UNIT – III

Complex Differentiation.

Functions of complex variable – analytic function – Necessary condition – CR eqn – sufficient conditions (excluding proof) – Properties of Analytic functions – Harmonic conjugate – construction of analytic functions – conformal mapping : $w = z+a$, $w = az$, $w = 1/z$, $w = z^2$ – Bilinear Transformation.

Complex Variable.

$x+iy$ is a complex variable and it is denoted by z.

(ie) $z = x+iy$ where $i = \sqrt{-1}$.

Function of a complex variable:

$w = u(x,y) + i\,v(x,y)$ is a function of the complex variable $z = x+iy$.

(ie) $w = f(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ is the real part and $v(x,y)$ is the imaginary part of the complex function $f(z)$. In general, $f(z) = u+iv$.

Analytic function / Regular fun/: / Holomorphic fn/:

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire function (Integral function)

A function is analytic every where in

the finite plane is called an entire function.

An entire function is analytic everywhere except at $z=\infty$.

eg: e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$.

Necessary condition for $f(z)$ to be analytic — Cauchy Riemann equations.

The necessary conditions for a complex variable functions $f(z) = u(x,y) + iv(x,y)$ to be analytic are,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

(ie) $$\boxed{u_x = v_y \quad \text{and} \quad v_x = -u_y}$$

Sufficient conditions for $f(z)$ to be analytic

If the partial derivatives u_x, u_y, v_x and v_y are all continuous in D. and $u_x = v_y$ and $u_y = -v_x$ Then the function $f(z)$ is analytic in a domain D

Polar form of C-R equations:

Let $z = re^{i\theta}$ and $f(z) = u+iv$

(ie) $u+iv = f(z) = f(re^{i\theta})$

we have $$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

(ie) $$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

Eqns are CR eqns in polar form.

Problems :

1) Test whether the function $f(z) = x^2 + iy^2$ is an analytic or not.

Soln :

Given $f(z) = x^2 + iy^2$

W.K.T $f(z) = u + iv$.

(ie) $u + iv = x^2 + iy^2$.

$u = x^2 \qquad v = y^2$

$u_x = 2x \qquad v_x = 0$.

$u_y = 0 \qquad v_y = 2y$.

$\Rightarrow \quad u_x \neq v_y \quad$ and $\quad v_x \neq -u_y$

$\therefore$ The given function is not an analytic function.

2) Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Soln :

Given $f(z) = \bar{z} = x - iy$

(ie) $u = x \quad , \quad v = -y$.

$\dfrac{\partial u}{\partial x} = 1 \qquad \dfrac{\partial v}{\partial x} = 0.$

$\dfrac{\partial u}{\partial y} = 0 \qquad \dfrac{\partial v}{\partial y} = -1$

$\Rightarrow \quad u_x \neq v_y$. and $v_x = -u_y$.

C-R eqns are not satisfied anywhere.

Hence, $f(z) = \bar{z}$ is not differentiable anywhere (or) nowhere differentiable.

3) Test the analyticity of the functions $f(z) = e^{x}(\cos y + i\sin y)$

soln:

Given $f(z) = e^{x}(\cos y + i\sin y)$

$= e^{x}\cos y + i e^{x}\sin y = u+iv$

$u = e^{x}\cos y$ $\qquad$ $v = e^{x}\sin y$

$\frac{\partial u}{\partial x} = e^{x}\cos y$ $\qquad$ $\frac{\partial v}{\partial x} = e^{x}\sin y$

$\frac{\partial u}{\partial y} = -e^{x}\sin y$ $\qquad$ $\frac{\partial v}{\partial y} = e^{x}\cos y$

Here, $u_x = v_y$ and $v_x = -u_y$.

(ie) C-R eqns satisfied.

Hence the given function is analytic.

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4) S.T $f(z) = |z|^2$ is differentiable at $z=0$ but not analytic at $z=0$.

soln:

Let $z = x+iy$

$\bar{z} = x-iy$.

$|z|^2 = z\bar{z} = (x+iy)(x-iy)$ $\qquad$ $(\because i^2 = -1)$

$f(z) = |z|^2 = x^2+y^2$.

(ie) $u+iv = x^2+y^2+i(0)$

$u = x^2+y^2$ $\qquad$ $v = 0$.

$\frac{\partial u}{\partial x} = 2x$. $\qquad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 2y$ $\qquad$ $\frac{\partial v}{\partial y} = 0$

$\therefore\ u_x \neq v_y$ and $u_y \neq v_x$. $\therefore f(z)$ is not analytic. (5)

No need: $\therefore$ [C-R eqns are not satisfied everywhere except at $z=0$.

(ie) $f(z)$ may be differentiable only at $z=0$

Now $u_x = 2x$, $u_y = 2y$, $v_x = 0$, $v_y = 0$ are continuous everywhere and in particular at $(0, 0)$.

$\therefore$ Sufficient conditions for differentiability are satisfied by $f(z)$ at $z=0$

(ie) $f(z)$ is differentiable at $z=0$ and not analytic at $z=0$]

Note: If $f(z)$ is differentiable at $z=z_0$, then $f(z)$ is continuous at $z=z_0$. This is nec. condn. for differentiability.

5) If $w = e^z$, find $\frac{dw}{dz}$ (or) S.T the function $w = e^z$ is analytic everywhere in the complex plane.

Soln: Let $z = x+iy$.

$$f(z) = e^z = e^{x+iy}$$

$$= e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

(ie) $u+iv = e^x \cos y + i e^x \sin y$.

$u = e^x \cos y$ $\quad\quad$ $v = e^x \sin y$

$u_x = e^x \cos y$ $\quad\quad$ $v_x = e^x \sin y$

$u_y = -e^x \sin y$ $\quad\quad$ $v_y = e^x \cos y$.

(ie) $u_x = v_y$ and $v_x = -u_y$.

$\therefore$ C-R equ: are satisfied.

$\because f(z) = u(x,y) + iv(x,y)$ be an analytic fn at the point z in a region R, then its derivative $f'(z)$ exits in R

$\Rightarrow \therefore f'(z) = u_x + iv_x = -iu_y + v_y$.

Let $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y$

$$= e^x (\cos y + i \sin y)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$\therefore \boxed{f'(z) = e^z}$$

$|||^{ly}$ $f'(z) = -iu_y + v_y = e^z$.

$\therefore$ $w = e^z$ is analytic everywhere in the complex plane.

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b) Test the analyticity of the function $w = \sin z$.

Soln:

$$w = f(z) = \sin z$$

$$u + iv = \sin(x+iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$\because$ $\boxed{\sin(ix) = i\sinh x \quad \text{and} \quad \cos(ix) = \cosh x}$

$$u + iv = (\sin x \cosh y) + i(\cos x \sinh y)$$

Now, $u = \sin x \cosh y$ $\quad \therefore v = \cos x \sinh y$.

$u_x = \cos x \cosh y$. $\quad v_x = -\sin x \sinh y$.

$u_y = \sin x \sinh y$ $\quad v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$. $\therefore$ C-R eqns are satisfied

$\therefore$ The fn: is analytic.

7) Determine whether the fn/: $2xy + i(x^2-y^2)$ is analytic or not.

Soln:

$$f(z) = 2xy + i(x^2-y^2)$$

$$u+iv = 2xy + i(x^2-y^2).$$

Take $u = 2x \cdot y$ $\quad\quad v = x^2-y^2.$

$u_x = 2y$ $\quad\quad v_x = 2x.$

$u_y = 2x$ $\quad\quad v_y = -2y.$

$\therefore$ $u_x \neq v_y$ and $v_x \neq -u_y$..

$\therefore$ $f(z)$ is not an analytic function.

8) If $f(z) = (x-y)^2 + 2i(x+y)$. S.T the C-R eqns are satisfied along the curve $x-y=1$.

Soln:

$$f(z) = (x-y)^2 + i(2(x+y))$$

$$u+iv = (x-y)^2 + i[2(x+y)]$$

Take $u = (x-y)^2$. $\quad\quad v = 2(x+y)$

$u_x = 2(x-y)$ $\quad\quad v_x = 2.$

$u_y = -2(x-y)$ $\quad\quad v_y = 2.$

$\Rightarrow$ $u_x = v_y$ $\quad\quad v_x = -u_y.$

$\cancel{2}(x-y) = \cancel{2}$ $\quad\quad 2 = -(-2(x-y))$

$\boxed{x-y=1}$. $\quad\quad \cancel{2} = \cancel{2}(x-y)$

$\boxed{1 = x-y.}$

$\therefore$ C-R eqns are satisfied along the curve $x-y=1$.

9) If $w = f(z)$ is analytic, Prove $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$.
where $z = x+iy$ and P.T $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$.

