A Constructive Logic with Classical Proofs and Refutations

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Introduction

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Propositions as Types: the λ^{PRK}-calculus

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The BHK Interpretation ($\{\wedge, \vee, \neg\}$ fragment)

$$\begin{aligned}(A \wedge B)^+ &\simeq A^+ \times B^+ \\ (A \vee B)^+ &\simeq A^+ \uplus B^+ \\ (\neg A)^+ &\simeq A^+ \Rightarrow \mathbf{0}\end{aligned}$$

$A^+ =$ "proofs of A"

Nelson's Strong Negation

(Nelson, 1949)

$$(A \wedge B)^+ \simeq A^+ \times B^+ \qquad (A \wedge B)^- \simeq A^- \uplus B^-$$

$$(A \vee B)^+ \simeq A^+ \uplus B^+ \qquad (A \vee B)^- \simeq A^- \times B^-$$

$$(\neg A)^+ \simeq A^- \qquad (\neg A)^- \simeq A^+$$

$A^+ =$ "proofs of A"

$A^- =$ "refutations of A"

Starting point: a BHK interpretation for classical logic

Can we recover classical logic by extending Nelson's system as follows?

$$A^{\oplus} \overset{?}{\simeq} A^{-} \Rightarrow A^{+} \qquad A^{\ominus} \overset{?}{\simeq} A^{+} \Rightarrow A^{-}$$

A^{+} = "strong proofs of A" A^{-} = "strong refutations of A"

$A^{\oplus}$ = "classical proofs of A" $A^{\ominus}$ = "classical refutations of A"

- From the strictly logical point of view, this gives us classical logic.
- From the point of view of proof normalization, it is not clear how to normalize a cut $A^{\oplus}$ / $A^{\ominus}$.

Our approach is based on the (mutually recursive!) equations:

$$A^{\oplus} \simeq A^{\ominus} \Rightarrow A^{+} \qquad A^{\ominus} \simeq A^{\oplus} \Rightarrow A^{-}$$

Starting point: a BHK interpretation for classical logic

$$
\begin{aligned}
(A \land B)^+ &\simeq A^\oplus \times B^\oplus & \qquad (A \land B)^- &\simeq A^\ominus \uplus B^\ominus \\
(A \lor B)^+ &\simeq A^\oplus \uplus B^\oplus & \qquad (A \lor B)^- &\simeq A^\ominus \times B^\ominus \\
(\neg A)^+ &\simeq A^\ominus & \qquad (\neg A)^- &\simeq A^\oplus \\
A^\oplus &\simeq A^\ominus \Rightarrow A^+ & \qquad A^\ominus &\simeq A^\oplus \Rightarrow A^-
\end{aligned}
$$

A^+ = "strong proofs of A"

$A^\oplus$ = "classical proofs of A"

A^- = "strong refutations of A"

$A^\ominus$ = "classical refutations of A"

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Natural DeductionNelson's strong negationSystem PRK

Pure propositions	A	$::=$	$\alpha \mid A \wedge A \mid A \vee A \mid \neg A$	
Propositions	P	$::=$	A^+	strong affirmation
		$\mid$	$A^- A^-$	strong denialstrong denia
		$\mid$	$A^\oplus A^\oplus$	classical affimationclassic
		$\mid$	$A^\ominus A^\ominus$	classical denialclassical d

Example rules

$$\frac{\Gamma \vdash A^+ \qquad \Gamma \vdash B^+}{\Gamma \vdash (A \wedge B)^+}\,\mathrm{I}\wedge^+ \quad \frac{\Gamma \vdash A^\oplus \qquad \Gamma \vdash B^\oplus}{\Gamma \vdash (A \wedge B)^+}\,\mathrm{I}\wedge^+ \quad \frac{\Gamma \vdash A^\oplus \qquad \Gamma \vdash B^\oplus}{\Gamma \vdash (A \wedge B)^+}\,\mathrm{I}\wedge^+$$

$$\frac{\Gamma \vdash (A_1 \wedge A_2)^+}{\Gamma \vdash {A_i}^+}\,\mathrm{E}\wedge_i^+ \quad \frac{\Gamma \vdash (A_1 \wedge A_2)^+}{\Gamma \vdash {A_i}^\oplus}\,\mathrm{E}\wedge_i^+ \quad \frac{\Gamma \vdash (A_1 \wedge A_2)^+}{\Gamma \vdash {A_i}^\oplus}\,\mathrm{E}\wedge_i^+$$

