The Many-Worlds Calculus: Representing Quantum Control

Kostia Chardonnet, M. Visme, B. Valiron, R. Vilmart

Univ. Paris Saclay, LMF
Univ. Paris Centre, IRIF

Buenos Aires, 17/10/2022

- Classical bits as vectors: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- Classical bits as vectors: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Arbitrary quantum bits (qubits): $\alpha\,|0\rangle + \beta\,|1\rangle$

- Classical bits as vectors: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Arbitrary quantum bits (qubits): $\alpha |0\rangle + \beta |1\rangle$
- Larger systems: $q_0 \otimes q_1$, $|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

- Classical bits as vectors: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- Arbitrary quantum bits (qubits): $\alpha\,|0\rangle + \beta\,|1\rangle$

- Larger systems: $q_0 \otimes q_1$, $|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad A \otimes B = \begin{pmatrix} a_{00}B & a_{01}B & \cdots \\ a_{10}B & \ddots & \\ \vdots & & \end{pmatrix}$

- Classical bits as vectors: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Arbitrary quantum bits (qubits): $\alpha\,|0\rangle + \beta\,|1\rangle$
- Larger systems: $q_0 \otimes q_1$, $|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad A \otimes B = \begin{pmatrix} a_{00}B & a_{01}B & \cdots \\ a_{10}B & \ddots & \\ \vdots & & \end{pmatrix}$
- Entangled state cannot be broken down as $q_0 \otimes q_1$

- Classical bits as vectors: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Arbitrary quantum bits (qubits): $\alpha\,|0\rangle + \beta\,|1\rangle$
- Larger systems: $q_0 \otimes q_1$, $|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad A \otimes B = \begin{pmatrix} a_{00}B & a_{01}B & \cdots \\ a_{10}B & \ddots & \\ \vdots & & \end{pmatrix}$
- Entangled state cannot be broken down as $q_0 \otimes q_1$
- Operation are *linear maps*

- $H := \frac{1}{\sqrt{2}} \begin{array}{c c} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array} = \begin{cases} |0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}} \end{cases}$ is unitary

- $H := \frac{1}{\sqrt{2}} \begin{array}{c c} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array} = \begin{cases} |0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}} \end{cases}$ is unitary

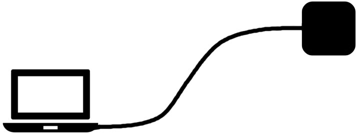
- $$\begin{array}{l} |+\rangle := H\,|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \\ |-\rangle := H\,|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} \end{array}$$

- $H := \frac{1}{\sqrt{2}} \begin{array}{c c} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array} = \begin{cases} |0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}} \end{cases}$ is unitary

- $|+\rangle := H\,|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$
  $|-\rangle := H\,|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

- $\mathrm{CNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{cases} |0x\rangle \mapsto |0x\rangle \\ |1x\rangle \mapsto |1\neg x\rangle \end{cases}$

- Classical Computer connected to coprocessor.
- Quantum circuit built **by** the classical computer.
- **Classical** control-flow of the program.
- Only **qubits** and tensors thereof.



Quantum Computation features **non-causal execution paths**:

$$\mathrm{QSwitch}(x, U, V) = \begin{cases} -\boxed{U}-\boxed{V}- & \text{if } x = |0\rangle \\ -\boxed{V}-\boxed{U}- & \text{if } x = |1\rangle \end{cases}$$

Quantum Computation features **non-causal execution paths**:

$$\mathrm{QSwitch}(x, U, V) = \begin{cases} -\boxed{U}-\boxed{V}- & \text{if } x = |0\rangle \\ -\boxed{V}-\boxed{U}- & \text{if } x = |1\rangle \end{cases}$$

Since x can be in **superposition** we get:

$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |y\rangle \mapsto \alpha\,|0\rangle \otimes (UV\,|y\rangle) + \beta\,|1\rangle \otimes (VU\,|y\rangle)$$

Quantum Computation features **non-causal execution paths**:

$$\mathrm{QSwitch}(x, U, V) = \begin{cases} -\boxed{U}-\boxed{V}- & \text{if } x = |0\rangle \\ -\boxed{V}-\boxed{U}- & \text{if } x = |1\rangle \end{cases}$$

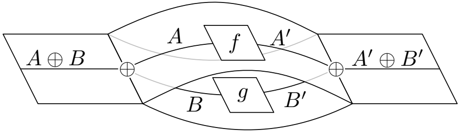
Since x can be in **superposition** we get:

$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |y\rangle \mapsto \alpha\,|0\rangle \otimes (UV\,|y\rangle) + \beta\,|1\rangle \otimes (VU\,|y\rangle)$$

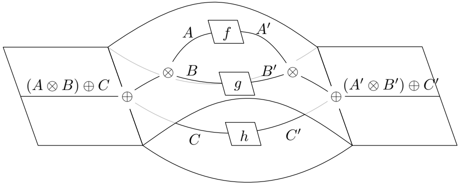
- Not realizable in Quantum Circuits with only one U and V.
- Physically implementable with Linear Optics [Arxiv 1810.09826].
- Alternative exists: addressable quantum gates, processes algebra [Arxiv 2109.08050, 2203.11245].
- Alternative still have limited **types** or **unclear semantics**.