Soln:

$$w = f(z) = u(x,y) + iv(x,y)$$

As $f(z)$ is analytic, $u_x = v_y$ and $v_x = -u_y$

$\therefore$ C-R eqns are satisfied.

Derivatives exists and it is continuous.

$$\frac{dw}{dz} = f'(z) = u_x + iv_x \quad (\text{Diff } f(z) \text{ w.r.to } 'x')$$

$$f'(z) = u_y + iv_y \quad (\text{Diff } f(z) \text{ w.r.to } 'y')$$

(ie) $\frac{dw}{dz} = f'(z) = u_x + iv_x$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial w}{\partial x}. \longrightarrow (1)$$

From C-R eqns /:

$$\frac{dw}{dz} = f'(z) = u_x + iv_x.$$

$$= v_y - iu_y = -i(iv_y + u_y)$$

$$= -i(u_y + iv_y) = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= -i\frac{\partial}{\partial y}(u+iv) = -i\frac{\partial w}{\partial y} \longrightarrow (2)$$

From (1) & (2),

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

W.K.T $\frac{\partial w}{\partial \bar{z}} = 0$

$\therefore \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$ And |||ly $\frac{\partial^2 w}{\partial \bar{z} \partial z} = 0$. //

Properties of Analytic functions :

Laplace Equation :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$ is known as Laplace Equation in two dimensions.

Laplacian Operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$ is called the Laplacian operator and is denoted by ∇^2.

Property - 1 :

The real and imaginary parts of an analytic fn: $w = u+iv$ satisfy the laplace equation in two dimensions $\nabla^2 u = 0$ and $\nabla^2 v = 0$.

Proof :

Let $f(z) = w = u+iv$ be analytic

To prove : u and v satisfy Laplace Eqn:

(ie) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Given, $f(z)$ is analytic.

$\therefore$ u and v satisfy C-R eqns.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (1) \quad \Big| \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \longrightarrow (2).$$

Diff (1) p.w.r.to 'x',

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \longrightarrow (3).$$

p.w.r.to 'y' in (2) $\Rightarrow$ $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \longrightarrow (4)$$

③ + ④ $\Rightarrow$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y}$$

$$= 0.$$

$\therefore$ u satisfies Laplace equation.

<u>To prove :</u> $\nabla^2 v = 0$.

Diff ① p.w.r.to 'y',

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} \longrightarrow ⑤$$

Diff ② p.w.r.to 'x',

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \longrightarrow ⑥.$$

⑤ + ⑥ $\Rightarrow$ $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y}$$

$$= 0.$$

$\therefore$ v satisfies Laplace equation.

<u>Harmonic function (or) Potential function.</u>

A real function of two ~~va~~ real variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a harmonic function. If u is harmonic, then

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

<u>Conjugate harmonic fn/:</u>

If u and v are harmonic function such that u+iv is analytic, then each is called the conjugate harmonic function of the other.

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Property - 2.

If $w = u(x,y) + iv(x,y)$ is an analytic function the curves of the family $u(x,y) = a$ and the curves of the family $v(x,y) = b$ act orthogonally, where a and b are varying constants

(or)

When the function $f(z) = u+iv$ is analytic, show that $u =$ constant and $v =$ constant are orthogonal.

Proof :

Given $f(z)$ is an analytic function.

$\therefore$ By C-R equations,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Given, $u(x,y) = a \rightarrow (1)$ and $v(x,y) = b \rightarrow (2)$.

Diff p.w.r.to 'x'.

$$(1) \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = m_1$$

$$(2) \Rightarrow \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y} = m_2.$$

$$\text{(ie)} \quad m_2 = \frac{\partial u/\partial y}{\partial u/\partial x} \quad \text{(from C-R eqns).}$$

Product of slopes at their point of intersection $= m_1 m_2$

$$= -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \times \frac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)} = -1.$$

$\therefore \boxed{m_1 m_2 = -1}$

Hence, the two family of curves form an orthogonal system.

Property - 3

An analytic function with constant modulus is constant.

Proof :

Let $f(z) = u + iv$ be analytic.

By C-R eqns/:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x .$$

$$|f(z)| = \sqrt{u^2+v^2} = c \neq 0 .$$

$$|f(z)|^2 = u^2+v^2 = c^2$$

(ie) $u^2 + v^2 = c^2 \longrightarrow (1).$

Diff (1) p.w.r.to 'x',

$$2u \cdot \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0$$

$\cancel{u+v}$ $\qquad u \cdot \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0 . \longrightarrow (2)$

Diff (2) p.w.r.to 'y',

$$2u \cdot \frac{\partial u}{\partial y} + 2v \cdot \frac{\partial v}{\partial y} = 0 .$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0.$$

By C-R eqns,

$$v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} = 0 \longrightarrow (3)$$

$(2) \times u + (3) \times v \Rightarrow$

$$u^2\frac{\partial u}{\partial x} + \cancel{uv\frac{\partial v}{\partial x}} + v^2\frac{\partial u}{\partial x} - \cancel{uv\frac{\partial v}{\partial x}} = 0$$

$$(u^2+v^2)\frac{\partial u}{\partial x} = 0.$$

$\because \quad u^2+v^2 \neq 0, \quad \frac{\partial u}{\partial x} = 0.$

$(2) \times v - (3) \times u \Rightarrow$

$$\cancel{uv\frac{\partial u}{\partial x}} + v^2\frac{\partial v}{\partial x} - \cancel{uv\frac{\partial u}{\partial x}} + u^2\frac{\partial v}{\partial x} = 0.$$

$$(u^2+v^2)\frac{\partial v}{\partial x} = 0.$$

$\because \quad u^2+v^2 \neq 0, \quad \frac{\partial v}{\partial x} = 0.$

If $f(z) = u + iv$.

$f'(z) = u_x + iv_x$.

$f'(z) = 0$.

$\Rightarrow \quad f(z) = c$ is constant.

Property -4

An analytic function whose real part is const. must itself a constant.

(or)

If $f(z)$ is analytic, show that $f(z)$ is constant if real part of $f(z)$ is constant.

Proof :

Let $f(z) = u + iv$ be an analytic function.

By C-R equations,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x .$$

Given, u is constant (ie) $u = c$.

To prove : $f(z)$ is a constant.

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0.$$

By C-R eqns,

$$u_x = 0 \Rightarrow v_y = 0.$$

$$u_y = 0 \Rightarrow v_x = 0.$$

If $f(z) = u + iv$

$$f'(z) = u_x + iv_x .$$

$$\therefore f'(z) = 0 .$$

$$\Rightarrow f(z) = c .$$

Hence, $f(z)$ is constant.

Problems :

1) S.T $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Soln:

Let $z = x + iy \longrightarrow (1)$

$\bar{z} = x - iy \longrightarrow (2)$

From ① and ②, We get.

$$z + \bar{z} = 2x.$$

$$\therefore \quad x = \frac{z+\bar{z}}{2}. \longrightarrow ③$$

$$z - \bar{z} = 2iy$$

$$y = \frac{z-\bar{z}}{2i} \times \left(\frac{-i}{-i}\right)$$

$$= -\frac{i(z-\bar{z})}{2} \longrightarrow ④.$$

Now, $\frac{\partial x}{\partial z} = \frac{1}{2}$, $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$

$$\frac{\partial y}{\partial z} = \frac{-i}{2}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$$

$$\therefore \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z}.$$

$$= \tfrac{1}{2}\left(\partial/\partial x\right) - \tfrac{i}{2}\left(\partial/\partial y\right)$$

$$\partial/\partial z. = \tfrac{1}{2}\left(\partial/\partial x - i\,\partial/\partial y\right) \longrightarrow ⑤$$

$$\partial/\partial \bar{z} = \partial/\partial x \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}\right) + \frac{i}{2}\left(\frac{\partial}{\partial y}\right)$$

$$\partial/\partial \bar{z} = \frac{1}{2}\left[\partial/\partial x + i\,\partial/\partial y\right] \longrightarrow ⑥$$

Now, $\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\right]$

$$= \tfrac{1}{2}\left(\partial/\partial x - i\,\partial/\partial y\right)\left[\tfrac{1}{2}\left(\partial/\partial x + i\,\partial/\partial y\right)\right]$$

$$= \tfrac{1}{4}\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right) \qquad \because (i^2 = -1)$$

$$\therefore \quad \partial^2/\partial x^2 + \partial^2/\partial y^2 = 4\left(\partial^2/\partial z\,\partial \bar{z}\right) \quad //$$

2) If $f(z)$ is a regular fn: of z, Prove,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

(ie) $$\nabla^2 |f(z)|^2 = 4|f'(z)|^2$$

Soln:

Let $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2+v^2}.$$

$$|f(z)|^2 = u^2 + v^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2+v^2) + \frac{\partial^2}{\partial y^2}(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) + \frac{\partial^2}{\partial x^2}(v^2) + \frac{\partial^2}{\partial y^2}(v^2) \longrightarrow (1)$$

Now, $$\frac{\partial}{\partial x}(u^2) = 2u \cdot \frac{\partial u}{\partial x}.$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] \longrightarrow (2)$$

III^{ly} $$\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right] \longrightarrow (3).$$

Adding (2) & (3),

$$\frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$\left(\because u \text{ is harmonic, } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right)$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[u_x^2 + u_y^2\right]$$

By C-R eqns/: $u_x = v_y \quad , \quad u_y = -v_x$.

$$= 2\left[u_x^2 + (-v_x)^2\right]$$

$$= 2\left[u_x^2 + v_x^2\right]$$

$\Big[$ W.K.T, $f(z) = u + iv$.

