

# Algorithm Details

In this article, we describe the algorithm in this project.

## 1 Introduction

Naive implementations of Kalman filters and smoothers have numerical instability issues. In this project, the square root formulation of the Kalman algorithm is implemented, which is more accurate and stable.

In case the dynamics are presented by multiple models, an approach to the multiple model fixed interval smoothing problem for the square root Kalman algorithms is examined. The approach is based on the interacting multiple models (IMM), except that we consider the square root Kalman algorithms.

## 2 Square Root Kalman Algorithm

In this section, we describe the square root Kalman algorithm described in [1].

Let  $X$  be a Gaussian random variable with mean  $m$  and covariance matrix  $V$ . The square root form of  $X$  is given by  $R$  and  $z$  where

$$\begin{aligned}(R^T R)^{-1} &= V, \\ Rm &= z.\end{aligned}$$

The square root formulation of the Kalman algorithms:

- square root information filter and
- square root information smoother

will be parametrized by Gaussian random variables in square root form.

Recall that in  $k$ th iteration, the true state  $x_k$  is evolved from the state  $x_{k-1}$  according to

$$(1) \quad x_k = \Phi_k x_{k-1} + G w_k$$

where

- $\Phi_k$  is the nonsingular transition matrix and
- $w_k$  is the process noise that are assumed to have zero mean and nonsingular covariance matrix  $Q_w(k) = (R_w^T(k)R_w(k))^{-1}$ .

An observation  $z_k$  of the true state  $x_k$  is made according to

$$z_k = Ax_k + v_k \quad \text{for } 1 \leq k \leq T$$

where  $v_k \in N(0, I)$ .

## 2.1 Square Root Information Filter

A filtering problem is to solve

$$\begin{aligned} & \underset{x_T}{\text{minimize}} && \|\tilde{R}_x(1)x_1 - \tilde{z}_x(1)\| + \sum_{k=1}^T \|Ax_k - z_k\|^2 \\ & \text{subject to} && \text{Equation 1.} \end{aligned}$$

### 2.1.1 Prediction Step

Let  $(\hat{R}_x(k-1), \hat{z}_x(k-1))$  be the square root form of the updated estimate in iteration  $k-1$ . The prediction step is to solve

$$\begin{aligned} & \underset{x_k}{\text{minimize}} && \|\hat{R}_x(k-1)x_{k-1} - \hat{z}_x(k-1)\|^2 + \|R_w(k)w_k - z_w(k)\|^2 \\ & \text{subject to} && \text{Equation 1.} \end{aligned}$$

Noted that  $x_{k-1} = \Phi_k^{-1}(x_k - Gw_k)$ , the above minimization problem is equivalent to

$$\underset{x_k, w_k}{\text{minimize}} \quad \left\| \begin{pmatrix} R_w(k) & 0 \\ -R_x^d(k)G & R_x^d(k) \end{pmatrix} \begin{pmatrix} w_k \\ x_k \end{pmatrix} - \begin{pmatrix} z_w(k) \\ \hat{z}_x(k-1) \end{pmatrix} \right\|^2$$

which can be solved by using the QR decomposition

$$\begin{pmatrix} R_w(k) & 0 & z_w(k) \\ -R_x^d(k)G & R_x^d(k) & \hat{z}_x(k-1) \end{pmatrix} = Q \begin{pmatrix} \tilde{R}_w(k) & \tilde{R}_{wx}(k) & \tilde{z}_w(k) \\ 0 & \tilde{R}_x(k) & \tilde{z}_x(k) \end{pmatrix}$$

where

- the right hand side is the QR decomposition of the left hand side,
- $(R_w(k), z_w(k))$  is the prior estimate of  $w_k$  in square root form,
- $R_x^d(k) = \hat{R}_x(k-1)\Phi_k^{-1}$  and
- $(\tilde{R}_x(k), \tilde{z}_x(k))$  is the predict estimate in square root form.

### 2.1.2 Update Step

Let  $(\tilde{R}_x(k), \tilde{z}_x(k))$  be the prior information obtained from prediction step, the update step is to solve

$$\underset{x_k}{\text{minimize}} \quad \left\| \tilde{R}_x(k)x_k - \tilde{z}_x(k) \right\|^2 + \|Ax_k - z_k\|^2.$$

The above minimization problem can be rewritten as

$$\underset{x_k}{\text{minimize}} \quad \left\| \begin{pmatrix} \tilde{R}_x(k) \\ \hat{R}_x(k) \end{pmatrix} x_k - \begin{pmatrix} \tilde{z}_x(k) \\ z_k \end{pmatrix} \right\|^2$$

which can be solved by using the QR decomposition

$$\begin{pmatrix} \tilde{R}_x(k) & \tilde{z}_x(k) \\ A & z_k \end{pmatrix} = Q \begin{pmatrix} \hat{R}_x(k) & \hat{z}_x(k) \\ 0 & e_k \end{pmatrix}$$

where

- the right hand side is the QR decomposition of the left hand side and
- $(\hat{R}_x(k), \hat{z}_x(k))$  is the updated estimate in square root form.

