Vector calculus

Point functions :

A function of the position of a point in space is called a point function.

when the position of points in space is specified by their rectangular cartesian co-ordinates x, y, z, then point fns will be fns of these independent variables x, y, z.

Examples

Most of physical quantities are pt fns/.

Temperature (s. pt. fn) at any point in space, Velocity of a fluid (v. pt. fn) in motion at any point, strength of a magnetic field at a point, are all point fns.

1) Point fns which are not associated with any direction in space are called scalar point fn.

2) Point fns which have a magnitude dependent on position of a point in space and also have specific direction associated with it are called Vector point functions.

Vector differential operator ∇.

Vector differential operator ∇ is defined as

$\nabla = \vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z$. where $\vec{i}, \vec{j}, \vec{k}$ are unit Vectors along the three rectangular axes OX, OY & OZ.

Gradient

Let $f(x,y,z)$ be a scalar point fn is continuously differentiable, then vector

$$\vec{\nabla} f = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) f$$

$$\nabla f = \vec{i}\,\partial f/\partial x + \vec{j}\,\partial f/\partial y + \vec{k}\,\partial f/\partial z .$$

is called gradient of the scalar fn f & is written as $(\text{grad}\,\phi = \nabla \phi)$

Problems :

1) If $\phi = x^2 + y - z - 1$, find grad ϕ at $(1,0,0)$

$$\text{grad}\,\phi = \nabla\phi = \left(\vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}\right)$$

$$\partial\phi/\partial x = 2x, \quad \partial\phi/\partial y = 1, \quad \partial\phi/\partial z = -1.$$

$$\text{grad}\,\phi\Big]_{(1,0,0)} = (2x)\vec{i} + \vec{j} + (-1)\vec{k}$$

$$= 2\vec{i} + \vec{j} - \vec{k}.$$

2) Find grad ϕ if $\phi = xyz$ at $(1,1,1)$

$$\text{grad}\,\phi\Big]_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}.$$

Directional Derivative

1) Directional derivative of $f = xyz$ at $(1,1,1)$ in the direction of $\vec{i} + \vec{j} + \vec{k}$.

Note : Directional Derivative of ϕ along $\vec{a}$ $\Big\}$ $= \nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$

(iii) D . D $= \nabla\phi \cdot \hat{a}$

$$= \frac{(\vec{i}+\vec{j}+\vec{k}) \cdot (\vec{i}+\vec{j}+\vec{k})}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

2) Find directional derivative $\phi = x^2yz + 4xz^2$ at $(1,1,1)$ in direction of $\vec{i}+\vec{j}-\vec{k}$.

D.D of ϕ along $\vec{a} = \frac{\nabla\phi \cdot \vec{a}}{|\vec{a}|}$.

$$\nabla\phi\Big]_{(1,1,1)} = (2xyz+4z^2)\vec{i} + (x^2z)\vec{j} + (x^2y + 8xz)\vec{k}.$$

$$= 6\vec{i}+\vec{j}+9\vec{k}.$$

$$D.D = \frac{(6\vec{i}+\vec{j}+9\vec{k})(\vec{i}+\vec{j}-\vec{k})}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{6+1-9}{\sqrt{3}} = \frac{-2}{\sqrt{3}} //$$

Unit normal Vector

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

1) Find a unit normal Vector to the surface $x^2-y^2+z=2$ at the point $(1,-1,2)$

$$\hat{n} = \frac{\nabla\phi]_{(1,-1,2)}}{|\nabla\phi|} = \frac{2\vec{i}+2\vec{j}+\vec{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2\vec{i}+2\vec{j}+\vec{k}}{\sqrt{9}}.$$

$$= \frac{2\vec{i}+2\vec{j}+\vec{k}}{3}.$$

2) Find unit normal vector to the surface $\phi = x^3 - xyz + z^3 - 1$ at the point $(1,1,1)$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} - \vec{j} + 2\vec{k}}{3}.$$

3) Surface . ϕ : $x^2y + 2xz = 4$ at $(2,-2,3)$. Find $\hat{n}$.

Angle between two surfaces ϕ_1 & ϕ_2

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

1) Find angle of intersection at the point $(2,-1,2)$ of the surfaces $x^2+y^2+z^2 = 9$ & $z = x^2+y^2-3$

Soln:

$$\phi_1 : x^2+y^2+z^2-9$$

$$\phi_2 : x^2+y^2-z-3$$

$$\nabla\phi_1\Big]_{(2,-1,2)} = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k}$$

$$= 4\vec{i} - 2\vec{j} + 4\vec{k}.$$

$$\nabla\phi_2\Big]_{(2,-1,2)} = (2x)\vec{i} + (2y)\vec{j} - (1)\vec{k}.$$

$$= 4\vec{i} - 2\vec{j} - \vec{k}.$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|} = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k})(4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{4^2+2^2+4^2}\ \sqrt{4^2+2^2+1^2}}.$$

$$= \frac{16 + \cancel{4} - \cancel{4}}{\sqrt{36}\ \sqrt{21}} = \frac{\cancel{16}\,8}{\cancel{6}_{3}\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}} \quad ; \quad \theta = \cos^{-1}\left(8/3\sqrt{21}\right)$$

Divergence and curl

Divergence of a vector fn $\vec{F}$ is defined as

$$\nabla \cdot \vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

(or) $$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

curl of $\vec{F}(x,y,z)$ is defined as

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note:

1) If $\nabla \cdot \vec{F} = 0$, then $\vec{F}$ is solenoidal
2) If $\nabla \times \vec{F} = 0$, then $\vec{F}$ is irrotational
3) curl $\vec{F}$ is a vector quantity
4) $\nabla \times \vec{F} = \left(\vec{i} \times \frac{\partial \vec{F}}{\partial x}\right) + \left(\vec{j} \times \frac{\partial \vec{F}}{\partial y}\right) + \left(\vec{k} \times \frac{\partial \vec{F}}{\partial z}\right)$

To find scalar potential ϕ

If $\nabla\phi = 2xyz\,\vec{i} + x^2 z\,\vec{j} + x^2 y\,\vec{k}$. Find Scalar potential ϕ

Gn, $\nabla\phi = 2xyz\,\vec{i} + x^2 z\,\vec{j} + x^2 y\,\vec{k}$

$$\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) = 2xyz\,\vec{i} + x^2 z\,\vec{j} + x^2 y\,\vec{k}$$

Equating, $\frac{\partial\phi}{\partial x} = 2xyz$, $\frac{\partial\phi}{\partial y} = x^2 z$, $\frac{\partial\phi}{\partial z} = x^2 y$

Integrating w.r.to x, y, z,

$$\int \partial\phi = \int 2xyz\,\partial x \Rightarrow 2(x^2/2)yz + f(y,z)$$

$$\phi = x^2yz + f(y,z)$$

$|||^{ly}$ $\phi = x^2yz + f(x,z)$

$$\phi = x^2yz + f(x,y)$$

Hence $\phi = x^2yz + c$.

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2) S.T the vector $\vec{F}$ is given by $\vec{F} = (x^2-yz)\vec{i} + (y^2-xz)\vec{j} + (z^2-xy)\vec{k}$ is irrotational. Find its scalar potential ϕ.