A strong affirmation A^+ is canonically proved with an introduction rule

System PRK – Noteworthy rules

Absurdity

$$\frac{\Gamma \vdash A^{+} \qquad \Gamma \vdash A^{-}}{\Gamma \vdash P}\ \textsc{Abs}$$

Negation

$$\frac{\Gamma \vdash A^{\ominus}}{\Gamma \vdash (\neg A)^{+}}\ \mathrm{I}\neg^{+} \qquad \frac{\Gamma \vdash A^{\oplus}}{\Gamma \vdash (\neg A)^{-}}\ \mathrm{I}\neg^{-}$$

$$\frac{\Gamma \vdash (\neg A)^{+}}{\Gamma \vdash A^{\ominus}}\ \mathrm{E}\neg^{+} \qquad \frac{\Gamma \vdash (\neg A)^{-}}{\Gamma \vdash A^{\oplus}}\ \mathrm{E}\neg^{-}$$

Classical formulas

$$\frac{\Gamma, A^{\ominus} \vdash A^{+}}{\Gamma \vdash A^{\oplus}}\ \mathrm{I}\bigcirc^{+} \qquad \frac{\Gamma, A^{\ominus} \vdash A^{+}}{\Gamma \vdash A^{\oplus}}\ \mathrm{I}\bigcirc^{+} \qquad \frac{\Gamma, A^{\oplus} \vdash A^{-}}{\Gamma \vdash A^{\ominus}}\ \mathrm{I}\bigcirc^{-}$$

$$\frac{\Gamma \vdash A^{\oplus} \qquad \Gamma \vdash A^{\ominus}}{\Gamma \vdash A^{+}}\ \mathrm{E}\bigcirc^{+} \qquad \frac{\Gamma \vdash A^{\ominus} \qquad \Gamma \vdash A^{\oplus}}{\Gamma \vdash A^{-}}\ \mathrm{E}\bigcirc^{-}$$

A classical affirmation $A^{\oplus}$ is canonically proved by assuming $A^{\ominus}$ and proving A^{+}.

System PRK – Admissible rules

Weakening

$$\frac{\Gamma \vdash P}{\Gamma, Q \vdash P}$$

Cut

$$\frac{\Gamma, P \vdash Q \qquad \Gamma \vdash P}{\Gamma \vdash Q}$$

Substitution

$$\frac{\Gamma \vdash Q}{\Gamma[\alpha := A] \vdash Q[\alpha := A]}$$

General absurdity

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash P^{\sim}}{\Gamma \vdash Q}$$

Contraposition

$$\frac{\Gamma, P \vdash Q \qquad P \text{ classical}}{\Gamma, Q^{\sim} \vdash P^{\sim}}$$

Strengthening

$$\frac{\Gamma, P^{\sim} \vdash P \qquad P \text{ classical}}{\Gamma \vdash P}$$

Where:

$$(A^{+})^{\sim} \stackrel{\text{def}}{=} A^{-} \qquad (A^{-})^{\sim} \stackrel{\text{def}}{=} A^{+} \qquad (A^{\oplus})^{\sim} \stackrel{\text{def}}{=} A^{\ominus} \qquad (A^{\ominus})^{\sim} \stackrel{\text{def}}{=} A^{\oplus}$$

System PRK – Properties

Theorem (Embedding + conservative extension)

$\vdash A$ holds classically   if and only if $\vdash A^{\oplus}$ holds in PRK

Strong propositions behave constructively

The classical excluded middle $\vdash (A \vee \neg A)^{\oplus}$ always holds.

