Linear Algebra

* **Identity matrix** is a square matrix where all values are $\emptyset$ except the diagonal

$A \cdot I = A$

examples

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* **Singular systems** contrary or redudant

→ the dog is blue
the dog is black

→ the dog is black
the cat is blue
the dog is black

- linear depedent
- multiple or no solutions
- determinant = $\emptyset$

* **Non-singular** carry as much information as possible
are complete

- determinant $\neq \emptyset$

Calculating the determinant

$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 3 \end{matrix}$

$(1 \cdot 1 \cdot 3) + (0 \cdot 0 \cdot 3) + (1 \cdot 0 \cdot 2)$

$= 0$

$(1 \cdot 1 \cdot 3) + (0 \cdot 2 \cdot 1) + (3 \cdot 0 \cdot 0)$

→ If all elements under the main diagonal are $\emptyset$,
then $\boxed{\text{det = product of main diagonal}}$

$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$ $\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$

$(a \cdot e \cdot i) + (b \cdot f \cdot g) + (c \cdot d \cdot h)$

$(c \cdot e \cdot g) + (f \cdot h \cdot a) + (i \cdot b \cdot d)$

Solving systems of equations

$$\begin{cases} 5a + b = 17 & :5 \\ 4a - 3b = 6 & :4 \end{cases}$$

→ Divide by coef. of a then subtract.

↳ $a + b/5 = 17/5$
$a - 3b/4 = 6/4$ subtract

$b/5 - (-3b/4) = 17/5 - 3/2$

Matrix row reduction - Row Echolon form

$$\begin{matrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

→ the right of a ∅ is another ∅

→ main diagonal has 1s or ∅s

↳ below main diagonal are all ∅s

$$\begin{matrix} 3 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & -4 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

→ this matrix is also in Echolon form because you can divide 1st row by 3 and 4th row by -4 to have 1s.

under a pivot, only zeros!

→ The 1st number after a zero in a row is a pivot and from top to bottom the pivots need to be always to the right (→) of the one above it.

(!) A matrix is non-singular only if the reduced row echolon form has only 1s in the main diagonal.

Rank of a matrix = number of pivots

= #rows − dimension of solution space

(the sum of 1s in the main diagonal of echolon's form)

the Gaussian Elimination Algorithm

1. Create the augmented matrix
2. Get matrix into reduced echolon form
3. Complete back substitution
4. Stop if you encounter row of zero and analyse

→ take a row as a pivot, transform the first value into 1 (the pivot) and then everything below it into ∅

→ when you reach the bottom of the matrix, go back and transform everything above pivot to zero

→ if you find a row of zero's your matrix is singular

0 0 0 1 0
(infinite solutions)

0 0 0 1 4
(no solutions)