$f'(z) = u_x + iv_x$.

$|f'(z)|^2 = u_x^2 + v_x^2$. $\Big]$

(ie) $\frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) = 2\,|f'(z)|^2 \longrightarrow (4)$

$|||^{ly}$ $\frac{\partial^2}{\partial x^2}(v^2) + \frac{\partial^2}{\partial y^2}(v^2) = 2\,|f'(z)|^2 \longrightarrow (5)$.

$\therefore$ Eqn (1) becomes .

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2 .$$

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3) If $f(z) = u + iv$ is analytic, P.T $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$

(ie) $\nabla^2 \log|f(z)| = 0$ (ie) $\log|f(z)|$ is harmonic .

Soln :

W.K.T, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial z\,\partial \bar{z}} \longrightarrow (1)$

$|f(z)|^2 = f(z)\, f(\bar{z})$

$|f(z)| = \left(f(z)\, f(\bar{z})\right)^{1/2}$.

$\log|f(z)| = \log\left(f(z)\, f(\bar{z})\right)^{1/2}$.

(ie) $\log|f(z)| = \frac{1}{2}[\log f(z) + \log f(\bar{z})]$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 2 \cdot \cancel{4} \frac{\partial^2}{\partial z \partial \bar{z}} \cancel{\tfrac{1}{2}}[\log f(z) + \log f(\bar{z})]$$

$(\because ①)$

$$= 2\frac{\partial^2}{\partial z \partial \bar{z}}[\log f(z) + \log f(\bar{z})]$$

$$= 2\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}[\log f(z) + \log f(\bar{z})]$$

$$= 2\frac{\partial}{\partial z}\left[0 + \frac{1}{f(\bar{z})} \cdot f'(\bar{z})\right]$$

$$= 2(0) = 0.$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0 \text{ //.}$$

---

4) If $f(z) = u+iv$ is a regular function of z, then S.T

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$$

Soln: W.K.T, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)$

x'ly by $|f(z)|^p$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)|f(z)|^p$$

Since, $z\bar{z} = |z|^2$ (or) $|z| = (z\bar{z})^{1/2}$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}[f(z) \cdot f(\bar{z})]^{p/2}$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}[(f(z))^{p/2} \cdot (f(\bar{z}))^{p/2}]$$

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$$= \cancel{4} \frac{\partial}{\partial z}\left[(\cancel{f}(z))^{\cancel{P/2}} \right] \frac{\partial}{\partial \cancel{\bar{z}}}$$

$$= 4 \frac{\partial}{\partial z}\left[(f(z))^{P/2} \cdot \frac{P}{2}\left(f(\bar{z})\right)^{P/2 - 1} f'(\bar{z}) \right]$$

$$= 4 \times \frac{P}{2}\left[\frac{P}{2}(f(z))^{P/2-1} f'(z) . (f(\bar{z}))^{P/2-1} f'(\bar{z}) \right]$$

$$= \cancel{4}\left(\frac{P}{\cancel{2}}\right)\left(\frac{P}{\cancel{2}}\right)\left[f'(z) f'(\bar{z}) \left(f(z) f(\bar{z})\right)^{P/2-1} \right]$$

$$= P^2 \left[|f'(z)|^2 \left(|f(z)|^2 \right)^{P/2-1} \right] \qquad \because (|z|^2 = z\bar{z})$$

$$= P^2 \left[|f'(z)|^2 \left(|f(z)|^{\cancel{2}} \right)^{\frac{P-2}{\cancel{2}}} \right]$$

$$= P^2 \, |f'(z)|^2 \, |f(z)|^{P-2}$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^P = P^2 \, |f(z)|^{P-2} \, |f'(z)|^2 \; //$$

Harmonic Functions.

Problem :

1). If $f(z) = e^z$, then S.T u and v are harmonic functions.

Soln:

$$f(z) = e^z$$

$$u + iv = e^{x+iy} = e^x e^{iy} \qquad \because (e^{ix} = \cos x + i \sin x)$$

$$= e^x (\cos y + i \sin y)$$

$$u + iv = e^x \cos y + i e^x \sin y$$

(ie) $u = e^x \cos y$ $\qquad$ $v = e^x \sin y$

$u_x = e^x \cos y$ $\qquad$ $v_x = e^x \sin y$

$u_{xx} = e^x \cos y$ $\qquad$ $v_{xx} = e^x \sin y$

And $u_y = -e^x \sin y$ $\qquad v_y = e^x \cos y$

$u_{yy} = -e^x \cos y$ $\qquad v_{yy} = -e^x \sin y$

Now, $u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0.$

$v_{xx} + v_{yy} = e^x \sin y - e^x \sin y = 0.$

$\therefore$ u and v are harmonic function.

Construction of Analytic functions.

Milne Thomsan Method.

(i) To find $f(z)$ when u is given,

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

(ie) $$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad (\text{by C-R})$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)}$$

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)}$$

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$\int f'(z)\,dz = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c.$$

where c is complex constant.

(ii) To find $f(z)$ when v is given,

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

(ie) $f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ (by C-R)

$$\phi_1(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)}$$

$$\phi_2(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$$

Now, $f'(z) = \phi_1(z,0) + i\,\phi_2(z,0)$.

$$\int f'(z)\,dz = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz.$$

$$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz + c$$

where c is a complex constant.

Problems :

1) Find an analytic function whose real part is $e^x(x\cos y - y\sin y)$

soln :

Given, $u = e^x(x\cos y - y\sin y)$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^x(\cos y + 0) + (x\cos y - y\sin y)e^x.$$

$$\phi_1(z,0) = e^z(1) + (z(1) - 0)e^z.$$

$$= e^z + ze^z.$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = e^x(-x\sin y - (y\cos y + \sin y)) + 0.$$

$$= -xe^x\sin y - e^x y\cos y - e^x\sin y.$$

$$\phi_2(z,0) = 0 - e^z(0) - 0 = 0.$$

By Milne's Thomson Method.

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int (e^z + ze^z)\,dz - i\int 0\,dz.$$

$$\int u\,dv = uv - \int v\,du$$

$$= \int (z+1)\,e^z\,dz.$$

$(u = z+1)$

$\int dv = e^z\,dz$

$(v = e^z)$

$du = 1$

$$= (z+1)\,e^z - (1+0)\,e^z + c$$

$$= ze^z + \cancel{e^z} - \cancel{e^z} + c$$

$$\therefore f(z) = ze^z + c.$$

---

8) Find a function w such that $w = u+iv$ is analytic, if $u = e^x \sin y$

soln:

$$u = e^x \sin y$$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^x \sin y + 0$$

$$\phi_1(z,0) = e^z(0) = 0$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = e^x \cos y.$$

$$\phi_2(z,0) = e^z.$$

By Milne's Thomson Method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz.$$

$$= 0 - i\int e^z\,dz.$$

$$= -i(e^z) + c.$$

---

3) construct the analytic function. $f(z)$ for which the real part is $e^x \cos y$.

soln:

Given $u = e^x \cos y$.

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^x \cos y.$$

$$\phi_1(z,0) = e^z(1) = e^z.$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = -e^x \sin y.$$

$$\phi_2(z,0) = 0.$$

By Milne's Thomson Method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int e^z\,dz - i(0)$$

$$\therefore f(z) = e^z + c$$

---

4) Determine the analytic fun/: whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

soln:

Given, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\phi_1(z,0) = 3z^2 + 6z.$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = -6xy - 6y.$$

$$\phi_2(z,0) = 0.$$

By Milne's Thomson Method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz.$$

$$= \int (3z^2 + 6z)\,dz - 0.$$

$$= 3\left(\frac{z^3}{3}\right) + 6\left(\frac{z^2}{2}\right) + c.$$

$$f(z) = z^3 + 3z^2 + c \,//.$$

5) Determine the analytic function whose real part is

$$\frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\frac{u}{v} = \frac{vu' - uv'}{v^2}$$

Soln: Given, $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}.$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2} \quad a^2 - b^2$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)\left[2\cos 2z - 2(1+\cos 2z)\right]}{(1-\cos 2z)^2}.$$

$$= \frac{\cancel{2\cos 2z} - 2 - \cancel{2\cos 2z}}{1-\cos 2z}$$

$$= \frac{-2}{1-\cos 2z}. = \frac{-1}{\left(\frac{1-\cos 2z}{2}\right)}$$

$$= \frac{-1}{\sin^2 z}$$

$$= -\operatorname{cosec}^2 z \,//.$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = 0$$

By Milne's Thomson Method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz.$$

$$= -\int \operatorname{cosec}^2 z\,dz + 0 \qquad \int + \operatorname{cosec}^2 x\,dx = -\cot x$$

$$f(z) = \cot z + c \;//$$

6) Find the analytic function $f(z) = u(x,y) + iv(x,y)$ whose real part is $y + e^x \cos y$.

soln:

Given, $u = y + e^x \cos y$.