## 2.2 Square Root Information Smoother

A smoothing problem is to solve

$$\begin{aligned} &\underset{x_k}{\text{minimize}} \quad \left\| \tilde{R}_x(1)x_1 - \tilde{z}_x(1) \right\| + \sum_{l=1}^K \|Ax_l - z_l\|^2 \\ &\text{subject to} \quad \text{Equation 1.} \end{aligned}$$

for  $1 \leq k \leq T$ .

### 2.2.1 Smooth Step

Let  $(R_x^*(k), z_x^*(k))$  be the smoothed estimate in iteration  $k$ . The smooth step is to solve

$$\begin{aligned} &\underset{x_{k-1}}{\text{minimize}} \quad \left\| \tilde{R}_w(k)w_k + \tilde{R}_{wx}(k)x_k - \tilde{z}_w(k) \right\|^2 + \left\| \tilde{R}_x(k)x_k - \tilde{z}_x(k) \right\|^2 \\ &\text{subject to} \quad \text{Equation 1.} \end{aligned}$$

Replacing  $x_k$  by  $\Phi_k x_{k-1} + Gw_k$ , the above minimization problem is equivalent to

$$\underset{x_{k-1}, w_k}{\text{minimize}} \quad \left\| \begin{pmatrix} \tilde{R}_w(k) + \tilde{R}_{wx}(k)G & \tilde{R}_{wx}(k)\Phi_k \\ R_x^*(k)G & R_x^*(k)\Phi_k \end{pmatrix} \begin{pmatrix} w_k \\ x_{k-1} \end{pmatrix} - \begin{pmatrix} \tilde{z}_w(k) \\ z_x^*(k) \end{pmatrix} \right\|^2$$

which can be solved by using the QR decomposition

$$\begin{pmatrix} \tilde{R}_w(k) + \tilde{R}_{wx}(k)G & \tilde{R}_{wx}(k)\Phi_j & \tilde{z}_w(k) \\ R_x^*(k)G & R_x^*(k)\Phi_j & z_x^*(k) \end{pmatrix} = Q \begin{pmatrix} R_w^*(k) & R_{wx}^*(k) & z_w^*(k) \\ 0 & R_x^*(k-1) & z_x^*(k-1) \end{pmatrix}$$

where

- the right hand side is the QR decomposition of the left hand side and
- $(R_x^*(k-1), z_x^*(k-1))$  is the smoothed estimate in square root form.

### 3 Square Root IMM Algorithm

Let  $M_k$  be the model in effect during the sampling period ending at time  $k$ . The event that model  $j$  ( $j = 1, \dots, n$ ) is in effect during the sampling period ending at time  $k$  will be denoted by  $M_k^j$ .

#### 3.1 Square Root IMM Filter

When applying the classical IMM filtering, attentions are needed for the followings:

- mixing the state estimates and
- model likelihood computation.

##### 3.1.1 Mixing The State Estimates

**Proposition 3.1** *Let  $a$ ,  $b$  and  $c$  be multivariate Gaussian distribution with square root form  $(R_a, z_a)$ ,  $(R_b, z_b)$  and  $(R_c, z_c)$  respectively. Suppose that  $c = a + Ab$  for some matrix  $A$ . Then*

$$\begin{pmatrix} R_b & 0 & z_b \\ -R_a A & R_a & z_a \end{pmatrix} = Q \begin{pmatrix} - & - & - \\ 0 & R_c & z_c \end{pmatrix}$$

*the right hand side is the QR decomposition of the left hand side. Moreover,*

$$Q^T \begin{pmatrix} 0 \\ z_a \end{pmatrix} = R_c m_a, \quad Q^T \begin{pmatrix} z_b \\ 0 \end{pmatrix} = R_c m_b.$$

### **3.1.2 Model Likelihood Computation**

## **References**

- [1] G.J. Bierman. *Factorization Methods for Discrete Sequential Estimation*. Dover Books on Mathematics Series. Dover Publications, 2006.