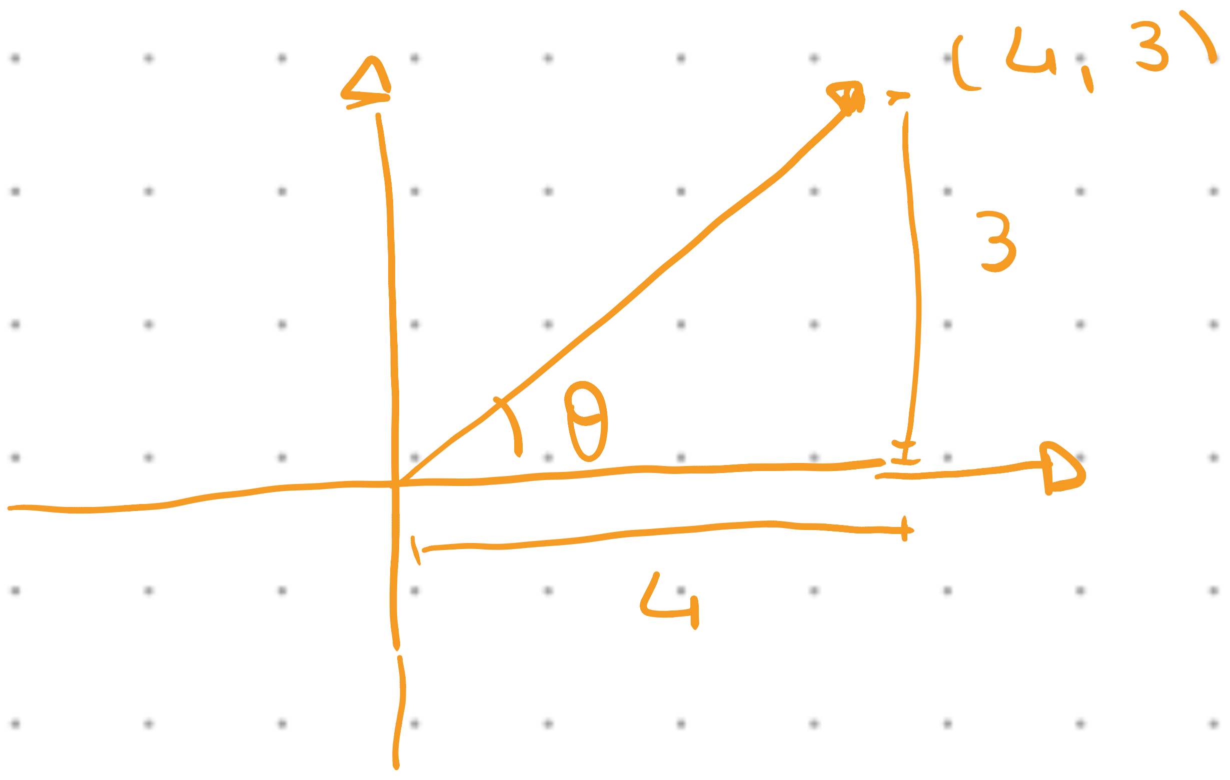
row manipulation can affect determinant (if non-zero) but not singularity!