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^x \cos y$$

$$\phi_1(z,0) = e^z.$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = 1 - e^x \sin y.$$

$$\phi_2(z,0) = 1$$

By Milne's Thomson Method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int e^z\,dz - i\int 1\,dz.$$

$$\therefore f(z) = e^z - iz + c.$$

7) If $f(z) = u+iv$ is an analytic function and $\cancel{u}-\cancel{v} = \cancel{e}^x$ $u-v = e^x(\cos y - \sin y)$ find $f(z)$ interms of $f(z)$

Soln:

$$f(z) = u+iv \longrightarrow (1)$$

(x'ly by(i)) $$i f(z) = iu - v \longrightarrow (2)$$

$$(1)+(2) \Rightarrow (1+i)f(z) = (u-v) + i(u+v).$$

$$\therefore F(z) = U + iV$$

where $F(z) = (1+i)f(z)$.

$$U = u-v$$

$$V = u+v$$

W.K.T, $U = u-v = e^x(\cos y - \sin y)$

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$$

$$\phi_1(z,0) = e^z$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

$$= -e^x(\sin y + \cos y)$$

$$\phi_2(z,0) = -e^z(1) = -e^z$$

By Milne's Thomson Method,

$$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz.$$

$$= \int e^z\,dz - i\int -e^z\,dz$$

$$= e^z + ie^z + c_1$$

$$\cancel{(1+i)} f(z) = e^z\cancel{(1+i)} + c_1$$

$$f(z) = e^z + c_1$$

8) Find the analytic function for which $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ is the ~~real~~ Img(real) part. Hence determine the analytic function $u+iv$ for which $u+v$ is the above function.

Soln:

$$\text{Let } f(z) = u+iv \longrightarrow (1)$$

$$i\,f(z) = iu - v \longrightarrow (2)$$

$$(1)+(2) \Rightarrow (1+i)f(z) = (u-v) + i\underbrace{(u+v)}_{V}$$

$$F(z) = U + i.V$$

Given, $$V = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\phi_1(x,y) = \frac{\partial V}{\partial y}$$

$$= \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}.$$

$$\phi_1(z,0) = 0.$$

$$\phi_2(x,y) = \frac{\partial V}{\partial x}.$$

$$= \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{\cancel{(1-\cos 2z)}\left[2\cos 2z - 2(1+\cos 2z)\right]}{(1-\cos 2z)^{\cancel{2}}}$$

$$= \frac{2\cancel{\cos 2z} - 2 - 2\cancel{\cos 2z}}{1-\cos 2z}.$$

$$= \frac{-2}{1-\cos 2z} = \frac{-1}{\left(\frac{1-\cos 2z}{2}\right)} = \frac{-1}{\sin^2 z}$$

$$= -\operatorname{cosec}^2 z$$

By Milne's Thomson Method,

$$F(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz.$$

$$= 0 + i\int -\operatorname{cosec}^2 z\,dz$$

$$= i\cot z + c_1$$

(ie) $(1+i)f(z) = i\cot z + c$

$$\therefore \quad f(z) = \left(\frac{i}{1+i}\right)\cot z + c.$$

$$= \left(\frac{i(1-i)}{(1+i)(1-i)}\right)\cot z + c$$

$$= \frac{(i+1)}{(1+1)}\cot z + c$$

$$f(z) = \left(\frac{1+i}{2}\right)\cot z + c.$$

9) Find the analytic function $f(z) = u+iv$ where $2u+v$

$2u+v = e^x(\cos y - \sin y)$

Soln:

Let $f(z) = u+iv \longrightarrow (1)$

$-if(z) = -iu+v \longrightarrow (2)$

$(1) \times 2 \Rightarrow 2u+i2v = 2f(z) \longrightarrow (3)$.

$(2) \Rightarrow v - iu = -if(z) \longrightarrow (4)$

$(3) + (4) \Rightarrow (2u+v) + i(2v-u) = (2-i)f(z)$

i.e) $F(z) = U+iV$

where, $F(z) = (2-i)f(z)$

$U = 2u+v$

$V = 2v-u$

Given, $2u+v = e^x(\cos y - \sin y)$

i.e) $U = e^x(\cos y - \sin y)$

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y).$$

$$\phi_1(z,0) = e^z.$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

$$\phi_2(z,0) = e^z(-1) = -e^z.$$

By Milne's Thomson Method,

$$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int e^z\,dz - i\int -e^z\,dz$$

$$F(z) = e^z + ie^z + c.$$

$$= (1+i).e^z + c$$

$$(2-i)\, f(z) = (1+i)\, e^z + c.$$

$$f(z) = \frac{(1+i)}{(2-i)}\, e^z + c.$$

$$= \frac{(1+i)(2+i)}{(4+1)}\, e^z + c.$$

$$= \frac{2+3i-1}{5}\, e^z + c.$$

(ie) $f(z) = \left(\frac{3i+1}{5}\right) e^z + c.$

---

10) Determine the analytic function $f(z) = u+iv$ given that $3u+2v = y^2 - x^2 + 16xy$.

soln:

Let $f(z) = u+iv. \longrightarrow (1).$

$$-i\, f(z) = -iu + v \longrightarrow (2).$$

$(1) \times 3 \Rightarrow 3f(z) = 3u + i3v \longrightarrow (3)$

$(2) \times 2 \Rightarrow -2i\, f(z) = -2iu + 2v \longrightarrow (4)$

$(3) + (4) \Rightarrow (3-2i) f(z) = (3u+2v) + i(3v-2u)$

$$F(z) = U + iV$$

where $F(z) = (3-2i)\, f(z)$

$$U = 3u+2v$$

$$V = 3v - 2u.$$

Given, $\quad 3u+2v = y^2 - x^2 + 16xy$.

(ie) $\quad U = y^2 - x^2 + 16xy$.

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = -2x + 16y$$

$$\phi_1(z,0) = -2z.$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = 2y + 16x.$$

$$\phi_2(z,0) = 16z.$$

By Milne's Thomson Method,

$$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz.$$

$$= \int -2z\,dz - i\int 16z\,dz$$

$$= -\cancel{2}\left(\frac{z^2}{\cancel{2}}\right) - i\,\overset{8}{\cancel{16}}\left(\frac{z^2}{\cancel{2}}\right) + c.$$

$$= -z^2 - i8z^2 + c.$$

$$= -z^2(1+8i) + c.$$

(ie) $\quad (3-2i)f(z) = -z^2(1+8i) + c.$

$$f(z) = -\frac{z^2(1+8i)}{3-2i} + c.$$

$$= \frac{-z^2(1+8i)(3+2i)}{9+4} + c.$$

$$= \frac{-z^2(26i-13)}{13} + c.$$

$$= \frac{-z^2(\cancel{13})(2i-1)}{\cancel{13}} + c.$$

$$= -z^2(2i-1) + c.$$

$$\therefore \quad f(z) = z^2(1-2i) + c \;//$$

CONFORMAL MAPPINGS :

A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense is said to be conformal at that point.

Isogonal :

A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be isogonal at that point.

Note :

(i) A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if $f'(z_0) \neq 0$.

(ii) The point at which the mapping $w = f(z)$ is not conformal, (ie) $f'(z) = 0$, is called a critical point of the mapping.

Fixed points of mapping.

Fixed or invariant point of a mapping $w = f(z)$ are points that are mapped onto themselves, are "kept fixed" under the mapping. Thus they are obtained from $w = f(z) = z$.

The identity mapping $w = z$ has every point as a fixed point.

The mapping $w = \bar{z}$ has indefinitely many fixed points.

$w = 1/z$ has two, a rotations has one and a translation has none in the ~~infif~~ infinite plane.

Standard Transformations :

(i) $w = c + z$ → This transformation transforms a circle to an equal circle.

(ii) $w = cz$ → circles will be mapped into circles by this transformation.

(iii) $w = 1/z$ → This transformation transforms a circle to a st. line. (passes — origin)

eqn: $x^2+y^2+z^2+2gx+2fy+c=0$.

