsymbol{p}^\pm = \overline{\boldsymbol{p}^\mp} \in \mathbb{C}^2$. As the notation suggests, we regard $\mathbb{V}$ as the "momentum space" of the dual $\mathfrak{n}^\vee$. Note that we do not explicitly incorporate the component corresponding to the central element T, as it will instead appear through the appropriate projective representation that we will construct, similarly to the

Moyal case. As an Abelian group, $\mathbb{V} \cong \mathbb{R}^5$ with the usual addition $+$ and identity $\mathbf{0}$. Corresponding to a deformation parameter $\theta \in \mathbb{R}$, we deform this Abelian Lie group structure to a generically non-Abelian one. The deformed composition law is denoted $\boxplus$. It is associative and in general will depend on θ. The identity element with respect to $\boxplus$ is still defined to be $\mathbf{0}$, and the inverse of any element $\boldsymbol{k} \in \mathbb{V}$ is denoted $\underline{\boldsymbol{k}}$, so that

$$\boldsymbol{k} \boxplus \underline{\boldsymbol{k}} = \underline{\boldsymbol{k}} \boxplus \boldsymbol{k} = \mathbf{0} \tag{4.54}$$

Being a deformation of the underlying Abelian group structure on $\mathbb{V}$ means that the composition of any two vectors $\boldsymbol{k}, \boldsymbol{q} \in \mathbb{V}$ has a formal small θ expansion of the form

$$\boldsymbol{k} \boxplus \boldsymbol{q} = \boldsymbol{k} + \boldsymbol{q} + O(\theta) \tag{4.55}$$

from which it follows that

$$\underline{\boldsymbol{k}} = -\boldsymbol{k} + O(\theta) \tag{4.56}$$

In other words, rather than introducing $\star$-products that deform the pointwise multiplication of functions on $\mathfrak{n}^\vee$, we now deform the "momentum space" of $\mathfrak{n}^\vee$ to a non-Abelian Lie group. We will see below that the five-dimensional group $(\mathbb{V}, \boxplus)$ is isomorphic to the original subgroup $\mathcal{S} \subset \mathrm{ISO}(4)$, and that the two notions of quantisation are in fact the same.

Given such a group, we now define a (generalised) Weyl system for the algebra $\mathfrak{n}$ as a quadruple $(\mathbb{V}, \boxplus, \mathsf{W}, \omega)$, where the map

$$\mathsf{W} : \mathbb{V} \longrightarrow \overline{U(\mathfrak{n})^{\mathbb{C}}} \tag{4.57}$$

is a projective representation of the group $(\mathbb{V}, \boxplus)$ with projective phase $\omega : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$. This means that for every pair of elements $\boldsymbol{k}, \boldsymbol{q} \in \mathbb{V}$ one has the composition rule

$$\mathsf{W}(\boldsymbol{k}) \cdot \mathsf{W}(\boldsymbol{q}) = e^{\frac{i}{2}\omega(\boldsymbol{k},\boldsymbol{q})\mathsf{T}} \cdot \mathsf{W}(\boldsymbol{k} \boxplus \boldsymbol{q}) \tag{4.58}$$

in the completed, complexified universal enveloping algebra of $\mathfrak{n}$. The associativity of $\boxplus$ and the relation (4.58) imply that the subalgebra $\mathsf{W}(\mathbb{V}) \subset \overline{U(\mathfrak{n})^{\mathbb{C}}}$ is associative if and only if

$$\omega(\boldsymbol{k} \boxplus \boldsymbol{p}, \boldsymbol{q}) = \omega(\boldsymbol{k}, \boldsymbol{p} \boxplus \boldsymbol{q}) + \omega(\boldsymbol{p}, \boldsymbol{q}) - \omega(\boldsymbol{k}, \boldsymbol{p}) \tag{4.59}$$

for all vectors $\boldsymbol{k},\boldsymbol{q},\boldsymbol{p}\in\mathbb{V}$. This condition means that ω defines a one-cocycle in the group cohomology of $(\mathbb{V},\boxplus)$. It is automatically satisfied if ω is a bilinear form with respect to $\boxplus$. We will in addition require that $\omega(\boldsymbol{k},\boldsymbol{q})=O(\theta)\ \forall\boldsymbol{k},\boldsymbol{q}\in\mathbb{V}$ for consistency with (4.55). The identity element of $\mathsf{W}(\mathbb{V})$ is $\mathsf{W}(\boldsymbol{0})$ while the inverse of $\mathsf{W}(\boldsymbol{k})$ is given by

$$\mathsf{W}(\boldsymbol{k})^{-1}=\mathsf{W}(\underline{\boldsymbol{k}}) \tag{4.60}$$

The standard Weyl system on $\mathbb{R}^{2n}$ takes $\boxplus$ to be ordinary addition and ω to be the Darboux symplectic two-form, so that $\mathsf{W}(\mathbb{R}^{2n})$ is a projective representation of the translation group, as is appropriate to the Moyal product.

Given a Weyl system defined as above, we can now introduce another isomorphism

$$\Pi:\mathrm{C}^\infty\left(\mathbb{R}^5\right)\longrightarrow\mathsf{W}\big(\mathbb{V}\big) \tag{4.61}$$

defined by the symbol

$$\Pi(f):=\int\limits_{\mathbb{R}^5}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^5}\tilde{f}(\boldsymbol{k})\,\mathsf{W}(\boldsymbol{k}) \tag{4.62}$$

where as before $\tilde{f}$ denotes the Fourier transform of $f\in\mathrm{C}^\infty(\mathbb{R}^5)$. This definition implies that

$$\Pi\big(e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\big)=\mathsf{W}(\boldsymbol{k}) \tag{4.63}$$

and that we may introduce a $\star$-involution $\dagger$ on both algebras $\mathrm{C}^\infty(\mathbb{R}^5)$ and $\mathsf{W}(\mathbb{V})$ by the formula

$$\Pi\big(f^\dagger\big)=\Pi\big(f\big)^\dagger:=\int\limits_{\mathbb{R}^5}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^5}\overline{\tilde{f}(\underline{\boldsymbol{k}})}\,\mathsf{W}(\boldsymbol{k}) \tag{4.64}$$

The compatibility condition

$$\big(\Pi(f)\cdot\Pi(g)\big)^\dagger=\Pi(g)^\dagger\cdot\Pi(f)^\dagger \tag{4.65}$$

with the product in $\overline{U(\mathfrak{n})^{\mathbb{C}}}$ imposes further constraints on the group composition law $\boxplus$ and cocycle ω [1]. From (4.58) we may thereby define a $\dagger$-hermitean $\star$-product of $f,g\in\mathrm{C}^\infty(\mathbb{R}^5)$ by the formula

$$f\star g:=\Pi^{-1}\big(\Pi(f)\cdot\Pi(g)\big)=\int\limits_{\mathbb{R}^5}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^5}\int\limits_{\mathbb{R}^5}\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^5}\tilde{f}(\boldsymbol{k})\tilde{g}(\boldsymbol{q})e^{\frac{i}{2}\omega(\boldsymbol{k},\boldsymbol{q})}\Pi^{-1}\circ\mathsf{W}(\boldsymbol{k}\boxplus\boldsymbol{q}) \tag{4.66}$$

and in this way we have constructed a quantisation of the algebra $\mathfrak{n}$ solely from the formal notion of a Weyl system. The associativity of $\star$ follows from associativity of $\boxplus$. We may also rewrite the $\star$-product (4.66) in terms of a bi-differential operator as

$$f \star g = f e^{\frac{i}{2}\omega(-i\overleftarrow{\boldsymbol{\partial}},-i\overrightarrow{\boldsymbol{\partial}})+i\boldsymbol{x}\cdot(-i\overleftarrow{\boldsymbol{\partial}}\boxplus -i\overrightarrow{\boldsymbol{\partial}}\boxplus +i\overleftarrow{\boldsymbol{\partial}}+i\overrightarrow{\boldsymbol{\partial}})} g \tag{4.67}$$

This deformation is completely characterised in terms of the new algebraic structure and its projective representation provided by the Weyl system. It is clear that the Lie algebra of $(\mathbb{V},\boxplus)$ coincides precisely with the original subalgebra $\mathfrak{s} \subset \mathrm{iso}(4)$, while the cocycle ω generates the central extension of $\mathfrak{s}$ to $\mathfrak{n}$ in the usual way. From (4.66) one may compute the $\star$-products of coordinate functions on $\mathbb{R}^5$ as

$$x_a \star x_b = x_a x_b - i\boldsymbol{x}\cdot \frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b}(\boldsymbol{k}\boxplus\boldsymbol{q})\Big|_{\boldsymbol{k}=\boldsymbol{q}=\boldsymbol{0}} - \frac{i}{2}\frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b}\omega(\boldsymbol{k},\boldsymbol{q})\Big|_{\boldsymbol{k}=\boldsymbol{q}=\boldsymbol{0}} \tag{4.68}$$

The corresponding $\star$-commutator may thereby be written as

$$[x_a, x_b]_\star = i\theta C_{ab}^c x_c + i\theta\xi_{ab} \tag{4.69}$$

where the relation

$$\theta C_{ab}^c = -\left(\frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b} - \frac{\partial}{\partial k^b}\frac{\partial}{\partial q^a}\right)(\boldsymbol{k}\boxplus\boldsymbol{q})^c\Big|_{\boldsymbol{k}=\boldsymbol{q}=\boldsymbol{0}} \tag{4.70}$$

gives the structure constants of the Lie algebra defined by the Lie group $(\mathbb{V},\boxplus)$, while the cocycle term

$$\theta\xi_{ab} = -\frac{1}{2}\left(\frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b} - \frac{\partial}{\partial k^b}\frac{\partial}{\partial q^a}\right)\omega(\boldsymbol{k},\boldsymbol{q})\Big|_{\boldsymbol{k}=\boldsymbol{q}=\boldsymbol{0}} \tag{4.71}$$

gives the usual form of a central extension of this Lie algebra. Demanding that this yield a deformation quantisation of the Kirillov-Kostant Poisson structure on $\mathfrak{n}^\vee$ requires that C_{ab}^c coincide with the structure constants of the subalgebra $\mathfrak{s} \subset \mathrm{iso}(4)$ of $\mathfrak{n}$, and also that $\xi_{p^-,p^+} = -\xi_{p^+,p^-} = 2t$ be the only non-vanishing components of the central extension.

It is thus possible to define a broad class of deformation quantisations of $\mathfrak{n}^\vee$ solely in terms of an abstract Weyl system $(\mathbb{V},\boxplus,\mathsf{W},\omega)$, without explicit realisation of the operators $\mathsf{W}(\boldsymbol{k})$. In the remainder of this section we will set $\Pi = \Omega$ above and describe the Weyl systems underpinning the various products that we constructed previously. This entails identifying the appropriate maps (4.57), which enables the calculation of the projective representations (4.58) and hence explicit realisations of the group composition laws $\boxplus$ in the various instances. This unveils a purely algebraic description of the $\star$-products which will be particularly useful for our later constructions, and enables one to make the equivalences between these products explicit.

4.3.1 Time Ordering

Setting $t = t' = 0$ in (4.24), we find the "time-ordered" non-Abelian group composition law $\circledast$ for any two elements of the form (4.53) to be given by

$$\boldsymbol{k} \circledast \boldsymbol{k}' = \begin{pmatrix} j + j' \\ \boldsymbol{p}^+ + e^{-\theta j} \boldsymbol{p}'^+ \\ \boldsymbol{p}^- + e^{\theta j} \boldsymbol{p}'^- \end{pmatrix} \tag{4.72}$$

From (4.72) it is straightforward to compute the inverse $\underline{\boldsymbol{k}}$ of a group element (4.53), satisfying (4.54), to be

$$\underline{\boldsymbol{k}} = - \begin{pmatrix} j \\ e^{\theta j} \boldsymbol{p}^+ \\ e^{-\theta j} \boldsymbol{p}^- \end{pmatrix} \tag{4.73}$$

The group cocycle is given by

$$\omega_*(\boldsymbol{k}, \boldsymbol{k}') = 2i\theta \left(e^{\theta j} \boldsymbol{p}^+ \cdot \boldsymbol{p}'^- - e^{-\theta j} \boldsymbol{p}^- \cdot \boldsymbol{p}'^+ \right) \tag{4.74}$$

and it defines the canonical symplectic structure on the $j =$ constant subspaces $\mathbb{C}^2 \subset \mathbb{V}$. Note that in this representation, the central coordinate function x^+ is not written explicitly and is simply understood as the unit element of $\mathbb{C}(\mathbb{R}^5)$, as is conventional in the case of the Moyal product. For $\boldsymbol{k} \in \mathbb{V}$ and $\mathsf{X}_a \in \mathfrak{s}$ the projective representation (4.58) is generated by the time-ordered group elements

$$\mathsf{W}_*(\boldsymbol{k}) = {}^*_* e^{ik^a \mathsf{X}_a} {}^*_* \tag{4.75}$$

defined in (4.19).

