Economic Growth

Exogenous Growth Models: The Solow Swan Model

The Solow Model: the Aggregate Production Function

$$Y = F(K, L)$$

Y : **Aggregate real output**

K : **Aggregate capital stock**

L : **Aggregate labour input**

- *Property 1: Positive, diminishing marginal products.*

- For all $K > 0$ and $L > 0$, $F(\cdot)$ exhibits positive and diminishing marginal products with respect to each input:

$$\frac{\partial F}{\partial K} > 0, \qquad \frac{\partial^2 F}{\partial K^2} < 0$$

$$\frac{\partial F}{\partial L} > 0, \qquad \frac{\partial^2 F}{\partial L^2} < 0$$

The Solow Model: the Aggregate Production Function

- *Property 2: Constant Returns to Scale.*

- The production function exhibits constant returns to scale, such that:

$$\lambda Y = F(\lambda K, \lambda L)$$

- We can thus write the function in terms of output per worker:

$$y \equiv \frac{Y}{L} = F\left(\frac{K}{L}, 1\right) \equiv f(k) \quad \text{where } k \equiv \frac{K}{L}$$

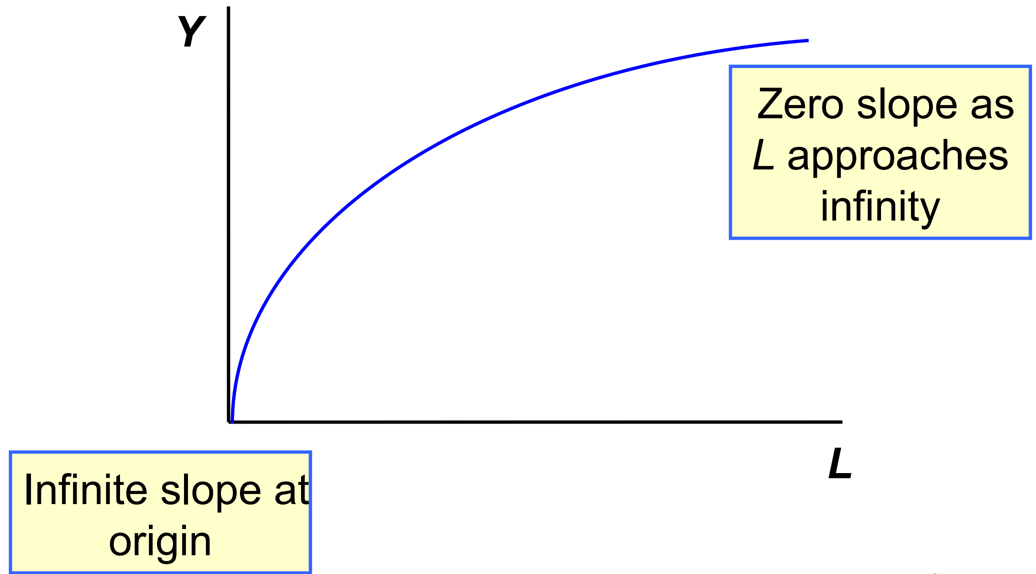
The Solow Model: the Aggregate Production Function

- *Property 3: Inada Conditions.*

- The marginal product of capital (or labour) approaches infinity as capital (or labour) approach 0. It approaches 0 as capital (or labour) approaches infinity:

$$\lim_{K \to 0}(F_K) = \lim_{L \to 0}(F_L) = \infty$$

$$\lim_{K \to \infty}(F_K) = \lim_{L \to \infty}(F_L) = 0$$



The Solow Model: the Aggregate Production Function

- The Cobb-Douglas production function has the three properties we require:

$$Y = AK^{\alpha}L^{1-\alpha} \qquad A > 0,\ 0 < \alpha < 1$$

$$y = \frac{AK^{\alpha}L^{1-\alpha}}{L} = Ak^{\alpha} = f(k)$$

Positive first derivative $\quad f'(k) = A\alpha k^{\alpha-1} > 0$

Negative second derivative $\quad f''(k) = -A\alpha(1-\alpha)k^{\alpha-2} < 0$

Inada Conditions

$$\lim_{k \to \infty} f'(k) = 0$$

$$\lim_{k \to 0} f'(k) = \infty$$

The Solow-Swan Model

- Original papers:
  - Robert Solow, ‘A Contribution to the Theory of Economic Growth’ *Quarterly Journal of Economics* (1956) and later in his book *Growth Theory* (1970)
  - Trevor Swan ‘Economic Growth and Capital Accumulation’, *Economic Record* (1956)).
- The model starts with the relationship between saving, investment and the change in the stock of capital over time:

$$\frac{dK}{dt} \equiv \dot{K} = \underbrace{I}_{\text{Investment}} - \underbrace{\delta}_{\text{Depreciation at rate } \delta} K = \underbrace{s}_{\text{Savings rate}} \cdot Y - \delta K = s \cdot F(K,L) - \delta K$$

