Contents

1 $\star$-Products for the Nappi-Witten Algebra

1.1 The $\star$-Product

star

The Weyl operator $\hat{\Omega}[f]$ can be thought of as a one-to-one map from local coordinates x^i (an algebra of fields on $\mathbb{R}^D$) to Hermitian operators $\hat{x}^i$ (on an appropriate Hilbert space) exhibiting the algebra

$$\left[\hat{x}^i, \hat{x}^j\right] = i\theta^{ij} \tag{1}$$

eq:algebra

where θ^{ij} is not necessarily constant. The inverse map $\hat{\Omega}^{-1}[f]$ pulls operators back to local coordinates; the functions obtained in this way are called Wigner distribution functions and their $\star$-commutator brackets, defined as

$$\left[x^i, x^j\right]_\star = x^i \star x^j - x^j \star x^i \tag{2}$$

eq:poisson

replicate the operator algebra

$$\left[x^i, x^j\right]_\star = i\theta^{ij} \tag{3}$$

eq:falgebra

where we understand that the θ in (3) may depend on Wigner distribution functions associated to the operators on the Hilbert space. If the fields live in an appropriate Schwartz space of functions, any function may be described by its Fourier transform

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} e^{-ik_i x^i} f(x) d^D x \tag{4}$$

eq:fourier

allowing us to define the symbol for $f(x)$

$$\hat{\Omega}[f] = \frac{1}{(2\pi)^D} \int_{-\infty}^{\infty} \widetilde{f}(k) {}^\circ_\circ e^{ik_i \hat{x}^i} {}^\circ_\circ d^D k \tag{5}$$

eq:weylsymb

We are free to choose a convenient ordering, denoted by ${}^\circ_\circ \; {}^\circ_\circ$. When the ordering is symmetric, $\hat{\Omega}[f]$ is often called the Weyl symbol. The products of $\hat{\Omega}[f]$ symbols may be computed allowing us to define the $\star$-product

$$\hat{\Omega}[f]\hat{\Omega}[g] = \hat{\Omega}[f \star g] \tag{6}$$

eq:weylproduct

$$f \star g = \iint \frac{\widetilde{f}(k)\widetilde{g}(k')}{(2\pi)^{2D}} \hat{\Omega}^{-1} [{}^\circ_\circ e^{ik_i \hat{x}^i} e^{ik'_i \hat{x}^i} {}^\circ_\circ] d^D k d^D k' \tag{7}$$

eq:starproduct

The product of exponentials is calculated by using the Baker-Campbell-Hausdorff (BCH) formula (which we will explain in the proceeding section).

The $\star$-product is a deformation of point-wise multiplication of functions on $\mathbb{R}^D$ to an associative but non-commutative algebra. For $\theta = 0$, the $\star$-product reduces to the ordinary product of functions and if θ is constant we may employ simplifications in the BCH formula such that we get the Groenewold-Moyal $\star$-product

$$\begin{aligned} f(x) \star g(x) &= f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_i \theta^{ij} \overrightarrow{\partial}_j\right) g(x) \\ &= f(x)g(x) + \sum_{n=1}^{\infty} \frac{i^n}{2^n n!} \theta^{i_1 j_1} \ldots \theta^{i_n j_n} \partial_{i_1} \ldots \partial_{i_n} f(x) \partial_{j_1} \ldots \partial_{j_n} g(x) \end{aligned} \tag{8}$$

eq:star:constantthet

For non-constant θ the expansion is generally more complicated. However, an easy example is for an algebra

$$[\hat{x}_i, \hat{x}_j] = f_{ij}^k \hat{x}_k \tag{9}$$

eq:example:nonconsta

where we might wish to find the $\star$-products between the local coordinates x_i. Since we know

$$\int x e^{ikx} dk = -i \frac{\partial}{\partial k} \delta(k) \tag{10}$$

eq:example:integral

we can insert this into (7) obtaining

$$x_i \star x_j = -\frac{\partial^2}{\partial k_i \partial k'_j} e^{F_n x^n} \Big|_{\boldsymbol{k}=\boldsymbol{k}'=0} \tag{11}$$

eq:example:star

where $F_n = F_n(\boldsymbol{k}, \boldsymbol{k}')$ are functions arising from the BCH formula.

More generally, there is a standard $\star$-product for Lie algebras [2]

$$f \star g(z) = f(z) \exp\left(i x^i \left[F_i - k_i - k'_i\right]\right) g(z) \tag{12}$$

eq:star:lie

which can easily be written in position space by replacing

$$\begin{aligned} k_\nu &\to -i \overleftarrow{\partial}_\nu \\ k'_\nu &\to -i \overrightarrow{\partial}_\nu \end{aligned} \tag{13}$$

eq:star:position

There is no general closed form for $F_n(\boldsymbol{k}, \boldsymbol{k}')$ (as we will see in the next section). It is often convenient to scale all of the generators $x^i \to \frac{1}{\alpha} x^i$ resulting in some non-trivial change in the F_n but allowing a power expansion of (12) in α. As $\alpha \to 0$ the normal commutative product is recovered.

1.2 Baker-Campbell-Hausdorff Formula

The BCH formula is used to calculate the product of exponentials when the exponents do not commute

$$e^A e^B = e^{\mathrm{C}(A:B)} \tag{14}$$

eq:BCH:define

where $\mathrm{C}(A:B)$ is often an infinite sum, calculated through the recursion formula

$$(n+1)\mathrm{C}_{n+1}(X:Y) = \frac{1}{2}[X-Y,\mathrm{C}_n(X:Y)] \tag{15}$$
$$+\sum_{p\geq 1,2p\leq n} K_{2p} \sum_{\substack{k_1,\ldots,k_{2p}>0\\ k_1+\ldots+k_{2p}=n}} [\mathrm{C}_{k_1}(X:Y),[\ldots,[\mathrm{C}_{k_{2p}}(X:Y),X+Y]\ldots]]$$

eq:BCH

where $(n\geq 1; X,Y\in\mathfrak{g})$ and $\mathrm{C}_1(X:Y)=X+Y$. The Ks are defined by

$$\frac{z}{1-e^{-z}}-\frac{z}{2}-1=\sum_{p=1} K_{2p}z^{2p} \tag{16}$$

eq:BCH:K

which are closely related to the Bernoulli numbers (B_n), defined by

$$\sum_{n=0}\frac{B_n x^n}{n!}=\frac{x}{e^x-1} \tag{17}$$

eq:BCH:bernoulli

The first few terms of the BCH formula may be written concisely as

$$\mathrm{C}_1(X:Y) = X+Y \tag{18}$$
eq:BCH:1
$$\mathrm{C}_2(X:Y) = \frac{1}{2}[X,Y] \tag{19}$$
eq:BCH:2
$$\mathrm{C}_3(X:Y) = \frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]] \tag{20}$$
eq:BCH:3
$$\mathrm{C}_4(X:Y) = -\frac{1}{24}[Y,[X,[X,Y]]] \tag{21}$$
eq:BCH:4

but beyond about 4th order, the terms get lengthy to write explicitly due to the partition sum in (15).

TODO: need to define F_i properly.

1.3 The Nappi-Witten Algebra

We wish to find the $\star$-product of the NW_4 algebra, a centrally extended Poincaré algebra. [1] define the algebra of anti-Hermitian operators as

$$\begin{aligned}\left[\hat{\mathsf{J}},\hat{\mathsf{P}}_i\right] &= \epsilon_{ij}\hat{\mathsf{P}}_j\\ \left[\hat{\mathsf{P}}_i,\hat{\mathsf{P}}_j\right] &= \epsilon_{ij}\hat{\mathsf{K}}\end{aligned} \tag{22}$$

eq:NW:origalg

which may be written as an algebra of Hermitian operators by replacing each operator with its complex equivalent: $\mathsf{X}_j\to i\mathsf{X}_j$

$$\begin{aligned}\left[\hat{\mathsf{J}},\hat{\mathsf{P}}_i\right] &= -i\epsilon_{ij}\hat{\mathsf{P}}_j\\ \left[\hat{\mathsf{P}}_i,\hat{\mathsf{P}}_j\right] &= -i\epsilon_{ij}\hat{\mathsf{K}}\end{aligned} \tag{23}$$

eq:NW:hermalg

By defining the conjugate operators

$$\begin{aligned}\hat{\mathsf{P}}^+ &= \hat{\mathsf{P}}_1 + i\hat{\mathsf{P}}_2 \\ \hat{\mathsf{P}}^- &= \hat{\mathsf{P}}_1 - i\hat{\mathsf{P}}_2\end{aligned} \tag{24}$$

eq:NW:defineP+-

we have an algebra which looks much like a centrally extended κ-Minkowski algebra*

$$\begin{aligned}\left[\hat{\mathsf{P}}^+, \hat{\mathsf{P}}^-\right] &= -2\hat{\mathsf{K}} \\ \left[\hat{\mathsf{J}}, \hat{\mathsf{P}}^\pm\right] &= \mp\hat{\mathsf{P}}^\pm\end{aligned} \tag{25}$$

eq:NW:algebra

Using the parameterisation (where u and v are real numbers, w is a complex with conjugate $\bar{w}$)

$$R(u) = e^{iu\hat{\mathsf{J}}} \tag{26}$$

eq:NW:param:R

$$T(w) = e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-} \tag{27}$$

eq:NW:param:T

$$Z(v) = e^{iv\hat{\mathsf{K}}} \tag{28}$$

eq:NW:param:Z

we know that $R(u)$ is the rotation by an angle u in the complex plane: $R(u)\cdot z = e^{iu}z$ and $T(w)$ is the translation by w: $T(w)\cdot z = z+w$. $Z(v)$ is a one-parameter subgroup acting as $Z(v)\cdot z = e^{iv}z$. The group multiplication law [3] tells us

$$R(u_1)R(u_2) = R(u_1+u_2) \tag{29}$$

eq:NW:mult:RR

$$R(u)T(w) = T(we^{iu})R(u) \tag{30}$$

eq:NW:mult:RT

$$T(w_1)T(w_2) = T(w_1+w_2)Z(iw\bar{w}' - i\bar{w}w') \tag{31}$$

eq:NW:mult:TT

which is invaluable when attempting to calculate exponential products such as

$$e^{iu\hat{\mathsf{J}}}e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-} = e^{i\bar{w}e^{-i\beta u}\hat{\mathsf{P}}^+ + iwe^{i\beta u}\hat{\mathsf{P}}^-}e^{iu\hat{\mathsf{J}}} \tag{32}$$

eq:NW:mult:JP

$$e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-}e^{iu\hat{\mathsf{J}}} = e^{iu\hat{\mathsf{J}}}e^{i\bar{w}e^{i\beta u}\hat{\mathsf{P}}^+ + iwe^{-i\beta u}\hat{\mathsf{P}}^-} \tag{33}$$

eq:NW:mult:PJ

$$e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\mathsf{P}^-}e^{i\bar{w}'\hat{\mathsf{P}}^+ + iw'\hat{\mathsf{P}}^-} = e^{i(\bar{w}+\bar{w}')\hat{\mathsf{P}}^+ + i(w+w')\hat{\mathsf{P}}^-}e^{(\bar{w}w' - w\bar{w}')\hat{\mathsf{K}}} \tag{34}$$

eq:NW:mult:PP

where we mean to expand the exponentials around $\beta = 1$.