4.3.2 Symmetric Time Ordering

In a completely analogous manner, inspection of (4.29) reveals the "symmetric time-ordered" non-Abelian group composition law $\boxdot$ defined by

$$\boldsymbol{k} \boxdot \boldsymbol{k}' = \begin{pmatrix} j + j' \\ e^{\frac{\theta}{2} j'} \boldsymbol{p}^{+} + e^{-\frac{\theta}{2} j} \boldsymbol{p}'^{+} \\ e^{-\frac{\theta}{2} j'} \boldsymbol{p}^{-} + e^{\frac{\theta}{2} j} \boldsymbol{p}'^{-} \end{pmatrix} \tag{4.76}$$

for which the inverse $\underline{\boldsymbol{k}}$ of a group element (4.53) is simply given by

$$\underline{\boldsymbol{k}} = -\boldsymbol{k} \tag{4.77}$$

The group cocycle is

$$\omega_{\bullet}(\boldsymbol{k}, \boldsymbol{k}') = 2i\theta \left(e^{\frac{\theta}{2}(j+j')} \boldsymbol{p}^{+} \cdot \boldsymbol{p}'^{-} - e^{-\frac{\theta}{2}(j+j')} \boldsymbol{p}^{-} \cdot \boldsymbol{p}'^{+} \right) \tag{4.78}$$

and it again induces the canonical symplectic structure on $\mathbb{C}^2 \subset \mathbb{V}$. The corresponding projective representation of $(\mathbb{V}, \boxdot)$ is generated by the symmetric time-ordered group elements

$$\mathsf{W}_{\bullet}(\boldsymbol{k}) = \vdots e^{i k^a \mathsf{x}_a} \vdots \tag{4.79}$$

defined in (4.28).

4.3.3 Weyl Ordering

Finally, we construct the Weyl system $(\mathbb{V}, \boxast, \mathsf{W}_{\star}, \omega_{\star})$ associated with the Weyl-ordered $\star$-product of Section 4.2.3. Starting from (4.47) we introduce the non-Abelian group composition law $\boxast$ by

$$\boldsymbol{k} \boxast \boldsymbol{k}' = \begin{pmatrix} j + j' \\ \frac{\phi_\theta(j) \boldsymbol{p}^{+} + e^{-\theta j} \phi_\theta(j') \boldsymbol{p}'^{+}}{\phi_\theta(j+j')} \\ \frac{\phi_{-\theta}(j) \boldsymbol{p}^{-} + e^{\theta j} \phi_{-\theta}(j') \boldsymbol{p}'^{-}}{\phi_{-\theta}(j+j')} \end{pmatrix} \tag{4.80}$$

from which we may again straightforwardly compute the inverse $\underline{\boldsymbol{k}}$ of a group element (4.53) simply as

$$\underline{\boldsymbol{k}} = -\boldsymbol{k} \tag{4.81}$$

When combined with the definition (4.64), one has $f^\dagger = \overline{f}\ \forall f \in \mathrm{C}^\infty(\mathbb{R}^5)$ and this explains the hermitean property (4.51) of the Weyl-ordered $\star$-product $\star$. This is also true of the product $\bullet$, whereas $*$ is only hermitean with respect to the modified involution $\dagger$ defined by (4.64) and (4.73). The group cocycle is given by

$$\begin{aligned}\omega_\star(\boldsymbol{k},\boldsymbol{k}') = -2i\theta\Big(&\phi_{-\theta}(j)\phi_{-\theta}(j')\boldsymbol{p}^+\cdot\boldsymbol{p}'^- - \phi_\theta(j)\phi_\theta(j')\boldsymbol{p}^-\cdot\boldsymbol{p}'^+ \\ &-\gamma_\theta(j+j')\big(\phi_\theta(j)\boldsymbol{p}^+ + e^{-\theta j}\phi_\theta(j')\boldsymbol{p}'^+\big)\cdot\big(\phi_{-\theta}(j)\boldsymbol{p}^- + e^{\theta j}\phi_{-\theta}(j')\boldsymbol{p}'^-\big) \\ &+\gamma_\theta(j)\phi_\theta(j)\phi_{-\theta}(j)\boldsymbol{p}^+\cdot\boldsymbol{p}^- + \gamma_\theta(j')\phi_\theta(j')\phi_{-\theta}(j')\boldsymbol{p}'^+\cdot\boldsymbol{p}'^-\Big)\end{aligned} \tag{4.82}$$

In contrast to the other cocycles, this does *not* induce any symplectic structure, at least not in the manner described earlier. The corresponding projective representation (4.58) is generated by the completely symmetrised group elements

$$\mathsf{W}_\star(\boldsymbol{k}) = e^{ik^a\mathsf{X}_a} \tag{4.83}$$

with $\boldsymbol{k} \in \mathbb{V}$ and $\mathsf{X}_a \in \mathfrak{s}$.

The Weyl system $(\mathbb{V}, \boxplus, \mathsf{W}_\star, \omega_\star)$ can be used to generate the other Weyl systems that we have found [1]. From (4.35) and (4.47) one has the identity

$$\mathsf{W}_*(j,\boldsymbol{p}^\pm) = \Omega_\star\left(e^{i(\boldsymbol{p}^+\cdot\overline{\boldsymbol{z}}+\boldsymbol{p}^-\cdot\boldsymbol{z})} \star e^{ijx^-}\right) \tag{4.84}$$

which implies that the time-ordered $\star$-product $*$ can be expressed by means of a choice of different Weyl system generating the product $\star$. Since $\Omega_\star$ is an algebra isomorphism, one has

$$\mathsf{W}_*(j,\boldsymbol{p}^\pm) = \mathsf{W}_\star(0,\boldsymbol{p}^\pm)\cdot\mathsf{W}_\star(j,\boldsymbol{0}) \tag{4.85}$$

This explicit relationship between the Weyl systems for the $\star$-products $*$ and $\star$ is another formulation of the statement of their cohomological equivalence, as established by other means in Section 4.2.3. Similarly, the symmetric time-ordered $\star$-product $\bullet$ can be expressed in terms of $\star$ through the identity

$$\mathsf{W}_\bullet(j,\boldsymbol{p}^\pm) = \Omega_\star\left(e^{\frac{i}{2}jx^-} \star e^{i(\boldsymbol{p}^+\cdot\overline{\boldsymbol{z}}+\boldsymbol{p}^-\cdot\boldsymbol{z})} \star e^{\frac{i}{2}jx^-}\right) \tag{4.86}$$

which implies the relationship

$$\mathsf{W}_\bullet(j, \boldsymbol{p}^\pm) = \mathsf{W}_\star(\frac{j}{2}, \mathbf{0}) \cdot \mathsf{W}_\star(0, \boldsymbol{p}^\pm) \cdot \mathsf{W}_\star(\frac{j}{2}, \mathbf{0}) \tag{4.87}$$

between the corresponding Weyl systems. This shows explicitly that the $\star$-products $\bullet$ and $\star$ are also equivalent.

4.4 Twisted Isometries

We will now start working our way towards the explicit construction of the geometric quantities required to define field theories on the noncommutative plane wave NW_6. We will begin with a systematic construction of derivative operators on the present noncommutative geometry, which will be used later on to write down kinetic terms for scalar field actions. In this section we will study some of the basic spacetime symmetries of the $\star$-products that we constructed in Section 4.2, as they are directly related to the actions of derivations on the noncommutative algebras of functions.

Classically, the isometry group of the gravitational wave NW_6 is the group $\mathcal{N}_{\mathrm{L}} \times \mathcal{N}_{\mathrm{R}}$ induced by the left and right regular actions of the Lie group $\mathcal{N}$ on itself. The corresponding Killing vector fields live in the 11-dimensional Lie algebra $\mathfrak{g} := \mathfrak{n}_{\mathrm{L}} \oplus \mathfrak{n}_{\mathrm{R}}$ (The left and right actions generated by the central element T coincide). This isometry group contains an SO(4) subgroup acting by rotations in the transverse space $\boldsymbol{z} \in \mathbb{C}^2 \cong \mathbb{R}^4$, which is broken to U(2) by the Neveu-Schwarz background (1.14). This symmetry can be restored upon quantisation by instead letting the generators of $\mathfrak{g}$ act in a twisted fashion [29, 25, 103], as we now proceed to describe.

The action of an element $\nabla \in U(\mathfrak{g})$ as an algebra automorphism $\mathrm{C}^\infty(\mathfrak{n}^\vee) \to \mathrm{C}^\infty(\mathfrak{n}^\vee)$ will be denoted $f \mapsto \nabla \triangleright f$. The universal enveloping algebra $U(\mathfrak{g})$ is given the structure of a cocommutative bialgebra by introducing the "trivial" coproduct $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined by the homomorphism

$$\Delta(\nabla) = \nabla \otimes 1 + 1 \otimes \nabla \tag{4.88}$$

which generates the action of $U(\mathfrak{g})$ on the tensor product $C^\infty(\mathfrak{n}^\vee) \otimes C^\infty(\mathfrak{n}^\vee)$. Since ∇ is an automorphism of $C^\infty(\mathfrak{n}^\vee)$, the action of the coproduct is compatible with the pointwise (commutative) product of functions $\mu : C^\infty(\mathfrak{n}^\vee) \otimes C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$ in the sense that

$$\nabla \triangleright \mu(f \otimes g) = \mu \circ \Delta(\nabla) \triangleright (f \otimes g) \tag{4.89}$$

For example, the standard action of spacetime translations is given by

$$\partial^a \triangleright f = \partial^a f \tag{4.90}$$

for which (4.89) becomes the classical symmetric Leibniz rule.

Let us now pass to a noncommutative deformation of the algebra of functions on NW_6 via a quantisation map $\Omega : C^\infty(\mathfrak{n}^\vee) \to \overline{U(\mathfrak{n})^{\mathbb{C}}}$ corresponding to a specific $\star$-product $\star$ on $C^\infty(\mathfrak{n}^\vee)$ (or equivalently a specific operator ordering in $U(\mathfrak{n})$). This isomorphism can be used to induce an action of $U(\mathfrak{g})$ on the algebra $\overline{U(\mathfrak{n})^{\mathbb{C}}}$ through

$$\Omega(\nabla_\star) \triangleright \Omega(f) := \Omega(\nabla \triangleright f) \tag{4.91}$$

which defines a set of quantised operators $\nabla_\star = \nabla + O(\theta) : C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$. However, the bialgebra $U(\mathfrak{g})$ will no longer generate automorphisms with respect to the noncommutative $\star$-product on $C^\infty(\mathfrak{n}^\vee)$. It will only do so if its coproduct can be deformed to a non-cocommutative one $\Delta_\star = \Delta + O(\theta)$ such that the covariance condition

$$\nabla_\star \triangleright \mu_\star(f \otimes g) = \mu_\star \circ \Delta_\star(\nabla_\star) \triangleright (f \otimes g) \tag{4.92}$$

is satisfied, where $\mu_\star(f \otimes g) := f \star g$. This deformation is constructed by writing the $\star$-product $f \star g = \hat{\mathcal{D}}(f, g)$ in terms of a bi-differential operator as in (4.13) or (4.67) to define an invertible Abelian Drinfeld twist [85] element $\hat{\mathcal{F}}_\star \in \overline{U(\mathfrak{g})^{\mathbb{C}}} \otimes \overline{U(\mathfrak{g})^{\mathbb{C}}}$ through

$$f \star g = \mu \circ \hat{\mathcal{F}}_\star^{-1} \triangleright (f \otimes g) \tag{4.93}$$

It obeys the cocycle condition

$$(\hat{\mathcal{F}}_\star \otimes 1)(\Delta \otimes 1)\hat{\mathcal{F}}_\star = (1 \otimes \hat{\mathcal{F}}_\star)(\Delta \otimes 1)\hat{\mathcal{F}}_\star \tag{4.94}$$

and defines the twisted coproduct through

$$\Delta_\star := \hat{\mathcal{F}}_\star \circ \Delta \circ \hat{\mathcal{F}}_\star^{-1} \tag{4.95}$$

where $(f \otimes g) \circ (f' \otimes g') := ff' \otimes gg'$. This new coproduct obeys the requisite coassociativity condition $(\Delta_\star \otimes \mathbb{1}) \circ \Delta_\star = (\mathbb{1} \otimes \Delta_\star) \circ \Delta_\star$. The important property of the twist element $\hat{\mathcal{F}}_\star$ is that it modifies only the coproduct on the bialgebra $U(\mathfrak{g})$, while leaving the original product structure (inherited from the Lie algebra $\mathfrak{g} = \mathfrak{n}_{\rm L} \oplus \mathfrak{n}_{\rm R}$) unchanged.

