Names:

Before you begin you should have read and worked through Lab 2.

All questions should be done by hand (not by computer). Show your steps!

1. Error terms in the backwards Euler

The forwards difference formula is given by:

$$T'(t_i) \approx \frac{T_{i+1} - T_i}{\Delta t}$$

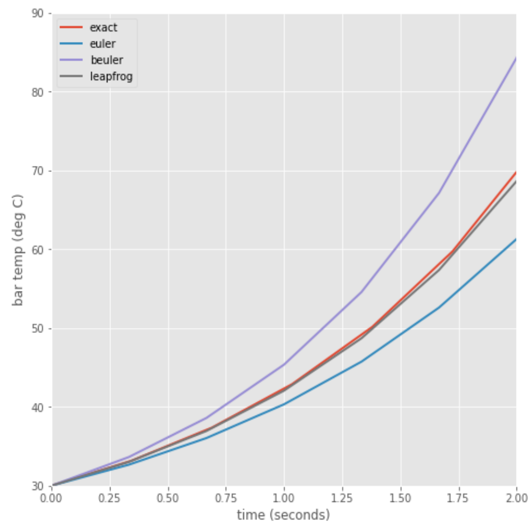
This can be re-arranged to give:

$$T(t_{i+1}) \approx T_i + \Delta t T'(t_i)$$

But note the use of $\approx$ to denote that this is not an exact identity. Indeed, terms have been missed out, and in the lab, it gives an example of expanding $T(t_{i+1}) = T(t_i + \Delta t)$ using Taylor expansion. The resulting identity can be rearranged to look like the forwards difference formula for $T(t_i)$, showing that the first missing term in our forward difference formula is $\frac{1}{2}(\Delta t)^2 T''(t_i)$, which is $\mathcal{O}(\Delta t^2)$, or second order. Since the forwards Euler method uses this formula, we say that the forward Euler method is first order accurate with errors of second order.

a. In a similar manner, derive the error term for the backward difference formula, by expanding $T(t_{i-1}) = T(t_i - \Delta t)$ using Taylor expansion. What is the order of the error term? What is the order of accuracy of the backwards Euler method?

b. How does the leading order error term you derived in part a. differ from that for the forward difference formula given in lab 2? How does this relate to the results plotted in the figure below (copied from Lab 2), where the forward and backwards Euler methods were used to solve the heat conduction problem (i.e., describe which method consistently overestimates, and which underestimates, and explain why this is the case for this particular problem).



2. An implicit scheme for $y = sin(\lambda t)$
For the equation $y = sin(\lambda t)$:

a. Derive dy/dt (== $y'(t)$):

b. Write down an approximation for $y'(t_i)$ that could be used in an implicit scheme (*hint: you've been working with an implicit scheme already in this worksheet!*)

c. Using the approximation $sin(\alpha - \beta) \approx sin(\alpha) - \beta cos(\alpha)$ (true when α and/or β follow a particular condition) show that your implicit scheme is consistent with the analytical solution.

d. What are the conditions under which the approximation $sin(\alpha - \beta) \approx sin(\alpha) - \beta cos(\alpha)$ holds? (do an internet search if none of your group knows offhand). What do these conditions mean (qualitatively) for the accuracy of the implicit scheme from section 2b?