# How many colours does Chromagon need?

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Figure 1: A 1D maze requiring 3 colours

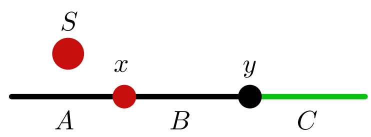
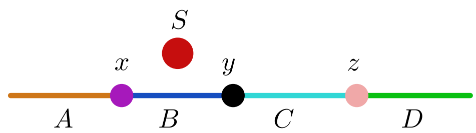



Figure 2: A section of a 1D maze showing every region that can interact with $y$




## 1 What is Chromagon?

See `http://foolswood.github.io/Hexagongame/` for an interactive version of the hexagonal case.

### 1.1 What is a necessary colour?

Mazes are equivalent if all their paths are the same. A trivial example of equivalent mazes would be if all the regions and lines (elements) of one colour were to swap colour with all the elements of another.

### 1.2 1D (straight line) mazes

The simplest maze type that the rules can be applied to is one where each region borders 2 others. These are equivalent to a single row maze of any other shape.

## 2 Proving a maze cannot be reduced

Consider the maze shown in figure 1 where $S$ is the starting colour, $A$, $B$ and $C$ are the colours of the corresponding regions and $x$ and $y$ are the colours of the dividers.

This maze can be expressed in terms of the set of equalities and inequalities that are required to impose the required movement restrictions:

$$S = x \tag{1}$$

$$A \neq x \tag{2}$$

$$A = y \tag{3}$$

$$B = y \tag{4}$$

$$C \neq y \tag{5}$$

$$C \neq x \tag{6}$$

Substituting equation 4 into 2 yields:

$$y \neq x \tag{7}$$

which when considered with equations 5 and 6 can be seen to require at least 3 colours to solve.

## 3 The maximum number of colours required for any maze

The number of colours required by a given maze is determined by the set of inequalities that restrict the available paths. Any equalities will force colour reuse (and therefore cannot increase the total number of required colours).

For instance to require 2 colours requires $a \neq b$, but to require 3 (as in section 2) requires a set of inequalities of the form:

$$a \neq b \quad \text{and} \quad b \neq c \quad \text{and} \quad c \neq a \tag{8}$$

Note that in order to have a set of equations like this it is necessary to have 2 things that are simultaneously $\neq a$.

More generally, in order to require a $n$ colours a single element must be able to participate simultaneously in $n-1$ inequalities.

### 3.1 1D

Consider the interactions of $y$ in figure 2.

For $C$ and $D$ to have any effect at $y$ the path must turn around at $D$. In order for this to be possible:

$$B = C \tag{9}$$

which implies that the possible inequalities $B \neq y$ and $C \neq y$ are redundant.

If $A \neq y$ and $S \neq y$ then $y$ cannot be traversed and thus no inequalities referring to $B$, $C$ or $D$ can take effect.

If $S \neq y$ then to traverse $y$ and bring the remaining potential inequalities ($B \neq y$ and $D \neq y$) into play requires the path to return to $B$ which requires $x = B = S$ making $B \neq y$ and $S \neq y$ redundant.

In order to return to the line from $A$, and thus apply $A \neq y$ requires that $x = B = S$, again this makes the constraint redundant.

Therefore the maximum number of simultaneous inequalities that can be affect $y$ in any maze is 2, and thus 3 colours is the maximum required to express any 1D maze.