Pigeon - Hole - Principle

* Show that, if 30 dictionaries in a library contain a total of 61327 pages, then one of the dictionary must have 2045 pages.

Sol. Using Pigeon Hole Principle,

$n = 61327, m = 30, n \geq m$; formula $= \left[\frac{n-1}{m}\right] + 1$

$$\Rightarrow \frac{61327 - 1}{30} + 1 = \frac{61326}{30} + 1 = 2044.2 + 1 = [2045.2]$$

$$= 2045 //$$

∴ it is proved that if 30 dictionaries in a library contain a total of 61327 pages, then one of the dictionary must have 2045 pages.

* prove that, in any group of 6 people, atleast three must be mutual friends (or) three must be mutual strangers.

Sol. Let A be one among the 6 people.

So, now we have 5 people remaining who can be further classifieds as friends of A (mutual friends) and strangers to A (mutual strangers).

using Pigeon Hole Principle, $n = 5, m = 2, n > m$,

$$\Rightarrow \left[\frac{5-1}{2}\right] + 1 = \frac{4}{2} + 1 = 2 + 1 = 3 //$$

∴ 3 can be mutual - friends / strangers.

* a man hiked for 10 hrs and covered a distance of 45 km in total. It is known that, he hiked 6 km in 1st hour and 3 km in last hour. Show that he must have hiked atleast 9 km within a certain period of 2 consecutive hours.

Sol. → man hiked 9 kms in 1st and 10th hour.

→ remaining distance = 45 - 9 = 36 km.

→ say, he hikes consecutive two hours of about 9 km, then the possibilities are - (2nd hr, 3rd hour), (4th hr, 5th hr), (6th hr, 7th hr) and (8th hr, 9th hr) } 4

→ using pigeon hole principle, $n = 36$, $m = 4$, $n > m$,

$$\Rightarrow \left[\frac{36-1}{4}\right]+1 = \frac{35}{4}+1 = 8.75+1 = [9.75] = 9$$

∴ we showed that he must have hiked, atleast 9 km within a certain period of 2 consecutive hours.

* find the no. of integers between 1 and 250 that are not divisible by integers 2, 3, 5, 7.

Inclusion & Exclusion Principle

Sol. let A, B, C, D are four "set" of integers divisible by 2, 3, 5, 7 that lie between 1 to 250.

$$\Rightarrow |A| = \left|\frac{250}{2}\right| = 125 \qquad \Rightarrow |B| = \left|\frac{250}{3}\right| = 83$$

$\Rightarrow |C| = \left|\frac{250}{5}\right| = 50$ $\Rightarrow |D| = \left|\frac{250}{7}\right| = 35$

Now, no. of integers not divisible by 2,3,5,7 } = total no. of integers 1-250 − no. of integers divisible by 2,3,5,7

∴ no. of integers divisible by 2,3,5,7 $\Rightarrow |A \cup B \cup C \cup D|$.

wkt, $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$

$\Rightarrow |A \cap B| = \left|\frac{n}{A \times B}\right| = \left|\frac{250}{2 \times 3}\right| = 41$ $\Rightarrow |A \cap C| = \left|\frac{250}{2 \times 5}\right| = 25$

$\Rightarrow |A \cap D| = \left|\frac{250}{2 \times 7}\right| = 17$ $\Rightarrow |B \cap C| = \left|\frac{250}{3 \times 5}\right| = 16$ $\Rightarrow |B \cap D| = \left|\frac{250}{3 \times 7}\right| = 11$

$\Rightarrow |C \cap D| = \left|\frac{250}{5 \times 7}\right| = 7$ $\Rightarrow |A \cap B \cap C| = \left|\frac{250}{2 \times 3 \times 5}\right| = 8$

$\Rightarrow |A \cap B \cap D| = \left|\frac{250}{2 \times 3 \times 7}\right| = 5$ $\Rightarrow |B \cap C \cap D| = \left|\frac{250}{3 \times 5 \times 7}\right| = 2$

$\Rightarrow |A \cap C \cap D| = \left|\frac{250}{2 \times 5 \times 7}\right| = 3$ $\Rightarrow |A \cap B \cap C \cap D| = \left|\frac{250}{2 \times 3 \times 5 \times 7}\right| = 1$

$\Rightarrow 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$

$\Rightarrow 193$ //

∴ no. of integers not divisible by 2,3,5,7 $= 250 - 193 = 57$ //

Permutation & Combination

* in how many ways, a commitee consisting of 5 men and 3 women can be choosen from 9 men and 12 women.

