Proofs and Refutations for Intuitionistic and Second-Order Logic

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Outline

PRK: typing

PRK: reduction rules

Properties

Conclusion

In previous work.

Pure propositions $A ::= \alpha \mid A \land A \mid A \lor A \mid \neg A$

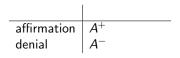
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 In this work.

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Most rules are derived mechanically from standard typing rules.

Example: rules for conjunction

Standard rule

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash s : B}{\Gamma \vdash \langle t, s \rangle : (A \land B)} I \land$$

$$\frac{\Gamma \vdash t : (A_1 \land A_2)}{\Gamma \vdash \pi_i(t) : A_i} E \land_i$$

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- Introduction rules have weak premises and strong conclusions.
- ▶ Elimination rules have strong premises and weak conclusions.

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Standard rule + weak/strong distinction

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : B^{\oplus}}{\Gamma \vdash \langle t, s \rangle^{+} : (A \land B)^{+}} I \land^{+}}$$

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- Most rules are derived mechanically from standard typing rules.
- Introduction rules have weak premises and strong conclusions.
- ▶ Elimination rules have strong premises and weak conclusions.
- ▶ Dual pairs of connectives are (\land/\lor) , (\to/\lor) , $(\forall/∃)$.

Example: rules for conjunction

Standard rule + weak/strong distinction + affirmation/denial distinction.

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : B^{\oplus}}{\Gamma \vdash \langle t, s \rangle^{+} : (A \land B)^{+}} I \wedge^{+} \qquad \frac{\Gamma \vdash t : A^{\ominus} \qquad \Gamma \vdash s : B^{\ominus}}{\Gamma \vdash \langle t, s \rangle^{-} : (A \land B)^{-}} I_{\vee}^{-}$$

$$\frac{\Gamma \vdash t : (A_{1} \land A_{2})^{+}}{\Gamma \vdash \pi_{i}^{+}(t) : A_{i}^{\oplus}} E \wedge_{i}^{+} \qquad \frac{\Gamma \vdash t : (A_{1} \lor A_{2})^{-}}{\Gamma \vdash \pi_{i}^{-}(t) : A_{i}^{\ominus}} E_{\vee i}^{-}$$

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Weak introduction and elimination

Typing rules for weak propositions are based on the following informal equations.

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Weak introduction and elimination

Typing rules for weak propositions are based on the following informal equations. Note that these equations are mutually recursive.

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Reduction rules

Simplification of intro./elim. pairs (Natural deduction-like)

In previous work:

$$(\bigcirc_{x}^{\pm}. t) \bullet^{\pm} s \xrightarrow{\beta_{\uparrow}^{+}/\beta_{\downarrow}^{-}} t\{x := s\}$$

$$\pi_{i}^{\pm}(\langle t_{1}, t_{2}\rangle^{\pm}) \xrightarrow{\beta_{\uparrow}^{+}/\beta_{\downarrow}^{-}} t_{i}$$

$$\delta^{\pm}(\mathsf{in}_{i}^{\pm}(t)) [_{x}.s_{1}][_{x}.s_{2}] \xrightarrow{\beta_{\downarrow}^{+}/\beta_{\downarrow}^{-}} s_{i}\{x := t\}$$

$$\mathsf{M}^{\pm}(\mathsf{N}^{\pm}t) \xrightarrow{\beta_{\downarrow}^{+}/\beta_{\downarrow}^{-}} t$$

New rules:

$$(\lambda_{x}^{\pm}.t) @^{\pm}s \xrightarrow{\beta_{\rightarrow}^{+}/\beta_{\rightarrow}^{-}} t\{x := s\}$$

$$\varrho^{\pm}(t; ^{\pm}s)[_{x;y}.u] \xrightarrow{\beta_{x}^{+}/\beta_{\rightarrow}^{-}} u\{x := t\}\{y := s\}$$

$$(\lambda_{\alpha}^{\pm}.t) @^{\pm}A \xrightarrow{\beta_{y}^{+}/\beta_{\exists}^{-}} t\{\alpha := A\}$$

$$\nabla^{\pm}\langle A, t\rangle^{\pm}[_{(\alpha,x)}.s] \xrightarrow{\beta_{\exists}^{+}/\beta_{\rightarrow}^{-}} s\{\alpha := A\}\{x := t\}$$

Reduction rules

Simplification of intro./intro. cuts

(Sequent calculus-like)