What we want:

- New formalism with **types** and **Quantum Control**.
- Allowing to represent both **pairing** and **case analysis**.
- Clear semantics of the objects.
- Usable as a model of programming languages.



Split over coproduct



Splits over coproduct and tensor

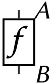
- Born from a token-based interpretation of the ZX-Calculus [Arxiv 2206.10916].
- Adding a $\oplus$ to the ZX-Calculus.
- Adding a $\otimes$ to tapes-diagrams / processes algebras.
- Features both a **token-based** semantics and **denotational** semantics.
- Equationnal Theory that is **sound** and **complete**.
- Case study into a typed quantum programming language.
- Colored Prop with two monoidal product, $\otimes$ and $\oplus$.

- Prop $(\mathcal{C}, \star, \boxtimes)$: product and permutations.
- Symmetric monoidal category with a single object $\star$
- Nice graphical representation (more on that later)
- Object of the form $\star \boxtimes \cdots \boxtimes \star$
- Objects = input and output, morphisms = diagrams.

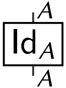
- Any morphism $f : A \to B$ seen as a box : $\overset{A}{\underset{B}{\boxed{f}}}$

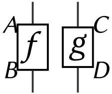
Graphical Representation for Props (1/2)

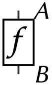
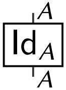
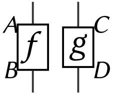
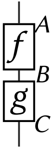
- Any morphism $f : A \to B$ seen as a box : $\boxed{f}^{A}_{B}$
- $\mathsf{Id}_A : \Big|^{A}_{A} := \boxed{\mathsf{Id}_A}^{A}_{A}$

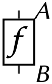
- Any morphism $f : A \to B$ seen as a box : 

- Id_A : 

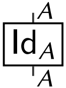
:= 

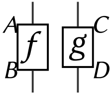
- $f : A \to B, g : C \to D$ then $f \otimes g : A \otimes C \to B \otimes D$ represented as 

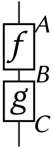
- Any morphism $f : A \to B$ seen as a box : 
- Id_A :  := 
- $f : A \to B, g : C \to D$ then $f \otimes g : A \otimes C \to B \otimes D$ represented as 
- Given $f : A \to B, g : B \to C$ we get: 

- Any morphism $f : A \to B$ seen as a box : 

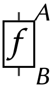
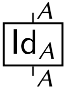
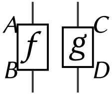
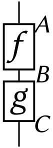
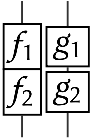
- Id_A : 

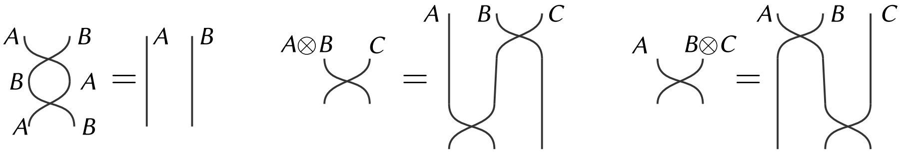
:= 

- $f : A \to B, g : C \to D$ then $f \otimes g : A \otimes C \to B \otimes D$ represented as 

- Given $f : A \to B, g : B \to C$ we get: 

- $(f_1 \circ f_2) \otimes (g_1 \circ g_2) = (f_1 \otimes g_1) \circ (f_2 \otimes g_2)$

- Any morphism $f : A \to B$ seen as a box : 
- Id_A :  $:=$ 
- $f : A \to B, g : C \to D$ then $f \otimes g : A \otimes C \to B \otimes D$ represented as 
- Given $f : A \to B, g : B \to C$ we get: 
- $(f_1 \circ f_2) \otimes (g_1 \circ g_2) = (f_1 \otimes g_1) \circ (f_2 \otimes g_2) =$ 



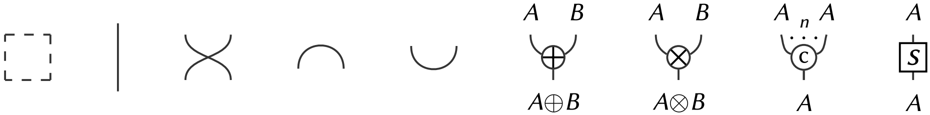
Any morphism can be represented by a diagram, any diagram represent a morphism.