The Solow-Swan Model

$$\boxed{\dot{K} = s \cdot F(K,L) - \delta K}$$

- Divide both sides by L:

$$\frac{\dot{K}}{L} = s \cdot \frac{F(K,L)}{L} - \delta \frac{K}{L} = s \cdot f(k) - \delta k$$

- Evolution of k (= K/L):

$$\frac{\dot{k}}{k} \equiv \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$$

Growth of k = Growth of K - Growth of L

- Multiply both sides by k

$$\dot{k} \equiv \frac{\dot{K}}{K}k - \frac{\dot{L}}{L}k = \frac{\dot{K}}{\cancel{K}}\frac{\cancel{K}}{L} - \frac{\dot{L}}{L}k$$

$$\text{or } \dot{k} = \frac{\dot{K}}{L} - nk$$

The Solow-Swan Model

$$\boxed{\frac{\dot{K}}{L} = s \cdot f(k) - \delta k \qquad \dot{k} = \frac{\dot{K}}{L} - nk}$$

- therefore: $$\dot{k} = s \cdot f(k) - (n+\delta)k$$
- Behaviour of *k* $$\dot{k} > 0 \text{ if } s \cdot f(k) > (n+\delta)k$$

$$\dot{k} < 0 \text{ if } s \cdot f(k) < (n+\delta)k$$

- ‘Steady-state’ equilibrium $$\dot{k} = 0 \text{ if } s \cdot f(k) = (n+\delta)k$$

The Solow-Swan Model

- Now ‘including’ technical progress
- Let L be labour measured in ‘*efficiency units*’ $L = AN$
    - where N is the number of workers
    - and A is a measure of their efficiency – ‘labour augmenting technical progress’
- Assume that: $\frac{\dot{A}}{A} = \lambda$ **is the growth rate of *A***
- so the growth rate of L ($= AN$) is now $n + \lambda$
- and the evolution of k is now $\dot{k} = s \cdot f(k) - (\underbrace{n + \lambda} + \delta)k$

The Solow-Swan Model

At k^*:

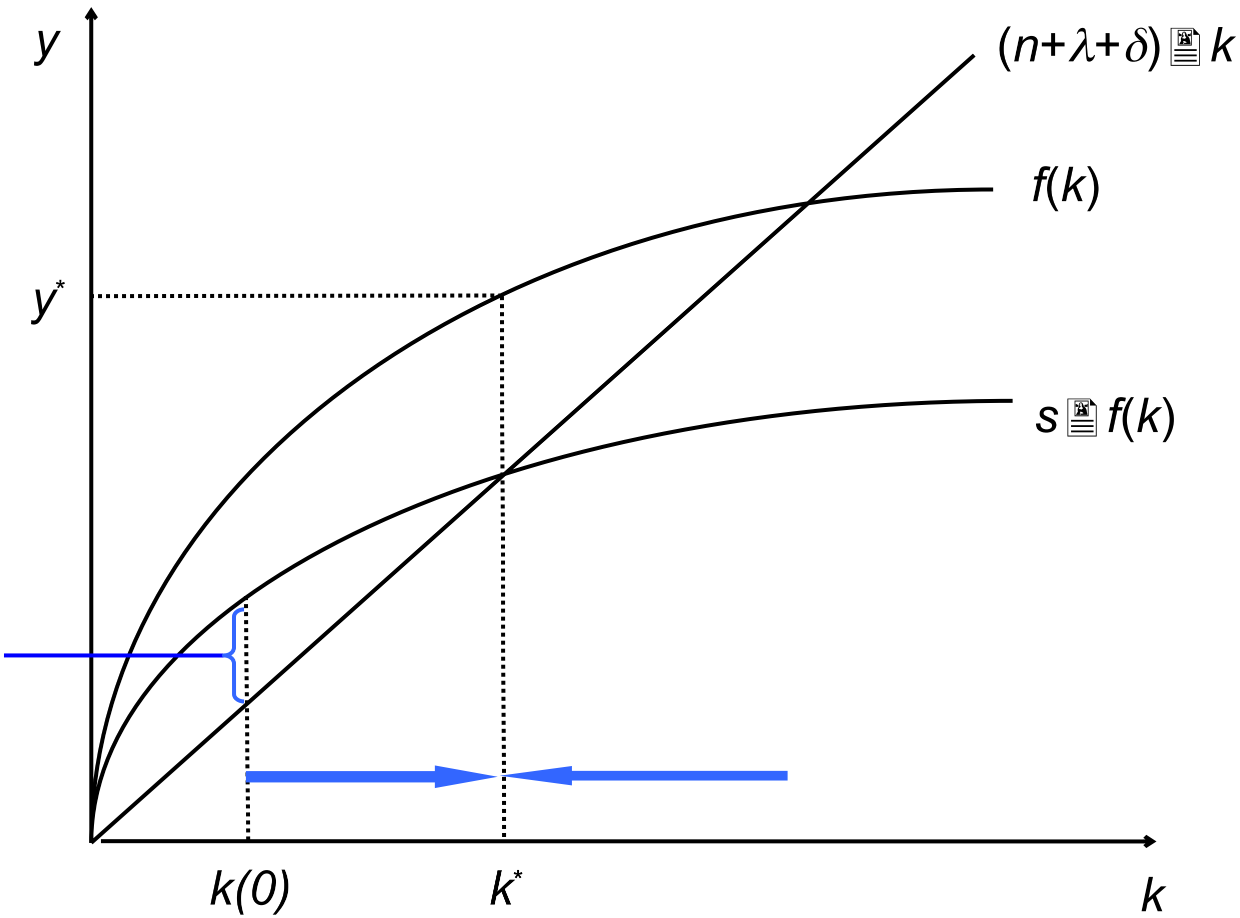
$s \cdot f(k^*) = (n + \lambda + \delta) k^*$