In previous work:

$$\langle t_1, t_2 \rangle^+ \bowtie \operatorname{in}_i^-(s) \xrightarrow{\bowtie_{\wedge}} (t_i \bullet^+ s) \bowtie (s \bullet^- t_i)$$

$$\operatorname{in}_i^+(t) \bowtie \langle s_1, s_2 \rangle^- \xrightarrow{\bowtie_{\vee}} (t \bullet^+ s_i) \bowtie (s_i \bullet^- t)$$

$$(\mathsf{N}^+ t) \bowtie (\mathsf{N}^- s) \xrightarrow{\bowtie_{\neg}} (s \bullet^+ t) \bowtie (t \bullet^- s)$$

New rules:

$$\lambda_{x}^{+}.t \bowtie (s; \overline{} u) \xrightarrow{\bowtie_{\rightarrow}} (t\{x := s\} \bullet^{+} u) \bowtie (u \bullet^{-} t\{x := s\})$$

$$(t; \overline{} s) \bowtie \lambda_{x}^{-}.u \xrightarrow{\bowtie_{\kappa}} (s \bullet^{+} u\{x := t\}) \bowtie (u\{x := t\} \bullet^{-} s)$$

$$(\lambda_{\alpha}^{+}.t) \bowtie \langle A, s \rangle^{-} \xrightarrow{\bowtie_{\rightarrow}} (t\{\alpha := A\} \bullet^{+} s) \bowtie (s \bullet^{-} t\{\alpha := A\})$$

$$\langle A, t \rangle^{+} \bowtie (\lambda_{\alpha}^{-}.s) \xrightarrow{\bowtie_{\exists}} (t \bullet^{+} s\{\alpha := A\}) \bowtie (s\{\alpha := A\} \bullet^{-} t)$$

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Theorem (Classical refinement)

The following are equivalent:

- $ightharpoonup A_1, \ldots, A_n \vdash B$ holds in classical second-order logic.
- ▶ There is a witness of $A_1^{\oplus}, \dots, A_n^{\oplus} \vdash B^{\oplus}$ in PRK.

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Theorem (Symmetry)

- 1. $\Gamma \vdash t : P$ if and only if $\Gamma^{\perp} \vdash t^{\perp} : P^{\perp}$
- 2. $t \rightarrow s$ if and only if $t^{\perp} \rightarrow s^{\perp}$

where $-^{\perp}$ flips all the signs and exchanges dual connectives.

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Theorem (Convergence)

PRK enjoys subject reduction, confluence, and strong normalization.

This provides a computational interpretation for classical logic.

Strong normalization

In previous work

We proved SN of the $\{\land,\lor,\lnot\}$ fragment by translating PRK to Mendler's extension of System F with non-strictly positive recursion.

The proof does not extend to second-order PRK.

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The proof does not extend to second-order PRK.

In this work

We prove SN of the $\{\land,\lor,\neg,\rightarrow,\ltimes,\forall,\exists\}$ fragment by constructing a reducibility model, adapting Mendler's proof.

The interesting part is the interpretation of the mutually recursive equations:

$$A^{\oplus} \simeq (A^{\ominus} \to A^+) \qquad A^{\ominus} \simeq (A^{\oplus} \to A^-)$$

respectively as a fixpoint/co-fixpoint in a complete lattice of reducibility candidates.

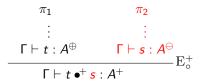
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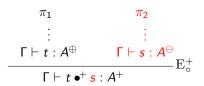
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- 1. Negative eliminations of \land , \rightarrow , \forall , and \neg are useless.
- 2. In any subterm \bigcirc_x^+ . t, free occurrences of x in t are useless.

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Theorem (Intuitionistic refinement)

The following are equivalent:

- $ightharpoonup A_1, \ldots, A_n \vdash B$ holds in intuitionistic second-order logic.
- ▶ There is an intuitionistic witness of $A_1^{\oplus}, \dots, A_n^{\oplus} \vdash B^{\oplus}$ in PRK.

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- ▶ We have extended system PRK to second-order logic.
- ▶ The good logical and computational properties remain.
- ► To prove SN we adapted Mendler's reducibility model.
- ▶ We identified an intuitionistic subset of PRK.
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Future work

- Study decidability results for fragments of PRK.
- Can PRK be related with existing classical calculi?
 (Parigot, Barbanera-Berardi, Curien-Herbelin, ...)
- Can PRK be understood through translations to linear logic?
- Can these ideas be extended to a dependently typed setting?