If two morphisms are equal, then so are their diagrams.

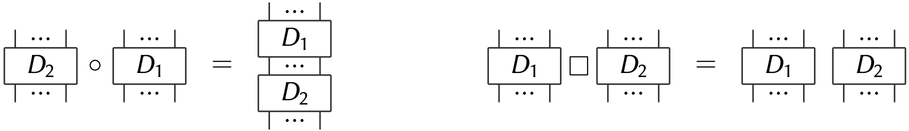
Generators

Many-Worlds, in Short

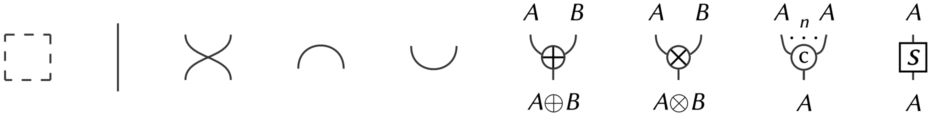
Generators



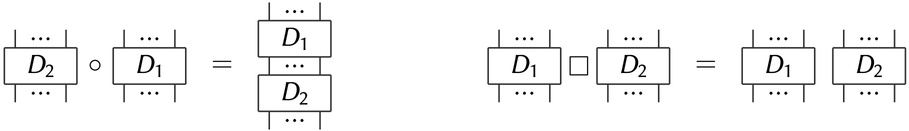
Compositions



Generators

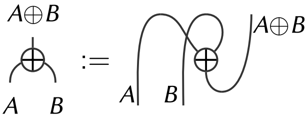


Compositions

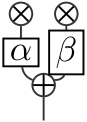


Standard Interpretation

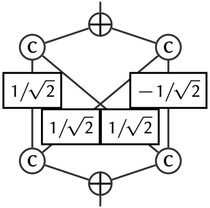
$$[\![.]\!] : \mathbf{MW} \to \mathbf{FdM}(R)$$



Mirrored versions

$\alpha\,|0\rangle + \beta\,|1\rangle \rightsquigarrow$ 

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \rightsquigarrow$$

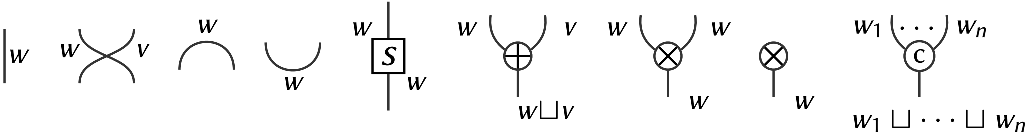


A Quantum Bit and the Hadamard Unitary

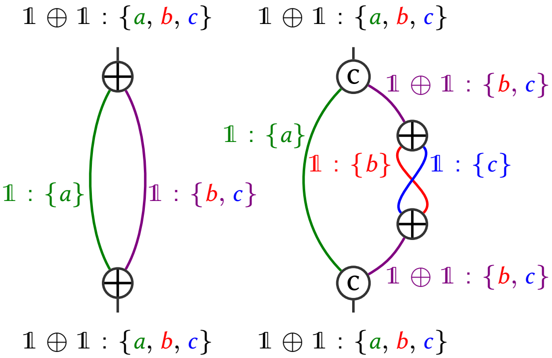
Not every diagram is **correct**: 

- Even those ”wrong” diagrams have a semantics (which is 0).
- Validity criterion to distinguish correct and incorrect diagrams.
- Akin to Proof Net validity criterion.

- Each wire have a *set* of worlds, $a, b, c \in w, v, \cdots \subseteq W$
- Can be empty.
- Some rewriting are not sound without the worlds systems.
- Similar rules for the mirrored versions.