Vectors

Norms

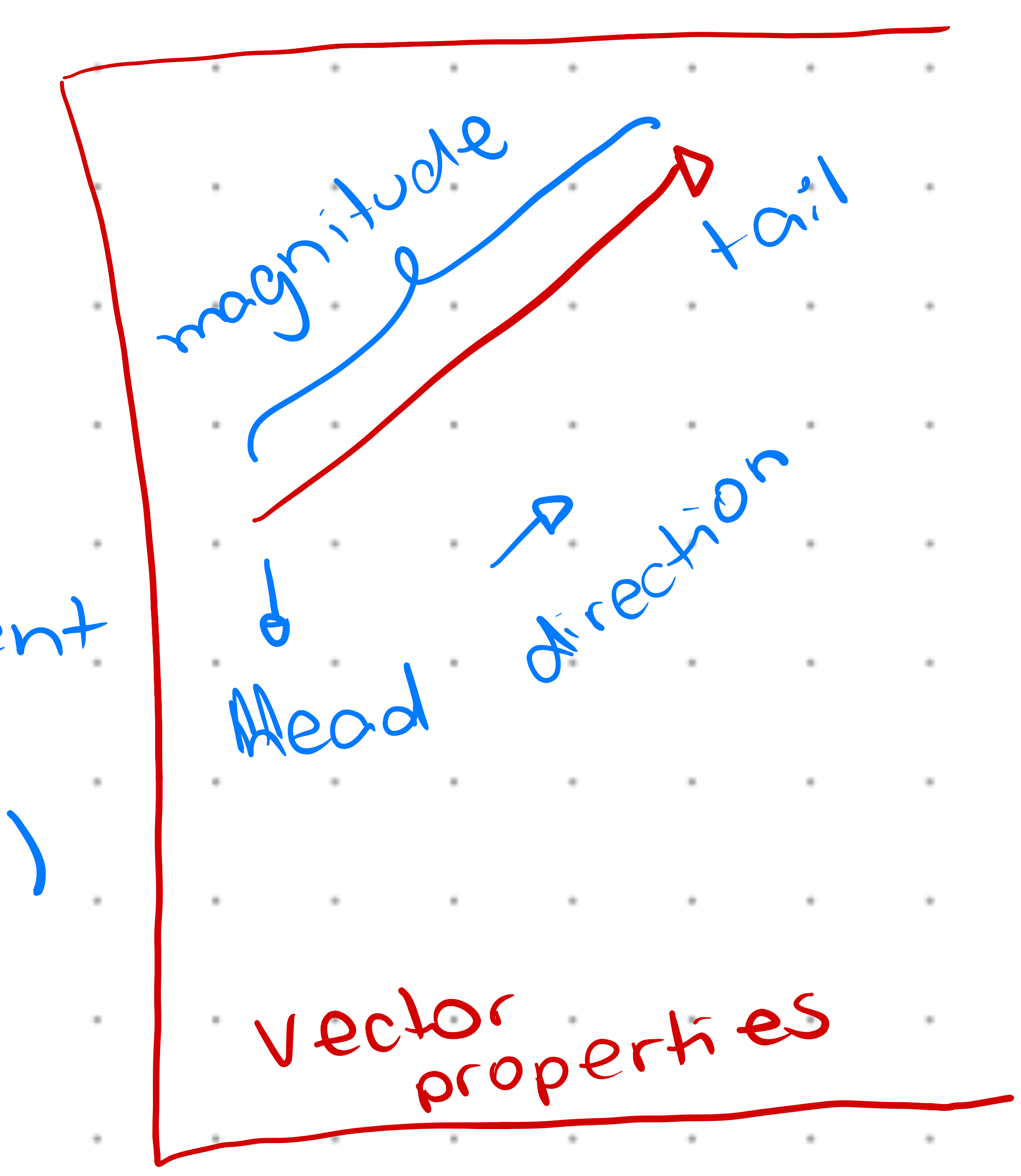
* L_1-norm $\|(a,b)\|_1 = |a|+|b|$ → $\|x\|_1 = \sum_1^n |x_n|$
* L_2-norm $\|(a,b)\|_2 = \sqrt{a^2+b^2}$ (this is the default) → $\|x_2\| = \sqrt{\sum_1^n x_n^2}$