TODO: add a note about the $\Im$ notation

The more general α-scaled generators of (12) are given by

$$\begin{aligned}\left[\hat{\mathsf{P}}^+, \hat{\mathsf{P}}^-\right] &= -2\alpha\hat{\mathsf{K}} \\ \left[\hat{\mathsf{J}}, \hat{\mathsf{P}}^\pm\right] &= \mp\alpha\hat{\mathsf{P}}^\pm\end{aligned}$$

*In fact, the algebra is so closely related that this approach may be used as an alternative derivation of the Weyl ordered κ-Minkowski $\star$-product, see Appendix B.

with the corresponding exponential products

$$\begin{aligned}
e^{iu\hat{\mathsf{J}}} e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-} &= e^{i\bar{w}e^{-i\alpha\beta u}\hat{\mathsf{P}}^+ + iwe^{i\alpha\beta u}\hat{\mathsf{P}}^-} e^{iu\hat{\mathsf{J}}} \\
e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-} e^{iu\hat{\mathsf{J}}} &= e^{iu\hat{\mathsf{J}}} e^{i\bar{w}e^{i\alpha\beta u}\hat{\mathsf{P}}^+ + iwe^{-i\alpha\beta u}\hat{\mathsf{P}}^-} \\
e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\mathsf{P}^-} e^{i\bar{w}'\hat{\mathsf{P}}^+ + iw'\hat{\mathsf{P}}^-} &= e^{i(\bar{w}+\bar{w}')\hat{\mathsf{P}}^+ + i(w+w')\hat{\mathsf{P}}^-} e^{\alpha(\bar{w}w' - w\bar{w}')\hat{\mathsf{K}}}
\end{aligned}$$

1.3.1 Time Ordering

As mentioned, we are free to choose a convenient exponential ordering for (7). Here we will use an ordering such that

$$ {}^*_* e^{ik_i\hat{x}^i} {}^*_* = e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-} e^{iu\hat{\mathsf{J}}} e^{iv\hat{\mathsf{K}}} \tag{35}$$ eq:time:defn

We call this "Time-Ordering" as the time coordinate (u) is always kept separated and to the right of the Ps. The group multiplication law is well studied and it is easy to calculate the exponential product explicitly

$$\begin{aligned}
&{}^*_* e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-} e^{iu\hat{\mathsf{J}}} e^{iv\hat{\mathsf{K}}} e^{i\bar{w}'\hat{\mathsf{P}}^+ + iw'\hat{\mathsf{P}}^-} e^{iu'\hat{\mathsf{J}}} e^{iv'\hat{\mathsf{K}}} {}^*_* = \\
&e^{i(\bar{w}+\bar{w}'e^{-iu})\hat{\mathsf{P}}^+ + i(w+w'e^{iu})\hat{\mathsf{P}}^-} e^{i(u+u')\hat{\mathsf{J}}} e^{\left\{i(v+v') + \bar{w}w'e^{iu} - w\bar{w}'e^{-iu}\right\}\hat{\mathsf{K}}}
\end{aligned} \tag{36}$$ eq:time:mult

so that the $*$-product between any two functions is

$$\begin{aligned}
f * g(z) &= f(z) e^{ix^i(F_i^* - k_i - k_i')} g(z) \\
F_1^* &= \bar{w} + \bar{w}'e^{-i\alpha u} \\
F_2^* &= w + w'e^{i\alpha u} \\
F_3^* &= u + u' \\
F_4^* &= v + v' - i\alpha\bar{w}w'e^{i\alpha u} + i\alpha w\bar{w}'e^{-i\alpha u}
\end{aligned} \tag{37}$$ eq:time:star

where k runs over $\bar{w}, w, u, v$ and x over $\mathsf{P}^+, \mathsf{P}^-, \mathsf{J}, \mathsf{K}$. We see from (11) and (12) that the operator algebra is recovered in the functional space with the $*$-product between the generators being

$$\begin{aligned}
\mathsf{P}^+ * \mathsf{P}^- &= \mathsf{P}^+\mathsf{P}^- - \mathsf{K} \\
\mathsf{P}^- * \mathsf{P}^+ &= \mathsf{P}^+\mathsf{P}^- + \mathsf{K} \\
\mathsf{J} * \mathsf{P}^+ &= \mathsf{J}\mathsf{P}^+ - \mathsf{P}^+ \\
\mathsf{P}^+ * \mathsf{J} &= \mathsf{J}\mathsf{P}^+ \\
\mathsf{J} * \mathsf{P}^- &= \mathsf{J}\mathsf{P}^- + \mathsf{P}^- \\
\mathsf{P}^- * \mathsf{J} &= \mathsf{J}\mathsf{P}^-
\end{aligned} \tag{38}$$ eq:time:star:generat

To second order in α and written in position space, the $*$-product between any two functions is

$$
\begin{aligned}
f * g(x) = fg \quad & +\alpha \left(\mathsf{K}\partial_- f \partial_+ g - \mathsf{K}\partial_+ f \partial_- g + \mathsf{P}^- \partial_\mathsf{J} f \partial_- g - \mathsf{P}^+ \partial_\mathsf{J} f \partial_+ g \right) \\
& +\alpha^2 \Bigg(\frac{1}{2}\mathsf{K}^2 \partial_-^2 f \partial_+^2 g - \mathsf{K}^2 \partial_+ \partial_- f \partial_+ \partial_- g + \frac{1}{2}\mathsf{K}^2 \partial_+^2 f \partial_-^2 g \\
& \qquad + \frac{1}{2}\mathsf{P}^{+2} d_\mathsf{J}^2 f \partial_+^2 g + \frac{1}{2}\mathsf{P}^+ \partial_\mathsf{J}^2 f \partial_+ g \\
& \qquad + \frac{1}{2}\mathsf{P}^- \partial_\mathsf{J}^2 f \partial_- g + \frac{1}{2}\mathsf{P}^{-2} \partial_\mathsf{J}^2 f \partial_-^2 g \\
& \qquad - \mathsf{P}^+ \mathsf{P}^- \partial_\mathsf{J}^2 f \partial_+ \partial_- g - \mathsf{K}\partial_- \partial_\mathsf{J} f \partial_+ g - \mathsf{K}\partial_+ \partial_\mathsf{J} f \partial_- g \\
& \qquad + \mathsf{K}\mathsf{P}^+ \partial_+ \partial_\mathsf{J} f \partial_+ \partial_- g - \mathsf{K}\mathsf{P}^+ \partial_- \partial_\mathsf{J} f \partial_+^2 g \\
& \qquad + \mathsf{K}\mathsf{P}^- \partial_- \partial_\mathsf{J} f \partial_+ \partial_- g - \mathsf{K}\mathsf{P}^- \partial_+ \partial_\mathsf{J} f \partial_-^2 g \Bigg) \\
& + \mathcal{O}(\alpha^3)
\end{aligned}
\tag{39}
$$

eq:time:positionspac

1.3.2 Symmetric Time Ordering

The ordering used in [4] is of interest as it allows us to recover the plane wave coordinates. Called “Symmetric Time Ordering” due to the time coordinate on either side of the translation

$$
{:}e^{ik_i \hat{x}^i}{:} = e^{i\frac{u}{2}\hat{\mathsf{J}}} e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-} e^{i\frac{u}{2}\hat{\mathsf{J}}} e^{iv\hat{\mathsf{K}}} \tag{40}
$$

eq:symtime:defn

From (32),(33) and (34) we know how to calculate all the exponential products required to find

$$
\begin{aligned}
& {:}e^{\frac{i}{2}u\hat{\mathsf{J}}} \; e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^-} e^{\frac{i}{2}u\hat{\mathsf{J}}} e^{iv\hat{\mathsf{K}}} e^{\frac{i}{2}u'\hat{\mathsf{J}}} e^{i\bar{w}'\hat{\mathsf{P}}^+ + iw'\hat{\mathsf{P}}^-} e^{\frac{i}{2}u'\hat{\mathsf{J}}} e^{iv'\hat{\mathsf{K}}}{:} = \\
& \quad e^{\frac{i}{2}(u+u')\hat{\mathsf{J}}} e^{i\left(\bar{w}e^{\frac{i}{2}u'} + \bar{w}'e^{-\frac{i}{2}u}\right)\hat{\mathsf{P}}^+ + i\left(we^{-\frac{i}{2}u'} + w'e^{\frac{i}{2}u}\right)\hat{\mathsf{P}}^-} \\
& \quad e^{\frac{i}{2}(u+u')\hat{\mathsf{J}}} e^{\left[i(v+v') + \bar{w}w'e^{\frac{i}{2}(u+u')} - w\bar{w}'e^{-\frac{i}{2}(u+u')}\right]\hat{\mathsf{K}}}
\end{aligned}
\tag{41}
$$

eq:symtime:mult

so that the $\bullet$-product between any two functions is

$$
\begin{aligned}
f \bullet g(z) &= f(z) e^{ix^i (F_i^\bullet - k_i - k_i')} g(z) \\
F_1^\bullet &= \bar{w}e^{\frac{i}{2}\alpha u'} + \bar{w}'e^{-\frac{i}{2}\alpha u} \\
F_2^\bullet &= we^{-\frac{i}{2}\alpha u'} + w'e^{\frac{i}{2}\alpha u} \\
F_3^\bullet &= u + u' \\
F_4^\bullet &= v + v' - i\alpha\bar{w}w'e^{\frac{i}{2}\alpha(u+u')} + i\alpha w\bar{w}'e^{-\frac{i}{2}\alpha(u+u')}
\end{aligned}
\tag{42}
$$

eq:symtime:star

where k runs over $\bar{w}, w, u, v$ and x over $\mathsf{P}^+, \mathsf{P}^-, \mathsf{J}, \mathsf{K}$. We see from (11) and (12) that the operator algebra is recovered in the functional space with the

$\bullet$-product between the generators being

$$\begin{aligned}
\mathsf{P}^+ \bullet \mathsf{P}^- &= \mathsf{P}^+\mathsf{P}^- - \mathsf{K} \\
\mathsf{P}^- \bullet \mathsf{P}^+ &= \mathsf{P}^+\mathsf{P}^- + \mathsf{K} \\
\mathsf{P}^+ \bullet \mathsf{J} &= \mathsf{J}\mathsf{P}^+ + \frac{1}{2}\mathsf{P}^+ \\
\mathsf{J} \bullet \mathsf{P}^+ &= \mathsf{J}\mathsf{P}^+ - \frac{1}{2}\mathsf{P}^+ \\
\mathsf{P}^- \bullet \mathsf{J} &= \mathsf{J}\mathsf{P}^- - \frac{1}{2}\mathsf{P}^- \\
\mathsf{J} \bullet \mathsf{P}^- &= \mathsf{J}\mathsf{P}^- + \frac{1}{2}\mathsf{P}^-
\end{aligned} \tag{43}$$

eq:symtime:generator

To second order in α and written in position space, the $\bullet$-product between any two functions is

$$\begin{aligned}
& f \bullet g(x) = fg \\
&+ \frac{\alpha}{2}\Big(-2\mathsf{K}\partial_+ f\partial_- g + 2\mathsf{K}\partial_- f\partial_+ g - \mathsf{P}^-\partial_- f\partial_{\mathsf{J}} g \\
&\qquad +\mathsf{P}^-\partial_{\mathsf{J}} f\partial_- g + \mathsf{P}^+\partial_+ f\partial_{\mathsf{J}} g - \mathsf{P}^+\partial_{\mathsf{J}} f\partial_+ g\Big) \\
&+ \frac{\alpha^2}{2}\Big(-\mathsf{K}\partial_- f\partial_+\partial_{\mathsf{J}} g - \mathsf{K}\partial_+ f\partial_-\partial_{\mathsf{J}} g - \mathsf{K}\partial_+\partial_{\mathsf{J}} f\partial_- g - \mathsf{K}\partial_-\partial_{\mathsf{J}} f\partial_+ g \\
&+\mathsf{K}\mathsf{P}^+\partial_+\partial_- f\partial_+\partial_{\mathsf{J}} g - \mathsf{K}\mathsf{P}^+\partial_-\partial_{\mathsf{J}} f\partial_+^2 g + \mathsf{K}\mathsf{P}^-\partial_-\partial_{\mathsf{J}} f\partial_+\partial_- g \\
&+\mathsf{K}\mathsf{P}^+\partial_+\partial_{\mathsf{J}} f\partial_+\partial_- g - \mathsf{K}\mathsf{P}^-\partial_-^2 f\partial_+\partial_{\mathsf{J}} g - \mathsf{K}\mathsf{P}^+\partial_+^2 f\partial_-\partial_{\mathsf{J}} g \\
&+\mathsf{K}\mathsf{P}^-\partial_+\partial_- f\partial_-\partial_{\mathsf{J}} g - \mathsf{K}\mathsf{P}^-\partial_+\partial_{\mathsf{J}} f\partial_-^2 g - \frac{1}{2}\mathsf{P}^+\mathsf{P}^-\partial_{\mathsf{J}}^2 f\partial_+\partial_- g \\
&-\frac{1}{2}\mathsf{P}^+\mathsf{P}^-\partial_+\partial_- f\partial_{\mathsf{J}}^2 g + \frac{1}{2}\mathsf{P}^+\mathsf{P}^-\partial_+\partial_{\mathsf{J}} f\partial_-\partial_{\mathsf{J}} g + \frac{1}{2}\mathsf{P}^+\mathsf{P}^-\partial_-\partial_{\mathsf{J}} f\partial_+\partial_{\mathsf{J}} g \\
&-\frac{1}{2}\mathsf{P}^{+2}\partial_+\partial_{\mathsf{J}} f\partial_+\partial_{\mathsf{J}} g - \frac{1}{2}\mathsf{P}^{-2}\partial_-\partial_{\mathsf{J}} f\partial_-\partial_{\mathsf{J}} g + \frac{1}{4}\mathsf{P}^{-2}\partial_{\mathsf{J}}^2 f\partial_-^2 g \\
&+\frac{1}{4}\mathsf{P}^{+2}\partial_+^2 f\partial_{\mathsf{J}}^2 g + \frac{1}{4}\mathsf{P}^{+2}\partial_{\mathsf{J}}^2 f\partial_+^2 g + \frac{1}{4}\mathsf{P}^{-2}\partial_-^2 f\partial_{\mathsf{J}}^2 g \\
&+\mathsf{K}^2\partial_+^2 f\partial_-^2 g + \mathsf{K}^2\partial_-^2 f\partial_+^2 g - 2\mathsf{K}^2\partial_+\partial_- f\partial_+\partial_- g \\
&+\frac{1}{4}\mathsf{P}^+\partial_{\mathsf{J}}^2 f\partial_+ g + \frac{1}{4}\mathsf{P}^+\partial_+ f\partial_{\mathsf{J}}^2 g + \frac{1}{4}\mathsf{P}^-\partial_- f\partial_{\mathsf{J}}^2 g + \frac{1}{4}\mathsf{P}^-\partial_{\mathsf{J}}^2 f\partial_- g\Big) \\
&+ \mathcal{O}(\alpha^3)
\end{aligned} \tag{44}$$