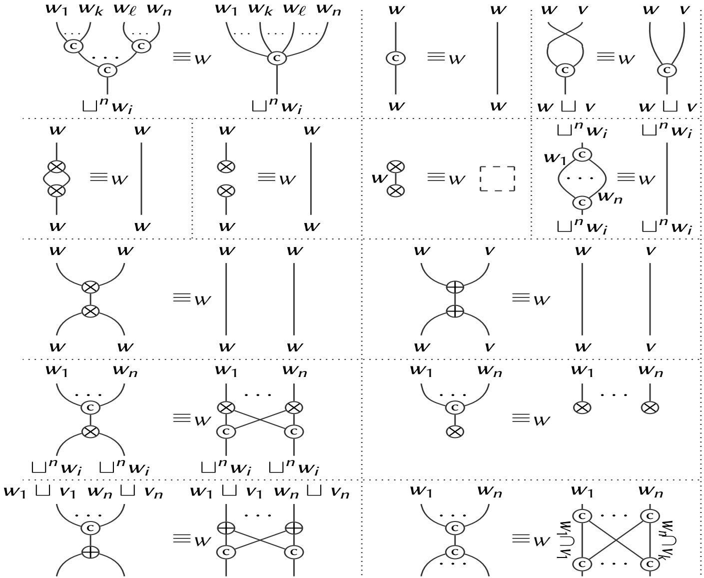


Example : CNOT (I hope no one is color blind)

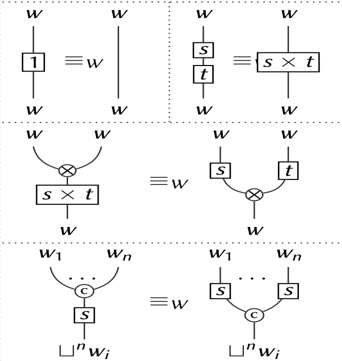


- Two sets of equations:
  - $\equiv_W$: for a fixed world set W.
  - $\equiv$: equations that can **change** the world set W.

- Two versions of $\equiv_W$
  - One with generalized contraction.
  - One where only unary / binary contraction.

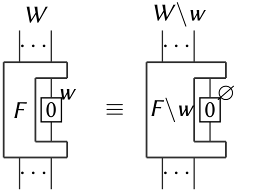


Worlds annotations on wires are ommited when uniquely determined. We assume that:
$w \cap v = \varnothing$
$w_i \cap w_j = \varnothing$ whenever $i \neq j$
$v_i \cap v_j = \varnothing$ whenever $i \neq j$

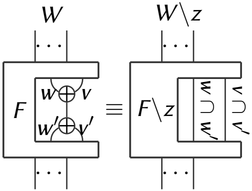


Mirrored up-down versions of those equations can be deduced from the compact closure.
Additional equations for the $\oplus$ and the scalars are provided in the

Worlds sets



Worlds sets

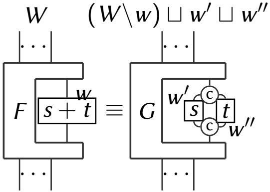


Where:

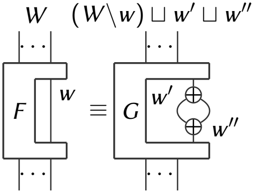
$$z = \begin{array}{l}(w \backslash w') \cup (w' \backslash w) \\ \cup (v \backslash v') \cup (v' \backslash v)\end{array}$$

$F \backslash w$ (resp. $F \backslash z$) is F where every world of w (resp. z) has been removed from the labels.

Worlds sets



Worlds sets



Where:

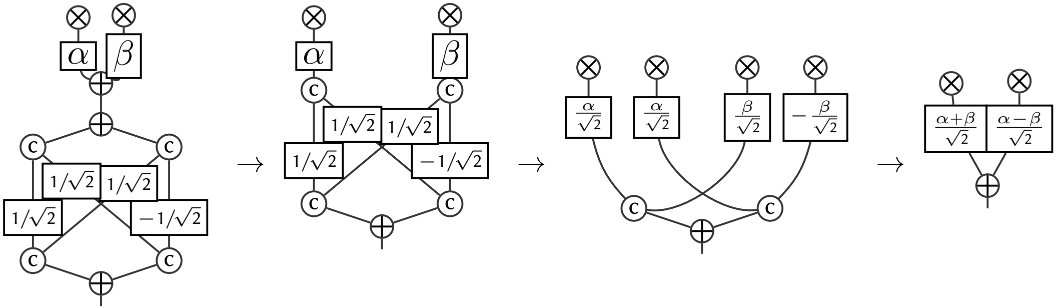
$w = \{a_1, \dots, a_n\}$

$w' = \{a'_1, \dots, a'_n\}$

$w'' = \{a''_1, \dots, a''_n\}$

G is F where every instance of the world a_i has been replaced by both a'_i and a''_i.