To prove : $\nabla \times \vec{F} = 0$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(z^2-xy) - \partial/\partial z\,(y^2-xz)\right] -$$

$$\vec{j}\left[\partial/\partial x\,(z^2-xy) - \partial/\partial z\,(x^2-yz)\right] +$$

$$\vec{k}\left[\partial/\partial x\,(y^2-xz) - \partial/\partial y\,(x^2-yz)\right]$$

$$= \vec{i}\left[\cancel{(-x)} - \cancel{(-x)}\right] - \vec{j}\left[\cancel{-y} - \cancel{(-y)}\right] + \vec{k}\left[\cancel{-z} - \cancel{(-z)}\right]$$

$$= 0.$$

$\vec{F}$ is irrotational

<u>To find ϕ :</u>

$$\vec{F} = \nabla\phi = (x^2-yz)\vec{i} + (y^2-xz)\vec{j} + (z^2-xy)\vec{k}$$

$$= \vec{i}\left(\partial\phi/\partial x\right) + \vec{j}\left(\partial\phi/\partial y\right) + \vec{k}\left(\partial\phi/\partial yz\right)$$

Equating,

$$\partial\phi/\partial x = x^2 - yz \quad , \quad \partial\phi/\partial y = y^2 - xz \quad , \quad \frac{\partial\phi}{\partial z} = z^2 - xy$$

$\int$ing, w.r.to x $\quad$ $\int$ ing w.r.to y $\quad$ $\int$ing w.r.to z.

$$\phi = \frac{x^3}{3} - xyz + c \qquad \phi = y^3/3 - xyz + c \qquad \phi = z^3/3 - xyz + c$$

$$\therefore \quad \phi = x^3/3 + y^3/3 + z^3/3 - xyz + c$$

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3) P.T the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

To prove : $\nabla \cdot \vec{F} = 0$.

$$\nabla \cdot \vec{F} = \partial \vec{F_1}/\partial x + \frac{\partial \vec{F_2}}{\partial y} + \partial\vec{F_3}/\partial z.$$

$$= \partial/\partial x(z) + \partial/\partial y(x) + \partial/\partial z(y)$$

$$= 0.$$

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4) Find the value of 'a' so that the vector $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal.

Given, $\vec{F}$ is solenoidal

$$\text{div } \vec{F} = 0. \quad (ie) \quad \nabla \cdot \vec{F} = 0.$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0.$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0.$$

$$1 + 1 + a = 0.$$

$$\boxed{a = -2}$$

Note: 1) $\nabla(\vec{F}\cdot\vec{G}) = \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) + (\vec{F}\cdot\nabla)\vec{G} + (\vec{G}\cdot\nabla)\vec{F}$.

2) $\nabla \times (\vec{F} \times \vec{G}) = \vec{F}(\nabla\cdot\vec{G}) - \vec{G}(\nabla\cdot\vec{F}) + (\vec{G}\cdot\nabla)\vec{F} - (\vec{F}\cdot\nabla)\vec{G}$.

5) S.T the vector $2xy\vec{i} + (x^2+2yz)\vec{j} + (y^2+1)\vec{k}$ is irrotational.

To prove : $\nabla \times \vec{F} = 0$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2+2yz & y^2+1 \end{vmatrix}$$

$$= \vec{i}[2y-2y] - \vec{j}[0-0] + \vec{k}[2x-2x]$$

$$= 0.$$

$\therefore$ $\nabla \times \vec{F}$ is irrotational.

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6) If $\vec{u}$ & $\vec{v}$ are irrotational vector then S.T $(\vec{u} \times \vec{v})$ is solenoidal vector.

Gn, $\vec{u}$ & $\vec{v}$ are irrotational vector.

$\nabla \times \vec{u} = 0$ & $\nabla \times \vec{v} = 0$.

To prove : $\nabla \cdot (\vec{u} \times \vec{v}) = 0$.

$$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \overset{0}{\cancel{(\nabla \times \vec{u})}} - \vec{u} \cdot \overset{0}{\cancel{(\nabla \times \vec{v})}}$$

$$= 0.$$

$\therefore$ $\vec{u} \times \vec{v}$ is solenoidal.

Conservative Vector field :

A vector field $\vec{F}$ is said to be conservative if $\exists$ a scalar potential fn $\phi \ni \vec{F} = \nabla \phi$

This scalar fn ϕ is called scalar potential of $\vec{F}$.

Note :

In conservative field $\vec{F} = \nabla\phi$.

$\nabla \times \vec{F} = \nabla \times \nabla\phi = \bar{0}$

$\Rightarrow \vec{F}$ is irrotational.

So, irrotational vector will have scalar potential.

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1) S.T $\vec{F} = (z^2+2x+3y)\vec{i} + (3x+2y+z)\vec{j} + (y+2zx)\vec{k}$ is irrotational, but not solenoidal. Find also its scalar potential.

$$\vec{F} = (z^2+2x+3y)\vec{i} + (3x+2y+z)\vec{j} + (y+2zx)\vec{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2+2x+3y & 3x+2y+z & y+2zx \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(y+2zx) - \partial/\partial z\,(3x+2y+z)\right]$$
$$- \vec{j}\left[\partial/\partial x\,(y+2zx) - \partial/\partial z\,(z^2+2x+3y)\right]$$
$$+ \vec{k}\left[\partial/\partial x\,(3x+2y+z) - \partial/\partial y\,(z^2+2x+3y)\right]$$

$$= \vec{i}\,[1-1] - \vec{j}\,[2z-2z] + \vec{k}\,[3-3]$$

$$= 0.$$

$\therefore \vec{F}$ is irrotational.

There exists a scalar fn $\phi \ni \vec{F} = \nabla\phi$.

$$(z^2+2x+3y)\vec{i} + (3x+2y+z)\vec{j} + (y+2zx)\vec{k} = \vec{i}\left(\partial\phi/\partial x\right) + \vec{j}\left(\partial\phi/\partial y\right) + \vec{k}\left(\partial\phi/\partial z\right).$$

$$\frac{\partial\phi}{\partial x} = z^2+2x+3y \quad , \quad \frac{\partial\phi}{\partial y} = 3x+2y+z \quad , \quad \frac{\partial\phi}{\partial z} = y+2zx .$$

$\int$ing w.r.to x, y, z.

$$\phi = z^2x + \cancel{2}(x^2/\cancel{2}) + 3xy + f(y,z)$$

$$\phi = 3xy + \cancel{2}(y^2/\cancel{2}) + yz + f(x,z)$$

$$\phi = yz + \cancel{2}(z^2/\cancel{2})x + f(x,y)$$

$$\Rightarrow \quad \phi = z^2x + \cancel{2z^2x} + 3xy + y^2 + zy + x^2 + c .$$

To prove : $\nabla\cdot\vec{F} = 0$.

$$\frac{\partial\vec{F_1}}{\partial x} + \frac{\partial\vec{F_2}}{\partial y} + \frac{\partial\vec{F_3}}{\partial z} = 0$$

$$= \partial/\partial x\,(z^2+2x+3y) + \partial/\partial y\,(3x+2y+z) + \partial/\partial z\,(y+2zx)$$

$$= 2 + 2 + 2x$$

$$= 4 + 2x .$$

Integration of Vector functions.

Line Integral

An integral evaluated over a curve c is called a line integral. We call, c as the path of integration.

(ie) A line integral of a vector point fn $\vec{F}(\vec{r})$ over a curve c, where $\vec{r}$ is the position vector of any point on c, defined as $\int_c \vec{F}\cdot d\vec{r}$.

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$.

and $\int_c \vec{F}\cdot d\vec{r} = \int_c (F_1\,dx + F_2\,dy + F_3\,dz)$

Problems :

1) If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, evaluate $\int_c \vec{F}\cdot d\vec{r}$, where c is the arc of the parabola, $y = 2x^2$ from the point $(0,0)$ to the point $(1,2)$

Gin $\vec{F} = 3xy\,\vec{i} - y^2\vec{j}$, $\vec{r} = x\vec{i} + y\vec{j}$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\vec{F}\cdot d\vec{r} = 3xy\,dx - y^2\,dy.$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_c 3xy\,dx - y^2\,dy$$

Given, $y = 2x^2$, $dy = 4x\,dx$

$$\vec{F}\cdot d\vec{r} = (3x)(2x^2)\,dx - 4x^4(4x)\,dx.$$

$$= 6x^3\,dx - 16x^5\,dx.$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5)\,dx.$$

$$= \left. 6\left(\frac{x^4}{4}\right) - 16\left(\frac{x^6}{6}\right)\right]_0^1$$

$$= \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

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2) If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\,\vec{k}$. Evaluate $\int_c \vec{F}\cdot d\vec{r}$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve c given by $x = t$, $y = t^2$, $z = t^3$.

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Gn, $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz\vec{k}$.

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$.

$\vec{F}\cdot d\vec{r} = (3x^2+6y)\,dx - (14yz)(dy) + 20xz^2\,dz$.

$x = t$, $y = t^2$, $z = t^3$.

$dx = dt$ $dy = 2t\,dt$, $dz = 3t^2 dt$.

$\vec{F}\cdot d\vec{r} = (3t^2 + 6t^2)dt - 14(t^3)(t^2)(2t\,dt) + 20(t)(3t^2)(t^6)\,dt$.