eq:symtime:positions

1.3.3 Weyl Ordering

The most obvious, but least physically enlightening parameterisation is given by the Weyl ordering

$$
{}^{\circ}_{\circ} e^{ik_i \hat{x}^i} {}^{\circ}_{\circ} = e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^- + iu\hat{\mathsf{J}} + iv\hat{\mathsf{K}}} \tag{45}
$$

eq:weyl:defn

It is a difficult task to calculate the product directly using the BCH formula (15) as it is a complicated, unrecognisable, infinite sum. We can however investigate the BCH formula for

$$
e^{ip_1\hat{\mathsf{P}}^+ + ip_2\hat{\mathsf{P}}^-} e^{ip_3\hat{\mathsf{J}}} = e^{G_1(\boldsymbol{p})\hat{\mathsf{P}}^+ + G_2(\boldsymbol{p})\hat{\mathsf{P}}^- + G_3(\boldsymbol{p})\hat{\mathsf{J}} + G_4(\boldsymbol{p})\hat{\mathsf{K}}} \tag{46}
$$

eq:weyl:start

where $G_i(\boldsymbol{p})$ may be calculated. This lets us write a time ordered product using the Weyl ordering coefficients. Using (36) the full exponential product can be calculated, and we may then use (46) again to get back to the Weyl ordering.

We notice that the only surviving $\hat{\mathsf{P}}^+$ terms are given by the commutators

$$
\left[\hat{\mathsf{J}}, \left[\ldots, \left[\hat{\mathsf{J}}, \hat{\mathsf{P}}^+\right]\right]\ldots\right]
$$

Gathering all the terms, we find

$$
\begin{aligned}
G_1 &= \sum_{n=0} \frac{B_n}{n!} \left[ip_3\hat{\mathsf{J}}_n, \left[\ldots, \left[ip_3\hat{\mathsf{J}}_1, ip_1\hat{\mathsf{P}}^+\right]\right]\ldots\right] \\
&= p_1 \sum_{n=0} \frac{B_n}{n!}(-ip_3)^n \\
&= \frac{ip_1}{\Phi(-ip_3)}
\end{aligned} \tag{47}
$$

eq:weyl:G1

where we have introduced the generating function

$$
\Phi(a) = \frac{e^{\beta a} - 1}{a} \tag{48}
$$

eq:weyl:Phi

Similarly for G_2

$$
\begin{aligned}
G_2 &= \sum_{n=0} \frac{B_n}{n!} \left[ip_3\hat{\mathsf{J}}_n, \left[\ldots, \left[ip_3\hat{\mathsf{J}}_1, ip_2\hat{\mathsf{P}}^-\right]\right]\ldots\right] \\
&= p_2 \sum_{n=0} \frac{B_n}{n!}(ip_3)^n \\
&= \frac{ip_2}{\Phi(ip_3)}
\end{aligned} \tag{49}
$$

eq:weyl:G2

The G_3 term is trivial

$$
G_3 = ip_3 \tag{50}
$$

eq:weyl:G3

For G_4 there are several non-vanishing commutators

$$\begin{aligned}
G_4 &= \sum_{n=1} \frac{B_{n+1}}{n!} \left[ip_1\hat{\mathsf{P}}^+, \left[ip_3\hat{\mathsf{J}}_n, \dots \left[ip_3\hat{\mathsf{J}}_1, ip_2\hat{\mathsf{P}}^-\right] \dots\right]\right] \\
&+ \sum_{n=1} \frac{B_{n+1}}{n!} \left[ip_2\hat{\mathsf{P}}^-, \left[ip_3\hat{\mathsf{J}}_n, \dots \left[ip_3\hat{\mathsf{J}}_1, ip_1\hat{\mathsf{P}}^+\right] \dots\right]\right] \\
&= -2p_1p_2 \sum_{n=1} \frac{B_{n+1}}{n!} (ip_3)^n \\
&= -2p_1p_2 \Upsilon(ip_3) \qquad (51)
\end{aligned}$$

eq:weyl:G4

where we have introduced the generating function (see Appendix A for a derivation)[†]

$$\Upsilon(a) = \frac{1}{2} + \frac{e^{\beta a} - 1 - \beta a e^{\beta a}}{\left(e^{\beta a} - 1\right)^2} \qquad (52)$$

eq:weyl:Upsilon

Now that we know the G_i in (46), we can rewrite the Weyl ordering as a product of exponentials. In preparation, we define $v = \widetilde{v} + q_4$ and write the Weyl ordering as

$$e^{i\bar{w}\hat{\mathsf{P}}^+ + iw\hat{\mathsf{P}}^- + iu\hat{\mathsf{J}} + i\widetilde{v}\hat{\mathsf{K}}} e^{iq_4\hat{\mathsf{K}}} \qquad (53)$$

eq:weyl:rewritten:1

We proceed to define qs by

$$\begin{aligned}
q_1 &= \bar{w}\Phi(-iu) \qquad (54) \\
q_2 &= w\Phi(iu) \\
q_3 &= u \\
q_4 &= v - 2i\bar{w}w\Upsilon(iu)\Phi(iu)\Phi(-iu)
\end{aligned}$$

eq:weyl:qs

allowing us to rewrite the Weyl ordering as

$$e^{iq_1\hat{\mathsf{P}}^+ + iq_2\hat{\mathsf{P}}^-} e^{iq_3\hat{\mathsf{J}}} e^{iq_4\hat{\mathsf{K}}} \qquad (55)$$

eq:weyl:time

It should be clear that $\bar{q}_1 = q_2$ and u is a real angle. Using (36) to calculate the exponential product of the time ordering form and then (46) to get back

[†]Note that the α-scaling also requires, in addition to the exponential product changes, that $\Phi(\lambda) \to \Phi(\alpha\lambda)$ and $\Upsilon(\lambda) \to \alpha\Upsilon(\alpha\lambda)$

into the Weyl ordering form, we find the $\star$-product to be

$$
\begin{aligned}
f \star g(z) &= f(z) e^{i x^i (F_i^\star - k_i - k_i')} g(z) \\
F_1^\star &= \frac{\bar{w}\Phi(-i\alpha u) + \bar{w}'\Phi(-i\alpha u')e^{-i\alpha u}}{\Phi(-i\alpha u - i\alpha u')} \\
F_2^\star &= \frac{w\Phi(i\alpha u) + w'\Phi(i\alpha u')e^{i\alpha u}}{\Phi(i\alpha u - i\alpha u')} \\
F_3^\star &= u + u' \\
F_4^\star &= -2i\alpha\bar{w}w\Phi(i\alpha u)\Phi(-i\alpha u)\Upsilon(i\alpha u) + i\alpha w\bar{w}'\Phi(-i\alpha u)\Phi(-i\alpha u') \\
&\quad -2i\alpha\bar{w}'w'\Phi(i\alpha u')\Phi(-i\alpha u')\Upsilon(i\alpha u') - i\alpha\bar{w}w'\Phi(i\alpha u)\Phi(i\alpha u') \\
&\quad +2i\alpha\left(\bar{w}\Phi(i\alpha u) + \bar{w}'\Phi(-i\alpha u')\right)\left(w\Phi(-i\alpha u) + w'\Phi(i\alpha u')\right) \\
&\qquad \times\Upsilon(i\alpha u + i\alpha u') + v + v'
\end{aligned} \tag{56}
$$

eq:weyl:star

where x runs over $\mathsf{P}^+, \mathsf{P}^-, \mathsf{J}, \mathsf{K}$. We see from (11) and (12) that the operator algebra is recovered in the functional space with the $\star$-product between the generators being

$$
\begin{aligned}
\mathsf{P}^+ \star \mathsf{P}^- &= \mathsf{P}^+\mathsf{P}^- - \mathsf{K} \\
\mathsf{P}^- \star \mathsf{P}^+ &= \mathsf{P}^+\mathsf{P}^- + \mathsf{K} \\
\mathsf{P}^+ \star \mathsf{J} &= \mathsf{J}\mathsf{P}^+ + \frac{1}{2}\mathsf{P}^+ \\
\mathsf{J} \star \mathsf{P}^+ &= \mathsf{J}\mathsf{P}^+ - \frac{1}{2}\mathsf{P}^+ \\
\mathsf{P}^- \star \mathsf{J} &= \mathsf{J}\mathsf{P}^- - \frac{1}{2}\mathsf{P}^- \\
\mathsf{J} \star \mathsf{P}^- &= \mathsf{J}\mathsf{P}^- + \frac{1}{2}\mathsf{P}^-
\end{aligned} \tag{57}
$$

eq:weyl:generators

To second order in α and written in position space, the $\star$-product between any two functions is