Problems :

1) Find the image of the circle $|z| = 1$ by the transformation $w = z + 2 + 4i$ $(w = c + z)$

soln:

Given, $w = z + 2 + 4i$

(ie) $u + iv = x + iy + 2 + 4i$

$u + iv = (x+2) + i(4+y)$

Equating real and imaginary parts,

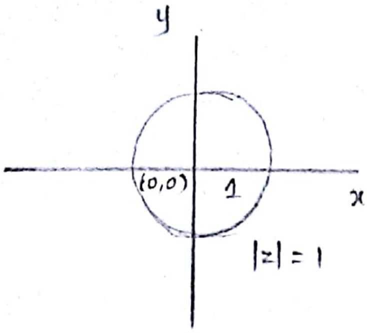
$u = x + 2$	$v = y + 4$
$\Rightarrow x = u - 2$	$y = v - 4$

Given $|z| = 1$,

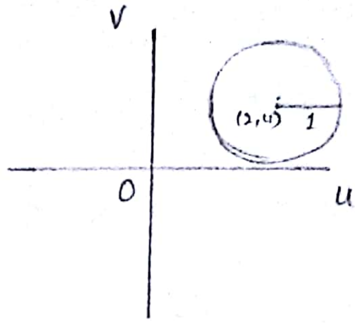
$$x^2 + y^2 = 1.$$

$$(u-2)^2 + (v-4)^2 = 1.$$

(ie) the circle $x^2+y^2=1$ is mapped to $(u-2)^2+(v-4)^2=1$ in w plane, is also a circle with $(2,4)$ & radius 1.



(Z-plane)



(w-plane)

2) Determine the region 'D' of the w-plane into which the triangular region D enclosed by the lines $x=0$, $y=0$, $x+y=1$ is transformed under the transformation $w = 2z$. (Magnification)

Soln:

Let $w = 2z$. $\quad (w = u+iv,\ z = x+iy)$

$$u+iv = 2(x+iy)$$

(ie) $u+iv = 2x + 2iy$.

Equating, $u = 2x$, $v = 2y$. $\Rightarrow (x = u/2,\ y = v/2)$

For $x = 0$, $\Rightarrow$ $u = 0$.

$y = 0$ $\Rightarrow$ $v = 0$.

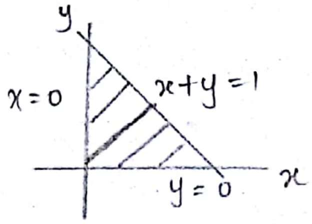
$x+y = 1$ $\Rightarrow$ $u/2 + v/2 = 1$.

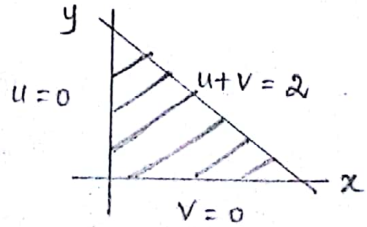
(ie) $u + v = 2$

In z plane, line $x=0$ is transformed to $u=0$ in w-plane.

In z plane, line $y=0$ is transformed to $v=0$ in w-plane.

In z plane, line $x+y=1$ is transformed to $u+v=2$ in w-plane.





3) Find the image of $|z-2i| = 2$ under the transformation $w = 1/z$. (Translation)

Soln: Given, $w = 1/z$. ($w = u+iv$, $z = x+iy$)

$$\Rightarrow z = 1/w$$

$$z = 1/w = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}.$$

(ie) $$x+iy = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} + \frac{i(-v)}{u^2+v^2}$$

Equating, $$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Given, $$|z-2i| = 2.$$

$$|x+iy-2i| = 2.$$

$$|x+i(y-2)| = 2.$$

Squaring, $$x^2+(y-2)^2 = 4. \quad (x=0,\ y=2,\ r=2)$$

$C(0,2)$, $r = 2$.

$$x^2+y^2-4y+\cancel{4} = \cancel{4}$$

$$x^2+y^2-4y = 0. \rightarrow \text{circle (In z plane)}$$

Sub x & y,

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0.$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0.$$

$$(u^2+v^2) + 4v(u^2+v^2) = 0.$$

$$(u^2+v^2)(1+4v) = 0.$$

$\because (u^2+v^2 \neq 0)$

$\therefore$ $1+4v = 0.$ $\rightarrow$ (St. line in w-plane)

$(u=0,\ v=-1/4)$

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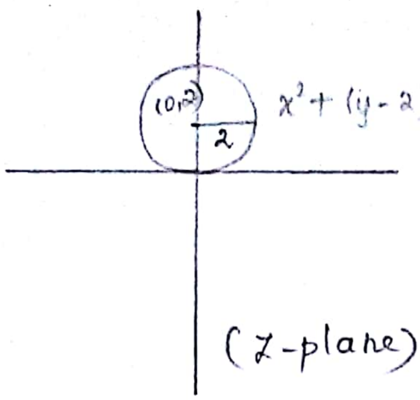
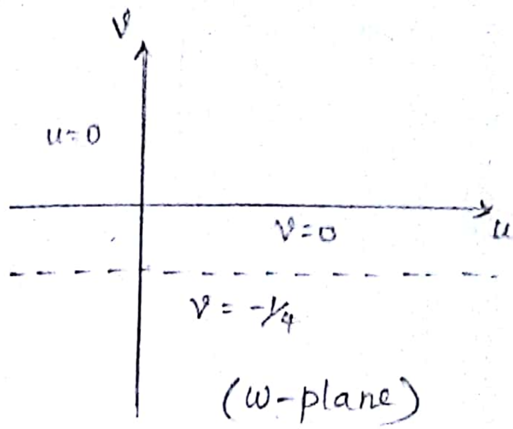




4) Find the image of the circle $|z-1|=1$ in the complex plane under the mapping $w = 1/z$.

Soln:

Given transformation, $w = 1/z$.

i.e) $z = 1/w$.

Let $w = u+iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}.$$

$$= \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

Equating real and imaginary parts,

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

$$\Rightarrow |z-1| = 1.$$

$$|x+iy-1| = 1.$$

$$|(x-1)+iy| = 1.$$

$$(x-1)^2+y^2 = 1. \quad (x=1,\ y=0)$$

$$x^2-2x+\cancel{1}+y^2 = \cancel{1}.$$

$$x^2-2x+y^2 = 0.$$

Sub ① & ②, we get,

$$\left(\frac{u}{u^2+v^2}\right)^2 - 2\left(\frac{u}{u^2+v^2}\right) + \left(\frac{-v}{u^2+v^2}\right)^2 = 0.$$

$$\frac{u^2}{(u^2+v^2)^2} - \frac{2u}{(u^2+v^2)} + \frac{v^2}{(u^2+v^2)^2} = 0.$$

$$\frac{u^2+v^2-2u(u^2+v^2)}{(u^2+v^2)^2} = 0.$$

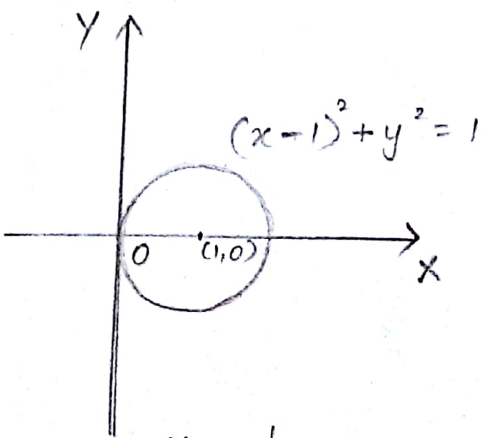
$$\frac{\cancel{(u^2+v^2)}(1-2u)}{\cancel{(u^2+v^2)}^2} = 0.$$

$$\frac{1-2u}{u^2+v^2} = 0.$$

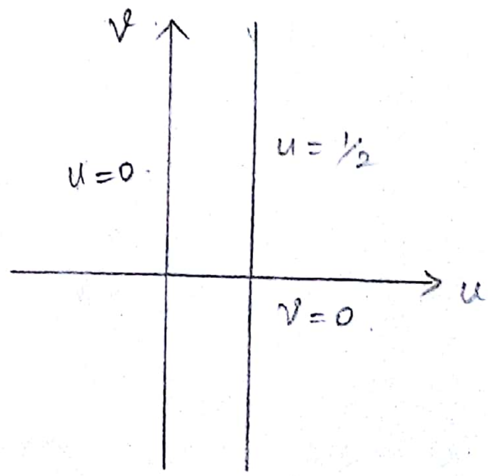
$$(1-2u) = 0. \qquad (\because u^2+v^2 \neq 0)$$

$$\therefore \boxed{u = 1/2.}$$

which is a straight in the w-plane.



z - plane



w- plane

5) Find the image of the infinite strips (i) $1/4 < y < 1/2$ (ii) $0 < y < 1/2$ under the transformation $w = 1/z$.