$= (9t^2 - 28t^6 + 60t^9)\,dt$

$\int_c \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$.

$= 9(t^3/3) - 28(t^7/7) + 60(t^{10}/10)\Big]_0^1$

$= (9/3 - 28/7 + 60/10)$

$= (3 - 4 + 6)$

$= 5$

Surface Integrals.

Any interval which is evaluated over a surface is called a surface integral.

Let $\vec{F}(x,y,z)$ be a vector point fn/- defined at each point of S. (where S is a surface

Let P be any point on the surfaces & let $\hat{n}$ be the outward unit normal at P.

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Then the surface integral of $\vec{F}$ over S is defined as $\iint_S \vec{F}\cdot\hat{n}\, ds$.

1) If the elementary area ds is projected on the xy-plane and if R is the region in the xy plane then $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n} \dfrac{dx\, dy}{|\hat{n}\cdot\vec{k}|}$

2) If the elementary area ds is projected on the yz-plane then $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n} \dfrac{dy\, dz}{|\hat{n}\cdot\vec{i}|}$

3) If the elementary area ds is projected on the zx-plane then $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n} \dfrac{dz\, dx}{|\hat{n}\cdot\vec{j}|}$

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1) Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$, where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is the part of the surface $x^2+y^2+z^2=1$. which lies in the first octant.

Given, $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k} \longrightarrow (1)$

Let $\phi = x^2+y^2+z^2-1. \longrightarrow (2)$.

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i}+y\vec{j}+z\vec{k})}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}} \quad (\text{by } (2))$$

$$\hat{n} = x\vec{i}+y\vec{j}+z\vec{k}.$$

Now, $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\, dx\, dy$

where R is the projection of S on the xy-plane.

The sphere lies in the first octant (XY plane)

$z = 0, \quad x^2 + y^2 = 1$

$$|\hat{n}\cdot\vec{k}| = (x\vec{i} + y\vec{j} + z\vec{k})\cdot\vec{k} = z.$$

$$\vec{F}\cdot\hat{n} = 3xyz.$$

$\because \vec{F}\cdot\hat{n} = (yz\vec{i} + zx\vec{j} + xy\vec{k})\cdot(x\vec{i} + y\vec{j} + z\vec{k})$

$= xyz + xyz + xyz$

$= 3xyz.$

x varies from 0 to 1.

y varies from 0 to $\sqrt{1-x^2}$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{3xy\cancel{z}}{\cancel{z}}\, dy\, dx.$$

$$= \int_0^1 3x\left(\frac{y^2}{2}\right)\Big]_0^{\sqrt{1-x^2}} dx.$$

$$= \frac{3}{2}\int_0^1 \left(x(1-x^2) - 0\right) dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{4}\right] = \frac{3}{8} \text{ sq. units}$$

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2) Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$, where $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$ and S is the surface of the cylinder $x^2 + y^2 = 1$. included in the first octant b/w the planes. $z = 0$ and $z = 2$.

$$\vec{F} = z\vec{i} - x\vec{j} - y^2 z\vec{k}$$

Let $\phi = x^2 + y^2 - 1.$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i} + y\vec{j})}{\sqrt{4x^2+4y^2}}$$

$$= \frac{\cancel{2}(x\vec{i} + y\vec{j})}{\cancel{2}\sqrt{x^2+y^2}} \qquad \because x^2+y^2=1.$$

$$= x\vec{i} + y\vec{j}.$$

$$\vec{F}\cdot\hat{n} = (z\vec{i} + x\vec{j} - y^2z\vec{k})(x\vec{i} + y\vec{j})$$

$$= xz + yx.$$

$$= x(z+y)$$

Now, $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\, dy\, dz.$

where R is projection of S on yz plane $(x=0)$

$$\hat{n}\cdot\vec{i} = x.$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\, dy\, dz$$

$$= \int_0^2\int_0^1 (z+y)\, dy\, dz.$$

$$= \int_0^2 \left(zy + y^2/2\right)_0^1 dz$$

$$= \int_0^2 (z + 1/2)\, dz.$$

$$= \left(z^2/2 + z/2\right)_0^2.$$

$$= (2+1) - 0$$

$$= 3$$

3) Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$, where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ as is is the part of the plane $2x+3y+6z=12$. which lies in the first Octant.

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Given, $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$

let $\phi = 2x + 3y + 6z - 12$.

$\nabla\phi = 2\vec{i} + 3\vec{j} + 6\vec{k}$

$|\nabla\phi| = \sqrt{4 + 9 + 36} = 7$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{7}$$

$$\vec{F}\cdot\hat{n} = \frac{(18z\vec{i} - 12\vec{j} + 3y\vec{k})(2\vec{i} + 3\vec{j} + 6\vec{k})}{7}$$

$$= \frac{36z - 36 + 18y}{7}$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\, dy\, dz.$$

$\hat{n}\cdot\vec{i} = 2/7$

Gn, $2x + 3y + 6z = 12$. In yz plane, $x = 0$.

$3y + 6z = 12$.

$y + 2z = 4$.

$y = 4 - 2z$

y varies from 0 to $4 - 2z$.

$x = 0$, $y = 0$., $6z = 12$., $z = 2$.

z varies from 0 to 2.

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \int_0^2\int_0^{4-2z} \frac{36z - 36 + 18y}{7} \cdot \frac{dy\, dz}{(2/7)}$$

$$= \int_0^2\int_0^{4-2z} (18z - 18 + 9y) \cdot dy\, dz.$$

$$= \int_0^2 \left. 18zy - 18y + 9y^2/2 \right)_0^{4-2z} dz.$$

$$= \int_0^2 18z(4-2z) - 18(4-2z) + \tfrac{9}{2}\underbrace{(4-2z)^2}_{16+4z^2-16z} dz$$

$$= \int_0^2 72z - 36z^2 - 72 + 36z + 9(8+2z^2-8z)\,dz.$$

$$= \int_0^2 \cancel{72z} - 36z^2 - \cancel{72} + 36z + \cancel{72} + 18z^2 - \cancel{72z}\,dz$$

$$= \int_0^2 (36z - 18z^2)\,dz$$

$$= 18\int_0^2 (2z - z^2)\,dz = 18\left[2\left(\tfrac{z^2}{2}\right) - \tfrac{z^3}{3}\right]_0^2$$

$$= 18\left[4 - \tfrac{8}{3}\right] = 18(4)\left(1 - \tfrac{2}{3}\right)$$

$$= \cancel{18}^{6} \times 4\left(\tfrac{1}{\cancel{3}}\right) = 24$$

---

i) <u>Line integral.</u>

If $\vec{F} = 2y\vec{i} - z\vec{j} + x\vec{k}$, Evaluate $\int_c \vec{F} \times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$

$$\vec{F} \times d\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & dz. \\ 2y & -z & x \end{vmatrix}$$

$$= \vec{i}\,[x\,dy + z\,dz] - \vec{j}\,[x\,dx - 2y\,dz] + \vec{k}\,[-z\,dx - 2y\,dy]$$

$$= \vec{i}\,[(\cos t)(\cos t)\,dt + (2\cos t)(-2\sin t)\,dt]$$
$$- \vec{j}\,[(\cos t)(-\sin t)\,dt + 2(\sin t)(2\sin t)\,dt]$$
$$+ \vec{k}\,[-2\cos t\,(-\sin t)\,dt - 2\sin t\cos t\,dt]$$

$= \vec{i}\left[\cos^2 t\, dt - 4\cos t \sin t\, dt\right] - \vec{j}\left[-\sin t \cos t\, dt + 4\sin^2 t\, dt\right] + \vec{k}\left[2\sin t/\cos t\, dt - 2\sin t/\cos t\, dt\right]$

$$\int_C \vec{F} \times d\vec{r} = \int_0^{\pi/2} \left[(\cos^2 t - 4\cos t \sin t)\vec{i} + (\sin t \cos t - 4\sin^2 t)\vec{j}\right] dt$$

$$= \vec{i}\left[\int_0^{\pi/2} \cos^2 t - \frac{4}{2}\int_0^{\pi/2} \sin 2t\, dt\right]$$

$$+ \vec{j}\left[\frac{1}{2}\int_0^{\pi/2} \sin 2t\, dt - 4\int_0^{\pi/2} \sin^2 t\right] dt.$$

$$= \vec{i}\left[\frac{1}{2}\cdot\frac{\pi}{2} + 2\left(\frac{1}{2}\right)(-1-1)\right]$$

$$+ \vec{j}\left[\frac{1}{2}\left(-\frac{1}{2}\right)(-1-1) - 4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)\right]$$

$$= \vec{i}\left[\frac{\pi}{4} - 2\right] + \vec{j}\left[\frac{1}{2} - \pi\right]$$

Volume Integrals.