As an example, let us illustrate how to compute the twisting of the quantised translation generators by the noncommutative geometry of NW_6. For this, we introduce a Weyl system $(\mathbb{V}, \boxplus, \mathrm{W}, \omega)$ corresponding to the chosen $\star$-product $\star$. With the same notations as in the previous section, for $a = 1, \dots, 5$ we may use (4.58), (4.64) with $\Pi = \Omega$, and (4.91) with $\nabla = \partial^a$ to compute

$$
\begin{aligned}
\Omega\big(\partial^a_\star\big) \triangleright \Omega\big(e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\big) \cdot \Omega\big(e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}\big) &= \Omega\big(\partial^a_\star\big) \triangleright e^{\frac{i}{2}\omega(\boldsymbol{k},\boldsymbol{k}')\mathsf{T}} \cdot \Omega\big(e^{i(\boldsymbol{k}\boxplus\boldsymbol{k}')\cdot\boldsymbol{x}}\big) \\
&= i e^{\frac{i}{2}\omega(\boldsymbol{k},\boldsymbol{k}')\mathsf{T}} \cdot \Omega\big((\boldsymbol{k}\boxplus\boldsymbol{k}')^a e^{i(\boldsymbol{k}\boxplus\boldsymbol{k}')\cdot\boldsymbol{x}}\big) \\
&= i \sum_i \Omega\big(d^a_{(1)i}(-i\boldsymbol{\partial}_\star)\big) \triangleright \Omega\big(e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\big) \\
&\qquad \cdot \Omega\big(d^a_{(2)i}(-i\boldsymbol{\partial}_\star)\big) \triangleright \Omega\big(e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}\big)
\end{aligned} \tag{4.96}
$$

where we have assumed that the group composition law of the Weyl system has an expansion of the form $(\boldsymbol{k} \boxplus \boldsymbol{k}')^a := \sum_i d^a_{(1)i}(\boldsymbol{k}) d^a_{(2)i}(\boldsymbol{k}')$. From the covariance condition (4.92) it then follows that the twisted coproduct assumes a Sweedler form

$$
\Delta_\star\left(\partial^a_\star\right) = i \sum_i d^a_{(1)i}(-i\boldsymbol{\partial}_\star) \otimes d^a_{(2)i}(-i\boldsymbol{\partial}_\star) \tag{4.97}
$$

Analogously, if we assume that the group cocycle of the Weyl system admits an expansion of the form $\omega(\boldsymbol{k}, \boldsymbol{k}') := \sum_i w^i_{(1)}(\boldsymbol{k}) w^i_{(2)}(\boldsymbol{k}')$, then a similar calculation gives the twisted coproduct of the quantised plane wave time derivative as

$$
\Delta_\star\left(\partial^\star_+\right) = \partial^\star_+ \otimes 1 + 1 \otimes \partial^\star_+ - \frac{1}{2} \sum_i w^i_{(1)}(-i\boldsymbol{\partial}_\star) \otimes w^i_{(2)}(-i\boldsymbol{\partial}_\star) \tag{4.98}
$$

Note that now the corresponding Leibniz rules (4.92) are no longer the usual ones associated with the product $\star$ but are the deformed, generically non-symmetric ones given by

$$
\begin{aligned}
\partial^a_\star \triangleright (f \star g) &= i \sum_i \big(d^a_{(1)i}(-i\boldsymbol{\partial}_\star) \triangleright f\big) \star \big(d^a_{(2)i}(-i\boldsymbol{\partial}_\star) \triangleright g\big) \\
\partial^\star_+ \triangleright (f \star g) &= \big(\partial^\star_+ \triangleright f\big) \star g + f \star \big(\partial^\star_+ \triangleright g\big) \\
&\quad -\frac{1}{2} \sum_i \big(w^i_{(1)}(-i\boldsymbol{\partial}_\star) \triangleright f\big) \star \big(w^i_{(2)}(-i\boldsymbol{\partial}_\star) \triangleright g\big)
\end{aligned} \tag{4.99}
$$

arising from the twisting of the coproduct. Thus these derivatives do *not* define derivations of the noncommutative algebra of functions, but rather implement the twisting of isometries of flat space appropriate to the plane wave geometry [77, 23, 16, 58].

In the language of quantum groups [80], the twisted isometry group of the spacetime NW_6 coincides with the quantum double of the cocommutative Hopf algebra $U(\mathfrak{n})$. The antipode $S_\star : U(\mathfrak{g}) \to U(\mathfrak{g})$ of the given non-cocommutative Hopf algebra structure on the bialgebra $U(\mathfrak{g})$ gives the dual action of the isometries of the noncommutative plane wave and provides the analog of inversion of isometry group elements. This analogy is made precise by computing $S_\star$ from the group inverses $\underline{\boldsymbol{k}}$ of elements $\boldsymbol{k} \in \mathbb{V}$ of the corresponding Weyl system. Symbolically, one has $S_\star(\boldsymbol{\partial}_\star) = \underline{\boldsymbol{\partial}_\star}$. In particular, if $\underline{\boldsymbol{k}} = -\boldsymbol{k}$ (as in the case of our symmetric $\star$-products) then $S_\star(\partial^a_\star) = -\partial^a_\star$ and the action of the antipode is trivial. In what follows we will only require the underlying bialgebra structure of $U(\mathfrak{g})$.

The generic non-triviality of the twisted coproducts (4.97) and (4.98) is consistent with and extends the fact that generic translations are not classically isometries of the plane wave geometry, but rather only appropriate twisted versions are [77, 23, 16, 58]. Similar computations can also be carried through for the remaining five isometry generators of $\mathfrak{g}$ and correspond to the right-acting counterparts of the derivatives above, giving the full action of the noncommutative isometry group on NW_6. We shall not display these formulas here. In the next section we will explicitly construct the quantised derivative operators $\partial^a_\star$ and $\partial^\star_+$ above. We now proceed to list the coproducts corresponding to our three $\star$-products.

4.4.1 Time Ordering

The Drinfeld twist $\hat{\mathcal{F}}_*$ for the time-ordered $\star$-product is the inverse of the exponential operator appearing in (4.26). Following the general prescription given above, from the group composition law (4.72) of the corresponding Weyl system we deduce the time-ordered coproducts

$$
\begin{aligned}
\Delta_*\left(\partial_-^*\right) &= \partial_-^* \otimes 1 + 1 \otimes \partial_-^* \qquad (4.100)\\
\Delta_*\left(\partial_*^i\right) &= \partial_*^i \otimes 1 + e^{i\theta\partial_-^*} \otimes \partial_*^i \\
\Delta_*\left(\overline{\partial}_*^i\right) &= \overline{\partial}_*^i \otimes 1 + e^{-i\theta\partial_-^*} \otimes \overline{\partial}_*^i
\end{aligned}
$$

while from the group cocycle (4.74) we obtain

$$
\Delta_*\left(\partial_+^*\right) = \partial_+^* \otimes 1 + 1 \otimes \partial_+^* + \theta e^{-i\theta\partial_-^*}\left(\boldsymbol{\partial}_*\right)^\top \otimes \overline{\boldsymbol{\partial}}_* - \theta e^{i\theta\partial_-^*}\left(\overline{\boldsymbol{\partial}}_*\right)^\top \otimes \boldsymbol{\partial}_* \qquad (4.101)
$$

The corresponding Leibniz rules read

$$
\begin{aligned}
\partial_-^* \triangleright (f * g) &= \left(\partial_-^* \triangleright f\right) * g + f * \left(\partial_-^* \triangleright g\right) \qquad (4.102)\\
\partial_+^* \triangleright (f * g) &= \left(\partial_+^* \triangleright f\right) * g + f * \left(\partial_+^* \triangleright g\right) \\
&\quad + \theta\left(e^{-i\theta\partial_-^*}(\boldsymbol{\partial}_*)^\top \triangleright f\right) * \left(\overline{\boldsymbol{\partial}}_* \triangleright g\right) - \theta\left(e^{i\theta\partial_-^*}(\overline{\boldsymbol{\partial}}_*)^\top \triangleright f\right) * \left(\boldsymbol{\partial}_* \triangleright g\right) \\
\partial_*^i \triangleright (f * g) &= \left(\partial_*^i \triangleright f\right) * g + \left(e^{i\theta\partial_-^*} \triangleright f\right) * \left(\partial_*^i \triangleright g\right) \\
\overline{\partial}_*^i \triangleright (f * g) &= \left(\overline{\partial}_*^i \triangleright f\right) * g + \left(e^{-i\theta\partial_-^*} \triangleright f\right) * \left(\overline{\partial}_*^i \triangleright g\right)
\end{aligned}
$$

4.4.2 Symmetric Time Ordering

The Drinfeld twist $\hat{\mathcal{F}}_\bullet$ associated to the symmetric time-ordered $\star$-product is given by the inverse of the exponential operator in (4.31). From the group composition law (4.76) of the corresponding Weyl system we deduce the symmetric time-ordered coproducts

$$
\begin{aligned}
\Delta_\bullet\left(\partial_-^\bullet\right) &= \partial_-^\bullet \otimes 1 + 1 \otimes \partial_-^\bullet \\
\Delta_\bullet\left(\partial_\bullet^i\right) &= \partial_\bullet^i \otimes e^{-\frac{i\theta}{2}\partial_-^\bullet} + e^{\frac{i\theta}{2}\partial_-^\bullet} \otimes \partial_\bullet^i \\
\Delta_\bullet\left(\overline{\partial}_\bullet^i\right) &= \overline{\partial}_\bullet^i \otimes e^{\frac{i\theta}{2}\partial_-^\bullet} + e^{-\frac{i\theta}{2}\partial_-^\bullet} \otimes \overline{\partial}_\bullet^i
\end{aligned} \tag{4.103}
$$

while from the group cocycle (4.78) we find

$$
\begin{aligned}
\Delta_\bullet\left(\partial_+^\bullet\right) &= \partial_+^\bullet \otimes 1 + 1 \otimes \partial_+^\bullet \\
&\quad + \theta e^{-\frac{i\theta}{2}\partial_-^\bullet}\left(\boldsymbol{\partial}_\bullet\right)^\top \otimes e^{-\frac{i\theta}{2}\partial_-^\bullet}\overline{\boldsymbol{\partial}}_\bullet - \theta e^{\frac{i\theta}{2}\partial_-^\bullet}\left(\overline{\boldsymbol{\partial}}_\bullet\right)^\top \otimes e^{\frac{i\theta}{2}\partial_-^\bullet}\boldsymbol{\partial}_\bullet
\end{aligned} \tag{4.104}
$$

The corresponding Leibniz rules are given by

$$
\begin{aligned}
\partial_-^\bullet \triangleright (f \bullet g) &= \left(\partial_-^\bullet \triangleright f\right) \bullet g + f \bullet \left(\partial_-^\bullet \triangleright g\right) \\
\partial_+^\bullet \triangleright (f \bullet g) &= \left(\partial_+^\bullet \triangleright f\right) \bullet g + f \bullet \left(\partial_+^\bullet \triangleright g\right) + \theta\left(e^{-\frac{i\theta}{2}\partial_-^\bullet}(\boldsymbol{\partial}_\bullet)^\top \triangleright f\right) \bullet \left(e^{-\frac{i\theta}{2}\partial_-^\bullet}\overline{\boldsymbol{\partial}}_\bullet \triangleright g\right) \\
&\quad - \theta\left(e^{\frac{i\theta}{2}\partial_-^\bullet}(\overline{\boldsymbol{\partial}}_\bullet)^\top \triangleright f\right) \bullet \left(e^{\frac{i\theta}{2}\partial_-^\bullet}\boldsymbol{\partial}_\bullet \triangleright g\right) \\
\partial_\bullet^i \triangleright (f \bullet g) &= \left(\partial_\bullet^i \triangleright f\right) \bullet \left(e^{-\frac{i\theta}{2}\partial_-^\bullet} \triangleright g\right) + \left(e^{\frac{i\theta}{2}\partial_-^\bullet} \triangleright f\right) \bullet \left(\partial_\bullet^i \triangleright g\right) \\
\overline{\partial}_\bullet^i \triangleright (f \bullet g) &= \left(\overline{\partial}_\bullet^i \triangleright f\right) \bullet \left(e^{\frac{i\theta}{2}\partial_-^\bullet} \triangleright g\right) + \left(e^{-\frac{i\theta}{2}\partial_-^\bullet} \triangleright f\right) \bullet \left(\overline{\partial}_\bullet^i \triangleright g\right)
\end{aligned} \tag{4.105}
$$

4.4.3 Weyl Ordering

Finally, for the Weyl-ordered $\star$-product (4.48) we read off the twist element $\hat{\mathcal{F}}_\star$ in the standard way, and use the associated group composition law (4.80) to write down the coproducts

$$
\begin{aligned}
\Delta_\star\left(\partial_-^\star\right) &= \partial_-^\star \otimes 1 + 1 \otimes \partial_-^\star \\
\Delta_\star\left(\partial_\star^i\right) &= \frac{\phi_{-\theta}\left(i\partial_-^\star\right)\partial_\star^i \otimes 1 + e^{i\theta\partial_-^\star} \otimes \phi_{-\theta}\left(i\partial_-^\star\right)\partial_\star^i}{\phi_{-\theta}\left(i\partial_-^\star \otimes 1 + 1 \otimes i\partial_-^\star\right)} \\
\Delta_\star\left(\overline{\partial}_\star^i\right) &= \frac{\phi_{\theta}\left(i\partial_-^\star\right)\overline{\partial}_\star^i \otimes 1 + e^{-i\theta\partial_-^\star} \otimes \phi_{\theta}\left(i\partial_-^\star\right)\overline{\partial}_\star^i}{\phi_{\theta}\left(i\partial_-^\star \otimes 1 + 1 \otimes i\partial_-^\star\right)}
\end{aligned} \tag{4.106}
$$