$$
\begin{aligned}
f\star g(x) = {} & fg + \frac{\alpha}{2}\Big(2\mathsf{K}\partial_- f\partial_+ g + \mathsf{P}^-\partial_\mathsf{J} f\partial_- g - \mathsf{P}^-\partial_- f\partial_\mathsf{J} g \\
& \qquad -2\mathsf{K}\partial_+ f\partial_- g - \mathsf{P}^+\partial_\mathsf{J} f\partial_+ g + \mathsf{P}^+\partial_+ f\partial_\mathsf{J} g\Big) \\
& + \frac{\alpha^2}{2}\Big(\mathsf{K}^2\partial_+^2 f\partial_-^2 g + \mathsf{K}\mathsf{P}^+\partial_+\partial_- f\partial_+\partial_\mathsf{J} g + \mathsf{K}^2\partial_-^2 f\partial_+^2 g \\
& +\frac{1}{6}\mathsf{P}^+\partial_\mathsf{J}^2 f\partial_+ g + \frac{1}{6}\mathsf{P}^+\partial_+ f\partial_\mathsf{J}^2 g + \frac{1}{6}\mathsf{P}^-\partial_- f\partial_\mathsf{J}^2 g \\
& -\mathsf{K}\mathsf{P}^+\partial_+^2 f\partial_-\partial_\mathsf{J} g + \mathsf{K}\mathsf{P}^-\partial_-\partial_\mathsf{J} f\partial_+\partial_- g - \mathsf{K}\mathsf{P}^-\partial_-^2 f\partial_+\partial_\mathsf{J} g \\
& -\mathsf{K}\mathsf{P}^+\partial_-\partial_\mathsf{J} f\partial_+^2 g - 2\mathsf{K}^2\partial_+\partial_- f\partial_+\partial_- g + \mathsf{K}\mathsf{P}^-\partial_+\partial_- f\partial_-\partial_\mathsf{J} g \\
& +\mathsf{K}\mathsf{P}^+\partial_+\partial_\mathsf{J} f\partial_+\partial_- g - \mathsf{K}\mathsf{P}^-\partial_+\partial_\mathsf{J} f\partial_-^2 g + \frac{1}{6}\mathsf{P}^-\partial_\mathsf{J}^2 f\partial_- g \\
& -\frac{1}{2}\mathsf{P}^{+2}\partial_+\partial_\mathsf{J} f\partial_+\partial_\mathsf{J} g - \frac{1}{2}\mathsf{P}^{-2}\partial_-\partial_\mathsf{J} f\partial_-\partial_\mathsf{J} g \\
& -\frac{1}{6}\mathsf{P}^-\partial_\mathsf{J} f\partial_-\partial_\mathsf{J} g - \frac{1}{6}\mathsf{P}^+\partial_+\partial_\mathsf{J} f\partial_\mathsf{J} g - \frac{1}{6}\mathsf{P}^+\partial_\mathsf{J} f\partial_+\partial_\mathsf{J} g \\
& -\frac{1}{3}\mathsf{K}\partial_+ f\partial_-\partial_\mathsf{J} g + \frac{2}{3}\mathsf{K}\partial_\mathsf{J} f\partial_+\partial_- g + \frac{2}{3}\mathsf{K}\partial_+\partial_- f\partial_\mathsf{J} g \\
& +\frac{1}{4}\mathsf{P}^{+2}\partial_\mathsf{J}^2 f\partial_+^2 g + \frac{1}{2}\mathsf{P}^+\mathsf{P}^-\partial_-\partial_\mathsf{J} f\partial_+\partial_\mathsf{J} g - \frac{1}{2}\mathsf{P}^+\mathsf{P}^-\partial_+\partial_- f\partial_\mathsf{J}^2 g \\
& +\frac{1}{4}\mathsf{P}^{-2}\partial_\mathsf{J}^2 f\partial_-^2 g + \frac{1}{4}\mathsf{P}^{-2}\partial_-^2 f\partial_\mathsf{J}^2 g - \frac{1}{6}\mathsf{P}^-\partial_-\partial_\mathsf{J} f\partial_\mathsf{J} g \\
& -\frac{2}{6}\mathsf{K}\partial_-\partial_\mathsf{J} f\partial_+ g - \frac{2}{6}\mathsf{K}\partial_- f\partial_+\partial_\mathsf{J} g - \frac{1}{3}\mathsf{K}\partial_+\partial_\mathsf{J} f\partial_- g \\
& +\frac{1}{2}\mathsf{P}^+\mathsf{P}^-\partial_+\partial_\mathsf{J} f\partial_-\partial_\mathsf{J} g + \frac{1}{4}\mathsf{P}^{+2}\partial_+^2 f\partial_\mathsf{J}^2 g - \frac{1}{2}\mathsf{P}^+\mathsf{P}^-\partial_\mathsf{J}^2 f\partial_+\partial_- g\Big) \\
& + \mathcal{O}(\alpha^3)
\end{aligned}
\tag{58}
$$

eq:weyl:positionspac

2 Weyl Systems

2.1 Standard Weyl Systems

A Weyl system is a map $\mathcal{W}$ from a real, finite dimensional, symplectic vector space S to the set of unitary operators on a suitable Hilbert space, with the property

$$\mathcal{W}(k)\,\mathcal{W}(k') = e^{i\omega(k,k')}\,\mathcal{W}(k')\,\mathcal{W}(k) \tag{59}$$

eq:nw:weyl:st:prop

where ω is the symplectic, translationally invariant form on S. According to Stone's theorem, $\mathcal{W}$ is the exponential of a Hermitian operator on $\mathcal{H}$

$$\mathcal{W}(ak) = e^{iaX(k)} \tag{60}$$

eq:nw:weyl:st:exp

The original motivation was to avoid the presence of unbounded operators in the Q.M. formalism and is satisfied by property (59) as it can be considered the exponentiated version of the commutation relations. The usual form of the commutators can be recovered with a series expansion

$$\big[X(k), X(k')\big] = -i\omega(k,k') \tag{61}$$

eq:nw:weyl:st:series

The usual identification has $S = \mathbb{R}^{2n}$ and $\omega = dq^i \wedge dp^j$ where (q^i, p_j) are canonical coordinates, giving the explicit realisation

$$\mathcal{W}(q,p) = e^{i\left(q^i Q_j + p_i P^j\right)} \tag{62}$$

eq:nw:weyl:st:realis

where Q and P are the usual operators that represent position and momentum observables.

It is then possible to formulate the Groenewold-Moyal $\star$-product given in section 1.1, which is the case when $\omega(k,k')$ and Θ are constant.

2.2 Generalised Weyl Systems

The previous construction of a $\star$-product for a Weyl system is not valid for the case when Θ is non-constant, simply because property (59) does not hold.

It is possible to define a class of deformed products in a set of functions on $\mathbb{R}^n$ without using an explicit realisation of $\mathcal{W}$, thus generalising the concept. A *generalised Weyl system* [5] is a map in the set of operators $\mathcal{W}$ with composition rule

$$\mathcal{W}(k)\,\mathcal{W}(k') = e^{\frac{i}{2}\omega(k,k')}\,\mathcal{W}(k \boxplus k') \tag{63}$$

eq:weyl:gen:prop

$\boxplus$ forms a group on $\mathbb{R}^n$ satisfying

$$(k \boxplus k') \boxplus k'' = k \boxplus (k' \boxplus k'') \tag{64}$$

eq:weyl:comp:group

$$k \boxplus \underline{k} = 0 \tag{65}$$

where $\underline{k}$ is the inverse of k and 0 is the neutral element. When $\boxplus$ is the normal addition on $\mathbb{R}^n$ this is a standard Weyl system.

In analogy to the standard Weyl system, we define the symbol

$$\hat{\Omega}[f] \equiv F = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \widetilde{f}(k)\, \mathcal{W}(k) dk \tag{66}$$

eq:weyl:gen:symb

(from here on we use capital Roman letters to denote elements of the deformed algebra). The product is

$$FG = \frac{1}{(2\pi)^{2n}} \int_{-\infty}^{\infty} \widetilde{f}(k)\widetilde{g}(k') e^{\frac{i}{2}\omega(k,k')}\, \mathcal{W}(k \boxplus k') dk dk' \tag{67}$$

eq:weyl:gen:prod

and it can be shown that

$$f \star g(x) = \frac{1}{(2\pi)^{2n}} \int_{-\infty}^{\infty} \widetilde{f}(k)\widetilde{g}(k') e^{\frac{i}{2}\omega(k,k')} e^{i(k \boxplus k')x} dk dk' \tag{68}$$

eq:weyl:gen:star

where

$$e^{ikx} = \hat{\Omega}^{-1}[\mathcal{W}(k)] \tag{69}$$

eq:weyl:gen:exp

Associativity of this product is a consequence of the associativity of $\boxplus$. The Hermitian conjugate of F is defined as

$$F^\dagger = \frac{1}{(2\pi)^{2n}} \int_{-\infty}^{\infty} \widetilde{f}^*(\underline{k})\, \mathcal{W}(k) dk \tag{70}$$

eq:weyl:gen:herm

and

$$(FG)^\dagger = G^\dagger F^\dagger \tag{71}$$

eq:weyl:gen:herm:2

an extra condition

$$\underline{(k \boxplus k')} = (\underline{k'} \boxplus \underline{k}) \tag{72}$$

eq:weyl:gen:lizzicon

is introduced in [5] which is automatically satisfied by the associativity of $\boxplus$ since

$$\begin{aligned} (k \boxplus k') \boxplus \underline{(k \boxplus k')} &= 0 \\ (k \boxplus k') \boxplus (\underline{k'} \boxplus \underline{k}) &= k \boxplus (k' \boxplus \underline{k'}) \boxplus \underline{k} = 0 \end{aligned} \tag{73}$$

eq:weyl:gen:assoc

2.3 Nappi-Witten Generalised Weyl System

As the Nappi-Witten algebra cannot satisfy property (59), it does not meet the conditions required to be a standard Weyl system. It can however be presented as a generalised Weyl system. We find the compositions $\boxplus$ for each ordering by inspection of the previously derived $\star$-products (37), (42) and (56). In each case $\omega(k, k') = 0$ and the composition can be shown to be associative.

2.3.1 Time Ordering

By inspection of (37), we find the composition $\boxast$ to be

$$\begin{aligned}
(k \boxast k')^1 &= \bar{w} + \bar{w}' e^{-i\alpha u} \\
(k \boxast k')^2 &= w + w' e^{i\alpha u} \\
(k \boxast k')^3 &= u + u' \\
(k \boxast k')^4 &= v + v' - i\alpha \bar{w} w' e^{i\alpha u} + i\alpha w \bar{w}' e^{-i\alpha u}
\end{aligned} \tag{74}$$

`eq:weyl:nw:time`

where we define k and it's inverse $\underline{k}$, component-wise as

$$\begin{aligned}
k &= (\bar{w}, w, u, v) \\
\underline{k} &= (-e^{iu}\bar{w}, -e^{-iu} w, -u, -v)
\end{aligned} \tag{75}$$

`eq:weyl:gen:time`

2.3.2 Symmetric Time Ordering

By inspection of (42), we find the composition $\boxdot$ to be

$$\begin{aligned}
(k \boxdot k')^1 &= \bar{w} e^{\frac{i}{2}\alpha u'} + \bar{w}' e^{-\frac{i}{2}\alpha u} \\
(k \boxdot k')^2 &= w e^{-\frac{i}{2}\alpha u'} + w' e^{\frac{i}{2}\alpha u} \\
(k \boxdot k')^3 &= u + u' \\
(k \boxdot k')^4 &= v + v' - i\alpha \bar{w} w' e^{\frac{i}{2}\alpha(u+u')} + i\alpha w \bar{w}' e^{-\frac{i}{2}\alpha(u+u')}
\end{aligned} \tag{76}$$

`eq:weyl:nw:symtime`

where we define k and it's inverse $\underline{k}$, component-wise as

$$\begin{aligned}
k &= (\bar{w}, w, u, v) \\
\underline{k} &= (-\bar{w}, -w, -u, -v)
\end{aligned} \tag{77}$$

`eq:weyl:gen:symtime`

2.3.3 Weyl Ordering

By inspection of (56), we find the composition $\boxstar$ to be

$$\begin{aligned}
(k \boxstar k')^1 &= \frac{\bar{w}\Phi(-i\alpha u) + \bar{w}'\Phi(-i\alpha u') e^{-i\alpha u}}{\Phi(-i\alpha u - i\alpha u')} \\
(k \boxstar k')^2 &= \frac{w\Phi(i\alpha u) + w'\Phi(i\alpha u') e^{i\alpha u}}{\Phi(i\alpha u - i\alpha u')} \\
(k \boxstar k')^3 &= u + u' \\
(k \boxstar k')^4 &= -2i\alpha \bar{w} w \Phi(i\alpha u)\Phi(-i\alpha u)\Upsilon(i\alpha u) + i\alpha w \bar{w}' \Phi(-i\alpha u)\Phi(-i\alpha u') \\
&\quad -2i\alpha \bar{w}' w' \Phi(i\alpha u')\Phi(-i\alpha u')\Upsilon(i\alpha u') - i\alpha \bar{w} w' \Phi(i\alpha u)\Phi(i\alpha u') \\
&\quad +2i\alpha \left(\bar{w}\Phi(i\alpha u) + \bar{w}'\Phi(-i\alpha u')\right)\left(w\Phi(-i\alpha u) + w'\Phi(i\alpha u')\right) \\
&\qquad \times \Upsilon(i\alpha u + i\alpha u') + v + v'
\end{aligned} \tag{78}$$

`eq:weyl:nw:weyl`

where we define k and it's inverse $\underline{k}$, component-wise as

$$\begin{aligned}
k &= (k_1, k_2, k_3, k_4) \\
\underline{k} &= (-k_1, -k_2, -k_3, -k_4)
\end{aligned} \tag{79}$$

`eq:weyl:gen:weyl`

3 Derivatives

3.1 Derivative Commutators

For any Lie Algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\lambda \hat{x}^\lambda \tag{80}$$ eq:lie

we may demand linear derivatives [7] which look like

$$\left[\hat{\partial}_\mu, \hat{x}^\nu\right] = \delta^\nu_\mu + i\rho^{\nu\lambda}_\mu \hat{\partial}_\lambda \tag{81}$$ eq:deriv