The volume integral of $F(x,y,z)$ over a region enclosing a volume V is given by $\iiint_V F(x,y,z)\, dV$ (or) $\iiint_V F(x,y,z)\, dx\, dy\, dz$

1) If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, evaluate $\iiint_V \nabla \times \vec{F}\, dV$ where V is the region bounded by $x=0$, $y=0$, $z=0$ & $2x + 2y + z = 4$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}(0) + \vec{j}(-3+4) + \vec{k}(-2y-0)$$

$$= \vec{j} - 2y\vec{k}$$

$$\iiint_V (\nabla\times\vec{F})\, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\vec{j} - 2y\vec{k})\, dz\, dy\, dx.$$

$$= \int_0^2 \int_0^{2-x} (z\vec{j} - 2yz\vec{k})_0^{4-2x-2y}\, dy\, dx.$$

$$= \int_0^2 \int_0^{2-x} [4-2x-2y]\vec{j} - 2y(4-2x-2y)\vec{k}]\, dy dx$$

$$= \int_0^2 \left[(4y - 2xy - y^2)\vec{j} - \left(4y^2 - 2xy^2 - \frac{4y^3}{3}\right)\vec{k}\right]_0^{2-x} dx$$

$$= \int_0^2 \Big[\left[4(2-x) - 2x(2-x) - (2-x)^2\right]\vec{j}$$

$$-\left[4(2-x)^2 - 2x(2-x)^2 - 4/3\,(2-x)^3\right]\vec{k}\Big]\, dx.$$

$$= \int_0^2 (8 - 4x - 4x + 2x^2 - 4 - x^2 + 4x)\vec{j} -$$

$$(16 - 16x + 4x^2 - 8x + 8x^2 - 2x^3).$$

$$- 4/3\,(8 - 12x + 6x^2 - x^3)\vec{k}\Big]\, dx.$$

$$= \int_0^2 \left[(4 - 4x + x^2)\vec{j} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3)\right] dx$$

$$= \left[4x - 2x^2 + \frac{x^3}{3}\right]\vec{j} - \frac{\vec{k}}{3}\left[16x - 12x^2 + 4x^3 - \frac{x^4}{2}\right]_0^2$$

$$= (8 - 8 + 8/3)\vec{j} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8)$$

$$= \frac{8}{3}(\vec{j} - \vec{k}).$$

1

Green's theorem in a plane

If $u, v, \partial u/\partial y, \partial v/\partial x$ are continuous and one-valued functions in the region R, enclosed by the curve C, then

$$\int_C u\,dx + v\,dy = \iint_R \left\{ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right\} dx\,dy$$

corollary - 1

If $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ the value of the integral $\int_c u\,dx + v\,dy$ is independent of the path of integration.

corollary - 2

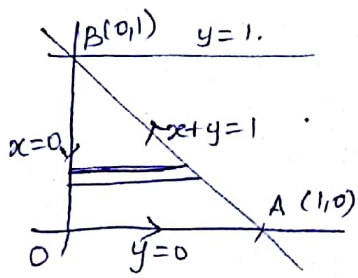
If R is a region bounded by a simply closed curve c then the area of R given by $\frac{1}{2}\int_c x\,dy - y\,dx$.

---

1) Verify Green's theorem in the XY plane for $\int_c \{(3x-8y^2)\,dx + (4y-6xy)\,dy\}$ where c is the boundary of the region given by $x=0$, $y=0$, $x+y=1$.

Green's theorem is

$$\int_c (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.$$



$u = 3x - 8y^2$, $\quad v = 4y - 6xy$

$\frac{\partial u}{\partial y} = -16y \qquad \frac{\partial v}{\partial x} = -6y.$

To evaluate $\int_c u\,dx + v\,dy$, Take c in 3 different paths,

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(i) Along OA $(y=0)$

(ii) Along AB $(x+y=1)$

(iii) Along BO $(x=0)$

1) Along OA $(y=0)$

$$\int_{OA} \{(3x-8y^2)dx + (4y-6xy)dy\}. = \int_{OA} 3x\,dx \qquad (\because y=0,\ dy=0)$$

$$= \int_0^1 3x\,dx.$$

$$= 3\left(\frac{x^2}{2}\right)_0^1$$

$$= \frac{3}{2}$$

2) Along AB $(x+y=1.$

$$\int_{AB} \{(3x-8y^2)dx + (4y-6xy)dy\}.$$

$$= \int_{AB} \left(3x-8(1-x^2)\right)dx + \left(4(1-x) - 6(x)(1-x)\right)(-dx)$$

$\because x+y=1,\quad y=1-x,\quad dy=-dx.$

$$\int_{AB} (3x-8-8x^2+16x-4+4x+6x-6x^2)\,dx.$$

$$\int_1^0 (-14x^2+29x-12)\,dx = \frac{13}{6}.$$

(3) Along BO :$(x=0)$

$$\int_{BO} \{(3x-8y^2)dx + (4y-6xy)\,dy\}$$

$$= \int_{BO} 4y\,dy \qquad (\because x=0, dx=0)$$

$$= 4\int_1^0 y\,dy, \qquad (y \text{ varies } 1 \text{ to } 0 \text{ for } BO)$$

$$= -2$$

Hence $\int_c u\,dx + v\,dy = \int_{OA} + \int_{AB} + \int_{BO} = 3/2 + 13/6 - 2 = 5/3$.

Evaluation : $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_R (-6y + 16y)\,dx\,dy$$

$$= \iint_R 10y\,dx\,dy = 10 \int_0^1 \int_0^{1-y} y\,dx\,dy$$

$$= 5/3.$$

Hence Green's theorem is verified

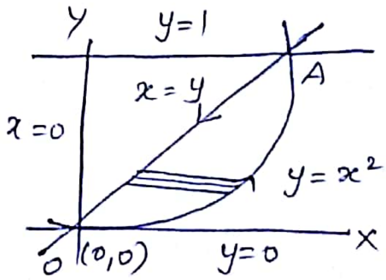
2) Verify Green's theorem in the XY plane for $\int_c \{(xy+y^2)dx + x^2 dy\}$, where c is the closed curve of the region bounded by $y = x$ & $y = x^2$.

Green's theorem in XY plane,

$$\int_c (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Here, $u = xy + y^2$ & $v = x^2$

$$\frac{\partial u}{\partial y} = x + 2y, \quad \frac{\partial v}{\partial x} = 2x.$$



Evaluation of $\int_c (u\,dx + v\,dy)$

Take C in 2 different paths.

1) Along OA $(y = x^2)$
2) Along AO $(y = x)$.

$$\int_c (u\,dx + v\,dy) = \int_{OA} + \int_{AO}.$$

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i) Along OA $[y = x^2,\ dy = 2x\,dx]$

$$\int_{OA} = \int_{OA} (x(x^2) + x^4)\,dx + x^2(2x)\,dx.$$

$$= \int_0^1 (x^3 + x^4)\,dx + 2x^3\,dx.$$

$$= \int_0^1 (3x^3 + x^4)\,dx.$$

$$= \frac{19}{20}.$$

Along AO $(y = x,\ dy = dx)$.

$$\int_{AO} = \int_{AO} (x^2 + x^2)\,dx + x^2\,dx.$$

$$= \int_1^0 3x^2\,dx, \quad (x \text{ varies from } 1 \text{ to } 0).$$

$$= -1.$$

Hence $\int_{OA} + \int_{AO} = \frac{19}{20} - 1 = \frac{-1}{20} \longrightarrow$ (LHS).