The remaining coproduct may be determined from the cocycle (4.82) as

$$
\begin{aligned}
\Delta_\star\left(\partial_+^\star\right) \;=\;& \partial_+^\star \otimes 1 + 1 \otimes \partial_+^\star \\
&+2i\theta\Big[\phi_\theta\left(i\partial_-^\star\right)\left(\boldsymbol{\partial}_\star\right)^\top \otimes \phi_\theta\left(i\partial_-^\star\right)\overline{\boldsymbol{\partial}}_\star - \phi_{-\theta}\left(i\partial_-^\star\right)\left(\overline{\boldsymbol{\partial}}_\star\right)^\top \otimes \phi_{-\theta}\left(i\partial_-^\star\right)\boldsymbol{\partial}_\star \\
&\quad+\big(\gamma_\theta(i\partial_-^\star)\otimes 1 - \gamma_\theta(i\partial_-^\star \otimes 1 + 1 \otimes i\partial_-^\star)\big)\big(\phi_\theta(i\partial_-^\star)\phi_{-\theta}(i\partial_-^\star)\overline{\boldsymbol{\partial}}_\star \cdot \boldsymbol{\partial}_\star \otimes 1\big) \\
&\quad+\big(1\otimes\gamma_\theta(i\partial_-^\star) - \gamma_\theta(i\partial_-^\star \otimes 1 + 1 \otimes i\partial_-^\star)\big)\big(1\otimes\phi_\theta(i\partial_-^\star)\phi_{-\theta}(i\partial_-^\star)\overline{\boldsymbol{\partial}}_\star \cdot \boldsymbol{\partial}_\star\big) \\
&\quad-\gamma_\theta\left(i\partial_-^\star \otimes 1 + 1 \otimes i\partial_-^\star\right)\big(e^{-i\theta\partial_-^\star}\phi_{-\theta}(i\partial_-^\star)(\boldsymbol{\partial}_\star)^\top \otimes \phi_\theta(i\partial_-^\star)\overline{\boldsymbol{\partial}}_\star \\
&\qquad\qquad+e^{i\theta\partial_-^\star}\phi_\theta(i\partial_-^\star)(\overline{\boldsymbol{\partial}}_\star)^\top \otimes \phi_{-\theta}(i\partial_-^\star)\boldsymbol{\partial}_\star\big)\Big]
\end{aligned} \tag{4.107}
$$

In (4.106) and (4.107) the functionals of the derivative operator $i\partial_-^\star \otimes 1 + 1 \otimes i\partial_-^\star$ are understood as usual in terms of the power series expansions given in section 4.2.3. This leads to the corresponding Leibniz rules.

$$
\begin{aligned}
\partial^{\star}_{-} \triangleright (f \star g) &= \big(\partial^{\star}_{-} \triangleright f\big) \star g + f \star \big(\partial^{\star}_{-} \triangleright g\big) \qquad (4.108)\\
\partial^{\star}_{+} \triangleright (f \star g) &= \big(\partial^{\star}_{+} \triangleright f\big) \star g + f \star \big(\partial^{\star}_{+} \triangleright g\big)\\
&\quad +2i\theta \Bigg\{ \Big(\frac{(1 - e^{-i\theta\partial^{\star}_{-}}(\boldsymbol{\partial}_{\star})^{\top}}{i\theta\partial^{\star}_{-}} \triangleright f \Big) \star \Big(\frac{(1 - e^{-i\theta\partial^{\star}_{-}}\overline{\boldsymbol{\partial}}_{\star}}{i\theta\partial^{\star}_{-}} \triangleright g \Big)\\
&\qquad - \Big(\frac{(1 - e^{i\theta\partial^{\star}_{-}}(\overline{\boldsymbol{\partial}}_{\star})^{\top}}{i\theta\partial^{\star}_{-}} \triangleright f \Big) \star \Big(\frac{(1 - e^{i\theta\partial^{\star}_{-}}\boldsymbol{\partial}_{\star}}{i\theta\partial^{\star}_{-}} \triangleright g \Big)\\
&\qquad + \Big(\Big[\frac{1}{2} + \frac{(1 + i\theta\partial^{\star}_{-})e^{-i\theta\partial^{\star}_{-}} - 1}{(e^{-i\theta\partial^{\star}_{-}} - 1)^2} \Big] \frac{\sin^2(\frac{\theta}{2}\partial^{\star}_{-})\overline{\boldsymbol{\partial}}_{\star} \cdot \boldsymbol{\partial}_{\star}}{(\theta\partial^{\star}_{-})^2} \triangleright f \Big) \star g\\
&\qquad + f \star \Big(\Big[\frac{1}{2} + \frac{(1 + i\theta\partial^{\star}_{-})e^{-i\theta\partial^{\star}_{-}} - 1}{(e^{-i\theta\partial^{\star}_{-}} - 1)^2} \Big] \frac{\sin^2(\frac{\theta}{2}\partial^{\star}_{-})\overline{\boldsymbol{\partial}}_{\star} \cdot \boldsymbol{\partial}_{\star}}{(\theta\partial^{\star}_{-})^2} \triangleright g \Big)\\
&\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{B_{n+1}(-i\theta)^{n-2}}{k!(n-k)!} \Big[\big((\partial^{\star}_{-})^{n-k-2} \sin^2(\frac{\theta}{2}\partial^{\star}_{-})\overline{\boldsymbol{\partial}}_{\star} \cdot \boldsymbol{\partial}_{\star} \triangleright f\big) \star \big((\partial^{\star}_{-})^{k} \triangleright g\big)\\
&\qquad + \big((\partial^{\star}_{-})^{n-k} \triangleright f\big) \star \big((\partial^{\star}_{-})^{k-2} \sin^2(\frac{\theta}{2}\partial^{\star}_{-})\overline{\boldsymbol{\partial}}_{\star} \cdot \boldsymbol{\partial}_{\star} \triangleright g\big)\\
&\qquad - \big((e^{-i\theta\partial^{\star}_{-}} - 1)(\partial^{\star}_{-})^{n-k-1}(\boldsymbol{\partial}_{\star})^{\top} \triangleright f\big) \star \big((e^{-i\theta\partial^{\star}_{-}} - 1)(\partial^{\star}_{-})^{k-1}\overline{\boldsymbol{\partial}}_{\star} \triangleright g\big)\\
&\qquad - \big((e^{i\theta\partial^{\star}_{-}} - 1)(\partial^{\star}_{-})^{n-k-1}(\overline{\boldsymbol{\partial}}_{\star})^{\top} \triangleright f\big) \star \big((e^{i\theta\partial^{\star}_{-}} - 1)(\partial^{\star}_{-})^{k-1}\boldsymbol{\partial}_{\star} \triangleright g\big) \Big]\\
\partial^{i}_{\star} \triangleright (f \star g) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{B_n(i\theta)^{n-1}}{k!(n-k)!} \big[\big((e^{i\theta\partial^{\star}_{-}} - 1)(\partial^{\star}_{-})^{n-k-1}\partial^{i}_{\star} \triangleright f\big) \star \big((\partial^{\star}_{-})^{k} \triangleright g\big)\\
&\qquad + \big(e^{i\theta\partial^{\star}_{-}}(\partial^{\star}_{-})^{n-k} \triangleright f\big) \star \big((e^{i\theta\partial^{\star}_{-}} - 1)(\partial^{\star}_{-})^{k-1}\partial^{i}_{\star} \triangleright g\big) \big]\\
\overline{\partial}^{i}_{\star} \triangleright (f \star g) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{B_n(-i\theta)^{n-1}}{k!(n-k)!} \Big[\big((e^{-i\theta\partial^{\star}_{-}} - 1)(\partial^{\star}_{-})^{n-k-1}\overline{\partial}^{i}_{\star} \triangleright f\big) \star \big((\partial^{\star}_{-})^{k} \triangleright g\big)\\
&\qquad + \big(e^{-i\theta\partial^{\star}_{-}}(\partial^{\star}_{-})^{n-k} \triangleright f\big) \star \big((e^{-i\theta\partial^{\star}_{-}} - 1)(\partial^{\star}_{-})^{k-1}\overline{\partial}^{i}_{\star} \triangleright g\big) \Big]
\end{aligned}
$$

Note that a common feature to all three deformations is that the coproduct of the quantisation of the light-cone position translation generator ∂_{-} coincides with the trivial one (4.88), and thereby yields the standard symmetric Leibniz rule with respect to the pertinent $\star$-product. This owes to the fact that the action of ∂_- on the spacetime NW_6 corresponds to the commutative action of the central Lie algebra generator T, whose left and right actions coincide. In the next section we shall see that the action of the quantised translations in x^- on $C^\infty(\mathfrak{n}^\vee)$ coincides with the standard commutative action (4.90). This is consistent with the fact that all frames of reference for the spacetime NW_6 possess an x^--translational symmetry, while translational symmetries in the other coordinates depend crucially on the frame and generally need to be twisted in order to generate an isometry of NW_6. Notice also that ordinary time translation invariance is always broken by the time-dependent Neveu-Schwarz background (1.14).

4.5 Derivatives

In this section we will systematically construct a set of quantised derivative operators $\partial_\star^a$, $a = 1, \ldots, 6$ satisfying the conditions of the previous section. In general, there is no unique way to build up such derivatives. To this end, we will impose some weak conditions, namely that the quantised derivatives be deformations of ordinary derivatives, $\partial_\star^a = \partial^a + O(\theta)$, and that they commute among themselves, $[\partial_\star^a, \partial_\star^b]_\star = 0$. The latter condition is understood as a requirement for the iterated action of the derivatives on functions $f \in C^\infty(\mathfrak{n}^\vee)$, $[\partial_\star^a, \partial_\star^b]_\star \triangleright f = 0$ or equivalently

$$\partial_\star^a \triangleright \left(\partial_\star^b \triangleright f\right) = \partial_\star^b \triangleright \left(\partial_\star^a \triangleright f\right) \tag{4.109}$$

For the former condition, the simplest consistent choice is to assume a linear derivative deformation on the coordinate functions, $[\partial_\star^a, x_b]_\star = \delta_b^a + i\theta\rho_{bc}^a\partial_\star^c$, which is understood as the requirement

$$[\partial_\star^a, x_b]_\star \triangleright f := \partial_\star^a \triangleright \left(x_b \star f\right) - x_b \star \left(\partial_\star^a \triangleright f\right) = \delta_b^a f + i\theta\rho_{bc}^a\partial_\star^c \triangleright f \tag{4.110}$$

A set of necessary conditions on the constant tensors $\rho_{bc}^a \in \mathbb{R}$ may be derived by demanding consistency of the derivatives with the original $\star$-commutators of coordinates (4.9). Applying the Jacobi identity for the $\star$-commutators between $\partial_\star^a$, x_b

and x_c leads to the relations

$$\begin{aligned} \rho^a_{bc} - \rho^a_{cb} &= C^a_{bc} \\ \rho^a_{bc}\rho^c_{de} - \rho^a_{dc}\rho^c_{be} &= C^c_{bd}\rho^a_{ce} \end{aligned} \tag{4.111}$$

With these requirements we now seek to find quantised derivative operators $\partial^a_\star$ as functionals of ordinary derivatives ∂^a acting on $C^\infty(\mathfrak{n}^\vee)$ as in (4.90). However, there are (uncountably) infinitely many solutions ρ^a_{bc} obeying (4.111) [37] with C^c_{ab} the structure constants of the Lie algebra $\mathfrak{n}$ given by (1.5). We will choose the simplest intuitive one defined by the $\star$-commutators

$$\begin{array}{llll} \left[\partial^\star_-, x^-\right]_\star = 1 & \left[\partial^\star_+, x^-\right]_\star = 0 & \left[\partial^i_\star, x^-\right]_\star = -i\theta\partial^i_\star & \left[\overline{\partial}^i_\star, x^-\right]_\star = i\theta\overline{\partial}^i_\star \\ \left[\partial^\star_-, x^+\right]_\star = 0 & \left[\partial^\star_+, x^+\right]_\star = 1 & \left[\partial^i_\star, x^+\right]_\star = 0 & \left[\overline{\partial}^i_\star, x^+\right]_\star = 0 \\ \left[\partial^\star_-, z_i\right]_\star = 0 & \left[\partial^\star_+, z_i\right]_\star = -i\theta\overline{\partial}^i_\star & \left[\partial^i_\star, z_j\right]_\star = \delta^i_j & \left[\overline{\partial}^i_\star, z_j\right]_\star = 0 \\ \left[\partial^\star_-, \overline{z}_i\right]_\star = 0 & \left[\partial^\star_+, \overline{z}_i\right]_\star = i\theta\partial^i_\star & \left[\partial^i_\star, \overline{z}_j\right]_\star = 0 & \left[\overline{\partial}^i_\star, \overline{z}_j\right]_\star = \delta^i_j \end{array} \tag{4.112}$$

whose $\mathcal{O}(\theta)$ parts mimick the structure of the Lie brackets (1.5). All other admissible choices for ρ^a_{bc} can be mapped into those given by (4.112) via non-linear redefinitions of the derivative operators $\partial^a_\star$ [37]. It is important to realise that the quantised derivatives do not generally obey the classical Leibniz rule, i.e. $\partial^a_\star \triangleright (fg) \neq f(\partial^a_\star \triangleright g) + (\partial^a_\star \triangleright f)g$ in general, but rather the generalised Leibniz rules spelled out in the previous section in order to achieve consistency for $\theta \neq 0$. Let us now construct the three sets of derivatives of interest to us here.

4.5.1 Time Ordering

For the time ordered case, we use (4.26) to compute the $*$-products

$$\begin{aligned} x^- * f &= \left(x^- - i\theta \boldsymbol{z} \cdot \boldsymbol{\partial} + i\theta \overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{\partial}}\right) f \\ x^+ * f &= x^+ f \\ z_i * f &= \left(z_i - i\theta x^+ \overline{\partial}^i\right) f \\ \overline{z}_i * f &= \left(\overline{z}_i + i\theta x^+ \partial^i\right) f \end{aligned} \tag{4.113}$$

Substituting these into (4.110) using (4.112) then shows that the actions of the $*$-derivatives simply coincide with the canonical actions of the translation generators on $C^\infty(\mathfrak{n}^\vee)$, so that

$$\partial_*^a \triangleright f = \partial^a f \tag{4.114}$$

Thus the time-ordered noncommutative geometry of NW_6 is invariant under *ordinary* translations of the spacetime in all coordinate directions, with the generators obeying the twisted Leibniz rules (4.102).