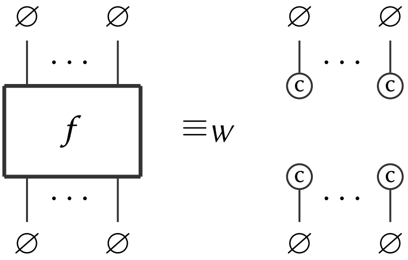
Example



α
β
c
$1/\sqrt{2}$
$-1/\sqrt{2}$
$\rightarrow$
$\frac{\alpha}{\sqrt{2}}$
$\frac{\beta}{\sqrt{2}}$
$-\frac{\beta}{\sqrt{2}}$
$\frac{\alpha+\beta}{\sqrt{2}}$
$\frac{\alpha-\beta}{\sqrt{2}}$

Example

Any diagram f where every world label is $\varnothing$:

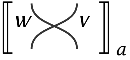


Two semantics:

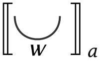
- $[\![\ ]\!]_a$: Where only wires with a a world on it, rest of disabled.
- $[\![\ ]\!]$: Semantics that consider all the possible enabling / disabling of worlds.

With :

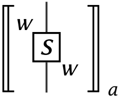
- $[\![g \circ f]\!]_a := [\![g]\!]_a \circ [\![f]\!]_a$.
- $[\![f \square g]\!]_a := [\![f]\!]_a \otimes [\![g]\!]_a$.



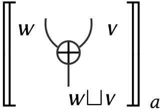
$$= \begin{cases} \mathrm{Id} & \in \mathbf{FdM}_R(\mathcal{M}_A, \mathcal{M}_A) & \text{if } a \in w \backslash v \\ \mathrm{Id} & \in \mathbf{FdM}_R(\mathcal{M}_B, \mathcal{M}_B) & \text{if } a \in v \backslash w \\ h \otimes h' \mapsto h' \otimes h & \in \mathbf{FdM}_R(\mathcal{M}_{A\otimes B}, \mathcal{M}_{B\otimes A}) & \text{if } a \in w \cap v \\ (1) & \in \mathbf{FdM}_R(R, R) & \text{otherwise} \end{cases}$$



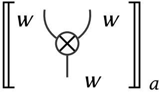
$$= \begin{cases} h \otimes h' \mapsto \langle h | h' \rangle & \in \mathbf{FdM}_R(\mathcal{M}_{A\otimes A}, R) & \text{if } a \in w \\ (1) & \in \mathbf{FdM}_R(R, R) & \text{otherwise} \end{cases}$$



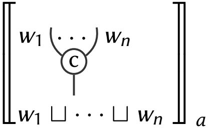
$$= \begin{cases} s \cdot \mathrm{Id} & \in \mathbf{FdM}_R(\mathcal{M}_A, \mathcal{M}_A) & \text{if } a \in w \\ (1) & \in \mathbf{FdM}_R(R, R) & \text{otherwise} \end{cases}$$



$$= \begin{cases} \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix} \in \mathbf{FdM}_R(\mathcal{M}_{A\oplus B}, \mathcal{M}_A) & \text{if } a \in w \\ \begin{pmatrix} 0 \\ \mathrm{Id} \end{pmatrix} \in \mathbf{FdM}_R(\mathcal{M}_{A\oplus B}, \mathcal{M}_B) & \text{if } a \in v \\ (1) \in \mathbf{FdM}_R(R, R) & \text{otherwise} \end{cases}$$

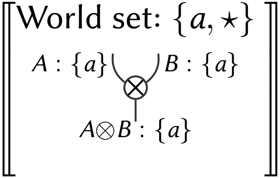


$$= \begin{cases} (\mathrm{Id}) \in \mathbf{FdM}_R(\mathcal{M}_{A\otimes B}, \mathcal{M}_{A\otimes B}) & \text{if } a \in w \\ (1) \in \mathbf{FdM}_R(R, R) & \text{otherwise} \end{cases}$$

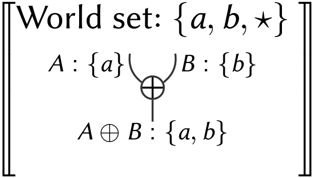


$$= \begin{cases} (\mathrm{Id}) \in \mathbf{FdM}_R(\mathcal{M}_A, \mathcal{M}_A) & \text{if } a \in w_i \\ (1) \in \mathbf{FdM}_R(R, R) & \text{otherwise} \end{cases}$$

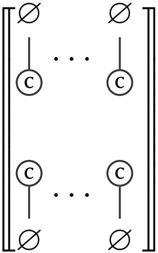
$$[\![[f]w]\!] := \left\{ \sum_{a \in W} [\![f]\!]_a \right\}$$



