<1.7>

(1) $$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}y^2\right) dx\,dy$$

$$= \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{1}{2\sigma^2}r^2\right) r\,dr\,d\theta$$

$x = r\cos\theta$, $y = \sin\theta$

$$\int_0^{\infty} \exp\left(\frac{-r^2}{2\sigma^2}\right) r\,dr = -\sigma^2 \exp\left(\frac{-r^2}{2\sigma^2}\right)\Big|_0^{+\infty}$$

$$= -\sigma^2(0-1) = \sigma^2$$

$$I^2 = \int_0^{2\pi} \sigma^2\,d\theta = 2\pi\sigma^2 \Rightarrow I = \sqrt{2\pi}\sigma$$

\#

(2) $$\int_{-\infty}^{\infty} N(x|u,\sigma^2)\,dx =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-u)^2\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy \quad (y = x-u)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy$$

$$= 1 \quad \#$$

<2.26>

right side

$$(A+BCD)(A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1})$$

$$= AA^{-1}+AA^{-1}BC(C^{-1}+DA^{-1}B)^{-1}DA^{-1}+BCDA^{-1}-BCDA^{-1}BC(C^{-1}DA^{-1}B)^{-1}DA^{-1}$$

$$= I + BC(C^{-1}+DA^{-1}B)^{-1}DA^{-1} + BCDA^{-1} + BC(C^{-1}DA^{-1}B)^{-1}DA^{-1} - BCDA^{-1}$$

$$= I$$

where $A = -BCDA^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$

$$= -BC(C^{-1}+C^{-1}+DA^{-1}B)(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$$

$$= (-BC)(-C^{-1})(C^{-1}+DA^{-1}B)^{-1}DA^{-1}+(-BC)(C^{-1}+DA^{-1}B)(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$$

$$= B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} - BCDA^{-1}$$

left side

$$(A+BCD)(A+BCD)^{-1} = I$$

$\Rightarrow$ correct #

⟨2.27⟩

$$E[x+z] = \iint (x+z)\, p(x,z)\, dx\, dz$$

$$= \iint (x+z)\, p(x)\, p(z)\, dx\, dz$$

$$= \iint x\, p(x)\, p(z)\, dx\, dz + \iint z\, p(x)\, p(z)\, dx\, dz$$

$$= \int x\, p(x)\, dx + \int z\, p(z)\, dz$$

$$= \underline{E[x] + E[z]} \quad \#$$

$$\mathrm{cov}[x+z] = \iint ((x+z) - E[x+z])(x+z - E[x+z])^T p(x,z)\, dx\, dz$$

$$= \iint ((x+z) - E[x] - E[z])(x+z - E[x] - E[z])^T p(x)p(z)\, dx\, dz$$

$$= \int (x - E[x])(x - E[x])^T p(x) + \int (z - E[z])(z - E[z])^T p(z)\, dz$$

$$= \mathrm{cov}[x] + \mathrm{cov}[z] \quad \#$$

(3. 6)

The log likelihood function:

$$\ln p(T|X, W, \beta) = \frac{-N}{2}\ln|\Sigma| - \frac{1}{2}\sum_{n=1}^{N}[t_n - W^T\phi(x_n)]^T\Sigma^{-1}[t_n - W^T\phi(x_n)]$$

Then set the derivative with respect to W to zero

$$0 = -\sum_{n=1}^{N}\Sigma^{-1}[t_n - W^T\phi(X_n)]\,\phi(X_n)^T$$

$$\Rightarrow \Sigma = \frac{1}{N}\sum_{n=1}^{N}(t_n - W_{ML}^T\phi(X_n)][t_n - W_{ML}^T\phi(X_n)]^T$$

\#

(from 2.125 in text book

...

(3.11)

$$S^{-1}_{N+1} = S_N^{-1} + \beta \phi(x_{N+1}) \phi(x_{N+1})^T$$

By using (3.110)

$$S_{N+1} = [S_N^{-1} + \beta \phi(x_{N+1}) \phi(x_{N+1})^T$$

$$= [S_N^{-1} + \sqrt{\beta} \phi(x_{N+1}) \sqrt{\beta} \phi(x_{N+1})^T]^{-1}$$

$$= S_N - \frac{S_N (\sqrt{\beta} \phi(x_{N+1}))(\sqrt{\beta} \phi(x_{N+1}))^T S_N}{1 + (\sqrt{\beta} \phi(x_{N+1}))^T S_N (\sqrt{\beta} \phi(x_{N+1}))}$$

$$= S_N - \frac{\beta S_N \phi(x_{N+1}) \phi(x_{N+1})^T S_N}{1 + \beta \phi(x_{N+1})^T S_N \phi(x_{N+1})}$$

$$\sigma^2_N(x) - \sigma^2_{N+1}(x) = \phi(x)^T (S_N - S_{N+1}) \phi(x)$$

$$= \phi(x)^T \frac{\beta S_N \phi(x_{N+1}) \phi(x_{N+1})^T S_N}{1 + \beta \phi(x_{N+1})^T S_N \phi(x_{N+1})} \phi(x)$$

$$= \frac{\phi(x)^T S_N \phi(x_{N+1}) \phi(x_{N+1})^T S_N \phi(x)}{1/\beta + \phi(x_{N+1})^T S_N \phi(x_{N+1})}$$

$$= \frac{[\phi(x)^T S_N \phi(x_{N+1})]^2}{1/\beta + \phi(x_{N+1})^T S_N \phi(x_{N+1})} \longrightarrow 1$$

$\because S_n \geq 0 \Rightarrow 1 \geq 0$

$\therefore \sigma^2_N(x) - \sigma^2_{N+1}(x) \geq 0$ #