The strong excluded middle $\vdash (A \vee \neg A)^{+}$ does not hold in general.

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The calculus λ^{PRK}

We assign explicit witnesses to proofs:

$$
\begin{array}{rcll}
t, s, u, \ldots & ::= & x & \text{variable} \\
& \mid & t \bowtie_P s & \text{absurdity} \\
& \mid & \langle t, s\rangle^{\pm} & \wedge^{+} \;/\; \vee^{-} \text{ introduction} \\
& \mid & \pi_i^{\pm}(t) & \wedge^{+} \;/\; \vee^{-} \text{ elimination} \\
& \mid & \mathrm{in}_i^{\pm}(t) & \vee^{+} \;/\; \wedge^{-} \text{ introduction} \\
& \mid & \mathrm{case}^{\pm}\, t\, [_{x:P}.s][_{y:Q}.u] & \vee^{+} \;/\; \wedge^{-} \text{ elimination} \\
& \mid & \nu^{\pm} t & \neg^{+} \;/\; \neg^{-} \text{ introduction} \\
& \mid & \mu^{\pm} t & \neg^{+} \;/\; \neg^{-} \text{ elimination} \\
& \mid & \bigcirc^{\pm}_{(x:P)}.\, t & \text{classical introduction} \\
& \mid & t \bullet^{\pm} s & \text{classical elimination}
\end{array}
$$

The calculus λ^{PRK}

Type system (excerpt)

$$\frac{\Gamma \vdash t : A^{+} \qquad \Gamma \vdash s : A^{-}}{\Gamma \vdash t \bowtie_{P} s : P}\ \textsc{Abs} \qquad \dots \qquad \frac{\Gamma \vdash t : A^{\ominus} \qquad \Gamma \vdash s : B^{\ominus}}{\Gamma \vdash \langle t, s\rangle^{-} : (A \vee B)^{-}}\ \mathrm{I}\vee^{-}$$

$$\frac{\Gamma, x : A^{\ominus} \vdash t : A^{+}}{\Gamma \vdash \bigcirc^{+}_{(x:A^{\ominus})}.\, t : A^{\oplus}}\ \mathrm{I}\bigcirc^{+} \qquad \dots \qquad \frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : A^{\ominus}}{\Gamma \vdash t \bullet^{+} s : A^{+}}\ \mathrm{E}\bigcirc^{+}$$

The calculus λ^{PRK}

Reduction rules

$$
\begin{array}{rcl}
\pi_i^{\pm}(\langle t_1, t_2\rangle^{\pm}) & \xrightarrow{\beta^+_\wedge \,/\, \beta^-_\vee} & t_i \\
\mathsf{case}^{\pm}(\mathsf{in}_i^{\pm}(t))\,[_x.s_1][_x.s_2] & \xrightarrow{\beta^+_\vee \,/\, \beta^-_\wedge} & s_i[x := t] \\
\mu^{\pm}(\nu^{\pm} t) & \xrightarrow{\beta^+_\neg \,/\, \beta^-_\neg} & t \\
(\bigcirc^{\pm}_x .\, t) \bullet^{\pm} s & \xrightarrow{\beta^+_\circ \,/\, \beta^-_\circ} & t[x := s] \\
\langle t_1, t_2\rangle^{+} \bowtie \mathsf{in}_i^{-}(s) & \xrightarrow{\bowtie_\wedge} & (t_i \bullet^{+} s) \bowtie (s \bullet^{-} t_i) \\
\mathsf{in}_i^{+}(t) \bowtie \langle s_1, s_2\rangle^{-} & \xrightarrow{\bowtie_\vee} & (t \bullet^{+} s_i) \bowtie (s_i \bullet^{-} t) \\
(\nu^{+} t) \bowtie (\nu^{-} s) & \xrightarrow{\bowtie_\neg} & (s \bullet^{+} t) \bowtie (t \bullet^{-} s) \\
 & & \\
\bigcirc^{\pm}_x .\, (t \bullet^{\pm} x) & \xrightarrow{\eta_\circ} & t \qquad \text{if } x \notin \mathsf{fv}(t)
\end{array}
$$

The calculus λ^{PRK}

Theorem (Subject Reduction)

If $\Gamma \vdash t : P$ and $t \to s$ then $\Gamma \vdash s : P$.