Soln:

$$w = 1/z \Rightarrow z = 1/w.$$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}.$$

$$x+iy = \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$

(ie) $x = \dfrac{u}{u^2+v^2}$, $y = \dfrac{-v}{u^2+v^2}$

(i) $1/4 < y < 1/2$.

$y = 1/4$

$$\frac{1}{4} = \frac{-v}{u^2+v^2}.$$

$$u^2+v^2 = -4v.$$

$$u^2+v^2+4v = 0.$$

$$u^2+v^2+4v+4-4 = 0.$$

$$u^2+(v+2)^2-4 = 0.$$

$$\Rightarrow u^2+(v+2)^2 = 4 \quad (u=0,\ v=-2,\ r=2)$$

is a circle with centre $(0,-2)$ and radius 2.

when $y = 1/2$

$$1/2 = \frac{-v}{u^2+v^2}.$$

$$u^2+v^2 = -2v.$$

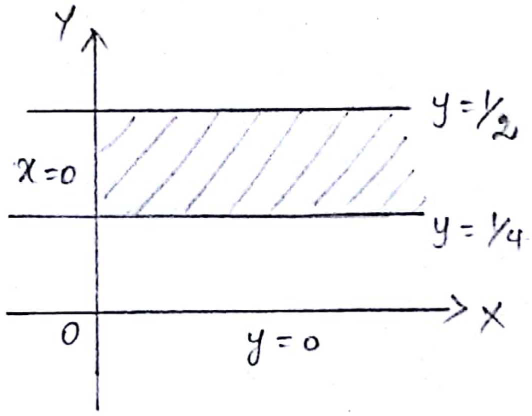
$$u^2+v^2+2v = 0.$$

$$u^2+v^2+2v+1-1 = 0.$$

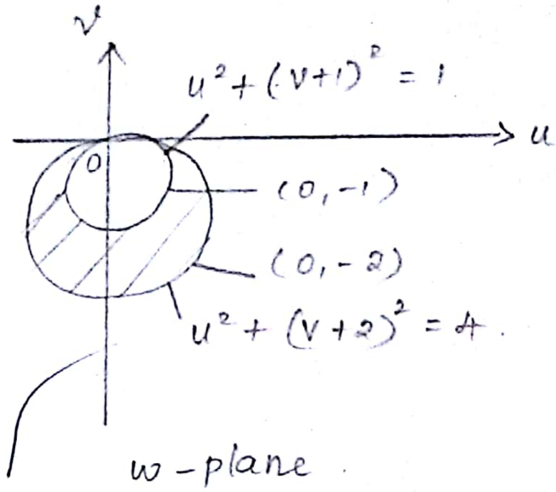
$$u^2+(v+1)^2-1 = 0.$$

$$u^2+(v+1)^2 = 1. \quad (u=0,\ v=-1,\ r=1)$$

which is a circle whose centre is at $(0,-1)$ in the w-plane and radius $= 1$.



z-plane



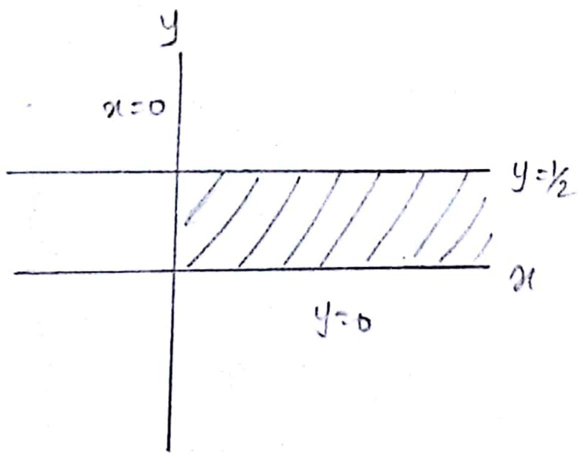
w-plane

Hence, $1/4 < y < 1/2$ is transformed into the region in between circles. $u^2+(v+1)^2 = 1$ & $u^2+(v+2)^2 = 4$ in the w-plane.

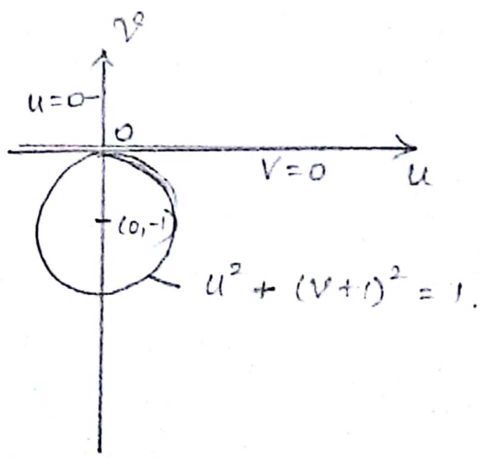
(ii) $0 < y < 1/2$.

when $y = 0$ $\Rightarrow$ $v = 0$.

when $y = 1/2$ $\Rightarrow$ $u^2+(v+1)^2 = 1$.



z-plane



w-plane

$\therefore$ $0 < y < 1/2$ is mapped into the region outside the circle $u^2+(v+1)^2 = 1$ in the w-plane.

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6) what will be the image of a circle passing through the origin in the XY plane under the transformation $w = 1/z$?

soln: Given $w = 1/z \Rightarrow z = 1/w$.

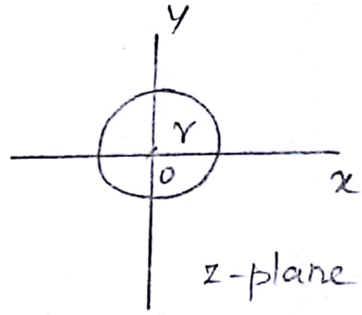
$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$

(ie) $x = \dfrac{u}{u^2+v^2}$, $y = \dfrac{-v}{u^2+v^2}$.

Given, $x^2+y^2 = r^2$ $(x=0, y=0, rad = r)$

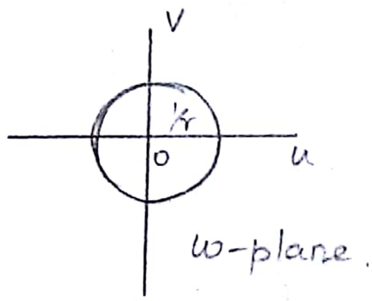
sub x and y, we get.



$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 = r^2.$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = r^2$$

$$\frac{\cancel{(u^2+v^2)}}{(u^2+v^2)^{\cancel{2}}} = r^2.$$



$$\frac{1}{u^2+v^2} = r^2.$$

$$u^2+v^2 = 1/r^2 \quad (u=0, v=0, rad = 1/r)$$

∴ The image of a circle passing through the origin in the XY plane is a circle passing through the origin in the w-plane and radius $= 1/r$.

7) Determine the image of $1 < x < 2$ under the mapping $w = 1/z$.

Soln. Given, $w = 1/z$.

$$\Rightarrow z = 1/w.$$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}.$$

$$x+iy = \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$

(ie) $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$.

Given, $1 < x < 2$

when $x = 1$,

$$1 = \frac{u}{u^2+v^2}.$$

$$u^2+v^2 = u.$$

$$u^2+v^2-u = 0.$$

$$(u^2-u)+v^2 = 0.$$

$$\Rightarrow u^2-u+\frac{1}{4}-\frac{1}{4}+v^2 = 0.$$

$$(u-1/2)^2+v^2-\frac{1}{4} = 0.$$

$$(u-1/2)^2+v^2 = \left(\frac{1}{2}\right)^2. \quad (u = 1/2, v = 0, r = 1/2)$$

is a circle whose centre is $(1/2, 0)$ and radius $= 1/2$.

when $x = 2$.