Direction of a vector



$tag(\theta) = 3/4$

then $\theta = \arctan(3/4) = 0.64 = 36.87^\circ$



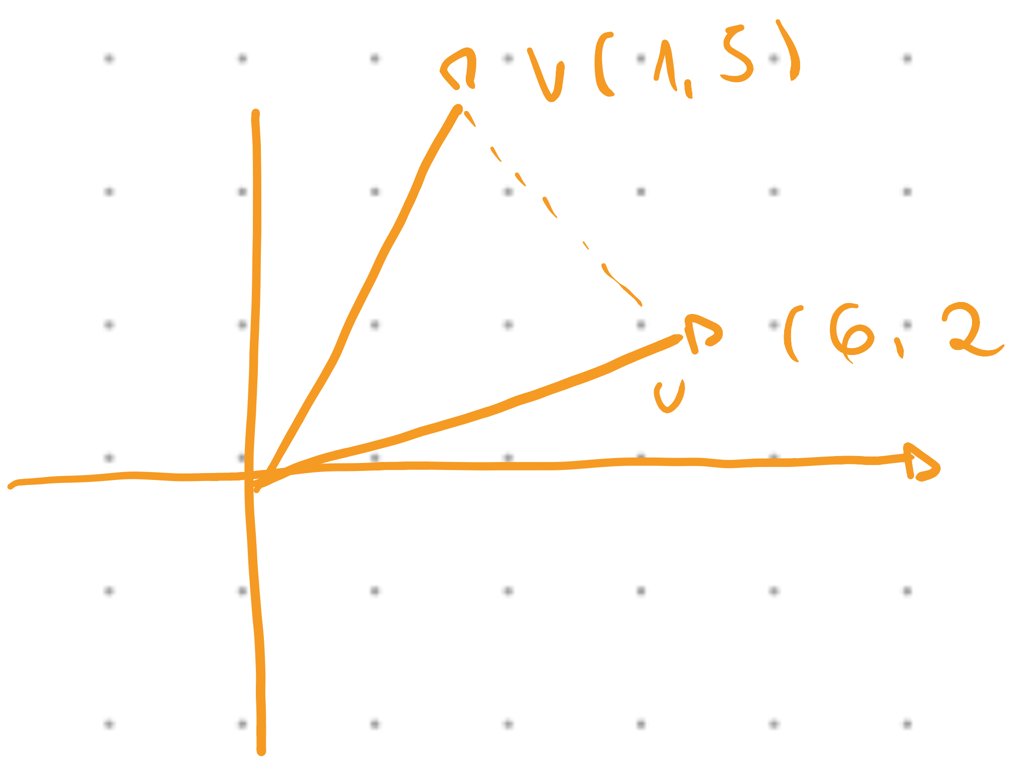
Sum and difference of vectors

Sum / subtract component by component

$x+y = (x_1+y_1 \quad x_2+x_2 \quad \dots \quad x_n+y_n)$

L_1 distance $|u-v|_1$

L_2 distance $|u-v|_2$



$L_1 = (|\underset{u_1}{6} - \underset{v_1}{1}| + |\underset{u_2}{2} - \underset{v_2}{5}|) = 8$

$L_2 = \sqrt{(6-1)^2 + (2-5)^2} = \sqrt{34} = 5.83$

→ in machine learning it's useful to know the difference to compare vectors

* the dot product

the L_2-norm is always the square root of the dot product of the vector by itself

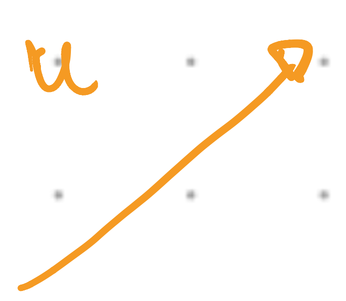
$$L_2 \text{norm} = \sqrt{\text{dot product}(u,u)}$$

$$|u|_2 = \sqrt{\langle u,u \rangle}$$

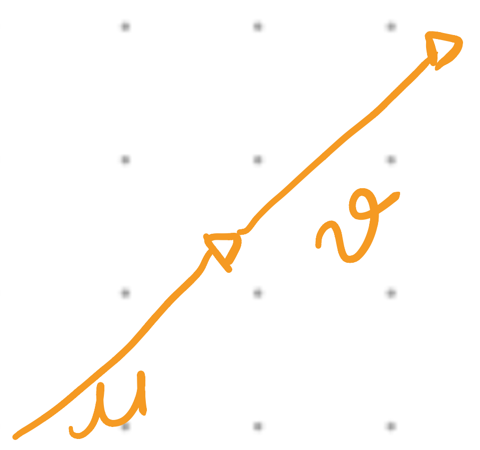
dot product

$$\boxed{x \cdot y = (x_1+y_1)+(x_2\, y_2) \dots + (x_n+y_n)}$$

* Vector transpose transforms columns into rows



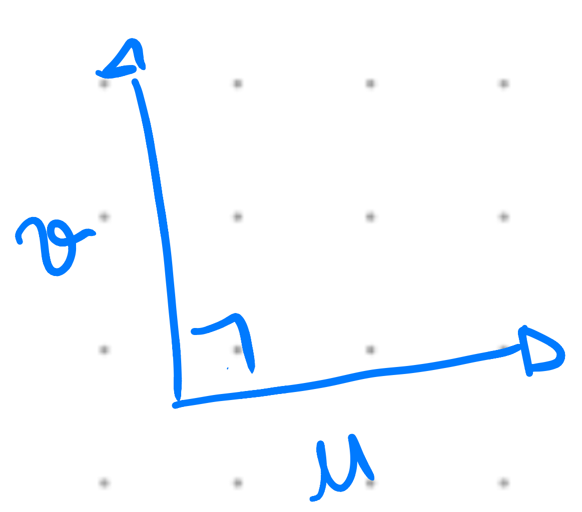
$$\langle u,u \rangle = |u|^2 = |u| \cdot |u|$$