and $\dot{k} = 0$

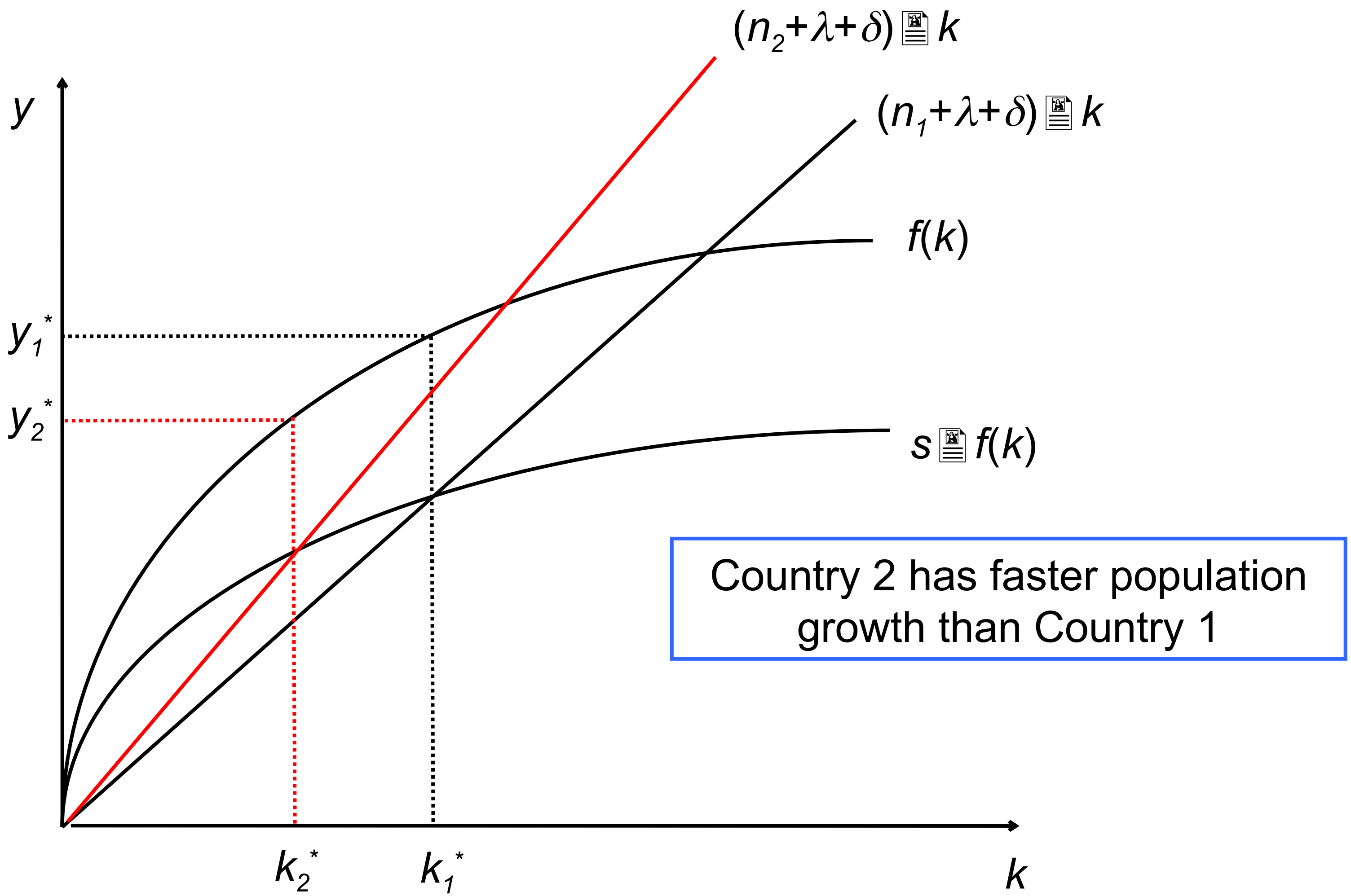
At $k(0)$:

$s \cdot f(k(0)) > (n + \lambda + \delta) k(0)$

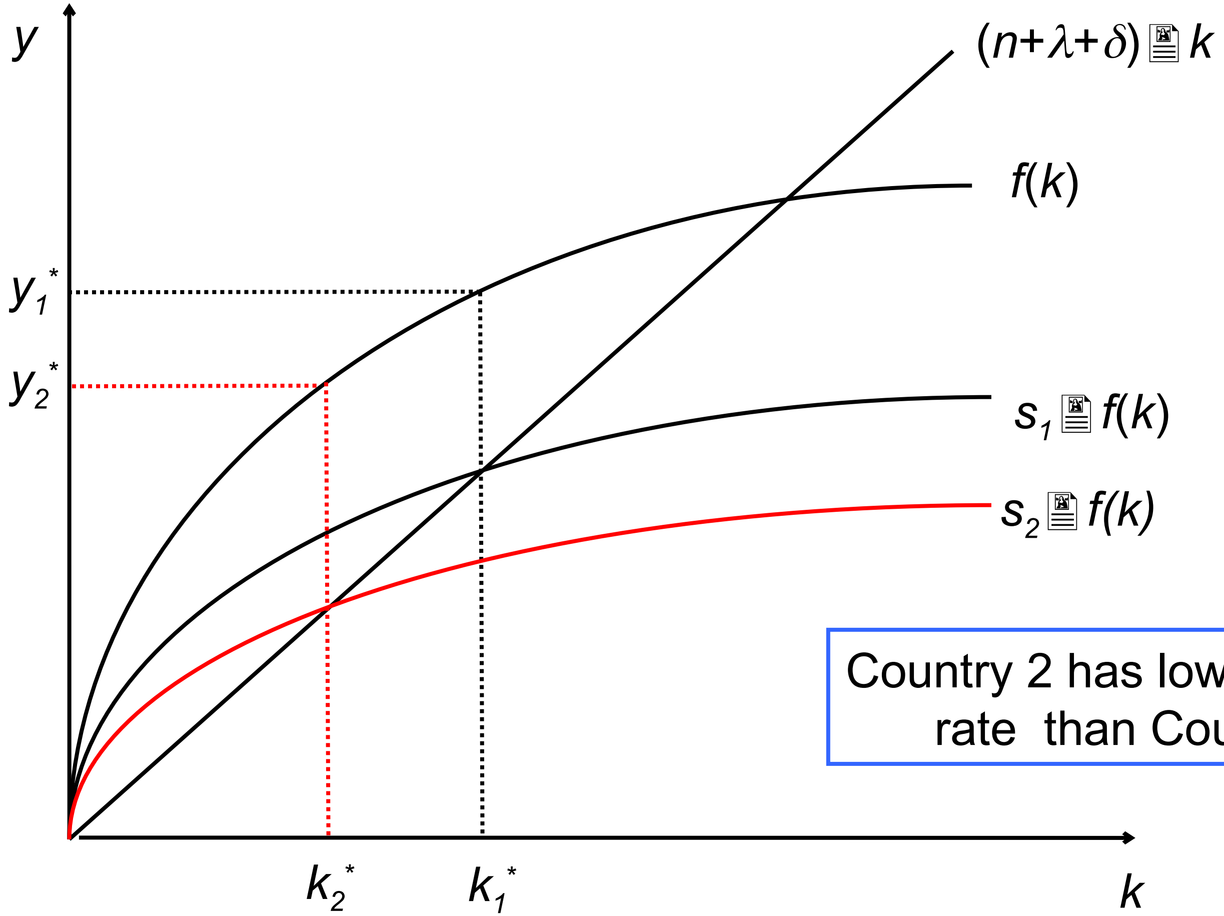
and $\dot{k} > 0$



The Solow-Swan Model



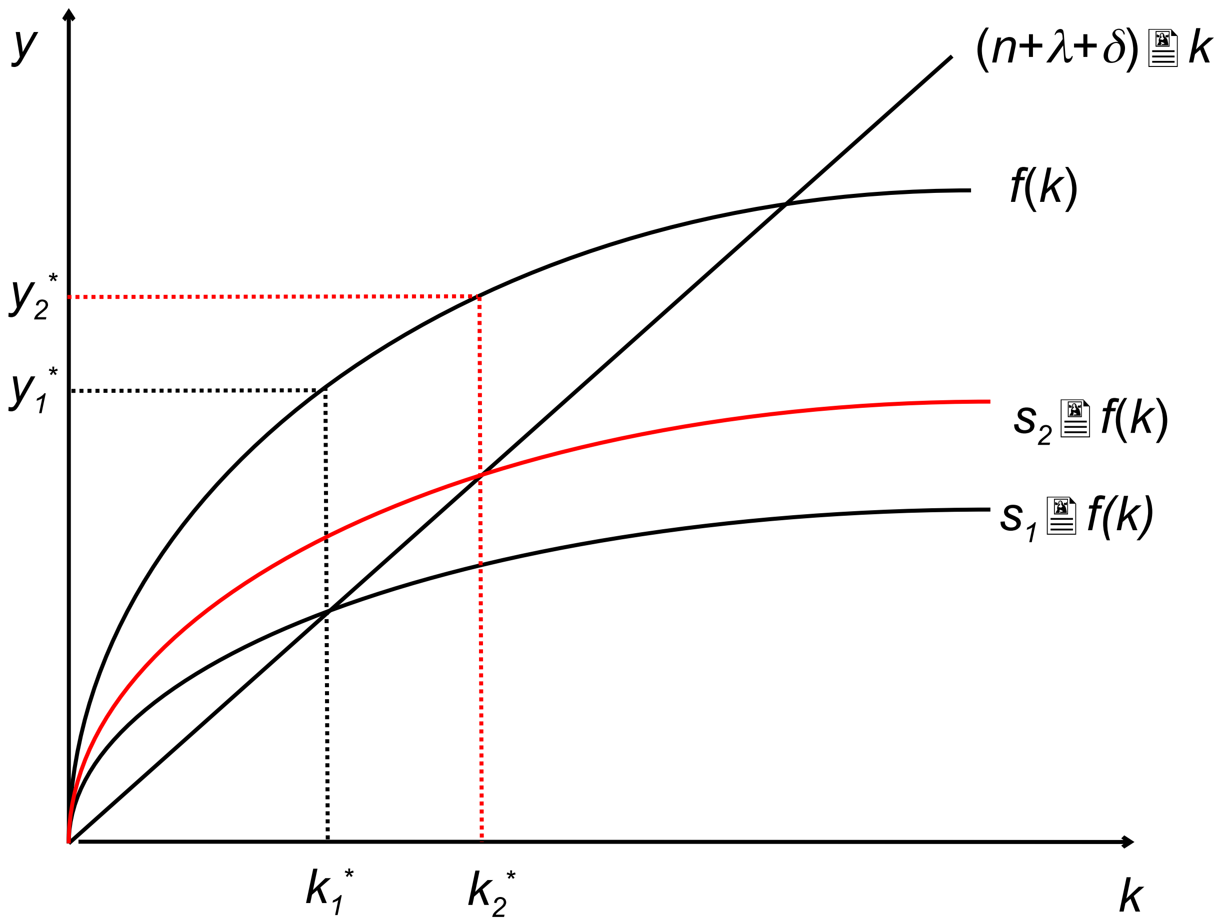
The Solow-Swan Model



Country 2 has lower savings rate than Country 1

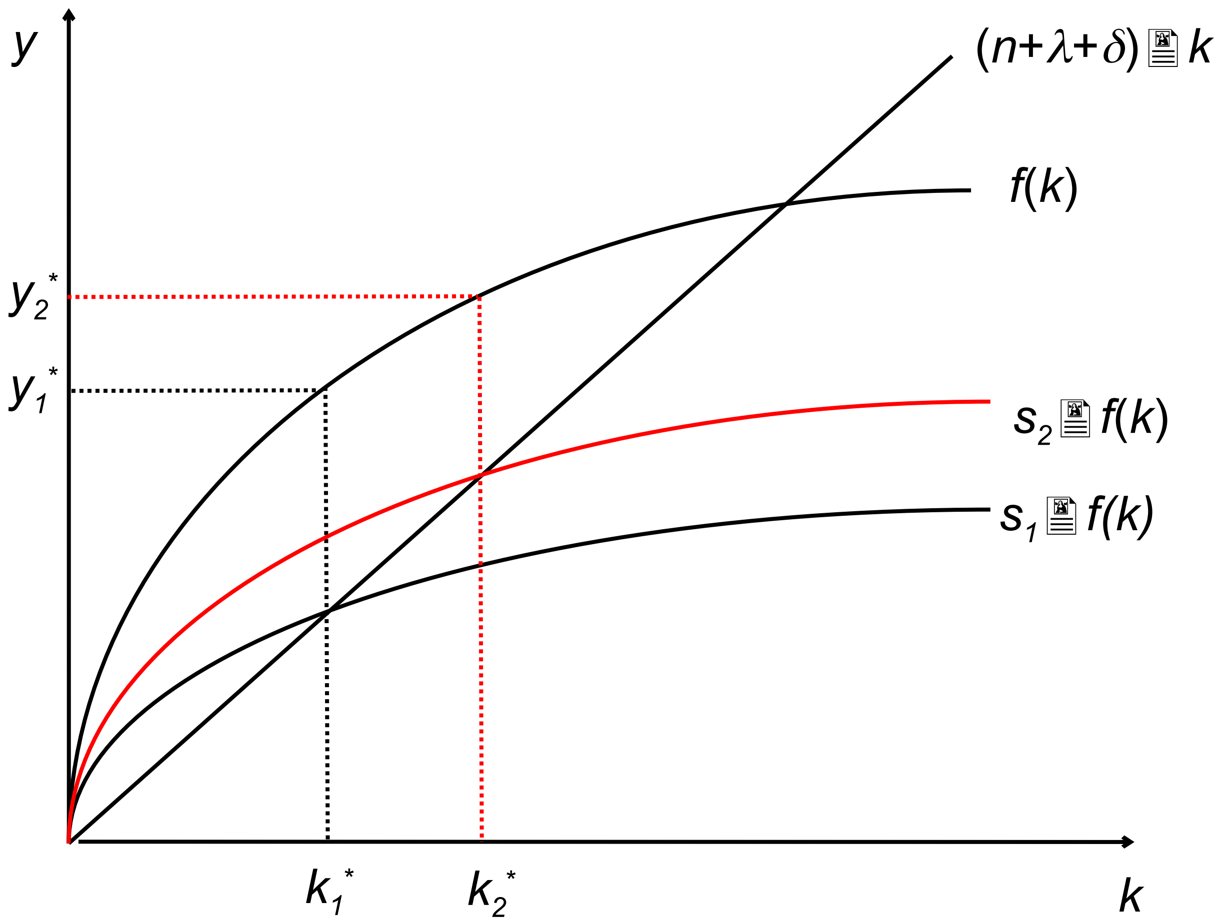
The Solow-Swan Model

- Consider the case of a country with a savings rate of s_1 and in steady state equilibrium.
- $y = Y/L$ & $k = K/L$ are constant so Y & K are growing at the same rate as L, *i.e. at rate* $n + \lambda$
- The savings rate rises to s_2 and there is an increase in the steady-state y and k
- At the new steady state, y & k are again constant (at higher levels) so Y and K *are* again growing at the same rate as L, *i.e. at rate* $n + \lambda$