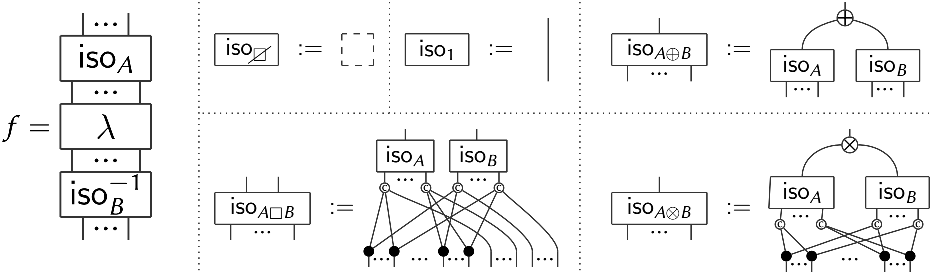
$$= \begin{array}{c} \\ A \otimes B \\ \bullet \end{array} \begin{array}{c} \begin{array}{cccc} A \square B & A \square \bullet & \bullet \square B & \bullet \square \bullet \end{array} \\ \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$



$$= \begin{array}{c} \\ A \oplus B \\ \bullet \end{array} \begin{array}{c} \begin{array}{cccc} A \square B & A \square \bullet & \bullet \square B & \bullet \square \bullet \end{array} \\ \begin{pmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & \begin{smallmatrix} \mathrm{Id} \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ \mathrm{Id} \end{smallmatrix} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$



$$= \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$



- **Theorem** : Any diagram can be put into a normal form.
- **Theorem** : The normal form is unique.

Universality

Any linear operator can be represented as a diagram with a world set W.

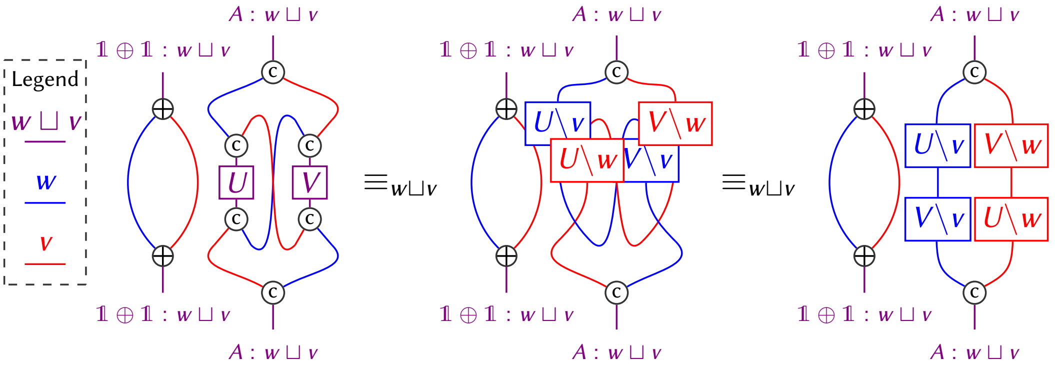
Soudness

If $D \equiv D'$ then $[\![D]\!] = [\![D']\!]$

Completeness

$D \equiv D'$ iff $[\![D]\!] = [\![D']\!]$

Representing Quantum Control: The Quantum Switch



The Many-Worlds as a model of Quantum programming

$$
\begin{array}{ll}
\text{(Base types)} & A, B ::= \mathbb{1} \mid A \oplus B \mid A \otimes B \\
\text{(Isos, first-order)} & T ::= A \leftrightarrow B
\end{array}
$$

$$
\begin{array}{ll}
\text{(Values)} & v ::= x \mid () \mid \mathtt{inj}_l\, v \mid \mathtt{inj}_r\, v \mid \langle v, v' \rangle \\
\text{(Expressions)} & e ::= v \mid e + e' \mid \alpha\ e \mid \mathtt{let}\, x = \omega\ v\ \mathtt{in}\ e
\end{array}
$$

The Many-Worlds as a model of Quantum programming

$$
\begin{array}{lrl}
\text{(Base types)} & A, B ::= & \mathbb{1} \mid A \oplus B \mid A \otimes B \\
\text{(Isos, first-order)} & T ::= & A \leftrightarrow B
\end{array}
$$

$$
\begin{array}{lrl}
\text{(Values)} & v ::= & x \mid () \mid \mathtt{inj}_l\, v \mid \mathtt{inj}_r\, v \mid \langle v, v' \rangle \\
\text{(Expressions)} & e ::= & v \mid e + e' \mid \alpha\ e \mid \mathtt{let}\, x = \omega\ v\ \mathtt{in}\ e \\
\text{(Isos)} & \omega ::= & \{ v_1 \leftrightarrow e_1 \dots v_n \leftrightarrow e_n \}
\end{array}
$$

$$
\alpha \left|0\right\rangle + \beta \left|1\right\rangle := \alpha \mathtt{inj}_l\, () + \beta \mathtt{inj}_r\, () : \mathbb{1} \oplus \mathbb{1}
$$

$$
H := \left\{ \begin{array}{lcl} \mathtt{inj}_l\, () & \leftrightarrow & \frac{1}{\sqrt{2}} \mathtt{inj}_l\, () + \frac{1}{\sqrt{2}} \mathtt{inj}_r\, () \\ \mathtt{inj}_r\, () & \leftrightarrow & \frac{1}{\sqrt{2}} \mathtt{inj}_l\, () + -\frac{1}{\sqrt{2}} \mathtt{inj}_r\, () \end{array} \right\} : \mathbb{1} \oplus \mathbb{1} \leftrightarrow \mathbb{1} \oplus \mathbb{1}
$$