$$\langle u,v \rangle = |u| \cdot |v|$$

$$\langle u,v \rangle = |u'| \cdot |v|$$

→ (you project the vector u into the vector v)



$$\langle u,v \rangle = 0$$

→ If the vectors are orthogonal (perpendicular) then the dot product is zero

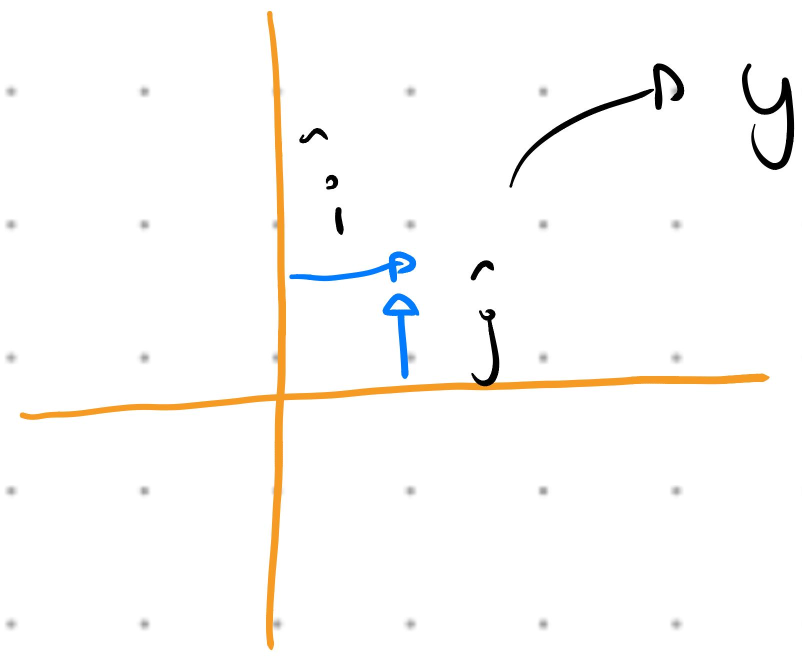
* the sign of the dot product of a vector corresponds to the region (left of the vector if negative / right if positive)

* the determinant → (the area)

$$\det(AB) = \det(A) \cdot \det(B) \qquad \det(A^{-1}) = (\det(A))^{-1}$$

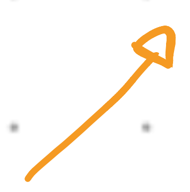
* Span the points you can get to using the directions from which the vector points to

- you can also go back

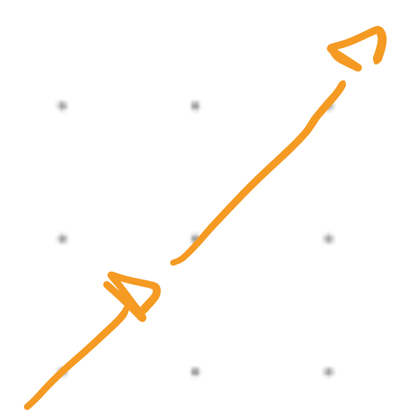


you can reach anywhere in the plane using $\hat{i}$ & $\hat{j}$

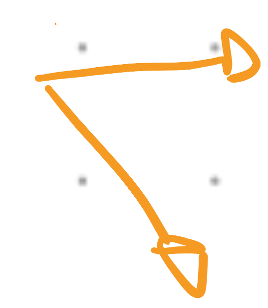
* A basis is a minimal spanning set



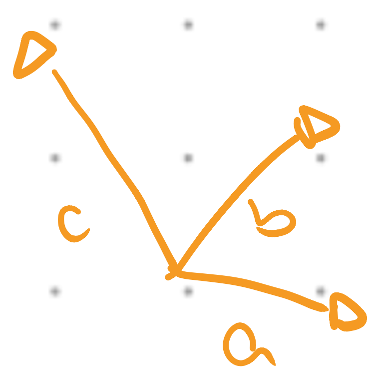
is a basis



not a basis



is a basis



Not a basis
(any subset of two vectors would be, but 3 make it redundant)
(linear dependent = you can get c from a+b)