<u>sol.</u> no. of ways of choosing 5 men from 9 men $\Rightarrow {}^9C_5$

" " " 3 women " 12 women $\Rightarrow {}^{12}C_3$

combination of 'r' objects from 'n' objects $\Rightarrow {}^nC_r$

$$\Rightarrow {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$\Rightarrow {}^9C_5 = \frac{9!}{5!(9-5)!} = 126 \text{ ways}$$

$$\Rightarrow {}^{12}C_3 = \frac{12!}{3!(12-3)!} = 220 \text{ ways.}$$

no. of ways of selecting 5 men & 3 women = m ways × n ways

= 126×220

= 27,720 ways

* in a dictionary, if all the permutations of the letters of the word "AGAIN" are arranged in an order, what is the 49th word?

<u>sol.</u> total possible arrangements of letters of AGAIN,

$\Rightarrow n=5, p_1=2(A), p_2=1(G), p_3=1(I), p_4=1(N)$

$$\Rightarrow {}^5P_{p_1} = \frac{5!}{2!\times1!\times1!\times1!} = \frac{5\times4\times3\times2\times1}{2\times1\times1\times1\times1} = 60$$

$\therefore$ 60 combinations are possible.

So, case i → letters starting with 'A' -

A _ _ _ _ $\Rightarrow \frac{4!}{1!\times1!\times1!} = \frac{4\times3\times2\times1}{1\times1\times1} \Rightarrow \underbrace{24 \text{ combinati}}_{1^{st} \text{ word}}$

$\underbrace{\quad}$ AGAIN

case ii → letters starting with 'G' -

G _ _ _ _ $\Rightarrow \frac{4!}{2!\times1!\times1!} = \frac{4\times3\times2\times1}{2\times1\times1\times1} = \underbrace{12 \text{ combination}}_{1^{st} \text{ word}}$

$\underbrace{\quad}$ GAAIN

48 words over (24+12+12)

case iii → letters starting with 'I' -

I _ _ _ _ $\Rightarrow \frac{4!}{2!\times1!\times1!} = \frac{4\times3\times2\times1}{2\times1\times1\times1} = \underbrace{12 \text{ combinations}}_{1st \text{ word}}$

$\underbrace{\quad}$ INAAG

case iv → letters starting with 'N' -

N _ _ _ _ $\Rightarrow \frac{4!}{2!\times1!\times1!} = \frac{4\times3\times2\times1}{2\times1\times1\times1} = \underbrace{12 \text{ combo.}}_{1^{st} \text{ word}}$

$\underbrace{\quad}$ NAAGI

$\therefore 49^{th}$ word $\Rightarrow$ NAAGI.

* determine the value of n if $4\,{}^{n}P_3 = {}^{(n+1)}P_3$.

sol. $4\,{}^{n}P_3 = {}^{(n+1)}P_3$

$\Rightarrow 4 \times {}^{n}P_3 = {}^{(n+1)} \times P_3$

$\Rightarrow 4 \times \frac{n!}{(n-3)!} = \frac{(n+1)\,n!}{(n+1-3)!}$

$\Rightarrow 4 \times \frac{n!}{(n-3)!} = \frac{(n+1)\,n!}{(n-2)!}$

$\left\{ \because (n-2)! = (n-2) \times (n-3) \times (n-4) \times \cdots (n-n) \times 0! \times 1! \times \cdots \right\}$

$\Rightarrow \frac{4 \times n!}{(n-3)!} = \frac{(n+1)\,n!}{(n-2)(n-3)!}$

$\Rightarrow 4 \times (n-2) = (n+1) \Rightarrow 4n - 8 = n+1$

$\Rightarrow 3n = 9 \Rightarrow \boxed{n = 3}$

Euclid's Alg.

* find the GCD (1819, 3587) using Euclid's Algorithm. Express the GCD as Linear combination of given numbers.

sol.

```
1819 ) 3587 ( 1
       1819
       ----
       1768 ) 1819 ( 1
              1768
              ----
                51 ) 1768 ( 34
                     1734
                     ----
                       34 ) 51 ( 1
                            34
                            --
                            17 ) 34 ( 2
                                 34
                                 --
                                  0
```

$\gcd(1819, 3587) \rightarrow$ $3587 = 1819 \times 1 + 1768$

$$1819 = 1768 \times 1 + 51$$
$$1768 = 51 \times 34 + 34$$
$$51 = 34 \times 1 + \underline{\underline{17}}$$
$$34 = 17 \times 2 + 0$$

$\gcd(1819, 3587) = 17$

linear combination -

$$17 = 51 - (34 \times 1)$$
$$17 = 51 - ((1768 - (51 \times 34)) \times 1)$$
$$17 = 51 - 1768 + 51 \times 34$$
$$17 = 35 \times 51 - 1768 \times 1$$
$$17 = 35 \times (1819 - 1768 \times 1) - 1768$$
$$17 = 35 \times 1819 - 36 \times 1768$$
$$17 = 35 \times 1819 - 36\,(3587 - 1819 \times 1)$$
$$17 = 35 \times 1819 - 36 \times 3587 + 36 \times 1819$$
$$17 = 71 \times 1819 - 36 \times 3587 \;//$$