4.5.2 Symmetric Time Ordering

Next, consider the case of symmetric time ordering. From (4.31) we compute the $\bullet$-products

$$\begin{aligned} x^- \bullet f &= \left(x^- - \frac{i\theta}{2} \boldsymbol{z} \cdot \boldsymbol{\partial} + \frac{i\theta}{2} \overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{\partial}}\right) f \\ x^+ \bullet f &= x^+ f \\ z_i \bullet f &= e^{\frac{i\theta}{2}\partial_-} \left(z_i - i\theta x^+ \overline{\partial}^i\right) f \\ \overline{z}_i \bullet f &= e^{-\frac{i\theta}{2}\partial_-} \left(\overline{z}_i + i\theta x^+ \partial^i\right) f \end{aligned} \tag{4.115}$$

Substituting (4.115) into (4.110) using (4.112) along with the derivative rule

$$e^{i\theta\partial_-} x^- = (x^- + i\theta) e^{i\theta\partial_-} \tag{4.116}$$

we find that the actions of the $\bullet$-derivatives on $C^\infty(\mathfrak{n}^\vee)$ are generically non-trivial and given by

$$
\begin{aligned}
\partial_-^\bullet \triangleright f &= \partial_- f \\
\partial_+^\bullet \triangleright f &= \partial_+ f \\
\partial_\bullet^i \triangleright f &= e^{-\frac{i\theta}{2}\partial_-}\partial^i f \\
\overline{\partial}_\bullet^i \triangleright f &= e^{\frac{i\theta}{2}\partial_-}\overline{\partial}^i f
\end{aligned}
\tag{4.117}
$$

Only the transverse space derivatives are modified: a consequence of the invariance of the Brinkman coordinate system under translations of the light-cone coordinates $x^\pm$. Again the twisted Leibniz rules (4.105) are straightforward to verify in this instance.

4.5.3 Weyl Ordering

Finally, from the Weyl-ordered $\star$-product (4.48) we compute

$$
\begin{aligned}
x^- \star f &= \Bigg[x^- + \left(1 - \frac{1}{\phi_{-\theta}(i\partial_-)}\right)\frac{\boldsymbol{z}\cdot\boldsymbol{\partial}}{\partial_-} + \left(1 - \frac{1}{\phi_{\theta}(i\partial_-)}\right)\frac{\overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}}{\partial_-} \\
&\qquad - 2\theta x^+\left(\frac{2}{\theta\partial_-} - \cot\left(\frac{\theta}{2}\partial_-\right)\right)\frac{\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}}{\partial_-}\Bigg] f \\
x^+ \star f &= x^+ f \\
z_i \star f &= \left[\frac{z_i}{\phi_{-\theta}(i\partial_-)} + 2x^+\left(1 - \frac{1}{\phi_{-\theta}(i\partial_-)}\right)\frac{\overline{\partial}^i}{\partial_-}\right] f \\
\overline{z}_i \star f &= \left[\frac{\overline{z}_i}{\phi_{\theta}(i\partial_-)} + 2x^+\left(1 - \frac{1}{\phi_{\theta}(i\partial_-)}\right)\frac{\partial^i}{\partial_-}\right] f
\end{aligned}
\tag{4.118}
$$

From (4.110), (4.112) and the derivative rule

$$
\phi_\theta(i\partial_-)x^- = \frac{e^{i\theta\partial_-} - \phi_\theta(i\partial_-)}{i\partial_-} + x^-\phi_\theta(i\partial_-) \tag{4.119}
$$

it then follows that the actions of the $\star$-derivatives on $C^\infty(\mathfrak{n}^\vee)$ are given by

$$\begin{aligned}
\partial_-^\star \triangleright f &= \partial_- f && (4.120)\\
\partial_+^\star \triangleright f &= \left[\partial_+ + 2\left(1-\frac{\sin(\theta\partial_-)}{\theta\partial_-}\right)\frac{\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}}{\partial_-}\right] f\\
\partial_\star^i \triangleright f &= -\frac{1-e^{i\theta\partial_-}}{i\theta\partial_-}\partial^i f\\
\overline{\partial}_\star^i \triangleright f &= \frac{1-e^{-i\theta\partial_-}}{i\theta\partial_-}\overline{\partial}^i f
\end{aligned}$$

Thus in the completely symmetric noncommutative geometry of NW_6 both the light-cone and the transverse space of the plane wave are generically only invariant under rather complicated twisted translations, obeying the involved Leibniz rules (4.108).

4.6 Integrals

The final ingredient required to construct noncommutative field theory action functionals is a definition of integration. At the algebraic level, we define an integral to be a trace on the algebra $\overline{U(\mathfrak{n})^{\mathbb{C}}}$, i.e. a map $⨍ : \overline{U(\mathfrak{n})^{\mathbb{C}}} \to \mathbb{C}$ which is linear,

$$⨍ \big(c_1\Omega(f) + c_2\Omega(g)\big) = c_1 ⨍ \Omega(f) + c_2 ⨍ \Omega(g) \tag{4.121}$$

for all $f, g \in C^\infty(\mathfrak{n}^\vee)$ and $c_1, c_2 \in \mathbb{C}$, and which is cyclic,

$$⨍ \Omega(f)\cdot\Omega(g) = ⨍ \Omega(g)\cdot\Omega(f) \tag{4.122}$$

We define the integral in the $\star$-product formalism using the usual definitions for the integration of commuting Schwartz functions in $C^\infty(\mathbb{R}^6)$. Then the linearity property (4.121) is automatically satisfied. To satisfy the cyclicity requirement (4.122), we introduce [32, 20, 2, 43, 49] a measure κ on $\mathbb{R}^6$ which deforms the flat space volume element $\mathrm{d}\boldsymbol{x}$ and define

$$⨍ \Omega(f) := \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})f(\boldsymbol{x}) \tag{4.123}$$

The measure κ is chosen in order to achieve the property (4.122), so that

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\kappa(\boldsymbol{x})(f \star g)(\boldsymbol{x}) = \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\kappa(\boldsymbol{x})(g \star f)(\boldsymbol{x}) \tag{4.124}$$

Such a measure always exists [32, 43, 49] and its inclusion in the present context is natural for the curved spacetime NW_6 which we are considering here. It is important note that, for the $\star$-products that we use, a measure which satisfies (4.124) gives the integral the additional property

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\kappa(\boldsymbol{x})(f \star g)(\boldsymbol{x}) = \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\kappa(\boldsymbol{x})f(\boldsymbol{x})g(\boldsymbol{x}) \tag{4.125}$$

providing an explicit realisation of the Connes-Flato-Sternheimer conjecture [49].

Since the coordinate functions x_a generate the noncommutative algebra, the cyclicity constraint (4.124) is equivalent to the $\star$-commutator condition

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\kappa(\boldsymbol{x})\left[(x_a)^n, f(\boldsymbol{x})\right]_\star = 0 \tag{4.126}$$

which must hold for arbitrary functions $f \in C^\infty(\mathbb{R}^6)$ (for which the integral makes sense) and for all $n \in \mathbb{N}$, $a = 1, \ldots, 6$. We may rewrite any commutator of the form $\left[\hat{x}^n, \hat{y}\right]$ using the following identity

$$\left[\hat{x}^{n+1}, \hat{y}\right] = \sum_{m=0}^{n} \binom{n}{m} \hat{x}^{n-m} \left[\hat{x}, \hat{y}\right] \hat{x}^m \tag{4.127}$$

allowing us to expand (4.126) to the form

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\kappa(\boldsymbol{x}) \sum_{m=0}^{n} \binom{n}{m} (x_a)^{n-m} \star \left[x_a, f(\boldsymbol{x})\right]_\star \star (x_a)^m = 0 \tag{4.128}$$

We may thus insert the explicit form of $[x_a, f]_\star$ for generic f and use the ordinary integration by parts property

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x} f(\boldsymbol{x})g(\boldsymbol{x})\partial_a^n h(\boldsymbol{x}) = (-1)^n \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\Big[f(\boldsymbol{x})\left(\partial_a^n g(\boldsymbol{x})\right)h(\boldsymbol{x}) + \left(\partial_a^n f(\boldsymbol{x})\right)g(\boldsymbol{x})h(\boldsymbol{x})\Big] \tag{4.129}$$

for Schwartz functions $f, g, h \in C^\infty(\mathbb{R}^6)$. This will lead to a number of constraints on the measure κ.

The trace (4.123) can also be used to define an inner product $(-,-) : C^\infty(\mathfrak{n}^\vee) \times C^\infty(\mathfrak{n}^\vee) \to \mathbb{C}$ through

$$(f,g) := \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})\big(\overline{f} \star g\big)(\boldsymbol{x}) \tag{4.130}$$

Note that this is different from the inner product introduced in Section 1.3.1. When we come to deal with the variational principle in the next section, we shall require that our $\star$-derivative operators $\partial_\star^a$ be anti-Hermitean with respect to the inner product (4.130), i.e. $(f, \partial_\star^a \triangleright g) = -(\partial_\star^a \triangleright f, g)$, or equivalently

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})\big(\overline{f} \star \partial_\star^a \triangleright g\big)(\boldsymbol{x}) = -\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})\big(\overline{\partial_\star^a \triangleright f} \star g\big)(\boldsymbol{x}) \tag{4.131}$$

This allows for a generalised integration by parts property [43] for our noncommutative integral. As always, we will now go through our list of $\star$-products to explore the properties of the integral in each case. We will find that the measure κ is not uniquely determined by the above criteria and that there is a large flexibility in the choices that can be made. We will also find that the derivatives of the previous section must be modified by a κ-dependent shift in order to satisfy (4.131).

4.6.1 Time Ordering

Using (4.113) along with the analogous $*$-products $f * x_a$

$$\begin{aligned} f * x^- &= x^- f \\ f * x^+ &= x^+ f \\ f * z_i &= (z_i + i\theta x^+ \overline{\partial}^i) e^{-i\theta\partial_-} f \\ f * \overline{z}_i &= (\overline{z}_i - i\theta x^+ \partial^i) e^{i\theta\partial_-} f \end{aligned} \tag{4.132}$$

we arrive at the $*$-commutators

$$\begin{aligned}
\left[x^-, f\right]_* &= i\theta\left(\overline{z}\cdot\overline{\boldsymbol{\partial}} - z\cdot\boldsymbol{\partial}\right)f \qquad (4.133)\\
\left[x^+, f\right]_* &= 0\\
\left[z_i, f\right]_* &= z_i\left(1-e^{-i\theta\partial_-}\right)f - i\theta x^+\left(1+e^{-i\theta\partial_-}\right)\overline{\partial}^i f\\
\left[\overline{z}_i, f\right]_* &= \overline{z}_i\left(1-e^{i\theta\partial_-}\right)f + i\theta x^+\left(1+e^{i\theta\partial_-}\right)\partial^i f
\end{aligned}$$

When inserted into (4.128), after integration by parts and application of the derivative rule (4.116) these expressions imply constraints on the corresponding measure κ_*. We highlight the steps involved in calculating the constraint for x^-

$$\begin{aligned}
i\theta\int_{\mathbb{R}^6}\mathrm{d}\boldsymbol{x}\kappa_*\sum_{m=0}^{n}\binom{n}{m}(x^-)^n\left(\overline{z}\cdot\overline{\boldsymbol{\partial}} - z\cdot\boldsymbol{\partial}\right)f &= 0 \qquad (4.134)\\
\Longrightarrow \int_{\mathbb{R}^6}\mathrm{d}\boldsymbol{x}\kappa_*\left(\overline{z}\cdot\overline{\boldsymbol{\partial}} - z\cdot\boldsymbol{\partial}\right)f &= 0\\
\cancel{\kappa_*} + z\cdot\boldsymbol{\partial}\kappa_* &= \cancel{\kappa_*} + \overline{z}\cdot\overline{\boldsymbol{\partial}}\kappa_*\\
z\cdot\boldsymbol{\partial}\kappa_* &= \overline{z}\cdot\overline{\boldsymbol{\partial}}\kappa_*
\end{aligned}$$

Clearly κ_* terms appear from both $z\cdot\boldsymbol{\partial}$ and $\overline{z}\cdot\overline{\boldsymbol{\partial}}$ parts of (4.133). As κ-Minkowski only consists of a single set of z_i (missing the complementary $\overline{z_i}$), this cancellation does not occur [2] and (4.134) does not simplify so elegantly. Unfortunately, this limits the generalisation outlined in this thesis and one must choose the appropriate $\kappa(\boldsymbol{x})$ for calculations on κ-Minkowski. This is true for all of our orderings.

The remaining constraints are

$$\begin{aligned}
\left(1+e^{i\theta\partial_-}\right)\overline{\partial}^i\kappa_* &= 0 \qquad (4.135)\\
\left(1-e^{-i\theta\partial_-}\right)\partial^i\kappa_* &= 0
\end{aligned}$$

It is straightforward to see that the equations (4.135) imply that the measure must be independent of both the light-cone position and transverse coordinates, so that

$$\partial_-\kappa_* = \partial^i\kappa_* = \overline{\partial}^i\kappa_* = 0 \qquad (4.136)$$

However, the derivative ∂_+^* in (4.114) does not satisfy the anti-hermiticity requirement (4.131). This can be remedied by translating it by a logarithmic derivative of the measure κ_* and defining the modified $*$-derivative

$$\widetilde{\partial}_+^* = \partial_+ + \frac{1}{2}\partial_+ \ln \kappa_* \tag{4.137}$$

The remaining $*$-derivatives in (4.114) are unaltered. While this redefinition has no adverse effects on the commutation relations (4.112), the action $\widetilde{\partial}_+^* \triangleright f$ contains an additional linear term in f even if the function f is independent of the time coordinate x^+.