Evaluation of $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.$

$$= \iint_R (2x - (x + 2y))\,dx\,dy$$

$$= \int_0^1 \int_y^{\sqrt{y}} (x - 2y)\,dx\,dy.$$

$$= \frac{-1}{20}. \quad \text{(RHS)}$$

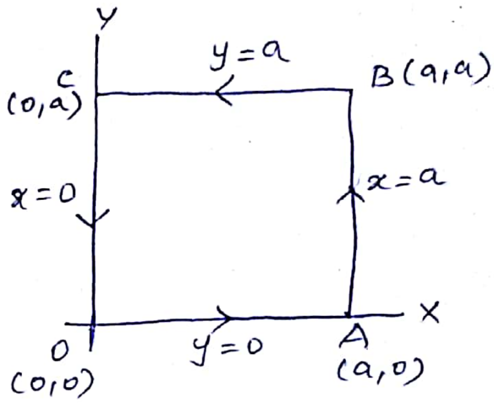
Hence Green's theorem is Verified.

---

3) Verify Green's theorem in the plane for $\int_c (x^2dx + xy\,dy)$ where c is the curve in XY plane given by $x = 0,\ y = 0,\ x = a,\ y = a\ (a > 0)$

By Green's theorem,

$$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$



$u = x^2$ | $v = xy$

$\dfrac{\partial u}{\partial y} = 0$ | $\dfrac{\partial v}{\partial x} = y$

Evaluation of $\int_C \{u\,dx + v\,dy\}$.

C in 4 different paths.

1) Along OA $(y=0)$
2) Along AB $(x=a)$
3) Along BC $(y=a)$
4) Along CO $(x=0)$

$$\int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA. $(y=0) \Rightarrow dy = 0$.

$$\int_C \{x^2 dx + xy\,dy\} = \int_{OA} x^2 dx = \int_0^a x^2 dx = a^3/3.$$

Along AB $(x = a,\ dx = 0)$

$$\int_C x^2 dx + xy\,dy = \int_{AB} x^2 dx + xy\,dy = \int_{AB} 0 + ay\,dy$$

$$= a\int_0^a y\,dy.$$

$$= a^3/2$$

Along BC $(y=a)\ (dy=0)$

$$\int_C \{x^2 dx + xy\,dy\} = \int_{BC} x^2 dx. = \int_a^0 x^2 dx.$$

$$= -\frac{a^3}{3}.$$

Along co $(x=0, dx=0)$.

$$\int_c x^2dx + xydy = \int_{co} (0+0) = 0.$$

$$\int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}.$$

$$= \cancel{a^3/3} + a^3/2 - \cancel{a^3/3} + 0$$

$$= a^3/2. \longrightarrow \text{(LHS)}$$

Evaluation of $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$.

$$= \iint_R y\,dx\,dy. = \int_0^a\int_0^a y\,dx\,dy.$$

$$= \int_0^a y\,(x)_0^a\,dy = a\int_0^a y\,dy$$

$$= a\left(\frac{y^2}{2}\right)_0^a = a^3/2 \longrightarrow \text{(RHS)}$$

Green's theorem is verified.

---

7) Verify Green's theorem in the plane for $\int_c (3x^2-8y^2)dx + (4y-6xy)dy$, where c is the boundary of the region defined by $x=y^2$, $y=x^2$.

Green's theorem states that

$$\int_c udx + vdy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Given : $\int_c (3x^2-8y^2)dx + (4y-6xy)\,dy$.

$u = 3x^2 - 8y^2$ $\qquad$ $v = 4y - 6xy$.

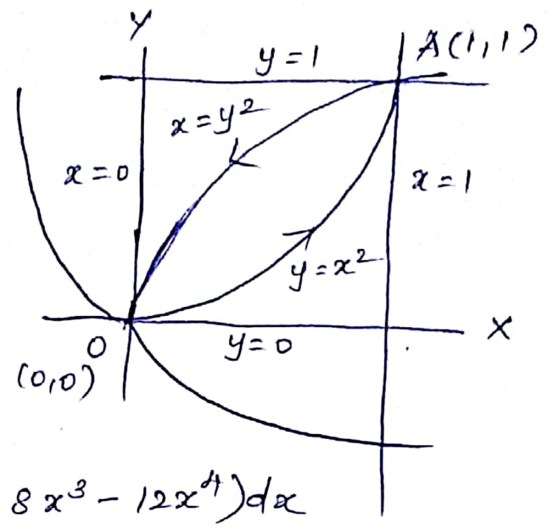
$\frac{\partial u}{\partial y} = -16y$ $\qquad$ $\frac{\partial v}{\partial x} = -6y$

To evaluate $\int_c u\,dx + v\,dy$,

c in 2 different paths

1) Along OA $(y = x^2)$

2) Along AO $(x = y^2)$



Along OA $(y = x^2)$

$$\int_{OA} u\,dx + v\,dy = \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= -1$$

Along AO $(x = y^2)$ $(2y\,dy = dx)$

$$\int_{AO} (3y^4 - 8y^2)\,dy + (4y - 6y\,y^2)\,dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy$$

$$= 5/2$$

$$\int_C = \int_{OA} + \int_{AO} = -1 + 5/2 = 3/2 \quad (LHS)$$

Evaluation of $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

$$= \iint_R (-6y + 16y)\,dx\,dy.$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (10y)\,dx\,dy$$

$$= 3/2 \quad (RHS)$$

Hence, Green's theorem is Verified.

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Stoke's theorem

The surface integral of the normal component of the curl of a vector fn F over an open surface S is equal to the line integral of the tangential component of F around the closed curve c bounding S.

$$\int_c \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds.$$

1) Verify stoke's thm for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$. where S is the upper half of the sphere $x^2+y^2+z^2=1$. and c is bdry in xy plane.

Bdry c of S is a circle in xy-plane of radius unity & centre at origin

Parametric eqn is $x=\cos\theta$, $y=\sin\theta$, $z=0$, $0\le\theta\le 2\pi$, $(r=1)$

By stoke's thm, $\iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds = \int_c \vec{F}\cdot d\vec{r}$.

LHS: $\iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz+2yz)\vec{i} - (0)\vec{j} + (1)\vec{k}$$

$$= \vec{k}.$$

$ds = dx\,dy$, $\hat{n} = \vec{k}$.

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint \vec{k}\cdot\vec{k}\,dx\,dy = \iint dx\,dy$$

$$= \text{Area of circle}.$$

$$= \pi(r^2) = \pi.$$

RHS : $\int_C \vec{F}\cdot d\vec{r}$.

$$\vec{F}\cdot d\vec{r} = \left[(2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}\right]\left[dx\vec{i}+dy\vec{j}+dz\vec{k}\right]$$

$$= \int_C (2x-y)\,dx \quad , \quad z=0,\ dz=0.$$

$$= -\int_0^{2\pi}(2\cos\theta - \sin\theta)\sin\theta\,d\theta.$$

$$= -\int_0^{2\pi} 2\sin\theta\cos\theta - \sin^2\theta\,d\theta.$$

$$= -\int_0^{2\pi} \sin 2\theta - \tfrac{1}{2} - \cos 2\theta/2\,d\theta.$$

$$= -\left[-\frac{\cos 2\theta}{2} - \frac{1}{2}(\theta) - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= -\left[\left(-\tfrac{1}{2} - \pi - 0\right) - \left(-\tfrac{1}{2}\right)\right]$$

$$= \pi.$$

LHS = RHS.

Stokes thm is Verified.

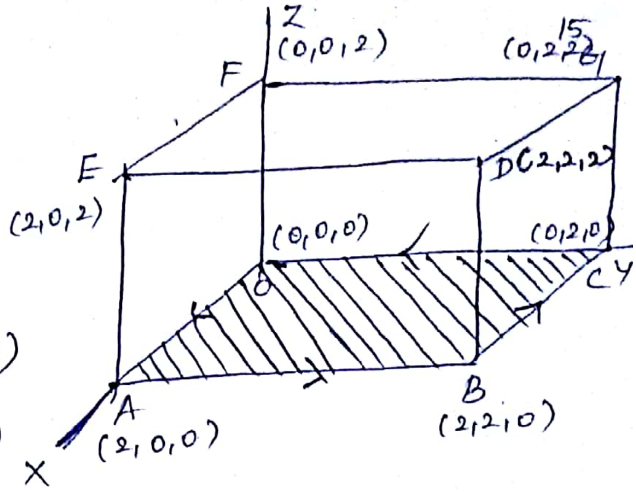
---

2) Verify stokes thm for $\vec{F} = (y-z+2)\vec{i} - (yz+4)\vec{j} - (xz)\vec{k}$ over the surface of the cube above XOY plane. $x=0,\ y=0,\ z=0,\ x=2,\ y=2,\ z=2$.

$$\vec{F}\cdot d\vec{r} = (y-z+2)\,dx + (yz+4)\,dy - xz\,dz$$

$$(\vec{r} = x\vec{i}+y\vec{j}+z\vec{k} \quad d\vec{r} = dx\vec{i}+dy\vec{j}+dz\vec{k}.)$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}.$$



4 different paths.