- if you have more vectors than dimensions, you always have linear dependent vectors

* Basis a basis is a set of vectors that
  - spans a vector space
  - is linear independent

(!) Not all sets of N-vectors are a basis of a N-dimensional space

Eigenbasis, eingenvalues & eigenvectors

1 $A v_1 = \lambda_1 v_1$

$A v_2 = \lambda_2 v_2$

eingenvalues

eingenvectors

eigenvalues the amount that gets stretched (or shrunk)

eigenvectors don't change direction when multiplied by a matrix

$$\det(A - \lambda I) = 0$$

example: find the eigenvalues of matrix

$$A = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

the det must be Ø

$$\begin{matrix} 9 & 4 \\ 4 & 3 \end{matrix} - \begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix} \rightarrow \begin{bmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} \therefore$$

$$(9-\lambda)(3-\lambda) - 16 = 0 \therefore 27 - 12\lambda + \lambda^2 - 16 = 0$$

$\boxed{\lambda = 11, 1}$ values

to find the vectors

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1 when you multiply v by A you get a stretched version of v

for $\lambda = 1$

$$\begin{bmatrix} 9-1 & 4 \\ 4 & 3-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$8x + 4y = 0 \qquad x = 1 \quad y = -2 \quad v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$4x + 2y = 0$$

for $\lambda = 11$

$$\begin{bmatrix} 9-11 & 4 \\ 4 & 3-11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad x = 2 \quad y = 1$$

Find eingenvalues

$$\det(A - \lambda I) = \emptyset$$ (magnitude)

Find eigenvectors

$$(A - \lambda I)\, v = \emptyset$$ (direction)

PCA for dimensionality reduction

Projections

to project a matrix A onto a vector v

$$A_p = A \frac{v}{\|v\|_2}$$

If A has $r \times c$ (rows, columns), v has $c \times 1$

to project a matrix A onto vectors v1 and v2 (2 dimensions)

$$A_p = A \underbrace{\left[\frac{v_1}{\|v_1\|_2} \quad \frac{v_2}{\|v_2\|_2} \right]}_{V} \quad \therefore \quad A_p = AV$$

Note: projecting to $x = -y$ $\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$2x = y$ $\therefore \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

multiply by

- Variance $(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \underbrace{\text{mean}(x)}_{\mu_x})^2$