- Usual one for values:

$$\frac{\Delta \vdash v : A}{\Delta \vdash \mathtt{inj}_l\ v : A \oplus B} \qquad \frac{\Delta_1 \vdash v_1 : A \qquad \Delta_2 \vdash v_2 : B}{\Delta_1, \Delta_2 \vdash \langle v_1, v_2 \rangle : A \otimes B}$$

- Less usual one for linear combination:

$$\frac{\Delta \vdash e : A}{\Delta \vdash \alpha\ e : A} \qquad \frac{\Delta \vdash e_1 \qquad \Delta \vdash e_2}{\Delta \vdash e_1 + e_2 : A}$$

- Weird one for isos:

$$\frac{\begin{array}{ccc} \Delta_1 \vdash v_1 : A & \dots & \Delta_n \vdash v_n : A \\ \Delta_1 \vdash e_1 : B & \dots & \Delta_n \vdash e_n : B \end{array} \qquad \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \text{ is unitary}}{\vdash \left\{ \begin{array}{ccc} v_1 & \leftrightarrow & \alpha_{11} \cdot e_1 + \cdots + \alpha_{1n} \cdot e_n \\ & \dots & \\ v_n & \leftrightarrow & \alpha_{n1} \cdot e_1 + \cdots + \alpha_{nn} \cdot e_n \end{array} \right\} : A \leftrightarrow B.}$$

- Rewriting system based on **pattern-matching**.
- Clauses **has** to be **exhaustive** and **non-overlapping**.
- Only represent **unitary** operations.
- If limited to type $\bigotimes^n \mathbb{1} \oplus \mathbb{1}$
  - Can encode any quantum circuit.
  - Can be represented into the ZX-Calculus.
- Can easily represent quantum control:

Let $\mathtt{tt} = \mathtt{inj}_l\,(), \mathtt{ff} = \mathtt{inj}_r\,()$ then:

$$\text{Gate} ::= \left\{ \begin{array}{l} \langle \mathtt{tt}, x\rangle \leftrightarrow \mathtt{let}\; y = \mathsf{H}\; x \;\mathtt{in}\; \frac{1}{\sqrt{2}}\langle \mathtt{tt}, y\rangle + \frac{1}{\sqrt{2}}\langle \mathtt{ff}, y\rangle \\ \\ \langle \mathtt{ff}, x\rangle \leftrightarrow \mathtt{let}\; y = \mathsf{Id}\; x \;\mathtt{in}\; \frac{1}{\sqrt{2}}\langle \mathtt{tt}, y\rangle - \frac{1}{\sqrt{2}}\langle \mathtt{ff}, y\rangle \end{array} \right\}$$

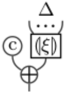
- Expression $A_1, \dots, A_n \vdash e : A$ diagram with input $A_1, \dots, A_n$ and output B.
- Iso $\{v_1 \;\leftrightarrow\; e_1 \dots v_n \;\leftrightarrow\; e_n\} : A \leftrightarrow B$ diagram with one input and one output of type A and B.

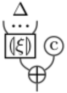
Encoding into the Many-Worlds (2/4)

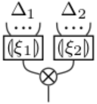
$$\left(\!\!\left(\overline{x : A \vdash_e x : A} \right)\!\!\right) = \big|^{A}$$

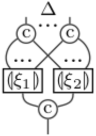
$$\left(\!\!\left(\overline{\vdash_e \langle\rangle : \mathbb{1}} \right)\!\!\right) =$$ 

$$\left(\!\!\left(\dfrac{\dfrac{\xi}{\Delta \vdash_e t : A}}{\Delta \vdash_e \alpha t : A} \right)\!\!\right) =$$ 

$$\left(\!\!\left(\dfrac{\dfrac{\xi}{\Delta \vdash_e t : A}}{\Delta \vdash_e \mathtt{inj}_r\, t : A \oplus B} \right)\!\!\right) =$$ 