1) Along OA $(y=0, dy=0, z=0, dz=0)$
2) Along AB $(x=2, dx=0, z=0, dz=0)$
3) Along BC $(y=2, dy=0, z=0, dz=0)$
4) Along CO $(x=0, dx=0, z=0, dz=0)$

Now along OA.

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^2 2\,dx = 4$$

Along AB, $\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^2 4\,dy = 4(y)_0^2 = 8$

Along BC, $\int_{BC} \vec{F}\cdot d\vec{r} = \int_2^0 4\,dx = -8$

Along CO, $\int_{CO} \vec{F}\cdot d\vec{r} = \int_2^0 4\,dy = -8$

$$\int_C \vec{F}\cdot d\vec{r} = 4 + \cancel{8} - 8 - \cancel{8} = -4 \quad (LHS)$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= (0-y)\vec{i} - (-z+1)\vec{j} + (0-1)\vec{k}$$

$$= -y\vec{i} + (z-1)\vec{j} - \vec{k}.$$

To integrate over 5 surfaces.

ABDE, OCGF, OAEF, BCGD, DEFG

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

S_6 is not applicable. (Gn condn in above XOY plane)

Surfaces	Faces	2D plane	$\hat{n}$	
S_1	ABDE	YZ	$\vec{i}$	$x=2$
S_2	OCGF	YZ	$-\vec{i}$	$x=0$
S_3	AOEF	XZ	$-\vec{j}$	$y=0$
S_4	BCGD	XZ	$\vec{j}$	$y=2$
S_5	EDGF	XY	$\vec{k}$	$z=2$

$$\iint_{S_1} \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_{S_1} (-y\vec{i} + (z-1)\vec{j} - \vec{k})\,\vec{i}\,dy\,dz \qquad (\hat{n}=\vec{i})$$

$$= \int_0^2\int_0^2 -y\,dy\,dz$$

$$= \left(-\frac{y^2}{2}\right)_0^2 (z)_0^2 = -4$$

$$\iint_{S_2} (\nabla\times\vec{F})\,\hat{n}\,ds = \iint (-y\vec{i} + (z-1)\vec{j} - \vec{k})(-\vec{i})\,dy\,dz.$$

$$= \int_0^2\int_0^2 y\,dy\,dz = \int_0^2 y\,dy \int_0^2 dz.$$

$$= 2\left(\frac{y^2}{2}\right) = 4$$

$$\iint_{S_3} (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint (-y\vec{i} + (z-1)\vec{j} - \vec{k})\,\vec{j}\,dx\,dz.$$

$$= \iint (z-1)\,dx\,dz.$$

$$= \int_0^2 dx \int_0^2 (z-1)\,dz$$

$$= x\Big]_0^2 \cdot \left[\frac{z^2}{2} - z\right]_0^2 = 0.$$

$$\iint_{S_4} (\nabla\times\vec{F})\cdot\hat{n}\, ds = \iint (-y\vec{i} + (z-1)\vec{j} - \vec{k})(\vec{j})\, dx\, dz.$$

$$= -\iint (z-1)\, dx\, dz.$$

$$= -\int_0^2 dx \int_0^2 (z-1) dz.$$

$$= -(x)_0^2 \left(\frac{z^2}{2} - z\right)_0^2 = 0.$$

$$\iint_{S_5} (\nabla\times\vec{F})\cdot\hat{n}\, ds = \iint (-y\vec{i} + (z-1)\vec{j} - \vec{k})\,\vec{k}\, dx\, dy$$

$$= -\int_0^2\int_0^2 dx\, dy.$$

$$= -4.$$

Total surfaces $= -\cancel{4} + \cancel{4} + 0 + 0 - 4 = -4.$

LHS = RHS.

Stoke's thm is verified.

---

3) Verify Stoke's thm for a vector fied defined by $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region XOY plane bounded by the lines $x=0$, $x=a$, $y=0$, $y=b$.

By Stokes thm, $\iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds = \int_c \vec{F}\cdot d\vec{r}.$

$$\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}.$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$

$$\vec{F}\cdot d\vec{r} = (x^2-y^2)dx + 2xy\, dy$$

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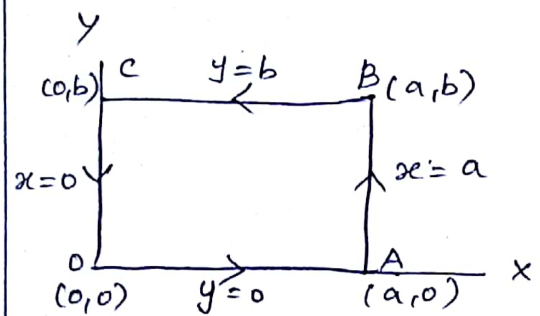
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y)$$

$$= 4y\vec{k}, \quad \text{where } \hat{n} = \vec{k}.$$

RHS: $\iint_S (\nabla \times \vec{F}) \hat{n}\, ds = \int_0^b \int_0^a 4y\vec{k} \cdot \hat{n}\, dx\, dy.$



$$= \int_0^b \int_0^a (4y\vec{k}\cdot\vec{k})\, dx\, dy$$

$$= \int_0^b \int_0^a 4y\, dx\, dy$$

$$= 2ab^2.$$

LHS: $\int_C \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}.$

Along OA, $y=0$, $dy=0$.

Along AB, $x=a$, $dx=0$

Along BC, $y=b$, $dy=0$

Along CO, $x=0$, $dx=0$.

$$\int_C \vec{F} \cdot d\vec{r} = \underbrace{\int_{OA} x^2 dx}_{x \to 0 \text{ to } a} + \underbrace{\int_{AB} 2ay\, dy}_{y \to 0 \text{ to } b} + \underbrace{\int_{BC} (x^2-b^2) dx}_{x \to a \text{ to } 0} + \int_{CO} 0.$$

$$= 2ab^2.$$

LHS = RHS.

Stokes theorem is verified.

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Gauss Divergence theorem

(Relation between surface & Volume integrals)

If $\vec{F}$ is a vector point function having continuous first order derivatives in the region V bounded by a closed surface S, then

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds,$$ $\hat{n}$ is outward drawn unit normal vector to the surface S.

---

1) Verify divergence thm for the fn $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ taken over the cube bounded by planes.

$x=0,\ y=0,\ z=0,\ x=1,\ y=1,\ z=1.$

By divergence thm, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$.

$$\nabla\cdot\vec{F} = \partial/\partial x\,(4xz) + \partial/\partial y\,(-y^2) + \partial/\partial z\,(yz)$$

$$= 4z - 2y + y$$

$$\nabla\cdot\vec{F} = 4z - y.$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz.$$

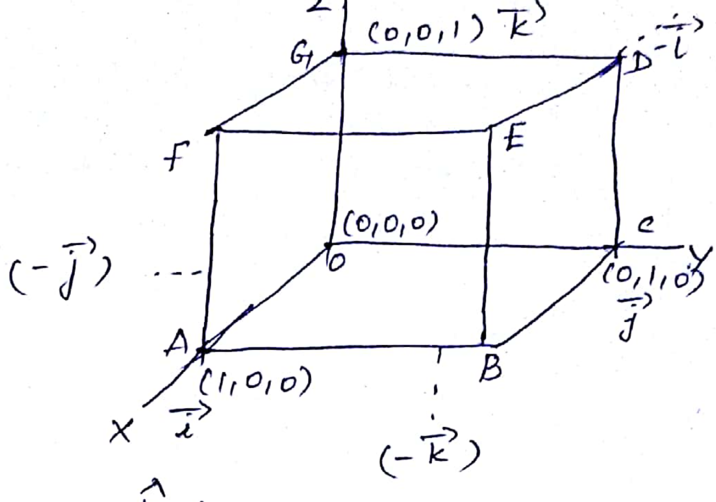
$$= \int_0^1\int_0^1 \left(4(z^2/2) - yz\right)_0^1 dy\,dz.$$

$$= \int_0^1\int_0^1 (2(1) - y)\,dy\,dz.$$

$$= 3/2$$

To find $\iint_S \vec{F}\cdot\hat{n}\,ds$.