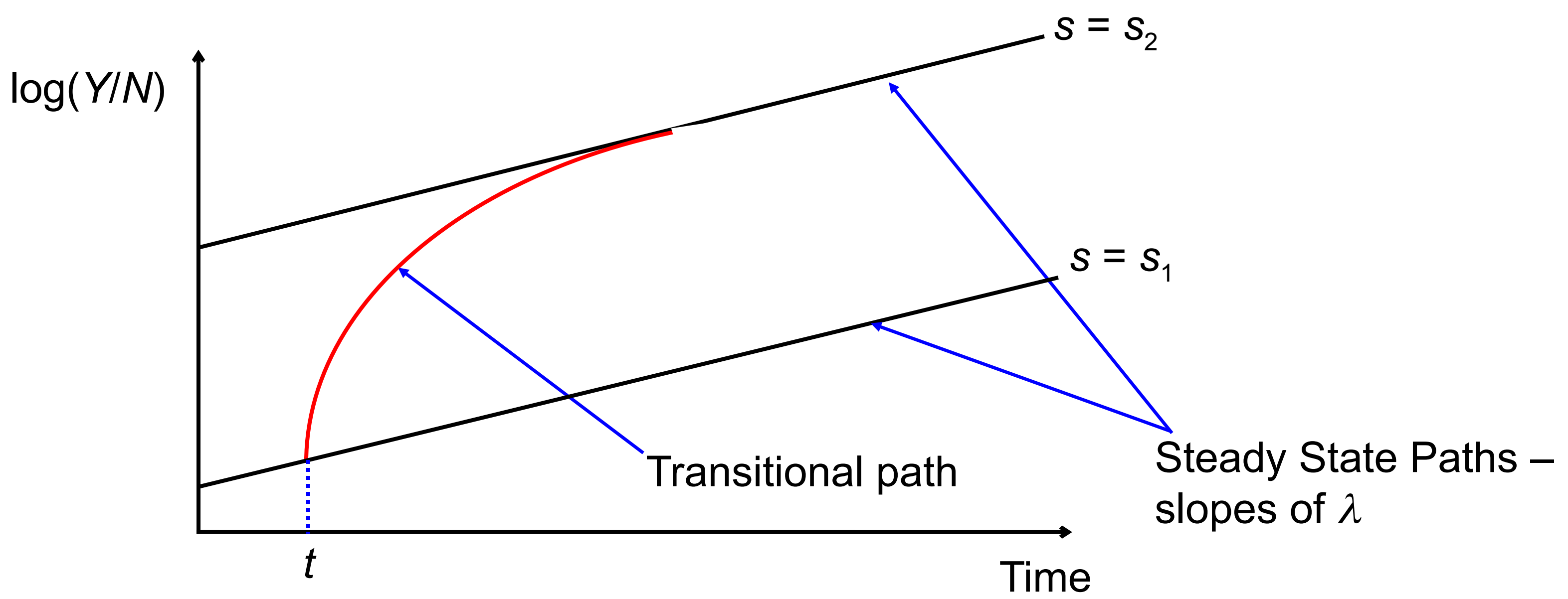


The Solow-Swan Model

- There are two steady-state equilibria with Y & K growing at $n + \lambda$, but one is at a *higher level.*
- To get from y_1^* to y_2^*, Y will have to grow faster than $n + \lambda$
- Growth of Y along the steady-state path is $n + \lambda$
- Growth of Y in transition between these paths is greater than $n + \lambda$
- If $Y/L = Y/(AN)$ is constant in steady state, Y/N is growing at the same rate as A, *i.e. at rate* λ



The Solow-Swan Model



- Until time t the economy is on low-level steady-state path labelled $s = s_1$
- At time t the savings rate rises from s_1 to s_2 giving the new path $s = s$
- Between the two, as k and y rise to new steady-state, output per worker grows faster than λ in the transitional phase