The Jacobi identity between $\hat{\partial}_\mu$, $\hat{x}^\nu$, $\hat{x}^\lambda$ supplies the sufficient conditions

$$\begin{aligned} \rho^{\mu\nu}_\lambda - \rho^{\nu\mu}_\lambda &= C^{\mu\nu}_\lambda \\ \rho^{\mu\nu}_\lambda \rho^{\kappa\sigma}_\nu - \rho^{\kappa\nu}_\lambda \rho^{\mu\sigma}_\nu &= C^{\mu\kappa}_\nu \rho^{\nu\sigma}_\lambda \end{aligned} \tag{82}$$ eq:deriv:rho

3.2 $\star$-Derivatives

This allows us to define a $\star$-derivative by elevating the $\hat{\partial}$ to $\partial^\star$ composed of normal derivatives and satisfying the following conditions

$$\begin{aligned} [x^\mu, x^\nu]_\star &= iC^{\mu\nu}_\lambda x^\lambda \\ \left[\partial^\star_\mu, x^\nu\right]_\star f(x) &= \partial^\star_\mu(x^\nu \star f(x)) - x^\nu \star (\partial^\star_\mu f(x)) \\ &= (\delta^\nu_\mu + i\rho^{\nu\lambda}_\mu \partial^\star_\lambda) f(x) \end{aligned} \tag{83}$$ eq:lie:starcomm

TODO: I see why the awful notation by the Germans exists now... using this cleaner notation means that strictly speaking we mean $\partial^\star_\mu \star (\ldots)$ instead of $\partial^\star_\mu(\ldots)$ in (83)

3.3 Leibniz Rule

These $\hat{\partial}$, $\partial^\star$ do not necessarily follow the normal Leibniz rule. However we may obtain a Leibniz rule from the co-product, given by the Generalised Weyl System

TODO: I am still unclear about the details of *why* this is true. speak to Berndt maybe?

For example, (74) implies the co-product

$$\Delta(\hat{\partial}_1) = \hat{\partial}_1 \otimes \mathbb{1} + e^{\alpha\hat{\partial}_3} \otimes \hat{\partial}_1 \tag{84}$$ eq:coproduct

giving the Leibniz rule

$$\begin{aligned} \hat{\nabla}_1(\hat{f}\hat{g}) &= i\mu\Delta(\hat{\partial}_1)(\hat{f} \otimes \hat{g}) \\ &= (\hat{\partial}_1 \hat{f})\hat{g} + (e^{\alpha\hat{\partial}_3}\hat{f})(\hat{\partial}_1 \hat{g}) \end{aligned} \tag{85}$$ eq:leibniz

with the rule for the $\star$-derivatives being inherited

$$\nabla_1^\star(f \star g) = (\partial_1^\star f) \star g + (e^{\alpha\partial_3^\star} f) \star (\partial_1^\star g) \tag{86}$$
`eq:leibtiz:star`

It is important to note that normal Leibniz does not necessarily hold even when fg is normal multiplication

$$\nabla^\star(fg) \neq f\nabla^\star g + g\nabla^\star f \tag{87}$$
`eq:leibniz:normal`

3.4 Nappi-Witten $\star$-Derivatives

There are an infinite number of solutions to $\rho_\lambda^{\mu\nu}$ (82), but we choose an intuitive one giving

$$\begin{array}{llll}
\left[\hat\partial_1, \hat x^1\right] = 1 & \left[\hat\partial_2, \hat x^1\right] = 0 & \left[\hat\partial_3, \hat x^1\right] = 0 & \left[\hat\partial_4, \hat x^1\right] = -\alpha\hat\partial_2 \\
\left[\hat\partial_1, \hat x^2\right] = 0 & \left[\hat\partial_2, \hat x^2\right] = 1 & \left[\hat\partial_3, \hat x^2\right] = 0 & \left[\hat\partial_4, \hat x^2\right] = \alpha\hat\partial_1 \\
\left[\hat\partial_1, \hat x^3\right] = -\alpha\hat\partial_1 & \left[\hat\partial_2, \hat x^3\right] = \alpha\hat\partial_2 & \left[\hat\partial_3, \hat x^3\right] = 1 & \left[\hat\partial_4, \hat x^3\right] = 0 \\
\left[\hat\partial_1, \hat x^4\right] = 0 & \left[\hat\partial_2, \hat x^4\right] = 0 & \left[\hat\partial_3, \hat x^4\right] = 0 & \left[\hat\partial_4, \hat x^4\right] = 1
\end{array} \tag{88}$$
`eq:rho:nw4`

3.4.1 Time Ordering

From (74) we deduce the co-products

$$\begin{aligned}
\Delta^*(\hat\partial_1) &= \hat\partial_1 \otimes \mathbb{1} + e^{\alpha\hat\partial_3} \otimes \hat\partial_1 \\
\Delta^*(\hat\partial_2) &= \hat\partial_2 \otimes \mathbb{1} + e^{-\alpha\hat\partial_3} \otimes \hat\partial_2 \\
\Delta^*(\hat\partial_3) &= \hat\partial_3 \otimes \mathbb{1} + \mathbb{1} \otimes \hat\partial_3 \\
\Delta^*(\hat\partial_4) &= \hat\partial_4 \otimes \mathbb{1} + \mathbb{1} \otimes \hat\partial_4 - i\alpha\hat\partial_1 e^{-\alpha\hat\partial_3} \otimes \hat\partial_2 + i\alpha\hat\partial_2 e^{\alpha\hat\partial_3} \otimes \hat\partial_1
\end{aligned} \tag{89}$$
`eq:deriv:leibniz:tim`

Using (37) we find the $x * f(x)$ products

$$\begin{aligned}
x^1 * f(x) &= (x^1 - \alpha x^4\partial_2) f(x) \\
x^2 * f(x) &= (x^2 + \alpha x^4\partial_1) f(x) \\
x^3 * f(x) &= (x^3 + \alpha x^2\partial_2 - \alpha x^1\partial_1) f(x) \\
x^4 * f(x) &= x^4 f(x)
\end{aligned} \tag{90}$$
`eq:time:xstar`

which we may insert into (83) to show that the $*$-derivatives are simply the normal derivatives on functions

$$\partial_i^* = \partial_i \tag{91}$$
`eq:time:derivs`

3.4.2 Symmetric Time Ordering

From (76) we deduce the co-products

$$
\begin{aligned}
\Delta^{\bullet}(\hat{\partial}_1) &= \hat{\partial}_1 \otimes e^{-\frac{\alpha}{2}\hat{\partial}_3} + e^{\frac{\alpha}{2}\hat{\partial}_3} \otimes \hat{\partial}_1 \\
\Delta^{\bullet}(\hat{\partial}_2) &= \hat{\partial}_2 \otimes e^{\frac{\alpha}{2}\hat{\partial}_3} + e^{-\frac{\alpha}{2}\hat{\partial}_3} \otimes \hat{\partial}_2 \\
\Delta^{\bullet}(\hat{\partial}_3) &= \hat{\partial}_3 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\partial}_3 \\
\Delta^{\bullet}(\hat{\partial}_4) &= \hat{\partial}_4 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\partial}_4 - i\alpha\hat{\partial}_1 e^{-\frac{\alpha}{2}\hat{\partial}_3} \otimes e^{-\frac{\alpha}{2}\hat{\partial}_3}\hat{\partial}_2 + i\alpha\hat{\partial}_2 e^{\frac{\alpha}{2}\hat{\partial}_3} \otimes e^{\frac{\alpha}{2}\hat{\partial}_3}\hat{\partial}_1
\end{aligned} \tag{92}
$$

eq:deriv:leibniz:sym

Using (42) we find the $x \bullet f(x)$ products

$$
\begin{aligned}
x^1 \bullet f(x) &= \left((x^1 - \alpha x^4 \partial_2) e^{\frac{\alpha}{2}\partial_3}\right) f(x) \\
x^2 \bullet f(x) &= \left((x^2 + \alpha x^4 \partial_1) e^{-\frac{\alpha}{2}\partial_3}\right) f(x) \\
x^3 \bullet f(x) &= (x^3 + \frac{\alpha}{2} x^2 \partial_2 - \frac{\alpha}{2} x^1 \partial_1) f(x) \\
x^4 \bullet f(x) &= x^4 f(x)
\end{aligned} \tag{93}
$$

eq:symtime:xstar

which we may insert into (83) to show that the $\bullet$-derivatives are non-trivial

$$
\begin{aligned}
\partial_1^{\bullet} &= \partial_1 e^{-\frac{\alpha}{2}\partial_3} \\
\partial_2^{\bullet} &= \partial_2 e^{\frac{\alpha}{2}\partial_3} \\
\partial_3^{\bullet} &= \partial_3 \\
\partial_4^{\bullet} &= \partial_4
\end{aligned} \tag{94}
$$

eq:symtime:derivs

3.4.3 Weyl Ordering

From (78) we deduce the co-products

$$
\begin{aligned}
\Delta^{\star}(\hat{\partial}_1) &= \frac{\hat{\partial}_1 \Phi(\alpha\hat{\partial}_3) \otimes \mathbb{1} + e^{\alpha\hat{\partial}_3} \otimes \hat{\partial}_2 \Phi(\alpha\hat{\partial}_3)}{\Phi(\alpha\hat{\partial}_3 \otimes \mathbb{1} + \mathbb{1} \otimes \alpha\hat{\partial}_3)} \\
\Delta^{\star}(\hat{\partial}_2) &= \frac{\hat{\partial}_2 \Phi(-\alpha\hat{\partial}_3) \otimes \mathbb{1} + e^{-\alpha\hat{\partial}_3} \otimes \hat{\partial}_1 \Phi(-\alpha\hat{\partial}_3)}{\Phi(-\alpha\hat{\partial}_3 \otimes \mathbb{1} + \mathbb{1} \otimes \alpha\hat{\partial}_3)} \\
\Delta^{\star}(\hat{\partial}_3) &= \hat{\partial}_3 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\partial}_3 \\
\Delta^{\star}(\hat{\partial}_4) &= -2\alpha\hat{\partial}_1\hat{\partial}_2 \Phi(-\alpha\hat{\partial}_3)\Phi(\alpha\hat{\partial}_3)\Upsilon(-\alpha\hat{\partial}_3) \otimes \mathbb{1} \\
&\quad -\mathbb{1} \otimes 2\alpha\hat{\partial}_2\hat{\partial}_1 \Phi(-\alpha\hat{\partial}_3)\Phi(\alpha\hat{\partial}_3)\Upsilon(-\alpha\hat{\partial}_3) \\
&\quad +\alpha\hat{\partial}_2 \Phi(\alpha\hat{\partial}_3) \otimes \hat{\partial}_1 \Phi(\alpha\hat{\partial}_3) - \alpha\hat{\partial}_1 \Phi(-\alpha\hat{\partial}_3) \otimes \hat{\partial}_2 \Phi(-\alpha\hat{\partial}_3) \\
&\quad +2\alpha \left(\hat{\partial}_1 \Phi(-\alpha\hat{\partial}_3) \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\partial}_1 \Phi(\alpha\hat{\partial}_3)\right) \\
&\qquad \times \left(\hat{\partial}_2 \Phi(\alpha\hat{\partial}_3) \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\partial}_2 \Phi(-\alpha\hat{\partial}_3)\right) \\
&\qquad \times \Upsilon(-\alpha\hat{\partial}_3 \otimes \mathbb{1} - \mathbb{1} \otimes \alpha\hat{\partial}_3) + \hat{\partial}_4 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\partial}_4
\end{aligned} \tag{95}
$$

eq:deriv:leibniz:wey

Using (56) we find the $x \star f(x)$ products

$$
\begin{aligned}
x^1 \star f(x) &= \left\{ \frac{x^1}{\Phi(-\alpha\partial_3)} + 2x^4 \left(1 - \frac{1}{\Phi(-\alpha\partial_3)}\right) \frac{\partial_2}{\partial_3} \right\} f(x) \\
x^2 \star f(x) &= \left\{ \frac{x^2}{\Phi(\alpha\partial_3)} + 2x^4 \left(1 - \frac{1}{\Phi(\alpha\partial_3)}\right) \frac{\partial_1}{\partial_3} \right\} f(x) \\
x^3 \star f(x) &= \left\{ x^1 \left(1 - \frac{1}{\Phi(-\alpha\partial_3)}\right) \frac{\partial_1}{\partial_3} + x^2 \left(1 - \frac{1}{\Phi(\alpha\partial_3)}\right) \frac{\partial_2}{\partial_3} \right. \\
&\qquad \left. + x^3 - 2\alpha x^4 \left(\frac{2}{\alpha\partial_3} - \frac{e^{\alpha\partial_3} + 1}{e^{\alpha\partial_3} - 1} \right) \frac{\partial_1\partial_2}{\partial_3} \right\} f(x) \\
x^4 \star f(x) &= x^4 f(x)
\end{aligned}
\tag{96}
$$