To evaluate $\iint_S$ in 6 different surfaces.



Surfaces	Faces	Planes	$\hat{n}$	
S_1	ABEF	YZ	$\vec{i}$	$x = 1$
S_2	OCDG	YZ	$-\vec{i}$	$x = 0$
S_3	CDEB	XZ	$\vec{j}$	$y = 1$
S_4	OGFA	XZ	$-\vec{j}$	$y = 0$
S_5	GFED	XY	$\vec{k}$	$z = 1$
S_6	OABC	XY	$-\vec{k}$	$z = 0$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} 4xz\,dy\,dz + \iint_{S_2} -4xz\,dy\,dz + \iint_{S_3} -y^2.dx\,dz + \iint_{S_4} y^2dx\,dz + \iint_{S_5} yz\,dx\,dy + \iint_{S_6} -yz\,dx\,dy.$$

$(x, y, z \longrightarrow 0 \text{ to } 1)$

① $\Rightarrow \int_0^1\int_0^1 4xz\,dy\,dz = \int_0^1 (4xyz)_0^1\,dz = 2.$

② $\Rightarrow \iint_0^1 -4xz\,dy\,dz = 0 \qquad (\because x = 0)$

③ $\Rightarrow \int_0^1\int_0^1 -y^2dx\,dz \quad (y=1), \quad \int_0^1 x]_0^1 = -1.$

④ $\Rightarrow \int_0^1\int_0^1 y^2\,dx\,dz\,,\quad y=0 \quad \Rightarrow \quad 0.$

⑤ $\Rightarrow \int_0^1\int_0^1 yz\,dx\,dy\,,\quad z=1\,,\quad \int_0^1\int_0^1 y\,]\,dx\,dy = \frac{1}{2}$

⑥ $\Rightarrow \int_0^1\int_0^1 -yz\,dx\,dy\,,\quad (z=0) \quad \Rightarrow \quad 0.$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 2+0-1+0+\frac{1}{2}+0 = \frac{3}{2}$$

LHS = RHS.

Divergence theorem is verified.

---

2) Verify divergence theorem for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$.

taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

By divergence theorem, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\operatorname{div}\vec{F})\,dv$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$= 2(x+y+z)$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz.$$

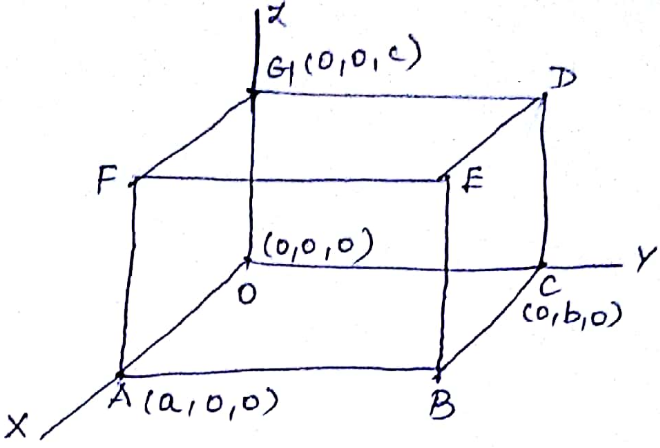
$$= 2\int_0^c\int_0^b \left(\frac{x^2}{2}+yx+zx\right)_0^a dy\,dz.$$

$$= 2\int_0^c\int_0^b \left(\frac{a^2}{2}+ya+za\right) dy\,dz$$

$$= 2\int_0^c \left(\frac{a^2y}{2}+a\left(\frac{y^2}{2}\right)+azy\right)_0^b dz.$$

$$= 2\int_0^c \left(\frac{a^2}{2}b+\frac{ab^2}{2}+abz\right) dz$$

$$= a^2bc + ab^2c + abc^2 = abc(a+b+c)$$



Surfaces	Faces	Planes	$\hat{n}$	
S_1	ACEF	YZ	$\vec{i}$	$x = a$
S_2	OCDG	YZ	$-\vec{i}$	$x = 0$
S_3	BDEC	XZ	$\vec{j}$	$y = b$
S_4	OCFA	XZ	$-\vec{j}$	$y = 0$
S_5	GFED	XY	$\vec{k}$	$z = c$
S_6	OACB	XY	$-\vec{k}$	$z = 0$.

$$\iint_S \vec{F} \cdot \hat{n}\, ds \Rightarrow S_1 + S_2 + \cdots + S_6 .$$

(1) $\Rightarrow \iint_S (x^2 - yz)\, dy\, dz$, $x = a$

$$\int_0^c \int_0^b (a^2 - yz)\, dy\, dz = \int_0^c \left(a^2 y - \tfrac{y^2}{2} z\right)_0^b dz .$$

$$= \int \left[a^2 b - \tfrac{b^2}{2} z \right] dz .$$

$$= \left(a^2 b z - \tfrac{b^2}{2} \left(\tfrac{z^2}{2}\right) \right)_0^c .$$

$$= a^2 bc - \tfrac{b^2}{4} (c^2)$$

$$= a^2 bc - \frac{b^2 c^2}{4}$$

② $\Rightarrow \; -\iint (x^2 - yz)\, dy\, dz \qquad (x = 0)$

$$= -\int_0^c \int_0^b -yz \; dy \; dz .$$

$$= \frac{b^2c^2}{4}$$

③ $\Rightarrow \; \iint (y^2 - zx)\, dx\, dz , \qquad y = b .$

$$= \int_0^c \int_0^a (b^2 - zx)\, dx\, dz .$$

$$= ab^2c - \frac{a^2c^2}{4}$$

④ $\Rightarrow \; \iint (y^2 - zx)\, dx\, dz , \qquad y = 0 .$

$$= \int_0^c \int_0^a zx \, dx \, dz . \; = \int_0^c z\left(x^2/2\right)\Big]_0^a dz$$

$$= \int_0^c z/2 \,(a^2)\, dz .$$

$$= \frac{a^2c^2}{4}$$

⑤ $\Rightarrow \; \iint (z^2 - xy)\, dx\, dy , \qquad z = c .$

$$= \int_0^b \int_0^a (c^2 - xy)\, dx\, dy .$$

$$= \frac{ac^2b}{\cancel{1/2}} - \frac{a^2b^2}{4}$$

⑥ $\Rightarrow \; -\iint (z^2 - xy)\, dx\, dy , \quad z = 0 .$

$$= \int_0^b \int_0^a (x\,y)\, dx\, dy$$

$$= \frac{a^2b^2}{4}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{a^2c^2}{4}}$$

$$+ \cancel{\frac{a^2c^2}{4}} - ac^2b - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}}$$

$$= a^2bc + ab^2c - ac^2b$$

$$= abc(a+b+c)$$

LHS = RHS.

Gauss divergence theorem is verified.

Additional Problems:

1) S.T $\vec{F} = (2xy+z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative force field. Find the scalar potential.

To prove : $\nabla\times\vec{F} = 0$.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\,[0-0] - \vec{j}\,[3z^2-3z^2] + \vec{k}\,[2x-2x]$$

$$= 0$$

$\therefore$ $\vec{F}$ is conservative force field.

$\vec{F} = \nabla\phi$.

$$(2xy+z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 2xy + z^3 \qquad \frac{\partial\phi}{\partial y} = x^2 \qquad \frac{\partial\phi}{\partial z} = 3xz^2$$

$$\phi = x^2y + z^3x + c_1, \qquad \phi = x^2y + c_2 \qquad \phi = xz^3 + c_3$$

$$\therefore \quad \phi = x^2y + xz^3 + c.$$

To prove $\nabla \cdot \vec{F} = 0$.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2xy + z^3) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(3xz^2).$$

$$= 2y \neq 0.$$

$\vec{F}$ is not solenoidal.