It is also unfortunate that the κ-Minkowski and NW measure constraints are mutually exclusive. It is interesting to note that the measure is much simpler for NW, an unexpected result.

4.6.2 Symmetric Time Ordering

Using (4.115) along with the corresponding $\bullet$-products $f \bullet x_a$

$$\begin{aligned}
f \bullet x^- &= \left(x^- + i\frac{\theta}{2} \boldsymbol{z} \cdot \boldsymbol{\partial} - i\frac{\theta}{2} \overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{\partial}} \right) f \qquad (4.138) \\
f \bullet x^+ &= x^+ f \\
f \bullet z_i &= \left(z_i + i\theta x^+ \overline{\partial}^i \right) e^{-i\frac{\theta}{2}\partial_-} f \\
f \bullet \overline{z}_i &= \left(\overline{z}_i - i\theta x^+ \partial^i \right) e^{i\frac{\theta}{2}\partial_-} f
\end{aligned}$$

we arrive at the $\bullet$-commutators

$$\begin{aligned}
\left[x^-, f \right]_\bullet &= i\theta \left(\overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{\partial}} - \boldsymbol{z} \cdot \boldsymbol{\partial} \right) f \qquad (4.139) \\
\left[x^+, f \right]_\bullet &= 0 \\
\left[z_i, f \right]_\bullet &= 2iz_i \sin\left(\frac{\theta}{2}\partial_- \right) f - 2i\theta x^+ \overline{\partial}^i \cos\left(\frac{\theta}{2}\partial_- \right) f \\
\left[\overline{z}_i, f \right]_\bullet &= -2i\overline{z}_i \sin\left(\frac{\theta}{2}\partial_- \right) f + 2i\theta x^+ \partial^i \cos\left(\frac{\theta}{2}\partial_- \right) f \qquad (4.140)
\end{aligned}$$

Substituting these into (4.128) and integrating by parts, we arrive at constraints on the measure $\kappa_\bullet$ given by

$$
\begin{aligned}
(1-\overline{\partial}^i)\sin\left(\frac{\theta}{2}\partial_-\right)\kappa_\bullet &= 0 \\
(1+\partial^i)\sin\left(\frac{\theta}{2}\partial_-\right)\kappa_\bullet &= 0 \\
\boldsymbol{z}\cdot\boldsymbol{\partial}\kappa_\bullet &= \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}\kappa_\bullet
\end{aligned}
\tag{4.141}
$$

which can be reduced to the conditions

$$
\boldsymbol{z}\cdot\boldsymbol{\partial}\kappa_\bullet = \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}\kappa_\bullet \quad \partial_-\kappa_\bullet = 0 \tag{4.142}
$$

Now the derivative operators $\partial_+^\bullet$, $\partial_\bullet^i$ and $\overline{\partial}_\bullet^i$ all violate the requirement (4.131). Introducing translates of $\partial_\bullet^i$ and $\overline{\partial}_\bullet^i$ analogously to what we did in (4.137) is problematic. While such a shift does not alter the canonical commutation relations between the coordinates and derivatives, i.e. the algebraic properties of the differential operators, it does violate the $\bullet$-commutator relationships (4.110) and (4.112) for generic functions f. Consistency between differential operator and function commutators would only be possible in this case by demanding that multiplication from the left follow a Leibniz-like rule for the translated part.

Thus in order to satisfy both sets of constraints, we are forced to further require that the measure $\kappa_\bullet$ depend only on the plane wave time coordinate x^+ so that (4.142) truncates to

$$
\partial^i\kappa_\bullet = \overline{\partial}^i\kappa_\bullet = \partial_-\kappa_\bullet = 0 \tag{4.143}
$$

The logarithmic translation of $\partial_+^\bullet$ must still be applied in order to ensure that the time derivative is anti-hermitean with respect to the noncommutative inner product. This modifies its action to

$$
\widetilde{\partial}_+^\bullet = \partial_+ + \frac{1}{2}\partial_+\ln\kappa_\bullet \tag{4.144}
$$

The actions of all other $\bullet$-derivatives are as in (4.117). Again this shifting has no adverse effects on (4.112), but it carries the same warning as in the time ordered case regarding extra linear terms from the action $\widetilde{\partial}_+^\bullet \triangleright f$.

4.6.3 Weyl Ordering

Finally, the Weyl ordered $\star$-products (4.118) along with the corresponding $f \star x_a$ products

$$
\begin{aligned}
f \star x^- &= \left\{ \left(1 - \frac{1}{\phi_\theta(i\partial_-)}\right) \frac{\boldsymbol{z} \cdot \boldsymbol{\partial}}{\partial_-} + \left(1 - \frac{1}{\phi_{-\theta}(i\partial_-)}\right) \frac{\overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{\partial}}}{\partial_-} \right. \\
&\quad \left. + x^- - 2\theta x^+ \left(\frac{2}{\theta\partial_-} - \cot\left(\frac{\theta}{2}\partial_-\right)\right) \frac{\boldsymbol{\partial} \cdot \overline{\boldsymbol{\partial}}}{\partial_-} \right\} f \\
f \star x^+ &= x^+ f \\
f \star z_i &= \left\{ \frac{z_i}{\phi_\theta(i\partial_-)} + 2x^+ \left(1 - \frac{1}{\phi_\theta(i\partial_-)}\right) \frac{\overline{\partial}^i}{\partial_-} \right\} f \\
f \star \overline{z}_i &= \left\{ \frac{\overline{z}_i}{\phi_{-\theta}(i\partial_-)} + 2x^+ \left(1 - \frac{1}{\phi_{-\theta}(i\partial_-)}\right) \frac{\partial^i}{\partial_-} \right\} f
\end{aligned} \tag{4.145}
$$

lead to the $\star$-commutators

$$
\begin{aligned}
\left[x^-, f\right]_\star &= i\theta \left(\overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{\partial}} - \boldsymbol{z} \cdot \boldsymbol{\partial}\right) f \\
\left[x^+, f\right]_\star &= 0 \\
\left[z_i, f\right]_\star &= i\theta \left(z_i \partial_- - 2x^+ \overline{\partial}^i\right) f \\
\left[\overline{z}_i, f\right]_\star &= i\theta \left(-\overline{z}_i \partial_- + 2x^+ \partial^i\right) f
\end{aligned} \tag{4.146}
$$

Substituting these commutation relations into (4.128), integrating by parts and using the derivative rule (4.116) along with (4.119) leads to the corresponding measure constraints

$$
\begin{aligned}
z_i \partial_- \kappa_\star &= 2x^+ \overline{\partial}^i \kappa_\star \\
\overline{z}_i \partial_- \kappa_\star &= 2x^+ \partial^i \kappa_\star \\
\boldsymbol{z} \cdot \boldsymbol{\partial} \kappa_\star &= \overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{\partial}} \kappa_\star
\end{aligned} \tag{4.147}
$$

Again these differential equations imply that the measure $\kappa_\star$ depends only on the plane wave time coordinate x^+ so that

$$\partial_- \kappa_\star = \partial^i \kappa_\star = \overline{\partial}^i \kappa_\star = 0 \tag{4.148}$$

Translating the derivative operator $\partial_+^\star$ as before in order to satisfy (4.131) yields the modified derivative

$$\widetilde{\partial}_+^\star = \partial_+ + 2\left(1 - \frac{\sin(\theta\partial_-)}{\theta\partial_-}\right)\frac{\overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}}{\partial_-} + \frac{1}{2}\partial_+ \ln \kappa_\star \tag{4.149}$$

with the remaining $\star$-derivatives in (4.120) unchanged. Once again this produces no major alteration to (4.112) but does yield extra linear terms in the actions $\widetilde{\partial}_+^\star \triangleright f$.

Chapter 5

Scalar Field Theory

We are now ready to apply the detailed constructions of the preceding sections to the analysis of noncommutative field theories on the plane wave NW_6, regarded as the worldvolume of a non-symmetric D5-brane [67]. In this paper we will only study the simplest example of free scalar fields, leaving the detailed analysis of interacting field theories and higher spin (fermionic and gauge) fields for future work. The analysis of this section will set the stage for more detailed studies of noncommutative field theories in these settings, and will illustrate some of the generic features that one can expect.

Given a real scalar field $\Phi \in C^\infty(\mathfrak{n}^\vee)$ of mass m, we define an action functional using the integral (4.123) by

$$S[\Phi] = \int\limits_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x}) \left[\frac{1}{2}\eta_{ab}\big(\widetilde{\partial}{}^a_\star \triangleright \Phi\big) \star \big(\widetilde{\partial}{}^b_\star \triangleright \Phi\big) + \frac{1}{2}m^2 \Phi \star \Phi\right] \tag{5.1}$$

where η_{ab} is the invariant Minkowski tensor induced by the inner product (1.7) with the non-vanishing components $\eta_{\pm\mp} = 1$ and $\eta_{z_i\bar{z}_j} = \frac{1}{2}\delta_{ij}$. The tildes on the derivatives in (5.1) indicate that the time component must be appropriately shifted as described in the previous section. Using the property (4.125) we may simplify the action to the form

$$S[\Phi] = \int\limits_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x}) \left[\frac{1}{2}\eta_{ab}\big(\widetilde{\partial}{}^a_\star \triangleright \Phi\big)\big(\widetilde{\partial}{}^b_\star \triangleright \Phi\big) + \frac{1}{2}m^2 \Phi^2\right] \tag{5.2}$$

By using the integration by parts property (4.131) on Schwartz fields Φ, we may easily compute the first order variation of the action (5.2) to be

$$\frac{\delta S[\Phi]}{\delta\Phi}\delta\Phi := S[\Phi+\delta\Phi]-S[\Phi] = -\int\limits_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})\left[\eta_{ab}\overline{\widetilde{\partial}{}^a_\star}\triangleright\left(\widetilde{\partial}{}^b_\star\triangleright\Phi\right)-m^2\Phi^2\right]\delta\Phi \quad (5.3)$$

Applying the variational principle $\frac{\delta S[\Phi]}{\delta\Phi}=0$ to (5.3) thereby leads to the noncommutative Klein-Gordon field equation

$$\square^\star\triangleright\Phi - m^2\Phi = 0 \quad (5.4)$$

where

$$\square^\star\triangleright\Phi := 2\partial_+\triangleright\partial_-\Phi + \boldsymbol{\partial}^\top\triangleright\overline{\boldsymbol{\partial}}\triangleright\Phi + \frac{1}{2}\partial_+\ln\kappa\,\partial_-\Phi \quad (5.5)$$

and we have used $\partial_-\kappa = 0$. The second order $\star$-differential operator $\square^\star$ should be regarded as a deformation of the covariant Laplace operator $\square_0^\star$ corresponding to the commutative plane wave geometry of NW_6. This Laplacian coincides with the quadratic Casimir element

$$\mathsf{C}_6 := \theta^{-2}\eta^{ab}\mathsf{X}_a\mathsf{X}_b = 2\mathsf{J}\mathsf{T} + \frac{1}{2}\sum_{i=1,2}\left(\mathsf{P}^i_+\mathsf{P}^i_- + \mathsf{P}^i_-\mathsf{P}^i_+\right) \quad (5.6)$$

of the universal enveloping algebra $U(\mathfrak{n})$, expressed in terms of left or right isometry generators for the action of the isometry group $\mathcal{N}_{\mathbb{L}}\times\mathcal{N}_{\mathbb{R}}$ on NW_6 [77, 23, 58].

However, in the manner which we have constructed things, this is not the case. Recall that the approximation in which our quantisation of the geometry of NW_6 holds is the small time limit $x^+\to 0$ in which the plane wave approaches flat six-dimensional Minkowski space $\mathbb{E}^{1,5}$. To incorporate the effects of the curved geometry of NW_6 into our formalism, we have to replace the derivative operators $\widetilde{\partial}{}^a_\star$ appearing in (5.1) with appropriate curved space analogs $\delta^a_\star$ [20, 63].

Recall that the derivative operators $\partial^a_\star$ are *not* derivations of the $\star$-product $\star$, but instead obey the deformed Leibniz rules (4.99). The deformation arose from twisting the co-action of the bialgebra $U(\mathfrak{g})$ so that it generated automorphisms of the noncommutative algebra of functions, i.e. isometries of the noncommutative plane wave. The basic idea is to now "absorb" these twistings into derivations $\delta^a_\star$ obeying the usual Leibniz rule

$$\delta^a_\star\triangleright(f\star g) = \left(\delta^a_\star\triangleright f\right)\star g + f\star\left(\delta^a_\star\triangleright g\right) \quad (5.7)$$

These derivations generically act on $C^\infty(\mathfrak{n}^\vee)$ as noncommutative $\star$-polydifferential operators

$$\delta^a_\star \triangleright f = \sum_{n=1}^\infty \xi_a^{a_1\cdots a_n} \star \left(\partial^\star_{a_1} \triangleright \cdots \triangleright \partial^\star_{a_n} \triangleright f\right) \tag{5.8}$$

with $\xi_a^{a_1\cdots a_n} \in C^\infty(\mathfrak{n}^\vee)$. Unlike the derivatives $\partial^a_\star$, these derivations will no longer $\star$-commute among each other. There is a one-to-one correspondence [69] between such derivations $\delta^\star_a$ and Poisson vector fields $E^a = E^a_b \partial^b$ on $\mathfrak{n}^\vee$ obeying

$$E^a \circ \Theta(f,g) = \Theta(E^a f, g) + \Theta(f, E^a g) \tag{5.9}$$

for all $f,g \in C^\infty(\mathfrak{n}^\vee)$. To leading order one has $\delta^a_\star \triangleright f = E^a_b \star (\partial^b_\star \triangleright f) + O(\theta)$. By identifying the Lie algebra $\mathfrak{n}$ with the tangent space to NW_6, at this order the vector fields E^a can be thought of as defining a natural local frame with flat metric η_{ab} and a curved metric tensor $G^\star_{ab} = \frac{1}{2}\eta_{cd}(E^c_a \star E^d_b + E^d_a \star E^c_b)$ on the noncommutative space NW_6. However, for our $\star$-products there are always higher order terms in (5.8) which spoil this interpretation. The noncommutative frame fields $\delta^a_\star$ describe the *quantum* geometry of the plane wave NW_6. In particular, the metric tensor $G^\star$ will in general differ from the classical open string metric $G_{\rm open}$. While the operators $\delta^a_\star$ always exist as a consequence of the Kontsevich formality map [69, 20], computing them explicitly is a highly difficult problem. We will see some explicit examples below, as we now begin to tour through our three $\star$-products. Throughout we shall take the natural choice of measure $\kappa = \sqrt{|\det G|} = \frac{1}{2}$, the constant Riemannian volume density of the NW_6 plane wave geometry.