`eq:weyl:xstar`

which we may insert into (83), showing that the following $\star$-derivatives are non-trivial

$$
\begin{aligned}
\partial_1^\star &= \Phi(-\alpha\partial_3)\partial_1 \\
\partial_2^\star &= \Phi(\alpha\partial_3)\partial_2 \\
\partial_3^\star &= \partial_3
\end{aligned}
\tag{97}
$$

`eq:weyl:derivs`

however the ansatz

$$
\partial_4^\star = \partial_4 + \frac{\partial_1\partial_2}{\partial_3} \left(2 - \Phi(-\alpha\partial_3) - \Phi(\alpha\partial_3)\right)
\tag{98}
$$

`eq:weyl:derivs:4`

is only confirmed for three of the $\partial_4^\star$ relationships given in (83). In order for (98) to be true we require

$$
\begin{aligned}
\left[\partial_4^\star, x^3\right]_\star f(x) &= \left\{ 2\left(1 - \Phi(-\alpha\partial_3) - \Phi(\alpha\partial_3)\right) + e^{\alpha\partial_3} + e^{-\alpha\partial_3} \right\} \frac{\partial_1\partial_2}{\partial_3^2} f(x) \\
&\quad + \frac{1}{\partial_3} \left(2 - \Phi(-\alpha\partial_3) - \Phi(\alpha\partial_3)\right) x^3 \partial_1\partial_2 f(x) \\
&\quad - x^3 \left(2 - \Phi(-\alpha\partial_3) - \Phi(\alpha\partial_3)\right) \frac{\partial_1\partial_2}{\partial_3} f(x) \\
&\overset{!}{=} 0
\end{aligned}
\tag{99}
$$

`eq:weyl:derivs:4:con`

which is very difficult to calculate due to the presence of an ill-defined $\frac{1}{\partial_3}$ term acting on $x^3\partial_i f(x)$.

4 Integrals

4.1 $\star$-Integral Measure

In order to derive field equations by means of a variational principle, we require an integral. Algebraically an integral is a linear map of the algebra into complex numbers

$$\int : \hat{\mathcal{A}}(\hat{x}) \to \mathbb{C} \tag{100}$$

eq:nw:int:alg

$$\int (c_1 \hat{f} + c_2 \hat{g}) = c_1 \int \hat{f} + c_2 \int \hat{g} \qquad \forall \hat{f}, \hat{g} \in \hat{\mathcal{A}}(\hat{x}), c_i \in \mathbb{C} \tag{101}$$

We also demand the cyclic property [6, 8]

$$\int \hat{f}\hat{g} = \int \hat{g}\hat{f} \tag{102}$$

eq:nw:int:cyc

We define the integral in the $\star$-product formalism using the usual definitions of an integral of commuting functions. It has been shown in [9] that a measure can be introduced to achieve property (102)

$$\int d^n x \mu(x) f(x) \star g(x) = \int d^n x \mu(x) g(x) \star f(x) \tag{103}$$

eq:nw:int:cyc:measur

It is important to note that a μ which satisfies (103) gives the integral the additional property TODO: show why this is true in more detail

$$\int d^n x \mu(x) f(x) \star g(x) = \int d^n x \mu(x) f(x) g(x) \tag{104}$$

eq:nw:int:cyc:nostar

Since the functions x^i form a basis, it is necessary for a cyclic action functional to satisfy the following condition

$$\int d^4 x \mu_\star(x) \left[(x^i)^n, f(x)\right]_\star = 0 \tag{105}$$

eq:nw:int:condition

for all the natural numbers n and $i = 1 \ldots 4$.

We may rewrite any commutator of the form $[\hat{x}^n, \hat{y}]$ using the following identity

$$\left[\hat{x}^{n+1}, \hat{y}\right] = \sum_{m=0}^{n} \binom{n}{m} \hat{x}^{n-m} \left[\hat{x}, \hat{y}\right] \hat{x}^m \tag{106}$$

eq:nw:int:samstheory

allowing us to simplify (105) to

$$\int d^4 x \mu_\star(x) \sum_{m=0}^{n} \binom{n}{m} (x^i)^{n-m} \left[x^i, f(x)\right]_\star (x^i)^m = 0 \tag{107}$$

eq:nw:int:condition2

It is then a matter of inserting the explicit form of $\left[x^i, f(x)\right]_\star$ to find restraints upon $\mu(x)$ for general $f(x)$, making use of integration by parts in the case where $\int d(fg) = 0$[‡].

$$\int \left[f g \partial^n h\right] dx = (-1)^n \int \left[f(\partial^n g) h + g(\partial^n f) h\right] dx \tag{108}$$

eq:nw:int:parts

[‡]As we are in a Schwartz space, all functions go to zero at infinity.

4.2 Anti-Hermitian Derivatives

TODO: this should not be its own section, encorporate this text into the previous section

The variational principle requires that the derivatives be anti-hermitian by the following definiton

$$\int d^n x \mu(x) \overline{f} \star \partial^\star g = - \int d^n x \mu(x) \overline{\partial^\star f} \star g \tag{109}$$ eq:nw:int:ah

Which allows for a generalised integration by parts.

4.3 Nappi-Witten $\star$-Integrals

4.3.1 Time Ordering

In addition to (90), the $f(x) * x^i$ products are

$$\begin{aligned} f(x) * x^1 &= (x^1 + \alpha x^4 \partial_2) e^{-\alpha \partial_3} f(x) \\ f(x) * x^2 &= (x^2 - \alpha x^4 \partial_1) e^{\alpha \partial_3} f(x) \\ f(x) * x^3 &= x^3 f(x) \\ f(x) * x^4 &= x^4 f(x) \end{aligned} \tag{110}$$ eq:time:starx

giving the following $*$-commutators

$$\begin{aligned} \left[x^1, f(x)\right]_* &= -\alpha x^4 \left(1 + e^{-\alpha \partial_3}\right) \partial_2 f(x) \\ \left[x^2, f(x)\right]_* &= \alpha x^4 \left(1 + e^{\alpha \partial_3}\right) \partial_1 f(x) \\ \left[x^3, f(x)\right]_* &= \alpha \left(x^2 \partial_2 - x^1 \partial_1\right) f(x) \\ \left[x^4, f(x)\right]_* &= 0 \end{aligned} \tag{111}$$ eq:time:comm

When inserted into (107), they give the restraints on $\mu_*(x)$

$$\begin{aligned} \left(1 + e^{\alpha \partial_3}\right) \partial_2 \mu_*(x) &= 0 \\ \left(1 - e^{-\alpha \partial_3}\right) \partial_1 \mu_*(x) &= 0 \\ x^1 \partial_1 \mu_*(x) &= x^2 \partial_2 \mu_*(x) \end{aligned} \tag{112}$$ eq:time:mu:all

satisfied by

$$\partial_{1,2} \mu_*(x) = 0 \tag{113}$$ eq:time:mu

However, the ∂_4^* derivative does not satisfy (109), but it may be remedied by introducing the translation $\partial_4 \to \partial_4 + \frac{\partial_4 \mu_*(x)}{2\mu_*(x)}$ so that the $*$-derivatives

(91) are now

$$\begin{aligned}
\widetilde{\partial}_1^* &= \partial_1 \\
\widetilde{\partial}_2^* &= \partial_2 \\
\widetilde{\partial}_3^* &= \partial_3 \\
\widetilde{\partial}_4^* &= \partial_4 + \frac{\partial_4 \mu_*(x)}{2\mu_*(x)}
\end{aligned} \qquad (114) \quad \boxed{\texttt{eq:time:d}}$$

with no adverse effects on (88) TODO: but affects linearity. go into more detail.

4.3.2 Symmetric Time Ordering

In addition to (93), the $f(x) \bullet x^i$ products are

$$\begin{aligned}
f(x) \bullet x^1 &= \left(x^1 + \alpha x^4 \partial_2\right) e^{-\frac{\alpha}{2}\partial_3} f(x) \\
f(x) \bullet x^2 &= \left(x^2 - \alpha x^4 \partial_1\right) e^{\frac{\alpha}{2}\partial_3} f(x) \\
f(x) \bullet x^3 &= -\alpha \left(x^3 + \frac{\alpha}{2} x^1 \partial_1 - \frac{\alpha}{2} x^2 \partial_2\right) f(x) \\
f(x) \bullet x^4 &= x^4 f(x)
\end{aligned} \qquad (115) \quad \boxed{\texttt{eq:symtime:starx}}$$

giving the following $\bullet$-commutators

$$\begin{aligned}
\left[x^1, f(x)\right]_\bullet &= \left(x^1 - \alpha x^4 \partial_2\right) \left(e^{\frac{\alpha}{2}\partial_3} - e^{-\frac{\alpha}{2}\partial_3}\right) f(x) \\
\left[x^2, f(x)\right]_\bullet &= \left(x^2 + \alpha x^4 \partial_1\right) \left(e^{-\frac{\alpha}{2}\partial_3} - e^{\frac{\alpha}{2}\partial_3}\right) f(x) \\
\left[x^3, f(x)\right]_\bullet &= \alpha \left(x^2 \partial_2 - x^1 \partial_1\right) f(x) \\
\left[x^4, f(x)\right]_\bullet &= 0
\end{aligned} \qquad (116) \quad \boxed{\texttt{eq:symtime:comm}}$$

When inserted into (107), they give the restraints on $\mu_\bullet(x)$

$$\begin{aligned}
(1 - \partial_2) \left(e^{-\frac{\alpha}{2}\partial_3} - e^{\frac{\alpha}{2}\partial_3}\right) \mu_\bullet(x) &= 0 \\
(1 + \partial_1) \left(e^{\frac{\alpha}{2}\partial_3} - e^{-\frac{\alpha}{2}\partial_3}\right) \mu_\bullet(x) &= 0 \\
x^1 \partial_1 \mu_\bullet(x) &= x^2 \partial_2 \mu_\bullet(x)
\end{aligned} \qquad (117) \quad \boxed{\texttt{eq:symtime:mu:all}}$$

satisfied by

$$x^1 \partial_1 \mu_\bullet(x) = x^2 \partial_2 \mu_\bullet(x) \qquad \partial_3 \mu_\bullet(x) = 0 \qquad (118) \quad \boxed{\texttt{eq:symtime:mu:rest}}$$

However, the $\partial^\bullet_{1,2,4}$ derivatives do not satisfy (109). The introduction of translations to $\partial_{1,2}$ breaks the derivative $\bullet$-commutator relationships§ (83,

§Article [6] suggests that such a change does not effect the canonical commutators, but this is not true for the function commutators. Consistency between operator and function space commutators would only be possible by demanding that multiplication from the left follow a Leibniz like rule for the translation part.

88) and in order to satisfy both sets of restraints, we are forced to further require $\mu_\bullet(x)$ to be independent of $x^{1,2}$

$$\partial_{1,2,3}\mu_\bullet(x) = 0 \tag{119}$$

eq:symtime:mu

The translation $\partial_4 \to \partial_4 + \frac{\partial_4\mu_\bullet(x)}{2\mu_\bullet(x)}$ must still be applied in order to ensure the derivatives are anti-Hermitian under integration. The $\bullet$-derivatives (94) are now

$$\begin{aligned}
\widetilde{\partial}_1^\bullet &= \partial_1 e^{-\frac{\alpha}{2}\partial_3} \\
\widetilde{\partial}_2^\bullet &= \partial_2 e^{\frac{\alpha}{2}\partial_3} \\
\widetilde{\partial}_3^\bullet &= \partial_3 \\
\widetilde{\partial}_4^\bullet &= \partial_4 + \frac{\partial_4\mu_\bullet(x)}{2\mu_\bullet(x)}
\end{aligned} \tag{120}$$

eq:symtime:d

with no adverse effects on (88).