2) Find the value of n, if $r^n \vec{r}$ is both solenoidal & irrotational when $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\text{div}(r^n \cdot \vec{r}) = \nabla r^n \cdot \vec{r} + r^n \cdot \nabla \vec{r}.$$

$$\because \text{div}(\phi \vec{u}) = \nabla \phi \cdot \vec{u} + \phi(\nabla \cdot \vec{u})$$

Gn $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\text{div}\,\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1+1+1 = 3.$$

$$\boxed{\text{div}\,\vec{r} = 3.}$$

$$\nabla r^n = \Sigma \vec{i} \frac{\partial}{\partial x}(r^n) = \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \Sigma \vec{i}\, n(r^{n-1}) \frac{\partial r}{\partial x}.$$

$$= n r^{n-1}\left[\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right]$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$r = \sqrt{x^2+y^2+z^2}.$$

(ie) $r^2 = x^2+y^2+z^2$.

Diff w.r.to x, $2r \cdot \frac{\partial r}{\partial x} = 2x$.

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}, \quad \text{III}^{ly} \quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}}$$

$$\text{grad}(r^n) = \nabla(r^n).$$

$$= \sum \vec{i}\,(n r^{n-1})\,(x/r)$$

$$= n r^{n-1}\left[\vec{i}\,(x/r) + \vec{j}\,(y/r) + \vec{k}\,(z/r)\right]$$

$$= \frac{n r^{n-1}}{r}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= n r^{n-2}\,(\vec{r})$$

$$\boxed{\nabla r^n = (n r^{n-2})\,\vec{r}.}$$

Now, $\text{div}(r^n \cdot \vec{r}) = (\nabla r^n)\vec{r} + r^n(\text{div}\,\vec{r})$

$$= ((n r^{n-2})\,\vec{r})\,\vec{r} + r^n(3)$$

$$= n r^{n-2}\, r^2 + 3r^n.$$

$$= n r^n + 3r^n.$$

$$\boxed{\text{div}(r^n \cdot \vec{r}) = r^n(n+3)}$$

$r^n\vec{r}$ is solenoidal means $\text{div}(r^n \cdot \vec{r}) = 0$ is possible only when $n = -3$.

<u>To prove :</u> $r^n\vec{r}$ is irrotational.

(ie) $\text{curl}(r^n\vec{r}) = 0$.

$$\text{curl}(\phi\vec{u}) = \nabla\phi \times \vec{u} + \phi\,\text{curl}\,\vec{u}.$$

(ie) $\text{curl}(r^n\vec{r}) = (\nabla r^n)\times\vec{r} + r^n\,\text{curl}\,\vec{r}.$

$$= (n r^{n-2}(\vec{r}) \times \vec{r}) + r^n\,\overset{0}{\cancel{\text{curl}}}\,\vec{r}$$

$$\nabla\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0.$$

$\therefore$ $\text{curl}(r^n\vec{r}) = 0$.

$\therefore$ $r^n\vec{r}$ is irrotational.

3) P.T curl grad $\phi = 0$.

$$\nabla\phi = \text{grad }\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\nabla\times\nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right] - \vec{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\,\partial x}\right] + \vec{k}\left[\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial x\,dy}\right]$$

$= 0$.

4) P.T div curl $\vec{F} = 0$

$$\nabla(\nabla\times\vec{F}) = 0$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\nabla(\nabla\times\vec{F}) = \frac{\partial}{\partial x}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial x\,\partial y} + \frac{\partial^2 F_1}{\partial y\,\partial z} + \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 z}{\partial z\partial y}$$

$= 0$.

i.e. $\nabla(\nabla\times\vec{F}) = 0$.

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5) P.T $\operatorname{div}(\phi \vec{u}) = \nabla\phi \cdot \vec{u} + \phi \operatorname{div} \vec{u}$.

W.K.T . $\operatorname{div} \vec{F} = \vec{i}\frac{\partial F_1}{\partial x} + \vec{j}\frac{\partial F_2}{\partial y} + \vec{k}\frac{\partial F_3}{\partial z}$

$= \sum \vec{i}\frac{\partial \vec{F}}{\partial x}$.

$\operatorname{div}(\phi\vec{u}) = \sum \vec{i}\frac{\partial}{\partial x}(\phi\vec{u})$

$= \sum \vec{i}\left[\vec{u}\left(\frac{\partial\phi}{\partial x}\right) + \phi\left(\frac{\partial\vec{u}}{\partial x}\right)\right]$

$= \sum \vec{i}\left(\frac{\partial\phi}{\partial x}\vec{u}\right) + \sum\vec{i}\left(\phi\frac{\partial\vec{u}}{\partial x}\right)$

$= \left(\sum\vec{i}\frac{\partial\phi}{\partial x}\right)\vec{u} + \phi\left(\sum\vec{i}\frac{\partial\vec{u}}{\partial x}\right)$

$= (\operatorname{div}\phi)\vec{u} + \phi(\operatorname{div}\vec{u})$

$\operatorname{div}(\phi\vec{u}) = (\nabla\phi)\vec{u} + \phi(\operatorname{div}\vec{u})$.

---

6) P.T $f(r)\vec{r}$ is irrotational vector.

$\operatorname{curl}(f(r)\vec{r}) = (\nabla f(r)) . \vec{r} + f(r)(\operatorname{curl}\vec{r})$

$\nabla f(r) = \vec{i}\,\partial/\partial x\, f(r) + \vec{j}\,\partial/\partial y\, f(r) + \vec{k}\,\partial/\partial z\, f(r)$

$= \vec{i} f'(r)\frac{\partial r}{\partial x} + \vec{j} f'(r)\,\partial r/\partial y + \vec{k} f'(r)\,\partial r/\partial z$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$\vec{r}.\vec{r} = r^2 = x^2+y^2+z^2$.

Diff p. w.r.to x, $2r \cdot \partial r/\partial x = 2x$.

$\partial r/\partial x = x/r$, $\partial r/\partial y = y/r$ $\partial r/\partial z = z/r$.

$\nabla f(r) = \frac{f'(r)}{r}[x\vec{i} + y\vec{j} + z\vec{k}]$

$$\nabla f(r) = \frac{f'(r)}{r} . \vec{r}.$$

And $\nabla \times \vec{r} = 0$.

$$\text{curl}\,(f(r)\vec{r}) = \nabla \times (f(r)\vec{r})$$

$$= \nabla f(r) \times \vec{r} + f(r)\, \cancel{\text{curl}}\, \vec{r}$$

$$= \frac{f'(r)}{r} (\vec{r} \times \vec{r})$$

$$= 0.$$

7) P.T $\nabla^2 r^n = n(n+1) r^{n-2}$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\nabla \phi = \Sigma \vec{i} \frac{\partial \phi}{\partial x}.$$

$$\nabla^2 \phi = \Sigma \vec{i} \frac{\partial^2 \phi}{\partial x^2}.$$

$$\frac{\partial r}{\partial x} = x/r.$$

$$\frac{\partial^2 r}{\partial x^2} = 1/r.$$

If $\nabla r = \Sigma \vec{i}\, \partial r/\partial x$.

$$\nabla^2 r^n = \partial^2/\partial x^2 (r^n).$$

$$= \partial/\partial x \left(n r^{n-1} \partial r/\partial x \right).$$

$$= \partial/\partial x \left(n r^{n-1} . x/r \right).$$

$$= \partial/\partial x \left(n r^{n-2} . x \right)$$

$$= \left[n(n-2) r^{n-3} \partial r/\partial x (x) + n r^{n-2} (1) \right]$$

$$= \left[n(n-2) r^{n-3} (x/r) x . + n r^{n-2} \right]$$

$$= \left[n(n-2) r^{n-4} x^2 + n r^{n-2} \right]$$

$$= \left[n(n-2) r^{n-4} r^2 \right] + \Sigma \left[n(r^{n-2}) \right]$$

$$= \left[n(n-2) r^{n-2} + n(r^{n-2}) 3 \right]$$

$$= n r^{n-2} (n-2+3)$$

$$= n(n+1) r^{n-2}.$$