$$2 = \frac{u}{u^2+v^2}$$

$$2(u^2+v^2)-u = 0.$$

$$\Rightarrow u^2+v^2-\frac{u}{2} = 0.$$

$$\Rightarrow u^2+v^2-u/2+1/16-1/16 = 0.$$

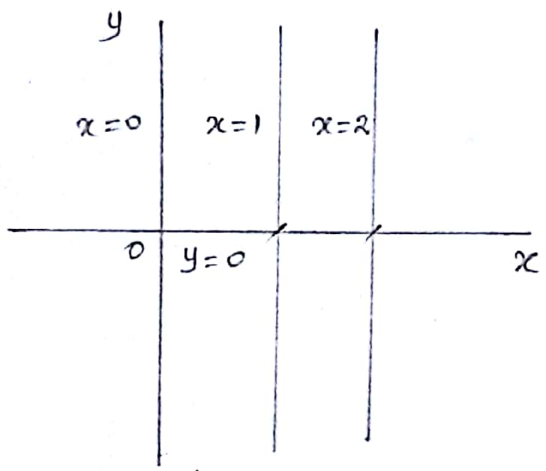
$$\Rightarrow \underbrace{\left(u^2 - \frac{u}{2} + \frac{1}{16}\right)} + v^2 - \frac{1}{16} = 0.$$

$$\left(u - \tfrac{1}{4}\right)^2 + v^2 - \left(\frac{1}{4}\right)^2 = 0.$$

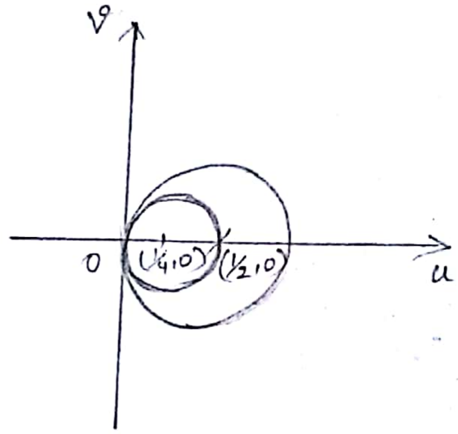
$$\left(u - \tfrac{1}{4}\right)^2 + v^2 = \left(\tfrac{1}{4}\right)^2 . \quad \left(u = \tfrac{1}{4},\ v = 0,\ r = \tfrac{1}{4}\right)$$

is a circle whose centre $(1/4, 0)$ and radius $= 1/4$.

$\therefore$ $1 < x < 2$ is transformed into the region is between the circles in the w-plane.



z - plane



w-plane

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8) S.T the transformation $w = 1/z$ transforms all circles and st. lines in the z-plane into circles or st. lines in the w-plane.

Soln: Given, $w = 1/z$. $\Rightarrow z = 1/w$.

Now, $w = u + iv \mid z \Rightarrow \frac{1}{u+iv}$.

$$= \frac{u - iv}{(u+iv)(u-iv)}$$

$$x + iy = \frac{u - iv}{u^2+v^2}.$$

$$\Rightarrow x + iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}.$$

General equation of circle,

$$a(x^2+y^2) + 2gx + 2fy + c = 0.$$

$$a\left(\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right) + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

$$a\left(\frac{\cancel{u^2+v^2}}{(u^2+\cancel{v^2})^2}\right) + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0.$$

$$a\left(\frac{1}{u^2+v^2}\right) + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0.$$

$$a + 2gu + (-)2fv + c(u^2+v^2) = 0$$

$$\Rightarrow \quad c(u^2+v^2) + 2gu - 2fv + a = 0.$$

is the transformed equation of circle.

---

8) P.T the transformation $w = \frac{z}{1-z}$ maps the upper half of the z-plane into the upper half of the w-plane. what is the image of the circle $|z|=1$ under this transformation?

Soln

$$w = \frac{z}{1-z}$$

$$u+iv = \frac{x+iy}{1-(x+iy)} = \frac{x+iy}{1-x-iy}$$

$$= \frac{(x+iy)}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$$

$$= \frac{x(1-x) + ixy + iy(1-x) - y^2}{(1-x)^2 + y^2}$$

$$= \frac{x - x^2 + i\cancel{xy} + iy - i\cancel{xy} - y^2}{(1-x)^2 + y^2}$$

$$= \frac{x - x^2 - y^2 + iy}{(1-x)^2 + y^2}.$$

(ie) $u = \dfrac{x - x^2 - y^2}{(1-x)^2 + y^2}$, $v = \dfrac{y}{(1-x)^2+y^2}$

upper half of z-plane is $y > 0$.

when $y > 0 \Rightarrow v > 0$. (for $(1-x)^2 + y^2 > 0$)

Thus, upper half of z-plane is mapped onto the upper half of the w-plane.

Image of $|z| = 1$:

Given $|z| = 1$.

(ie) $|x+iy| = 1$.

$x^2 + y^2 = 1$. $\longrightarrow$ (1)

W.K.T $u = \dfrac{x - x^2 - y^2}{(1-x)^2 + y^2}$ $v = \dfrac{y}{(1-x)^2+y^2}$

$= \dfrac{x - (x^2+y^2)}{1 + x^2 - 2x + y^2}$ $= \dfrac{y}{1 - 2x + x^2 + y^2}$.

$u = \dfrac{x - 1}{1 - 2x + 1}$ & $v = \dfrac{y}{1 - 2x + 1}$ (by (1))

$= \dfrac{x-1}{2-2x}$. $= \dfrac{y}{2-2x}$.

$= \dfrac{\cancel{x-1}}{-2(\cancel{x-1})}$ $= \dfrac{y}{-2(x-1)}$

$u = -\frac{1}{2}$ and. $v = \dfrac{-y}{2(x-1)}$

The region $|z| < 1$ transforms to $u > -\frac{1}{2}$.

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9) Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$.

Soln:

$$w = 5z$$

$$u+iv = 5(x+iy)$$

$$u = 5x \Rightarrow x = \frac{u}{5}$$

$$v = 5y \Rightarrow y = \frac{v}{5}$$

Given, $|z| = \lambda$.

$$|x+iy| = \lambda$$

$$(x^2+y^2)^{1/2} = \lambda$$

$$x^2+y^2 = \lambda^2.$$

$$\Rightarrow \left(\frac{u}{5}\right)^2 + \left(\frac{v}{5}\right)^2 = \lambda^2.$$

$$u^2+v^2 = 25\lambda^2.$$

$$\Rightarrow u^2+v^2 = (5\lambda)^2.$$

Image of $|z| = \lambda$ in z-plane is transformed into $u^2+v^2 = (5\lambda)^2$ in the w-plane.

---

10) Find the map of the line $x = c$ (a non-zero constant) under the transformation $w = \sin z$.

Soln:

$$w = \sin z$$

$$u+iv = \sin(x+iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$u+iv = \sin x \cosh y + i\cos x \sinh y$$

$$\Rightarrow u = \sin x \cosh y$$

$$v = \cos x \sinh y.$$

W.K.T $\cosh^2 y - \sinh^2 y = 1.$

$$\left(\frac{u}{\sin x}\right)^2 - \left(\frac{v}{\cos x}\right)^2 = 1.$$

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1.$$

when $x = c$, we get,

$$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1.$$

(ie) $\dfrac{u^2}{a^2} - \dfrac{v^2}{b^2} = 1$. where $a = \sin c$, $b = \cos c$.

which is hyperbola at $(1,0)$ and $(-1,0)$.

Bilinear transformation:

The transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$, where a, b, c, d are complex numbers. is called a bilinear transformation.

Fixed points (or) Invariant points.

The fixed points of the transformation,

$w = \dfrac{az+b}{cz+d}$ is obtained from

$z = \dfrac{az+b}{cz+d}$. (or)

$$cz^2 + (d-a)z - b = 0.$$

The bilinear trans/: which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Suppose $z_1 = \infty$, we replace $\dfrac{z-z_1}{z_2-z_1}$ by 1.

Problems:

1) Find the invariant points of the transformation $w = \dfrac{2z-4i}{iz+5}$

Soln:

The invariant points of the transformation is given by

$$z = \frac{2z-4i}{iz+5}$$

$$z(iz+5) = 2z-4i$$

$$iz^2+5z-2z+4i = 0.$$

$$iz^2+3z+4i = 0.$$

(x'y by i), $\quad i^2z^2+3iz+4i^2 = 0.$

$$-z^2+3iz-4 = 0.$$

$$\Rightarrow \quad z^2-3iz+4 = 0.$$

4

-4 1

$$\Rightarrow \quad z^2-4iz+iz+4 = 0$$

$$\Rightarrow \quad z(z+i)-4i(z+i) = 0.$$

$$\Rightarrow \quad (z-4i)(z+i) = 0.$$

$\therefore$ $z = 4i, -i$ are the invariant points.

---

2) Find the fixed points of $w = \dfrac{z-1}{z+1}$

Soln:

The invariant points of the transformation is

$$z = \frac{z-1}{z+1}$$

$$z(z+1) = z-1.$$

$$z^2+z = z-1.$$

$$z^2+\cancel{z}-\cancel{z}+1 = 0$$

$$z^2+1 = 0 \Rightarrow z^2 = -1.$$

$$z = \pm\sqrt{-1}.$$

$$\boxed{z = \pm i.}$$

3) Find the invariant points of the transformation $w = 2 - 2/z$.

Soln:

consider, $z = 2 - 2/z = \frac{2z - 2}{z}$.

$z^2 - 2z + 2 = 0$.

$z = \frac{2 \pm \sqrt{4 - 8}}{2}$.

$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$.

$\therefore \; z = 1 \pm i$

---

4) Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, i, -i$ respectively.

Soln:

Given, $z_1 = -2$, $z_2 = 0$ $z_3 = 2$.

