1 Penrose-Güven Limits and Isometric Embeddings

1.1 The Penrose-Güven Plane Wave Limit

In [1], Penrose showed that any Lorentzian spacetime has a limiting spacetime which is a plane wave. This limit can be thought of as a "first order approximation" along a null geodesic. The limiting spacetime depends on the choice of null geodesic, and hence any spacetime can have more than one Penrose limit. More recently, Güven extended Penrose's argument to show that any solution of a supergravity theory has plane wave limits which are also solutions [2].

Let (M, G) be a D-dimensional Lorentzian spacetime. By [3] we can always introduce local coordinates (U, V, Y^i) in the neighbourhood of a portion of a null geodesic γ which contains no conjugate points. Such a metric takes the form

$$G = \left(dU + \alpha dV + \beta_i dY^i\right) dV + C_{ij} dY^i dY^j \tag{1}$$

where α, β_i and C_{ij} are functions of the coordinates and C_{ij} is a symmetric positive-definite matrix, $i, j = 1, 2, \ldots, D-2$. The coordinates break down when $\det C = 0$, signalling the existence of a conjugate point. This coordinate system has the advantage that a null geodesic congruence is singled out by $V, Y^i \to$ constant, with U being an affine parameter along these geodesics. The geodesic γ is the one for which $V = Y^i = 0$.

In supergravity theories there are also general p-form potentials A with $(p+1)$-form field strengths $F = dA$. The potentials are defined up to gauge transformations $A \mapsto A + d\Lambda$ in such a way that F is gauge invariant. Güven extended the Penrose limit to supergravity theories by noting that in order to ensure well defined potentials in the target space, a gauge must be chosen for the potentials such that

$$A_{U(i_1 \ldots i_{p-1})} = 0 \tag{2}$$

By imposing this condition on A, we can now write general potential and associated field strengths on M

$$\begin{aligned} A &= a_{i_1 \ldots i_p} dV \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_{p-1}} \qquad (3) \\ &\quad + b_{i_1 \ldots i_p} dY^{i_1} \wedge \ldots \wedge dY^{i_p} \\ &\quad + c_{i_1 \ldots i_p} dU \wedge dV \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_{p-2}} \\ F &= \left(\frac{\partial b_{i_1 \ldots i_{p+1}}}{\partial U}\right) dU \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_p} \qquad (4) \\ &\quad + d_{i_1 \ldots i_{p+1}} dY^{i_1} \wedge \ldots \wedge dY^{i_{p+1}} \\ &\quad + e_{i_1 \ldots i_{p+1}} dU \wedge dV \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_{p-1}} \\ &\quad + f_{i_1 \ldots i_{p+1}} dV \wedge dY^{i_1} \wedge \ldots \wedge dY^{i_p} \end{aligned}$$

Where a, b, c, d, e and f are functions of the coordinates. The Penrose limit is in practise the rescaling of the coordinates

$$U \to u \qquad V \to \Omega^2 v \qquad Y^i \to \Omega y^i \tag{5}$$

where $\Omega \to 0$. Let us define the new fields on the target space as

$$\begin{aligned}\widetilde{G} &= \lim_{\Omega\to 0} \Omega^{-2}\psi^* G \\ \widetilde{A} &= \lim_{\Omega\to 0} \Omega^{-p}\psi^* A \\ \widetilde{F} &= \lim_{\Omega\to 0} \Omega^{-p}\psi^* F\end{aligned} \tag{6}$$

where ψ^* denotes the local diffeomorphism defined by (5). These new fields are related to the original ones by a diffeomorphism, a rescaling and (in the case of the potentials) possibly a gauge transformation. We obtain the metric and fields in Rosen [4] coordinates (u, v, y^i)

$$\widetilde{G} = dudv + C(u)_{ij}dy^i dy^j \tag{7}$$

$$\begin{aligned}\widetilde{A} &= b(u)_{i_1\ldots i_p} dy^{i_1} \wedge \ldots \wedge dy^{i_p} \\ &\quad + c(u)_{i_1\ldots i_p} du \wedge dv \wedge dy^{i_1} \wedge \ldots \wedge dy^{i_{p-2}}\end{aligned} \tag{8}$$

$$\widetilde{F} = \frac{\partial b(u)_{i_1\ldots i_{p+1}}}{\partial u} du \wedge dy^{i_1} \wedge \ldots \wedge dy^{i_p} \tag{9}$$

Due to (5), we can see that $C(u)_{ij} = C(U, 0, 0)_{ij}$ which is just the value of C along the geodesic γ. This is also true of $b(u)$ and $c(u)$.