4.3.3 Weyl Ordering

In addition to (96), the $f(x)\star x^i$ products are

$$\begin{aligned}
f(x)\star x^1 &= \left\{\frac{x^1}{\Phi(\alpha\partial_3)} + 2x^4\left(1 - \frac{1}{\Phi(\alpha\partial_3)}\right)\frac{\partial_2}{\partial_3}\right\} f(x) \\
f(x)\star x^2 &= \left\{\frac{x^2}{\Phi(-\alpha\partial_3)} + 2x^4\left(1 - \frac{1}{\Phi(-\alpha\partial_3)}\right)\frac{\partial_1}{\partial_3}\right\} f(x) \\
f(x)\star x^3 &= \left\{x^1\left(1 - \frac{1}{\Phi(\alpha\partial_3)}\right)\frac{\partial_1}{\partial_3} + x^2\left(1 - \frac{1}{\Phi(-\alpha\partial_3)}\right)\frac{\partial_2}{\partial_3}\right. \\
&\quad \left. + x^3 - 2\alpha x^4\left(\frac{2}{\alpha\partial_3} - \frac{e^{\alpha\partial_3}+1}{e^{\alpha\partial_3}-1}\right)\frac{\partial_1\partial_2}{\partial_3}\right\} f(x) \\
f(x)\star x^4 &= x^4 f(x)
\end{aligned} \tag{121}$$

eq:weyl:starx

giving the following $\star$-commutators

$$\begin{aligned}
\left[x^1, f(x)\right]_\star &= \alpha\left(x^1\partial_3 - 2x^4\partial_2\right) f(x) \\
\left[x^2, f(x)\right]_\star &= \alpha\left(x^2\partial_3 + 2x^4\partial_1\right) f(x) \\
\left[x^3, f(x)\right]_\star &= \alpha\left(x^2\partial_2 - x^1\partial_1\right) f(x) \\
\left[x^4, f(x)\right]_\star &= 0
\end{aligned} \tag{122}$$

eq:weyl:comm

When inserted into (107), they give the restraints

$$\begin{aligned}
x^1\partial_3\mu_\star(x) &= 2x^4\partial_2\mu_\star(x) \\
x^2\partial_3\mu_\star(x) &= -2x^4\partial_1\mu_\star(x) \\
x^1\partial_1\mu_\star(x) &= x^2\partial_2\mu_\star(x)
\end{aligned} \tag{123}$$

eq:weyl:mu:all

satisfied by

$$\partial_{1,2,3}\mu_\star(x) = 0 \tag{124}$$

eq:weyl:mu

However, the $\partial_4^\star$ derivative does not satisfy (109), but it may be remedied by introducing the translation $\partial_4 \to \partial_4 + \frac{\partial_4\mu_\star(x)}{2\mu_\star(x)}$ so that the $\star$-derivatives (97) are now

$$\begin{aligned}
\widetilde{\partial}_1^\star &= \Phi(-\alpha\partial_3)\partial_1 \\
\widetilde{\partial}_2^\star &= \Phi(\alpha\partial_3)\partial_2 \\
\widetilde{\partial}_3^\star &= \partial_3 \\
\widetilde{\partial}_4^\star &= \partial_4 + \frac{\partial_4\mu_\star(x)}{2\mu_\star(x)} + (2 - \Phi(-\alpha\partial_3) - \Phi(\alpha\partial_3))\frac{\partial_1\partial_2}{\partial_3}
\end{aligned} \tag{125}$$

eq:weyl:d

with no adverse effects on (88).

5 Free Scalar Field Theory

We now have all the necessary tools in order to construct a free scalar field theory using the action principle for the non-commuting Nappi-Witten space-time. For our Lagrangian density, we use the familiar

$$\mathcal{L} = \frac{1}{2}\partial_\mu^\star\phi \star \partial_\mu^\star\phi + \frac{1}{2}m^2\phi \star \phi \tag{126}$$

eq:lagdens

where we have replaced multiplication by $\star$-multiplication, and derivatives by $\star$-derivatives. Using (104), the Lagrangian under integration $\mathcal{L}_\star$ simplifies to

$$\mathcal{L}_\star = \frac{1}{2}(\partial_\mu^\star\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{127}$$

eq:lagdens:simp

The action is defined with an appropriate $\mu_\star(x)$

$$\begin{aligned}
S[\phi] &= \int d^4x\mu_\star(x)\mathcal{L}_\star \\
&= \frac{1}{2}\int d^4x\mu_\star(x)\left\{(\partial_\mu^\star\phi)^2 + m^2\phi^2\right\}
\end{aligned} \tag{128}$$

eq:action

We should be able to recover a familiar $\star$-Klein-Gordon equation by finding the extremum of S by demanding the functional derivative be zero

$$\frac{\delta S[\phi]}{\delta\phi} = 0 \tag{129}$$

eq:action:zero

We define the functional derivative by

$$S[\phi, \delta\phi] - S[\phi] \equiv \frac{\delta S[\phi]}{\delta\phi}\delta\phi \tag{130}$$

eq:action:funderiv

With this definition, we get to first order in variation

$$\begin{aligned} S[\phi, \delta\phi] - S[\phi] &= \frac{1}{2} \int d^4x \mu_\star(x) \left\{ \left(\partial^\star_\mu(\phi + \delta\phi)\right)^2 + m^2(\phi+\delta\phi)^2 \right. \\ &\qquad \left. -(\partial^\star_\mu \phi)^2 - m^2\phi^2 \right\} \\ &= \int d^4x \mu_\star(x) \left\{ \partial^\star_\mu \phi \partial^\star_\mu \delta\phi + m^2 \phi \delta\phi \right\} \\ &= - \int d^4x \mu_\star(x) \left\{ \overline{\partial^\star_\mu} \partial^\star_\mu \phi - m^2\phi \right\} \delta\phi \end{aligned} \qquad (131)$$

eq:action:min

In the final line we make use of the anti-hermitian property (109) which is integration by parts when the boundary terms are zero.

This allows us to identify

$$\begin{aligned} \frac{\delta S[\phi]}{\delta\phi} &= 0 \\ &= \overline{\partial^\star_\mu} \partial^\star_\mu \phi - m^2\phi \end{aligned} \qquad (132)$$

eq:nw:starminaction

and therefore define[¶]

$$\Box^\star \phi \equiv \overline{\partial^\star_\mu} \partial^\star_\mu \phi \qquad (133)$$

eq:nw:kg

We may now explicitly calculate the box operator for all three of our orderings, where, for brevity, we use $\partial_\star$ to mean $\widetilde{\partial_\star}$. TODO: this could be fixed by using a better notation earlier on.

5.1 Time Ordering

Using (114), and making the simplificion

$$\eta_*(x) = \frac{\partial_4 \mu_*(x)}{2\mu_*(x)} \qquad (134)$$

eq:time:eta

we can write out the box operator

$$\Box^* \phi = \partial^2_\mu \phi + (\overline{\eta_*(x)} + \eta_*(x))\partial_4\phi + \phi \partial_4 \eta_*(x) + |\eta_*(x)|^2 \phi \qquad (135)$$

eq:time:box

¶Note that this differs from the definition of $\Box^\star$ in the κ-Minkowski literature [6]. Using this definition we obtain

$$\Box^\star_\kappa \phi = \frac{2}{\lambda^2 \partial_n^2} \left(1 - \cos(\lambda \partial_n)\right) \partial_i^2 \phi + \partial_n^2 \phi$$

whereas the literature obtains the similar

$$\Box^\star_\kappa \phi = \frac{2}{\lambda^2 \partial_n^2} \left(1 - \cos(\lambda \partial_n)\right) \left(\partial_i^2 + \partial_n^2\right) \phi$$

as a result of defining the box operator in terms of the invariant "Dirac" derivative. The standard box operator is recovered for both definitions in the commutative geometry $\lambda \to 0$ limit.

Example: $\mu_*(x) = 1$

$$\Box^* \phi = \partial_\mu^2 \phi \tag{136}$$

`eq:time:box:unitmeas`

5.2 Symmetric Time Ordering

Using (120), and making the simplificion

$$\eta_\bullet(x) = \frac{\partial_4 \mu_\bullet(x)}{2\mu_\bullet(x)} \tag{137}$$

`eq:symtime:eta`

we can write out the box operator for $\alpha \in \mathbb{R}$

$$\begin{aligned}\Box^\bullet_\alpha \phi \;=\; & e^{-\alpha\partial_3}\partial_1^2\phi + e^{\alpha\partial_3}\partial_2^2\phi + \partial_3^2\phi + \partial_4^2\phi \\ & +(\overline{\eta_\bullet(x)} + \eta_\bullet(x))\partial_4\phi + \phi\partial_4\eta_\bullet(x) + |\eta_\bullet(x)|^2\phi\end{aligned} \tag{138}$$

`eq:symtime:box:1`

and for the Wick rotated $\alpha \to i\alpha$, $\alpha \in \mathbb{R}$

$$\Box^\bullet_{i\alpha}\phi = \partial_\mu^2\phi + (\overline{\eta_\bullet(x)} + \eta_\bullet(x))\partial_4\phi + \phi\partial_4\eta_\bullet(x) + |\eta_\bullet(x)|^2\phi \tag{139}$$

`eq:symtime:box:2`

Example: $\mu_\bullet(x) = 1$

For $\alpha \in \mathbb{R}$

$$\Box^\bullet_\alpha \phi = e^{-\alpha\partial_3}\partial_1^2\phi + e^{\alpha\partial_3}\partial_2^2\phi + \partial_3^2\phi + \partial_4^2\phi \tag{140}$$

`eq:symtime:box:1:uni`

and for the Wick rotated $\alpha \to i\alpha$, $\alpha \in \mathbb{R}$

$$\Box^\bullet_{i\alpha}\phi = \partial_\mu^2\phi \tag{141}$$

`eq:symtime:box:2:uni`

The standard box operator is recovered in the commutative geometry $\alpha \to 0$ limit.

5.3 Weyl Ordering

Using (125), and making the simplificions

$$\begin{aligned}\eta_\star(x) \;&=\; \frac{\partial_4\mu_\star(x)}{2\mu_\star(x)} \\ \partial_\triangle \;&=\; (2 - \Phi(-\alpha\partial_3) - \Phi(\alpha\partial_3))\,\frac{\partial_1\partial_2}{\partial_3}\end{aligned} \tag{142}$$

`eq:weyl:etasigma`

we can write out the box operator for $\alpha \in \mathbb{R}$

$$\begin{aligned}\Box^\star_\alpha\phi \;=\; & \frac{2}{\alpha^2\partial_3^2}\,(1-\cos(\alpha\partial_3))\left(e^{-\alpha\partial_3}\partial_1^2\phi + e^{\alpha\partial_3}\partial_2^2\phi\right) \\ & +\partial_3^2\phi + \partial_4^2\phi \\ & + \left(2\partial_\triangle + \eta_\star(x) + \overline{\eta_\star(x)}\right)\partial_4\phi + |\eta_\star(x)|^2\phi + \phi\partial_4\eta_\star(x) \\ & +(\eta_\star(x) + \overline{\eta_\star(x)})\partial_\triangle\phi + (\partial_\triangle)^2\phi\end{aligned} \tag{143}$$

`eq:weyl:box:1`

and for the Wick rotated $\alpha \to i\alpha$, $\alpha \in \mathbb{R}$

$$\begin{aligned}\Box^{\star}_{i\alpha}\phi = & \frac{2}{\alpha^2\partial_3^2}\left(1-\cos(\alpha\partial_3)\right)\left(\partial_1^2\phi+\partial_2^2\phi\right)+\partial_3^2\phi+\partial_4^2\phi \\ & +\left(\partial_\triangle+\overline{\partial_\triangle}+\eta_\star(x)+\overline{\eta_\star(x)}\right)\partial_4\phi+|\eta_\star(x)|^2\phi+\phi\partial_4\eta_\star(x) \\ & +(\eta_\star(x)\overline{\partial_\triangle}+\overline{\eta_\star(x)}\partial_\triangle)\phi+\partial_\triangle\overline{\partial_\triangle}\phi\end{aligned} \quad (144) \quad \boxed{\texttt{eq:weyl:box:2}}$$

which is hard to solve, again due to $\frac{1}{\partial_3}$ terms.