5.1 Time Ordering

In the case of time ordering, we use (4.114) to compute

$$\Box^* \triangleright \Phi = \left(2\partial_+\partial_- + \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}\right)\Phi \tag{5.10}$$

and thus the equation of motion coincides with that of a free scalar particle on flat Minkowski space $\mathbb{E}^{1,5}$ (Deviations from flat spacetime can only come about here by choosing a time-dependent measure κ_*). This illustrates the point made above that the treatment of this thesis tackles only the semi-classical flat space limit of the spacetime NW_6. The appropriate curved geometry for this ordering corresponds to the global coordinate system (1.11) in which the classical Laplace operator is given by

$$\Box_0^* = 2\partial_+\partial_- + \left|\boldsymbol{\partial} + \frac{i}{2}\theta\overline{z}\partial_-\right|^2 \tag{5.11}$$

so that the free wave equation $(\Box_0^* - m^2)\Phi = 0$ is equivalent to the Schrödinger equation for a particle of charge p^+ (the momentum along the x^- direction) in a constant magnetic field of strength θ. A global pseudo-orthonormal frame is provided by the commutative vector fields

$$\begin{aligned} E_-^* &= \partial_- \\ E_+^* &= \partial_+ - i\theta\left(\boldsymbol{z}\cdot\boldsymbol{\partial} - \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}\right) \\ E_*^i &= \partial^i \\ \overline{E}_*^i &= \overline{\partial}^i \end{aligned} \tag{5.12}$$

Determining the derivations δ_*^a corresponding to the commuting frame (5.12) on the quantum space is in general rather difficult. Evidently, from the coproduct structure (4.102) the action along the light-cone position is given by

$$\delta_-^* \triangleright f = \partial_- f \tag{5.13}$$

This is simply a consequence of the fact that translations along x^- generate an automorphism of the noncommutative algebra of functions, i.e. an isometry of the noncommutative geometry. From the Hopf algebra coproduct (4.100) we have

$$\Delta_*\left(e^{i\theta\partial_-}\right) = e^{i\theta\partial_-} \otimes e^{i\theta\partial_-} \tag{5.14}$$

and consequently

$$e^{i\theta\partial_-} \triangleright (f * g) = \left(e^{i\theta\partial_-} \triangleright f\right) * \left(e^{i\theta\partial_-} \triangleright g\right) \tag{5.15}$$

On the other hand, the remaining isometries involve intricate twistings between the light-cone and transverse space directions. For example, let us demonstrate how to unravel the coproduct rule for ∂_+^* in (4.102) into the desired symmetric Leibniz rule (5.7) for δ_+^*. This can be achieved by exploiting the $*$-product identities

$$
\begin{aligned}
z_i * f &= \left(e^{i\theta\partial_-} f\right) * z_i - 2i\theta x^+ \overline{\partial}^i f \\
\overline{z}_i * f &= \left(e^{-i\theta\partial_-} f\right) * \overline{z}_i + 2i\theta x^+ \partial^i f
\end{aligned}
\tag{5.16}
$$

along with the commutativity properties $[\partial_-^*, z_i]_* = [\partial_-^*, \overline{z}_i]_* = 0$ for $i = 1, 2$ and for arbitrary functions f. Using in addition the modified Leibniz rules (4.102) along with the $*$-multiplication properties (4.113) we thereby find

$$
\delta_+ \triangleright f = \left[x^+\partial_+ + \frac{1}{2i}\left(\boldsymbol{z}\cdot\overline{\boldsymbol{\partial}} + \overline{\boldsymbol{z}}\cdot\boldsymbol{\partial}\right)\right] f \tag{5.17}
$$

This action mimics the form of the classical frame field E_+^* in (5.12).

Finally, for the transversal isometries, one can attempt to seek functions $g^i \in C^\infty(\mathfrak{n}^\vee)$ such that $g^i * f = (e^{-i\theta\partial_-} f) * g_i$ in order to absorb the light-cone translation in the Leibniz rule for ∂_*^i in (4.102). This would mean that the x^- translations are generated by *inner* automorphisms of the noncommutative algebra. If such functions exist, then the corresponding derivations are given by $\delta_*^i \triangleright f = g^i * \partial_*^i f$ (no sum over i) and similarly for $\overline{\delta}_*^i$. However, it is doubtful that such inner derivations exist and the transverse space frame fields are more likely to be given by higher-order $*$-polyvector fields. For example, using similar steps to those which led to (5.17), one can show that the actions

$$
\begin{aligned}
\delta_* \triangleright f &:= \left(\overline{\boldsymbol{z}}\cdot\boldsymbol{\partial} + 2ix^+\partial_+ - i\theta x^+\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}\right) f \\
\overline{\delta}_* \triangleright f &:= \left(\boldsymbol{z}\cdot\overline{\boldsymbol{\partial}} - 2ix^+\partial_+ + i\theta x^+\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}\right) f
\end{aligned}
\tag{5.18}
$$

define derivations of the $*$-product on NW_6, and hence naturally determine elements of a noncommutative transverse frame.

The action of the corresponding noncommutative Laplacian $\eta_{ab}\delta_*^a \triangleright (\delta_*^b \triangleright \Phi)$ deforms the harmonic oscillator dynamics generated by (5.11) by non-local higher spatial derivative terms. These extra terms will have significant ramifications at large energies for motion in the transverse space. This could have profound physical effects in the interacting noncommutative quantum field theory. In particular, it may alter the UV/IR mixing story [74] in an interesting way. For time-dependent noncommutativity with standard tree-level propagators, UV/IR mixing becomes intertwined with violations of energy conservation in an intriguing way [14, 82], and it would be interesting to see how our modified free field propagators affect this analysis. It would also be interesting to see if and how these modifications are related to the generic connection between wave propagation on homogeneous plane waves and the Lewis-Riesenfeld theory of time-dependent harmonic oscillators [16].

5.2 Symmetric Time Ordering

The analysis in the case of symmetric time ordering is very similar to that just performed, so we will be very brief and only highlight the essential changes. From (4.117) we find once again that the Laplacian (5.5) coincides with the flat space wave operator

$$\square^{\bullet} \triangleright \Phi = \Big(2\partial_+\partial_- + \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}\Big)\,\Phi \tag{5.19}$$

The relevant coordinate system in this case is given by the Brinkman metric (1.12) for which the classical Laplace operator reads

$$\square_0^{\bullet} = 2\partial_+\partial_- + \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial} - \frac{1}{4}\theta^2|\boldsymbol{z}|^2\partial_-^2 \tag{5.20}$$

A global pseudo-orthonormal frame in this case is provided by the vector fields

$$\begin{aligned} E_-^{\bullet} &= \partial_- \\ E_+^{\bullet} &= \partial_+ + \frac{1}{8}\theta^2|\boldsymbol{z}|^2\partial_- \\ E_{\bullet}^{i} &= \partial^i \\ \overline{E}_{\bullet}^{i} &= \overline{\partial}^i \end{aligned} \tag{5.21}$$

The corresponding twisted derivations $\delta_{\bullet}^{a}$, which symmetrise the Leibniz rules (4.105) can be constructed analogously to those of the time ordering case in Section 5.1 above.

5.3 Weyl Ordering

Finally, the case of Weyl ordering is particularly interesting because the effects of curvature are present even in the flat space limit. Using (4.120) we find the Laplacian

$$\Box^{\star} \triangleright \Phi = \left(2\partial_+\partial_- + 2\left[2\left(1 - \frac{\sin(\theta\partial_-)}{\theta\partial_-}\right) + \frac{1-\cos(\theta\partial_-)}{\theta^2\partial_-^2}\right]\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}\right)\Phi \tag{5.22}$$

which coincides with the flat space Laplacian only at $\theta = 0$. To second order in the deformation parameter θ, the equation of motion (5.4) thereby yields a second order correction to the usual flat space Klein-Gordan equation given by

$$\left[\left(2\partial_+\partial_- + \overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial} - m^2\right) + \frac{7}{12}\theta^2\partial_-^2\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial} + O\left(\theta^4\right)\right]\Phi = 0 \tag{5.23}$$

Again we find that only the transverse space motion is altered by noncommutativity, but this time through a non-local dependence on the light-cone momentum p^+ yielding a drastic modification of the dispersion relation for free wave propagation in the noncommutative spacetime. This dependence is natural. The classical mass-shell condition for motion in the curved background is $2p^+p^- + |4\theta p^+\boldsymbol{\lambda}|^2 = m^2$, where $\boldsymbol{\lambda} \in \mathbb{C}^2$ represents the position and radius of the circular trajectories in the background magnetic field [23]. Thus the quantity $4\theta p^+\boldsymbol{\lambda}$ can be interpreted as the momentum for motion in the transverse space. The operator (5.22) incorporates the appropriate noncommutative deformation of this motion. It illustrates the point that the fundamental quanta governing the interactions in the present class of noncommutative quantum field theories are not likely to involve the particle-like dipoles of the flat space cases, but more likely string-like objects owing to the non-vanishing H-flux in (1.14). These open string quanta become polarised as dipoles experiencing a net force due to their couplings to the non-uniform B-field. It is tempting to speculate that, in contrast to the other orderings, the Weyl ordering naturally incorporates the new vacua corresponding to long string configurations which are due entirely to the time-dependent nature of the background Neveu-Schwarz field [11].

While the Weyl ordered $\star$-product is natural from an algebraic point of view, it does not correspond to a natural coordinate system for the plane wave NW_6 due to the complicated form of the group product rule (4.47) in this case. In particular, the frame fields in this instance will be quite complicated. Computing the corresponding twisted derivations $\delta^a_\star$ directly would again be extremely cumbersome, but luckily we can exploit the equivalence between the $\star$-products $\star$ and $*$ derived in Section 4.2.3. Given the derivations δ^a_* constructed in Section 5.1 above, we may use the differential operator (4.46) which implements the equivalence (4.34) to define

$$\delta^a_\star \triangleright f := \mathcal{G}_\Omega \circ \delta^a_* \triangleright \big(\mathcal{G}_\Omega^{-1}(f)\big) \tag{5.24}$$

These noncommutative frame fields will lead to the appropriate curved space extension of the Laplace operator in (5.22).

5.4 Worldvolume Field Theories

In this final section we will describe how to build noncommutative field theories on regularly embedded worldvolumes of D-branes in the spacetime NW_6 using the formalism described above. We shall describe the general technique on a representative example by comparing the noncommutative field theory on NW_6 to that of the noncommutative D3-branes which was constructed in Chapter 3. We shall do so in a general fashion which illustrates how the construction extends to generic D-branes. This will provide further perspective on the natures of the different quantisations we have used throughout, and also illustrate the overall consistency of our results. As we will now demonstrate, we can view the noncommutative geometry of NW_6, in the manner constructed above, as a collection of all euclidean noncommutative D3-branes taken together. This is done by restricting the geometry to obtain the usual quantisation of coadjoint orbits in $\mathfrak{n}^\vee$ (as opposed to all of $\mathfrak{n}^\vee$ as described above). This restriction defines an alternative and more geometrical approach to the quantisation of these branes which does not rely upon working with representations of the Lie group $\mathcal{N}$, and which is more adapted to the flat space limit $\theta \to 0$.

This procedure can be thought of as somewhat opposite to the philosophy of

[58], which quantised the geometry of a non-symmetric D5-brane wrapping NW_6 [67] by viewing it as a noncommutative foliation by these euclidean D3-branes. Here the quantisation of the spacetime-filling brane in NW_6 has been carried out independently leading to a much simpler noncommutative geometry which correctly induces the anticipated worldvolume field theories on the $\mathbb{E}^4$ submanifolds of NW_6.

The euclidean D3-branes of interest wrap the non-degenerate conjugacy classes of the group $\mathcal{N}$ and are coordinatised by the transverse space $\boldsymbol{z} \in \mathbb{C}^2 \cong \mathbb{E}^4$ [87]. They are defined by the spacelike hyperplanes of constant time in NW_6 given by the transversal intersections of the null hypersurfaces

$$\begin{aligned} x^+ &= \text{constant} \\ x^- + \frac{1}{4}\theta|\boldsymbol{z}|^2 \cot\left(\frac{1}{2}\theta x^+\right) &= \text{constant} \end{aligned} \tag{5.25}$$

independently of the chosen coordinate frame. This describes the brane worldvolume as a wavefront expanding in a sphere in the transverse space. In the semi-classical flat space limit $\theta \to 0$, the second constraint in (5.25) to leading order becomes

$$C := 2x^+x^- + |\boldsymbol{z}|^2 = \text{constant} \tag{5.26}$$

The function C on $\mathfrak{n}^\vee$ corresponds to the Casimir element (5.6) and the constraint (5.26) is analogous to the requirement that Casimir operators act as scalars in irreducible representations. Similarly, the constraint on the time coordinate x^+ in (5.25) is analogous to the requirement that the central element T act as a scalar operator in any irreducible representation of $\mathcal{N}$.