Theorem (Duality)

1. $\Gamma \vdash t : P$ if and only if $\Gamma^{\perp} \vdash t^{\perp} : P^{\perp}$
2. $t \to s$ if and only if $t^{\perp} \to s^{\perp}$

where $-^{\perp}$ flips all the signs and exchanges dual connectives $(\wedge, \vee)$.

Theorem (Convergence)

λ^{PRK} is confluent and strongly normalizing.

- The main difficulty in the SN proof is how to deal with the mutually recursive types $A^{\oplus} \simeq A^{\ominus} \Rightarrow A^{+}$ and $A^{\ominus} \simeq A^{\oplus} \Rightarrow A^{-}$.
- The SN proof is via a translation to System F with non-strictly positive recursive types, relying on a result by Mendler.

The calculus λ^{PRK}

There have been many computational interpretations of classical logic:

1. Parigot's $\lambda\mu$.
2. Barbanera and Berardi's symmetric λ-calculus.
3. Curien and Herbelin's $\bar{\lambda}\mu\tilde{\mu}$.
4. Krivine's λ_c.
5. ...

λ^{PRK} provides a new computational interpretation for classical logic.

The calculus λ^{PRK}

Example: conjunction

Taking:

$$\begin{aligned}
\langle t, s\rangle &\stackrel{\text{def}}{=} \bigcirc^{+}_{(_:(A\wedge B)^{\ominus})}.\,\langle t, s\rangle^{+} \\
\pi_i(t) &\stackrel{\text{def}}{=} \bigcirc^{+}_{(x:A_i{}^{\ominus})}.\,\pi_i^{+}(t \bullet^{+} \bigcirc^{-}_{(_:(A_1\wedge A_2)^{\oplus})}.\,\mathsf{in}_i^{-}(x)) \bullet^{+} x
\end{aligned}$$

Classical introduction and elimination of conjunction can be derived:

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : B^{\oplus}}{\Gamma \vdash \langle t, s\rangle : (A \wedge B)^{\oplus}} \qquad \frac{\Gamma \vdash t : (A_1 \wedge A_2)^{\oplus}}{\Gamma \vdash \pi_i(t) : A_i{}^{\oplus}}$$

The standard computation rule for projection can be recovered:

$$\pi_i(\langle t_1, t_2\rangle) \to^{*} t_i$$

The calculus λ^{PRK}

A more interesting example: implication

In classical logic, implication is derivable from negation and disjunction. This can be extended to the computational level.

Let $(A \to B) \stackrel{\text{def}}{=} (\neg A \vee B)$.
Abstraction and application can be defined with their expected types:

$$
\begin{array}{rcl}
\lambda x.\, t & \stackrel{\text{def}}{=} & \bigcirc^{+}_{(y:(A \Rightarrow B)^{\ominus})}.\, \mathsf{in}_2^{+}(t[x := \mathbf{X}_y]) \\
\mathbf{X}_y & \stackrel{\text{def}}{=} & \bigcirc^{+}_{(z:A^{\ominus})}.\, (\mu^{-}(\mathbf{X'}_{y,z} \bullet^{-} \bigcirc^{+}_{(\_:(\neg A)^{\ominus})}.\, \nu^{+} z)) \bullet^{+} z \\
\mathbf{X'}_{y,z} & \stackrel{\text{def}}{=} & \pi_1^{+}(y \bullet^{-} \bigcirc^{+}_{(\_:(A \Rightarrow B)^{\ominus})}.\, \mathsf{in}_1^{+}(\bigcirc^{+}_{(\_:(\neg A)^{\ominus})}.\, \nu^{+} z)) \\
t \,@\, s & \stackrel{\text{def}}{=} & \quad \mathsf{IC}^{+}_{(x:B^{\ominus})}. \\
& & \qquad \mathsf{case}^{+}\ (t \bullet^{+} \bigcirc^{-}_{(\_:(A \to B)^{\oplus})}.\, \langle (\bigcirc^{-}_{(\_:(\neg A)^{\oplus})}.\, \nu^{-} s), x \rangle^{-}) \\
& & \qquad\qquad [_{(y:(\neg A)^{\oplus})}.s \bowtie_{B^{+}} \mu^{-}(y \bullet^{+} \bigcirc^{-}_{(\_:(\neg A)^{\oplus})}.\, \nu^{-} x)] \\
& & \qquad\qquad [_{(z:B^{\oplus})}.z \bullet^{+} x]
\end{array}
$$

The standard β-reduction rule can be recovered:

$$(\lambda x.\, t)\,@\, s \to^{*} t[x := s]$$

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Kripke Semantics

A Kripke model for PRK is a structure $\mathcal{M} = (\mathcal{W}, \leq, \mathcal{V}^+, \mathcal{V}^-)$.

(Enjoying appropriate technical conditions).

Forcing (excerpt)

$$
\begin{array}{lcl}
\mathcal{M}, w \Vdash \alpha^+ & \iff & \alpha \in \mathcal{V}_w^+ \\
\mathcal{M}, w \Vdash \alpha^- & \iff & \alpha \in \mathcal{V}_w^- \\
\vdots & & \\
\mathcal{M}, w \Vdash (A \vee B)^- & \iff & \mathcal{M}, w \Vdash A^\ominus \text{ and } \mathcal{M}, w \Vdash B^\ominus \\
\vdots & & \\
\mathcal{M}, w \Vdash A^\oplus & \iff & \mathcal{M}, w' \nVdash A^- \text{ for all } w' \geq w \\
\vdots & &
\end{array}
$$

Theorem (Soundness and Completeness)

$\Gamma \vdash P$ if and only if $\Gamma \Vdash P$

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Further Extensions

Second Order λ^{PRK}

We have extended λ^{PRK} with implication, co-implication, and second-order quantifiers:

$$\text{Pure propositions} \quad A \quad ::= \quad \ldots \mid A \to A \mid A \ltimes A \mid \forall\alpha.\, A \mid \exists\alpha.\, A$$

- All of the previous results can be extended to this setting.
- The SN proof requires a completely different strategy, using reducibility candidates.

Intuitionistic λ^{PRK}

We have identified an intuitionistic subset of λ^{PRK}.
The key is, essentialy, to identify $A^{\oplus}$ with A^{+} rather than with $A^{\ominus} \to A^{+}$.

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Contributions

- We studied an extension of the BHK interpretation.

  Key idea: $A^{\oplus} \simeq A^{\ominus} \Rightarrow A^{+}$

  A classical proof of A is a transformation that converts classical refutations of A into strong proofs of A.

- This interpretation motivates the logical system PRK.
  PRK is a **conservative extension** of classical logic.

- **Propositions-as-types.**
  PRK corresponds to a confluent and terminating calculus λ^{PRK}.
  It has been extended to second-order logic.
  An intuitionistic fragment of λ^{PRK} has been identified.

- **Kripke semantics.**
  PRK is sound and complete w.r.t. a notion of Kripke model.

Future Work

- Relate λ^{PRK} with existing classical calculi.
- Extend λ^{PRK} with dependent types.
- In System F, $\{\exists, \wedge, \vee, \bot, \top, \neg\}$ can be derived from $\{\forall, \to\}$.
  This is not true in second-order PRK (!)
  Can we identify subsets of "computationally adequate" connectives?