$w_1 = 0$ $w_2 = i$ $w_3 = -i$

Let the transformation be,

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{(w - 0)(i - (-i))}{(w - (-i))(i - 0)} = \frac{(z - (-2))(0 - 2)}{(z - 2)(0 - (-2))}$$

$$\frac{w(2\cancel{i})}{(w + i)\cancel{i}} = \frac{(z + 2)(-\cancel{2})}{(z - 2)(\cancel{2})}$$

$$\frac{2w}{w + i} = -\frac{(z + 2)}{(z - 2)}$$

$2w(z - 2) = -(z + 2)(w + i)$

$2wz - 4w = -(zw + iz + 2w + 2i)$.

$2wz - 4w = -zw - iz - 2w - 2i$

$$2wz - 4w + zw + 2w = -zi - 2i.$$

$$w(2z - 4 + z + 2) = -i(z+2)$$

$$w(3z-2) = -i(z+2).$$

$$w = \frac{-i(z+2)}{(3z-2)}$$

5) Determine the bilinear transformation that maps the $-1, 0, 1$ in the z-plane onto the points $0, i, 3i$ in the w-plane.

Soln:

Given $z_1 = -1$, $z_2 = 0$, $z_3 = 1$

$w_1 = 0$, $w_2 = i$, $w_3 = 3i$

Let the transformation be,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2\cancel{i})}{(w-3i)\cancel{i}} = \frac{(z+1)(-\cancel{1})}{(z-1)(\cancel{1})}$$

$$\frac{\cancel{-}2w}{w-3i} = \frac{\cancel{-}(z+1)}{(z-1)}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}.$$

$$2w(z-1) = (w-3i)(z+1).$$

$$2wz - 2w = wz + w \underbrace{- 3iz - 3i}.$$

$$2wz - 2w - wz - w = -3i(z+1)$$

$$w(2z - 2\cancel{z} - z - 1) = -3i(z+1)$$

$$w(z-3) = -3i(z+1)$$

$$w = -\frac{3i(z+1)}{z-3}$$

6) Find the bilinear transformation that maps the points $\infty, i, 0$ into $0, i, \infty$ respectively.

Soln:

Given, $z_1 = \infty$, $z_2 = i$, $z_3 = 0$

$w_1 = 0$, $w_2 = i$, $w_3 = \infty$

Let the transformation be,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

$$\frac{(w-w_1)\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{\cancel{w_3}\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\cancel{z_1}\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\cancel{z_1}\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)\left(\frac{i}{\infty}-1\right)}{\left(\frac{w}{\infty}-1\right)(i-0)} = \frac{\left(\frac{z}{\infty}-1\right)(i-0)}{(z-0)\left(\frac{i}{\infty}-1\right)}$$

$$\frac{(w-0)(0\cancel{-1})}{(0\cancel{-1})(i-0)} = \frac{(0\cancel{-1})(i-0)}{(z-0)(0\cancel{-1})}$$

$$\frac{w}{i} = \frac{i}{z}.$$

$$w = \frac{i^2}{z} = -1/z.$$

$$\boxed{w = -1/z}$$

8) Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$. S.T the transformation maps the interior of the unit circle of the z-plane onto the upper half of the w-plane.

soln:

Given, $z_1 = 1$, $z_2 = i$, $z_3 = -1$.

$w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

Transformation be,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{\cancel{w_3}\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(0-\cancel{1})}{(0-\cancel{1})(1-0)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w = \frac{(z-1)(i+1)}{(z+1)(i-1)} \times \frac{(i+1)}{(i+1)}$$

$$= \frac{(z-1)}{(z+1)} \times \left(\frac{\cancel{1}+i+i+\cancel{1}}{-1-1}\right)$$

$$= \frac{(z-1)}{(z+1)}\left(\frac{\cancel{2}i}{-\cancel{2}}\right)$$

$$w = -i\frac{(z-1)}{(z+1)}$$

$w(z+1) = -i(z-1)$

$wz + w + iz \cancel{+} i = 0$

$(w+i)z = i - w$.

$z = \frac{-(w-i)}{w+i}$

Thus region $|z|<1$ gives onto $\left|\frac{w-i}{w+i}\right| < 1$

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Let $w = u+iv$, we get $\left|\dfrac{u+iv-i}{u+iv+i}\right| < 1.$

$$|u+iv-i| < |u+iv+i|.$$

$$|u+i(v-1)| < |u+i(v+1)|.$$

$$\cancel{u^2} + (v-1)^2 < \cancel{u^2} + (v+1)^2.$$

$$(v-1)^2 < (v+1)^2.$$

$$\cancel{v^2} - 2v + \cancel{1} < \cancel{v^2} + 2v + \cancel{1}$$

$$-4v < 0.$$

$$\text{(ie)} \quad v > 0.$$

∴ Interior of unit circle is mapped onto the upper half of the w-plane.

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9) Find the bilinear transformation which maps $z=0$ onto $w=-i$ and has -1 and 1 as the invariant points. Also S.T under this transformation the upper half of the z-plane maps onto the interior of the unit circle in the w-plane.

Soln:

$z_1 = 0. \quad z_2 = -1 \quad z_3 = 1$

$w_1 = -i \quad w_2 = -1 \quad w_3 = 1.$

Transformation be,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

$$\frac{(w+i)(-1-1)}{(w-1)(-1-(-i))} = \frac{(z-0)(-1-1)}{(z-1)(-1-0)}$$

$$\frac{(w+i)(-\cancel{2})}{(w-1)(-1+i)} = \frac{z(-\cancel{2})}{(z-1)(-1)}$$

$$\frac{w+i}{(w-1)(-1+i)} = \frac{-z}{z-1.}$$

$$(w+i)(z-1) = \underbrace{-z(w-1)(-1+i)} = (-zw+z)(-1+i)$$

$$w\cancel{z} - w + i\cancel{z} - i = z\cancel{w} - izw - z + iz.$$

$$w(\cancel{z}-1-\cancel{z}+iz) = i - \cancel{iz} - z + \cancel{iz}.$$

$$-w-i = -izw-z$$

$$-w+izw = i-z$$

$$-w(1-iz) = -(z-i)$$

$$w(iz-1) = (i-z)$$

$$\cancel{-}w(1-iz) = \cancel{-}(z-i)$$

$$w = \frac{z-i}{1-iz}$$

Let $z=i$ upper half of the z-plane.

$$w = \frac{\cancel{i}-\cancel{i}}{1+1} = 0.$$

$\therefore$ upper half of the z-plane maps onto the interior of the unit circle in the w-plane.

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10) Find the bilinear trans/: that maps the points $1+i, -i, 2-i$ of the z-plane into the points $0, 1, i$ of the w-plane.

Soln:

$z_1 = 1+i$, $\quad z_2 = -i \quad z_3 = 2-i$

$w_1 = 0 \quad w_2 = 1 \quad w_3 = i$

Transformation be,

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

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$$\frac{(w-0)(1-i)}{(w-i)(1-0)} = \frac{(z-(1+i))(-\cancel{1}-(2-\cancel{1}))}{(z-(2-i))(-i-(1+i))}$$

$$\frac{w(1-i)}{(w-i)} = \frac{(z-1-i)(\cancel{-}2)}{(z-2+i)(\cancel{-}(2i+1))}$$

$$\frac{w-wi}{w-i} = \frac{2z-2-2i}{2iz+z-4i\underbrace{-2+2(i^2)}_{-4}+i}$$

$$\frac{w-wi}{w-i} \times \frac{2z-2-2i}{2iz+z-3i-4}$$

$$2iwz+wz-3iw-4w+2wz-iwz-3w+4iw$$

$$-(2wz-2w-2iw-2iz+2i-2) = 0.$$

$$wzi+3wi-5w+wz+2iz-2i+2 = 0.$$

$$wzi+3wi-5w+wz = -2iz+2i-2$$

$$w(zi+3i-5+z) = -2i(z-1-i)$$

$$\cancel{-}w(-z+5-3i-zi) = \cancel{-}2i(z-1-i)$$

$$w = \frac{2i(z-1-i)}{-z+5-3i-zi}$$

UNIT - IV.

Complex Integration

Complex integration - Statement and applications of cauchy's integral theorem and cauchy's integral formula - Taylor and Laurent expansions - Singular points - Residues - Residue Theorem - Application of Residue theorem. to evaluate real integrals - unit circle and Semi-circular contour (excluding poles on boundaries).

Cauchy's theorem (or) cauchy's Integral Theorem (or) Cauchy's Fundamental Theorem.

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simply closed curve c, then,

$$\boxed{\int_c f(z)\,dz = 0}$$

Cauchy's integral formula.

If $f(z)$ is analytic inside and on a closed curve c of a simply connected region R and if 'a' is any point with in c, then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz ,}$$

the integration around c being taken in the positive direction.