1.2 Isometric Embedding Diagrams

If we wish to generate a commutative isometric embedding diagram, whereby the embeddings of N's into the M's are denoted by vertical arrows and Penrose-Güven limits (PGL) by horizontal arrows, we would have the following picture:

$$\begin{array}{ccc} M & \xrightarrow{\text{PGL}} & \widetilde{M} \\ \imath\big\uparrow & & \big\uparrow\widetilde{\imath} \\ N & \xrightarrow{\text{PGL}} & \widetilde{N} \end{array} \tag{10}$$

In order to ensure that the metric and fields commute in such a diagram (i.e. that $M \to \widetilde{M} \to \widetilde{N}$ yields the same result as $M \to N \to \widetilde{N}$), we must place some restrictions on the kind of projections $\imath^*$, $\widetilde{\imath}^*$ we can use. If we choose to set m coordinates to zero[1] in each projection, then we can define

$$\begin{aligned} \imath^* &: Y^{q_1}, \ldots, Y^{q_m} \to 0 && (11) \\ \widetilde{\imath}^* &: y^{r_1}, \ldots, y^{r_m} \to 0 && (12) \end{aligned}$$

If $I = \{1, 2, \ldots, D-2\}$ is the set of all indexes and the sets of indexes of the coordinates set to zero are $Q = \{q_1, \ldots, q_m\}$ and $R = \{r_1, \ldots, r_m\}$, then $Q, R \subset I$. We can now define two new sets of indexes, which are the indexes of the coordinates which are not set to zero; $S = I \backslash Q = \{s_1, \ldots, s_{D-2-m}\}$ and $T = I \backslash R = \{t_1, \ldots, t_{D-2-m}\}$.

The following condition is required for the metric to be commutative in such a diagram

$$C(U,0,0)_{is} = C(U,0,0)_{i\pi(t)} \tag{13}$$

where $\pi(t)$ is any permutation of the indexes t. A similar condition exists for field strengths

$$b(U,0,0)_{(i_1 \ldots i_{p+1})s} = b(U,0,0)_{(i_1 \ldots i_{p+1})\pi(t)} \tag{14}$$

For potentials we require the condition (14) (with indexes up to i_p rather than i_{p+1}) and also a second condition

$$c(U,0,0)_{(i_1 \ldots i_p)s} = c(U,0,0)_{(i_1 \ldots i_p)\pi(t)} \tag{15}$$

It is the particular pairing of S and T which we must choose carefully in order to satisfy (13), (14) and (15). It may even be possible that no such choice exists, in which case such a commutative diagram cannot be constructed. If we wish the metric and fields to both commute simultaneously, we require the pairing of S and T to be the same in all the conditions.

These conditions are essentially just a trivial statement that $\imath^*$ and $\widetilde{\imath}^*$ must be equivalent.

1.3 An Example

1.3.1 Broken Commutativity

We will now show an example of a diagram such as (10) which does not commute

$$\begin{array}{ccc} \mathrm{AdS}_3 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_6 \\ \imath \Big\uparrow & & \Big\uparrow \widetilde{\imath} \\ \mathrm{AdS}_2 \times \mathrm{S}^2 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_4 \end{array} \tag{16}$$

[1]We require the coordinates to go to zero in order that the PGL is along the same null geodesic in each case, but requiring the coordinates to go constant rather than zero is often sufficient.

This diagram was originally studied in [5, 6] and was conjectured to be commutative. As we will now show, the metric is commutative but the fields are not.

The restrictions of AdS_3 and S^3 in $\mathbb{R}^4$ are separately given by

$$
\begin{aligned}
\left(X^0\right)^2+\left(X^1\right)^2-\left(X^2\right)^2-\left(X^3\right)^2 &= R \qquad (17)\\
\left(X^4\right)^2+\left(X^5\right)^2+\left(X^6\right)^2+\left(X^7\right)^2 &= R \qquad (18)
\end{aligned}
$$

We choose to keep the radii R the same for each space with the explicit parameterisation

$$
\begin{array}{llll}
X^0 & =\sqrt{1+r^2}\sin\tau & X^1 & =\cos\tau\cosh\beta \\
X^2 & =\cos\tau\sinh\beta & X^3 & =r\sin\tau \\
X^4 & =\cos\phi\sin\theta & X^5 & =\chi\sin\phi \\
X^6 & =\sqrt{1-\chi^2}\sin\phi & X^7 & =\cos\phi\cos\theta
\end{array} \qquad (19)
$$

Using these coordinates, we may write the metric, NS-NS B field and strength H on $\mathrm{AdS}_3\times\mathrm{S}^3$ as

$$
\begin{aligned}
G &= -d\tau^2+\sin^2\tau\left(\frac{dr^2}{1+r^2}+\left(1+r^2\right)d\beta^2\right) \qquad (20)\\
&\quad +d\phi^2+\sin^2\phi\left(\frac{d\chi^2}{1-\chi^2}+\left(1-\chi^2\right)d\theta^2\right)\\
-H/2 &= \cos^2\tau d\tau\wedge dr\wedge d\beta+\sin^2\phi d\phi\wedge d\chi\wedge d\theta \qquad (21)\\
-2B &= \left(\sin 2\tau+2\tau\right)dr\wedge d\beta+\left(\sin 2\phi-2\phi\right)d\chi\wedge d\theta \qquad (22)
\end{aligned}
$$