the average squared distance from the mean

- Covariance

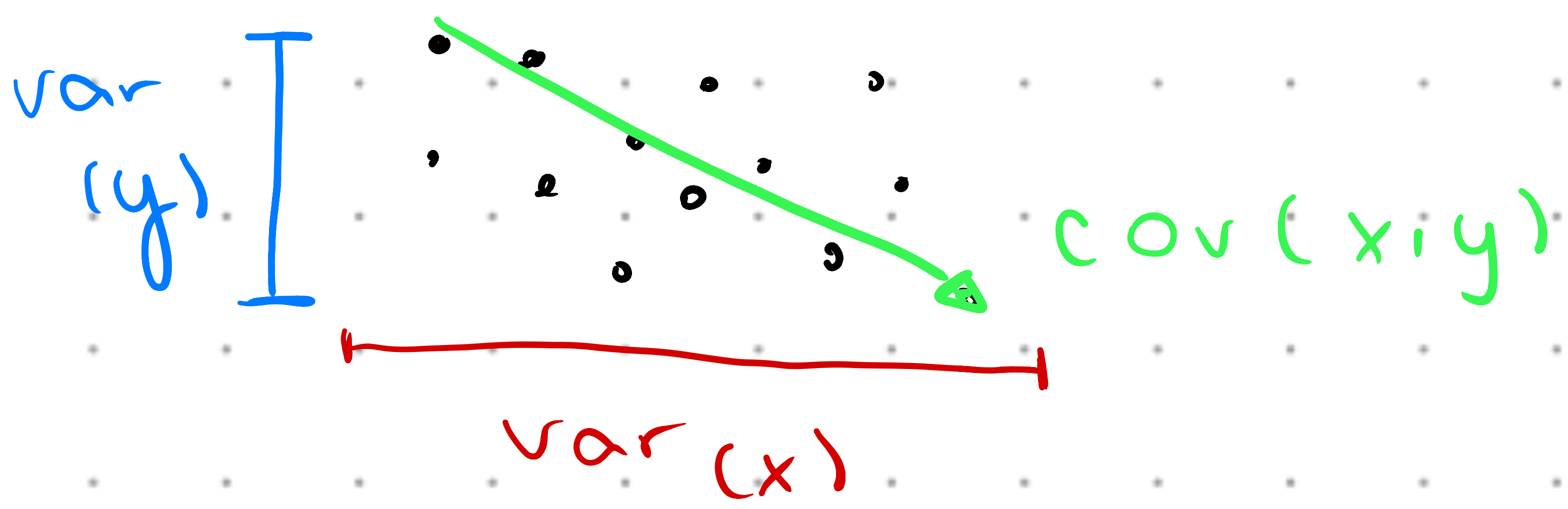
$$\text{Cov}(x,y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)$$

it tells how two variables vary in relation to each other

neg cov = negative trend ; positive cov = positive trend

close to zero = flat trend / no relationship

- Covariance matrix



$$C = \begin{array}{c|cc} & x & y \\ \hline x & \multicolumn{2}{c}{\left[\begin{matrix} var(x) & Cov(x,y) \\ Cov(y,x) & var(y) \end{matrix}\right]} \\ y & & \end{array}$$

$$\boxed{Cov(x,x) = var(x)}$$

$$C = \frac{1}{n-1} (A-\mu)^T (A-\mu)$$

→ in matrix, it's represented as above

PCA
- projections
- eigenvalues / eigenvectors
- covariance matrix

→ goal: reduce dimensionality by projecting data in the direction with biggest variability

Steps:
- start by centering the data (value - mean of column)
- you first find the covariance matrix
- find the eigenvalues/vectors of covariance matrix
- sort the eigenvectors from bigger to smaller
- keep the n-pairs (according to n = the # of dimensions you want)
- use the vectors to project your original matrix

$$\boxed{X_{PCA} = (X-\mu)V}$$

Random notes

- Matrix/vector transformations, read from right to left

$C_3 \; B_2 \; A_1$

- Vector $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$

2nd transformed vector

1st transformed vector

- A vector has a basis if linear independent / non-singular

- Discrete system - Markov chain

values in each column sum to 1

	0.6	0.25	0.10
	0.2	0.45	0.30
	0.2	0.30	0.60
	1	1	1

Markov-matrixes always have an eingenvalue of 1

- Matrix multiplication in Python uses @

- Example eigenvalues

$M = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$ $\quad \begin{matrix} 3-\lambda & 0 \\ -2 & 1-\lambda \end{matrix}$

$(3-\lambda)(1-\lambda) - (0 \cdot -2) = 0$

$(3-\lambda)(1-\lambda) = 0$

$\lambda = 3, \lambda = 1$

for $\lambda = 3$ value

$\begin{bmatrix} 3-3 & 0 \\ -2 & +1-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{matrix} 0x + 0y = 0 \\ -2x - 2y = 0 \end{matrix}$ $\quad x = -y \quad \begin{bmatrix} k \\ -k \end{bmatrix}$ vector