Let $\pi : \mathrm{NW}_6 \to \mathbb{E}^4$ be the projection of the six-dimensional plane wave onto the worldvolume of the symmetric D3-branes. Let $\pi^\sharp : \mathrm{C}^\infty(\mathbb{E}^4) \to \mathrm{C}^\infty(\mathrm{NW}_6)$ be the induced algebra morphism defined by pull-back $\pi^\sharp(f) = f \circ \pi$. To consistently reduce the noncommutative geometry from all of NW_6 to its conjugacy classes, we need to ensure that the candidate $\star$-product on $\mathfrak{n}^\vee$ respects the Casimir property of the functions x^+ and C, i.e. that x^+ and C $\star$-commute with every function $f \in \mathrm{C}^\infty(\mathfrak{n}^\vee)$. Only in that case can the $\star$-product be consistently restricted from all of NW_6 to a

$\star$-product $\star_{x^+}$ on the conjugacy classes $\mathbb{E}^4$ defined by

$$f \star_{x^+} g := \pi^\sharp(f) \star \pi^\sharp(g) \tag{5.27}$$

Then one has the compatibility condition

$$\iota^\sharp(f \star g) = \iota^\sharp(f) \star_{x^+} \iota^\sharp(g) \tag{5.28}$$

where $\iota^\sharp : C^\infty(NW_6) \to C^\infty(\mathbb{E}^4)$ is the pull-back induced by the inclusion map $\iota : \mathbb{E}^4 \hookrightarrow NW_6$. In this case one has an isomorphism $C^\infty(\mathbb{E}^4) \cong C^\infty(NW_6)/\mathcal{J}$ of associative noncommutative algebras [102], where $\mathcal{J}$ is the two-sided ideal of $C^\infty(NW_6)$ generated by the Casimir constraints ($x^+ -$ constant) and ($C -$ constant). This procedure is essentially a noncommutative version of the Poisson reduction, with the Poisson ideal $\mathcal{J}$ implementing the geometric requirement that the Seiberg-Witten bi-vector Θ be tangent to the conjugacy classes.

From the $\star$-commutators (4.133), (4.139) and (4.146) we see that $[x^+, f]_\star = 0$ for all three of our $\star$-products. However, the condition $[C, f]_\star = 0$ is *not* satisfied. Although classically one has the Poisson commutation $\Theta(C, f) = 0$, one can only consistently restrict the $\star$-products by first defining an appropriate projection of the algebra of functions on $\mathfrak{n}^\vee$ onto the $\star$-subalgebra $\mathcal{C}$ of functions which $\star$-commute with the Casimir function C. One easily computes that $\mathcal{C}$ naturally consists of functions f which are independent of the light-cone position, i.e. $\partial_- f = 0$. Then the projector $\iota^\sharp$ above may be applied to the subalgebra $\mathcal{C}$ on which it obeys the requisite compatibility condition (5.28). The general conditions for reduction of Kontsevich star-products to D-submanifolds of Poisson manifolds are described in [34, 24].

With these projections implicitly understood, one straightforwardly finds that all three $\star$-products (4.26), (4.31) and (4.48) restrict to

$$f \star_{x^+} g = \mu \circ \exp\left[i\theta x^+ \left(\boldsymbol{\partial}^\top \otimes \overline{\boldsymbol{\partial}} - \overline{\boldsymbol{\partial}}^\top \otimes \boldsymbol{\partial}\right)\right] f \otimes g \tag{5.29}$$

for functions $f, g \in C^\infty(\mathbb{E}^4)$. This is just the Moyal product, with noncommutativity parameter θx^+, on the noncommutative euclidean D3-branes. It is cohomologically equivalent to the Voros product (3.55) which arises from quantising the conjugacy classes through endomorphism algebras of irreducible representations of the twisted Heisenberg algebra $\mathfrak{n}$, with a normal or Wick ordering prescription for the generators $\mathsf{P}^i_\pm$. In this case, the noncommutative euclidean space arises from a projection of $U(\mathfrak{n})$ in the discrete representation V^{p^+,p^-} whose second Casimir invariant (5.6) is given in terms of light-cone momenta as $\mathsf{C} = -2p^+(p^- + \theta)$ and with $\mathsf{T} = \theta p^+$. In this approach the noncommutativity parameter is naturally the *inverse* of the effective magnetic field $p^+\theta$. On the other hand, the present analysis is a more geometrical approach to the quantisation of symmetric D3-branes in NW_6 which deforms the euclidean worldvolume geometry by a time parameter θx^+ without resorting to endomorphism algebras. The relationship between the two sets of parameters is given by $x^+ = p^+\tau$, where τ is the proper time coordinate for geodesic motion in the pp-wave geometry of NW_6.

In contrast to the coadjoint orbit quantisation [58], the noncommutativity found here matches exactly that predicted from string theory in the semi-classical limit [35], which asserts that the Seiberg-Witten bi-vector on the D3-branes is given by $\Theta_{x^+} = \frac{i}{2}\sin(\theta x^+)\,\boldsymbol{\partial}^\top \wedge \overline{\boldsymbol{\partial}}$. Note that the present analysis also covers as a special case the degenerate cylindrical null branes located at time $x^+ = 0$ [87], for which (5.29) becomes the ordinary pointwise product $f \star_0 g = f\,g$ of worldvolume fields and as expected these branes support a *commutative* worldvolume geometry. In contrast, the commutative null branes correspond to the class of continuous representations of the twisted Heisenberg algebra having quantum number $p^+ = 0$ which must be dealt with separately [58].

It is elementary to check that the rest of the geometrical constructs of this thesis reduce to the standard ones appropriate for a Moyal space. By defining

$$\partial^a_{\star_{x^+}} \triangleright f := \iota^\sharp \circ \partial^a_\star \triangleright \big(\pi^\sharp(f)\big) \tag{5.30}$$

one finds that the actions of the derivatives constructed in Section 4.5 all reduce to the standard ones of flat noncommutative euclidean space, i.e. $\partial^i_{\star_{x^+}} \triangleright f = \partial^i f$, $\overline{\partial}^i_{\star_{x^+}} \triangleright f = \overline{\partial}^i f$ for $f \in \mathrm{C}^\infty(\mathbb{E}^4)$. From Section 4.4 one recovers the standard Hopf algebra of these derivatives with trivial coproducts $\Delta_{\star_{x^+}}$ defined by

$$\Delta_{\star_{x^+}}(\nabla_{\star_{x^+}}) \triangleright (f \otimes g) := \big(\iota^\sharp \otimes \iota^\sharp\big) \circ \Delta_\star(\nabla_\star) \triangleright \big(\pi^\sharp(f) \otimes \pi^\sharp(g)\big) \tag{5.31}$$

and hence the symmetric Leibniz rules appropriate to the translational symmetry of field theory on Moyal space. Consistent with the restriction to the conjugacy classes, one also has $\partial_{\pm}^{\star_{x^+}} \triangleright f = 0$.

However, from (4.101), (4.104) and (4.107) one finds a non-vanishing co-action of time translations given by

$$\Delta_{\star_{x^+}}\big(\partial_{+}^{\star_{x^+}}\big) = \theta\big(\boldsymbol{\partial}_{\star_{x^+}}^{\top} \otimes \overline{\boldsymbol{\partial}}_{\star_{x^+}} - \overline{\boldsymbol{\partial}}_{\star_{x^+}}^{\top} \otimes \boldsymbol{\partial}_{\star_{x^+}}\big) \tag{5.32}$$

This formula is very natural. The isometries of NW_6 in $\mathfrak{g} = \mathfrak{n}_{\mathrm{L}} \oplus \mathfrak{n}_{\mathrm{R}}$ corresponding to the number operator J of the twisted Heisenberg algebra are generated by the vector fields $J_{\mathrm{L}} = \theta^{-1}\partial_+$ and $J_{\mathrm{R}} = -\theta^{-1}\partial_+ - i(\boldsymbol{z}\cdot\boldsymbol{\partial} - \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}) = \theta^{-1}E_+^*$ (in Brinkman coordinates). The vector field $J_{\mathrm{L}} + J_{\mathrm{R}}$ generates rigid rotations in the transverse space. Restricted to the D3-brane worldvolume, the time translation isometries thus truncate to rotations of $\mathbb{E}^4$ in so(4). The coproduct (5.32) gives the standard twisted co-action of rotations for the Moyal algebra which define quantum rotational symmetries of noncommutative euclidean space [29, 25, 103]. This discussion also drives home the point made earlier that our derivative operators $\partial_\star^a$ indeed do generate, through their twisted co-actions (Leibniz rules), quantum isometries of the full noncommutative plane wave.

Finally, a trace on $C^\infty(\mathbb{E}^4)$ is induced from (4.123) by restricting the integral to the submanifold $\iota : \mathbb{E}^4 \hookrightarrow \mathrm{NW}_6$ and using the induced measure $\iota^\sharp(\kappa)$. For the measures constructed in Section 4.6, $\iota^\sharp(\kappa)$ is always a constant function on $\mathbb{E}^4$ and hence the integration measures all restrict to the constant volume form of $\mathbb{E}^4$. Thus noncommutative field theories on the spacetime NW_6 consistently truncate to the anticipated worldvolume field theories on noncommutative euclidean D3-branes in NW_6, together with the correct twisted implementation for the action of classical worldvolume symmetries. The advantage of the present point of view is that many of the novel features of these canonical Moyal space field theories naturally originate from the pp-wave noncommutative geometry when the Moyal space is regarded as a regularly embedded coadjoint orbit in $\mathfrak{n}^\vee$, as described above. Furthermore, the method detailed in this thesis allows a more systematic construction of the deformed worldvolume field theories of *generic* D-branes in NW_6 in the semi-classical regime, and not just the symmetric branes analysed here. For instance, the analysis can in principle be applied to describe the dynamics of symmetry-breaking D-branes which localise along products of twisted conjugacy classes in the Lie group $\mathcal{N}$ [81]. However, these branes have yet to be classified in the case of the gravitational wave NW_6.

Chapter 6

Conclusions

In this thesis, we have studied symmetric and non-symmetric (twisted) D-branes of the six dimensional Nappi-Witten spacetime and discovered that the majority of the noncommutative geometries are simpler than one would have expected; the symmetric branes all observe a flat space with zero NS flux, allowing standard open string models to be used to calculate the noncommutativity parameter. We found a class of commutative null branes and a class of noncommutative Euclidean D3-branes, analogous to branes in a constant magnetic field.

Central to the discoveries of the noncommutative nature of branes in NW_6 was the ability to perform Penrose-Güven limits on embedded spaces and keep track of the supergravity fields in isometric embedding diagrams. To achieve this, we produced formal coordinate free restrictions that one must observe in order to be guaranteed a commuting diagram.

The most interesting non-symmetric branes were found to be Hpp-waves *without* the characteristic Nappi-Witten NS flux, and hence classified as Cahen-Wallach spacetimes, not Nappi-Witten spacetimes as previously thought. One of these non-symmetric branes was found to possess a complicated worldvolume flux that corresponds to an electric field and hence does not exhibit a well defined decoupling of the massive string states. However, by choosing an appropriate gauge for the NW_6 B field, we have been able to discover a spatially varying noncommutativity, analogous to that of the Dolan-Nappi model. This gauge choice results in non-vanishing Poisson brackets that reproduce the Nappi-Witten Lie algebra in the small time limit.

Inspired by this noncommutative geometry that reproduces the Nappi-Witten Lie algebra, we proceeded to obtain closed and explicit forms of the $\star$-products for several physically important orderings; corresponding to the global and Brinkman coordinatisations of the spacetime and the Weyl symmetric ordering. By using the formalism of generalised Weyl systems, we were able to calculate the Hopf algebra of twisted isometries.

By placing linear constraints on the $\star$-derivatives, we restricted our analysis to the flat space limit; the trade off being that we were able to obtain Leibniz rules for the derivatives and therefore proceed in a systematic manner. After formalising the rules for integration we advanced to the scalar field theory, confirming that our analysis is consistent with a flat space limit. We document the pseudo-orthonormal frames and twisted derivatives that deform the commutative Laplacian, finding that only transverse space motion is effected by the commutativity.

Restricting the algebraic analysis of the Nappi-Witten Lie algebra to embedded worldvolumes, we confirmed the previous analysis of the noncommutative geometries and developed a method that allows a more systematic construction of the deformed worldvolume field theories of generic D-branes in NW_6 in the semi-classical regime.

All techniques throughout this thesis have been presented in such a manner that they may be applied to a broad range of homogeneous pp-waves supported by a constant Neveu-Schwarz flux.

Further research in this area could involve the study of an interacting Φ^4 theory. In canonical noncommutative field theory, we calculate products such as $e^{iq_i x^i} \star e^{iq'_i x^i}$ at each vertex in a Feynman diagram. We now know that such an exponential product will be spacetime-dependent, meaning that planar Feynman diagrams will be granted a phase shift analogous to those observed in non-planar quantum field theories (such that result in UV/IR mixing). We must also face the possibility of violations in energy and momentum conservation.

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