where AdS_3 has coordinates (τ,r,β) and S^3 has (ϕ,χ,θ) so that $D=6$. Making a change of coordinates so that we can write everything in the generalised form of (1), (3) and (4)

$$
\begin{array}{ll}
2\tau=U-V & 2\phi=U+V \\
r=Y^1 \quad \beta=Y^2 & \chi=Y^3 \quad \theta=Y^4
\end{array} \qquad (23)
$$

we can now express the metric and fields along the geodesic γ as

$$
\begin{aligned}
C(u)_{ii} &= \sin^2\frac{u}{2} \qquad (24)\\
b(u)_{12} &= -\frac{\sin u+u}{2}\\
b(u)_{34} &= \frac{u-\sin u}{2}
\end{aligned}
$$

with everything else zero. The projection $\imath^*$ consists of setting $Y^q\to 0$ where $q=1,3$ and $\widetilde{\imath}$ as $y^r\to 0$ where $r=1,2$. We can now observe the sets $S=\{2,4\}$, $T=\{3,4\}$ and we see clearly that there is no such pairing between S and T such that (14) is satisfied, as it will result in requiring $b(u)_{12} = b(u)_{1j}$ where j is either 3 or 4. We can however see that (13) will always be satisfied since all the components of $C(u)_{ii}$ are identical. So we expect the metric but not the fields to commute in such a diagram.

We can confirm these predictions by following each path explicitly[2]. Firstly, the path $\mathrm{AdS}_3 \times \mathrm{S}^3 \to \mathrm{CW}_6 \to \mathrm{CW}_4$ yields

$$\begin{aligned} \widetilde{G} &= 2dx^-dx^+ + \frac{\vec{z}^2}{4}\left(dx^-\right)^2 + (d\vec{z})^2 && (26)\\ \widetilde{H} &= -dx^- \wedge dz^1 \wedge dz^2 && (27)\\ \widetilde{B} &= -x^- dz^1 \wedge dz^2 && (28)\end{aligned}$$

which is in fact, the Nappi-Witten WZW model [8]. The alternative path $\mathrm{AdS}_3 \times \mathrm{S}^3 \to \mathrm{AdS}_2 \times \mathrm{S}^2 \to \mathrm{CW}_4$ yields

$$\begin{aligned} \widetilde{G} &= 2dx^-dx^+ + \frac{\vec{z}^2}{4}\left(dx^-\right)^2 + (d\vec{z})^2 && (29)\\ \widetilde{H} &= d\widetilde{B} = 0 && (30)\end{aligned}$$

confirming the prediction that the metric commutes, but the fields do not.

1.3.2 Working Commutativity

We can now see why (16) breaks down and we may attempt to remedy it. The essential problem is in choosing a suitable $\imath^*$ projection where we may pair up the elements of S and T. The easiest and most obvious way to fix this is to make the projection $\widetilde{\imath}^*$ equivalent to $\imath^*$, i.e. so that we choose $\imath^* : Y^l \to 0$ where $l = 1, 2$. Since the sets $S = \{3, 4\}$ and $T = \{3, 4\}$ are now identical, the commutative conditions (13), (14), (15) are trivially satisfied. This has however changed the diagram somewhat, from (16) into the following

$$\begin{array}{ccc} \mathrm{AdS}_3 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_6 \\ \imath \big\uparrow & & \big\uparrow \widetilde{\imath} \\ \mathbb{R}^1 \times \mathrm{S}^3 & \xrightarrow{\mathrm{PGL}} & \mathrm{NW}_4 \end{array} \qquad (31)$$

[2]It is easier to work in Brinkman [7] coordinates (x^-, x^+, z^i) at this stage, we have made the transformation from Rosen coordinates by

$$u = 2x^- \qquad v = x^+ + \frac{\cot x^-}{2}\vec{z}^2 \qquad y^i = \frac{z^i}{\sin x^-} \qquad (25)$$

The final space of each path is the NW_4 model and both metric and fields commute in the diagram. It is interesting to note that the PGL on the bottom of the diagram has also been studied in [9], albeit with slight technical differences from the PGL used here.

We could also have fixed (16) by choosing the projection $\widetilde{\imath}^* : y^l \to 0$ where $l = 1, 3$ so that it is equivalent to $\imath$ and that the diagram will again trivially commute. Again this involves a change in the diagram

$$\begin{tikzcd} \mathrm{AdS}_3 \times \mathrm{S}^3 \arrow[r, "\mathrm{PGL}"] & \mathrm{NW}_6 \\ \mathrm{AdS}_2 \times \mathrm{S}^2 \arrow[u, "\imath"] \arrow[r, "\mathrm{PGL}"] & \mathrm{CW}_4 \arrow[u, "\widetilde{\imath}"'] \end{tikzcd} \tag{32}$$

where CW_N is used to denote a Cahen-Wallach space [10] in N dimensions. This diagram more closely resembles (16), since NW_4 is just CW_4 at the level of the geometry, neglecting the supergravity fields.

References

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