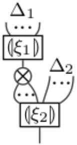
$$\left(\!\!\left(\dfrac{\dfrac{\xi}{\Delta \vdash_e t : B}}{\Delta \vdash_e \mathtt{inj}_l\, t : A \oplus B} \right)\!\!\right) =$$ 

$$\left(\!\!\left(\dfrac{\dfrac{\xi_1}{\Delta_1 \vdash_e t_1 : A} \qquad \dfrac{\xi_2}{\Delta_2 \vdash_e t_2 : B}}{\Delta_1, \Delta_2 \vdash_e \langle t_1, t_2 \rangle : A \otimes B} \right)\!\!\right) =$$ 

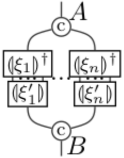
$$\left(\!\!\left(\dfrac{\dfrac{\xi_1}{\Delta \vdash_e t_1 : A} \qquad \dfrac{\xi_2}{\Delta \vdash_e t_2 : A}}{\Delta \vdash_e t_1 + t_2 : A} \right)\!\!\right) =$$ 

Encoding into the Many-Worlds (3/4)

$$\left(\!\!\left(\dfrac{\dfrac{\xi_1}{\Delta_1 \vdash_e t_1 : A_1 \otimes \cdots \otimes A_n} \qquad \dfrac{\xi_2}{x_1 : A_1, \ldots, x_n : A_n, \Delta_2 \vdash_e t_2 : B}}{\Delta_1, \Delta_2 \vdash_e \texttt{let}\ \langle x_1, \ldots, x_n \rangle = t_1\ \texttt{in}\ t_2 : B} \right)\!\!\right) =$$

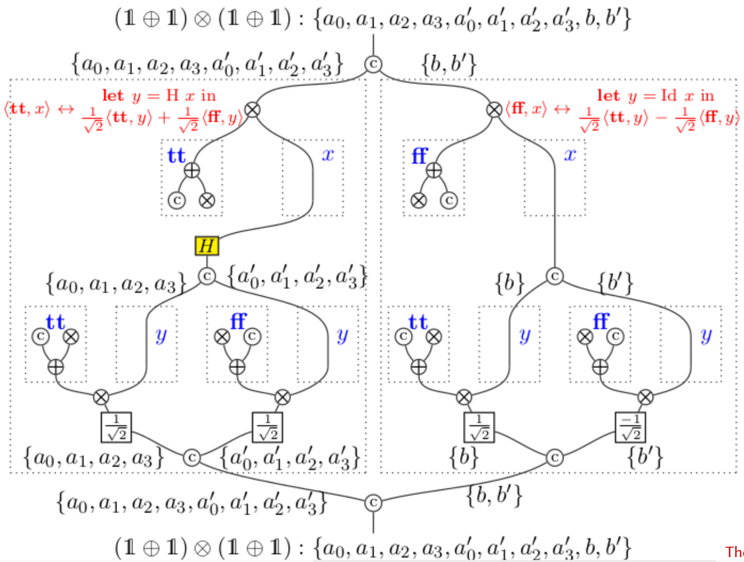


$$\left(\!\!\left(\dfrac{\dfrac{\xi_i}{\Delta_i \vdash_e v_i : A} \qquad \dfrac{\xi_i'}{\Delta_i \vdash_e e_i : B}}{\vdash_e \{ v_1 \leftrightarrow e_1 \mid \cdots \mid v_n \leftrightarrow e_n \} : A \leftrightarrow B} \right)\!\!\right) =$$



$$\left(\!\!\left(\dfrac{\dfrac{\xi_1}{\vdash_\omega \omega : A \leftrightarrow B} \qquad \dfrac{\xi_2}{\Delta \vdash_e t : A}}{\Delta \vdash_e \omega\ t : B} \right)\!\!\right) =$$

Encoding into the Many-Worlds (4/4)



Well-defined

Given a well-typed term, the translation is well-defined.

Soudness

If $t \to^* t'$ then $(\!|t|\!) = (\!|t'|\!)$

Conclusion

- Graphical language based on Colored-Prop with both a $\otimes$ and $\oplus$.
- Can represent computation with algebraic effect (non-deterministic, probabilistic, quantum, $\ldots$)
- Typed language.
- Denotational & Operational (GoI) Semantics.
- Used as a model for a programming language.

Conclusion

- Graphical language based on Colored-Prop with both a $\otimes$ and $\oplus$.
- Can represent computation with algebraic effect (non-deterministic, probabilistic, quantum, ...)
- Typed language.
- Denotational & Operational (GoI) Semantics.
- Used as a model for a programming language.

Future Work

- Extension to mixed-processes & Induction types and recursion.
- Closer relation to Linear Logic (Nouvelle syntaxe, MALL / μMALL Proof Nets, ...).

Paper available with (almost) everything: Arxiv 2206.10234