Example: $\mu_\star(x) = 1$

For $\alpha \in \mathbb{R}$

$$\begin{aligned}\Box^{\star}_{\alpha}\phi = & \Phi(-\alpha\partial_3)^2\partial_1^2\phi+\Phi(\alpha\partial_3)^2\partial_2^2\phi+\partial_3^2\phi+\partial_4^2\phi \\ & +2\partial_\triangle\partial_4\phi+(\partial_\triangle)^2\phi\end{aligned} \quad (145) \quad \boxed{\texttt{eq:weyl:box:1:unitme}}$$

and for the Wick rotated $\alpha \to i\alpha$, $\alpha \in \mathbb{R}$

$$\begin{aligned}\Box^{\star}_{i\alpha}\phi = & \frac{2}{\alpha^2\partial_3^2}\left(1-\cos(\alpha\partial_3)\right)\left(\partial_1^2\phi+\partial_2^2\phi\right)+\partial_3^2\phi+\partial_4^2\phi \\ & +\left(\partial_\triangle+\overline{\partial_\triangle}\right)\partial_4\phi+\partial_\triangle\overline{\partial_\triangle}\phi\end{aligned} \quad (146) \quad \boxed{\texttt{eq:weyl:box:2:unitme}}$$

The standard box operator is recovered in the commutative geometry $\alpha \to 0$ limit.

A Generating Function of Equation (51)

app:generating

$$\sum_{n=1} \frac{B_{n+1}y^n}{n!} = \sum_{n=1} \frac{(n+1)B_{n+1}}{(n+1)!}\frac{y^{n+1}}{y} \qquad (147)$$

eq:app:1

$$= \frac{1}{y}\sum_{m=2}\frac{mB_m}{m!}y^m$$

$$= \frac{d}{dy}\sum_{m=2}\frac{B_m}{m!}y^m$$

where $m = n+1$. Since

$$\sum_{m=2}\frac{B_m}{m!}y^m = -B_0 - B_1 y + \sum_{m=0}\frac{B_m}{m!}y^m \qquad (148)$$

eq:app:2

$$= -B_0 - B_1 y + \frac{y}{e^{xy}-1}$$

$$= \frac{y}{e^{xy}-1} + \frac{y}{2} - 1$$

around $x = 1$ and $B_0 = 1$, $B_1 = -\frac{1}{2}$, we can calculate the generating function of the original sum

$$\sum_{n=1}\frac{B_{n+1}y^n}{n!} = \frac{1}{2} + \frac{d}{dy}\left(\frac{y}{e^{xy}-1}\right) \qquad (149)$$

eq:app:3

$$= \frac{1}{2} + \frac{e^{xy} - 1 - ye^{xy}}{(e^{xy}-1)^2}$$

B κ-Minkowski Weyl Ordered $\star$-Product

app:kappaminsk

A κ-Minkowski algebra is one that satisfies

$$[\hat{x}_0, \hat{x}_i] = \lambda\hat{x}_i \qquad (150)$$

eq:app:kmink

$$[\hat{x}_i, \hat{x}_j] = 0$$

Such an algebra can be thought of as a simplified Nappi-Witten algebra (25) with $\hat{\mathsf{K}} = 0$, $\hat{x}_0$ taking the role of $\hat{\mathsf{J}}$ and the $\hat{x}_i$ acting as independent $\hat{\mathsf{P}}^-$s. The only non-trivial exponential products for an n-dimensional algebra with a representation similar to the Nappi-Witten scenario in the main text are

$$e^{ik_l\hat{x}^l}e^{ik_0\hat{x}^0} = e^{ik_0\hat{x}^0}e^{ik_l e^{-i\beta\lambda k_0}\hat{x}^l} \qquad (151)$$

eq:app:kmink:mult

$$e^{ik_0\hat{x}^0}e^{ik_l\hat{x}^l} = e^{ik_l e^{i\beta\lambda k_0}\hat{x}^n}e^{ik_0\hat{x}^0}$$

In analogy to (46) we find the BCH formula for

$$e^{ik_l\hat{x}^l}e^{ik_0\hat{x}^0} = e^{G_0\hat{x}^0 + G_l\hat{x}^l} \qquad (152)$$

eq:app:kmink:start

$$G_0 = ik_0$$

$$G_l = \frac{ik_l}{\Phi(i\lambda k_0)}$$

where we use the same Φ as in (48)

$$\Phi(a) = \frac{e^{\beta a} - 1}{a}$$

and define

$$\begin{aligned} q_0 &= k_0 \\ q_l &= k_l \Phi(i\lambda k_0) \end{aligned} \tag{153}$$

eq:app:kmink:qs

allowing us to rewrite the Weyl ordering in terms of the qs

$${}^\circ_\circ e^{ik_l\hat{x}^i} {}^\circ_\circ = e^{ik_0\hat{x}^0 + ik_l\hat{x}^l} = e^{iq_l\hat{x}^l} e^{iq_0\hat{x}^0} \tag{154}$$

eq:app:kmink:orderin

Using (151),(152), we calculate the Weyl ordered $\star$-product for κ-Minkowski as

$$\begin{aligned} f \star g(z) &= f(z) e^{ix^i(F_i - k_i - k'_i)} g(z) \\ F_0 &= k_0 + k'_0 \\ F_l &= \frac{k_l \Phi(i\lambda k_0) + k'_l \Phi(i\lambda k'_0) e^{i\lambda k_0}}{\Phi(i\lambda(k_0 + k'_0))} \end{aligned} \tag{155}$$

eq:app:kmink:star

where l runs over $1, \ldots, n-1$. This result may be compared with [5]. We see from (11) and (12) that the operator algebra is recovered in the functional space with deformed $\star$-product between the generators being

$$\begin{aligned} x_l \star x_0 &= x_l x_0 - \frac{\lambda}{2} x_l \\ x_0 \star x_l &= x_0 x_l + \frac{\lambda}{2} x_l \end{aligned} \tag{156}$$

eq:app:kmink:generat

Using (12) to second order in λ[‖] and written in position space, the $\star$-product between any two functions for $n = 3$ is

$$
\begin{aligned}
f \star g(x) = fg \; &+ \frac{\lambda}{2}\Big(x_1\partial_0 f \partial_1 g - x_1\partial_1 f\partial_0 g + x_2\partial_0 f\partial_2 g - x_2\partial_2 f\partial_0 g \Big) \\
&+ \frac{\lambda^2}{4}\Big(\frac{1}{3}x_1\partial_1 f\partial_0^2 g + \frac{1}{3}x_2\partial_0^2 f\partial_2 g + \frac{1}{3}x_2\partial_2 f\partial_0^2 g \\
&\qquad - x_1^2\partial_0\partial_1 f\partial_0\partial_1 g - \frac{1}{3}x_1\partial_0 f\partial_0\partial_1 g - \frac{1}{3}x_2\partial_0\partial_2 f\partial_0 g \\
&\qquad + \frac{1}{2}x_2^2\partial_0^2 f\partial_2^2 g + x_1x_2\partial_0^2 f\partial_1\partial_2 g - x_1x_2\partial_0\partial_1 f\partial_0\partial_2 g \\
&\qquad + \frac{1}{2}x_2^2\partial_2^2 f\partial_0^2 g + \frac{1}{2}x_1^2\partial_0^2 f\partial_1^2 g - \frac{1}{3}x_1\partial_0\partial_1 f\partial_0 g \\
&\qquad + \frac{1}{2}x_1^2\partial_1^2 f\partial_0^2 g - x_2^2\partial_0\partial_2 f\partial_0\partial_2 g + x_1x_2\partial_1\partial_2 f\partial_0^2 g \\
&\qquad + \frac{1}{3}x_1\partial_0^2 f\partial_1 g - \frac{1}{3}x_2\partial_0 f\partial_0\partial_2 g - x_1x_2\partial_0\partial_2 f\partial_0\partial_1 g \Big) \\
&+ \mathcal{O}(\lambda^3)
\end{aligned}
\tag{157}
$$

eq:app:kmink:positio

In this proof, we have used a real Lie algebra (150), as opposed to the anti-Hermitian definition often used. It should be noted that in order to directly compare this $\star$-product with those given in [6],[7] then the definition of the algebra should be changed (using $\lambda \to i\Lambda$) to

$$
\begin{aligned}
[\hat{x}_0, \hat{x}_i] &= i\Lambda\hat{x}_i \\
[\hat{x}_i, \hat{x}_j] &= 0
\end{aligned}
\tag{158}
$$

eq:app:kmink:alt

giving

$$
\begin{aligned}
f \star g(z) &= f(z)e^{ix^i(F_i - k_i - k_i')}g(z) \\
F_0 &= k_0 + k_0' \\
F_l &= \frac{k_l\Phi(-\Lambda k_0) + k_l'\Phi(-\Lambda k_0')e^{-\Lambda k_0}}{\Phi(-\Lambda(k_0 + k_0'))}
\end{aligned}
\tag{159}
$$

eq:app:kmink:star:al

C A "Not so Obvious" Mistake

A mistake which is not so obvious to spot is in assuming

$$e^{i\hat{A}}e^{i\hat{B}} = e^{iC(\hat{A}:\hat{B})}$$

‖ κ-Minkowski is by definition a scaled algebra, so there is no need to introduce a parameter such as α

To show why this is not the case, a simple counterexample is used. For an algebra $[A, B] = c$, we would have

$$\begin{aligned} e^{i\hat{A}}e^{i\hat{B}} &= e^{i\hat{A}+i\hat{B}+\frac{1}{2}[i\hat{A},i\hat{B}]} \\ &= e^{i\hat{A}+i\hat{B}-\frac{c}{2}} \end{aligned}$$

whereas

$$e^{iC(\hat{A}:\hat{B})} = e^{i(\hat{A}+\hat{B}+\frac{c}{2})}$$

which is not equivalent.

D Some Non-Trivial Derivative Rules

we note that

$$\begin{aligned} \partial_0^n x^0 \partial_1 f &= \partial_0^{n-1}(\partial_1 f + x^0 \partial_0 \partial_1 f) \\ &= \partial_0^{n-1}\partial_1 f + \partial_0^{n-2}(\partial_0\partial_1 f + x^0 \partial_0^2 \partial_1 f) \\ &\ \ \vdots \\ &= n\partial_0^{n-1}\partial_1 f + x^0 \partial_0^n \partial_1 f \end{aligned}$$

allowing us to construct the following rules

$$\begin{aligned} e^{\alpha\partial_0} x^0 \partial_1 f &= \left(\sum_{n=0}^{\infty} \frac{(\alpha\partial_0)^n}{n!}\right) x^0 \partial_1 f \\ &= \left(1 + \sum_{n=1}^{\infty} \frac{(\alpha\partial_0)^n}{n!}\right) x^0 \partial_1 f \\ &= x^0 \partial_1 f + \sum_{n=1}^{\infty} \frac{1}{n!} \left(n\alpha^n \partial_0^{n-1} \partial_1 f + x^0 (\alpha\partial_0)^n \partial_1 f\right) \\ &= \left(\alpha + x^0\right) e^{\alpha\partial_0} \partial_1 f \end{aligned}$$

$$\begin{aligned} \Phi(\alpha\partial_0) x^0 \partial_1 f &= \left(\frac{e^{\alpha\partial_0} - 1}{\alpha\partial_0}\right) x^0 \partial_1 f \\ &= \left(\sum_{n=1}^{\infty} \frac{(\alpha\partial_0)^{n-1}}{n!}\right) x^0 \partial_1 f \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left((n-1)\alpha^{n-1}\partial_0^{n-2} + x^0 \alpha^{n-1} \partial_0^{n-1}\right) \partial_1 f \\ &= \left\{ \frac{e^{\alpha\partial_0}}{\partial_0} - \frac{\Phi(\alpha\partial_0)}{\partial_0} + x^0 \Phi(\alpha\partial_0) \right\} \partial_1 f \end{aligned}$$

E Some Helpful Identities

$$
\begin{aligned}
\Phi(a)e^{-a} &= \Phi(-a) \\
\Phi(ia)\Phi(-ia) &= \frac{2}{a^2}\,(1-\cos(a))
\end{aligned}
$$

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