Time Domain Analysis of CT systems

There are 2 basic methods for analysing an LTIC system for a given input.

- In the first method, we develop a mathematical description for CT system. It is an ordinary linear differential eqn. with constant coeffs. of the form.

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

- For a given input, the o/p of the system can be obtained by solving diff. eqn.

- The soln. of diff eqn. consists of 2 parts.

  (i) Zero state response
  (ii) Zero input response.

① Zero state response
 - response due to the input when the initial state of the system is zero.

② Zero Input response.
 - response due to initial state of the s/m

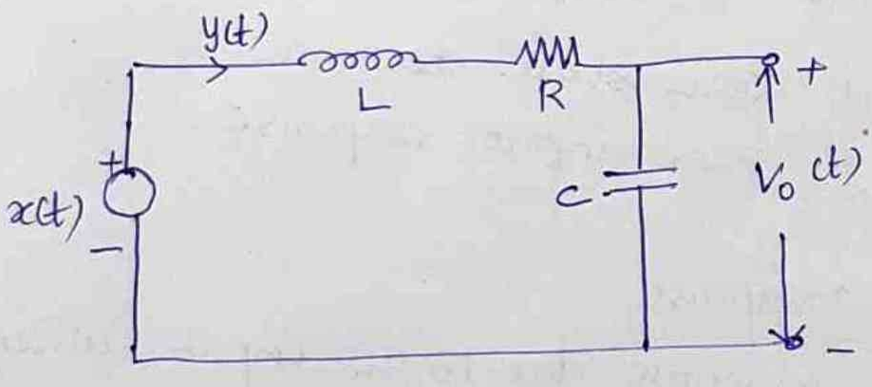
- In the 2^{nd} method, the o/p of an LTIC s/m can be obtained using convolution integral.

Solution of Differential equations (classical method)

- developed Mathematical models for diff. types of continuous time systems.
- These models give I/O relationship interms of differential equations.

ex The I/p & O/p eqn. relating the input voltage to the o/p current for a series RLC ckt is given by

$$x(t) = L\frac{dy(t)}{dt} + Ry(t) + \frac{1}{C}\int_{-\infty}^{t} y(t)\,dt \quad \text{—— (1)}$$



- Differentiating both sides of eqn. ① w.r.t, t,

$$L\frac{d^2y(t)}{dt} + \frac{Rdy(t)}{dt} + \frac{1}{C}y(t) = \frac{dx(t)}{dt} \quad \text{—— ②}$$

- The o/p current $y(t)$ can be obtained by solving the diff. equation for the given input and initial conditions.

Here we consider classical method of solving differential equations.

The general form of N^{th} order diff. eqn. is given by

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1}\frac{d^{M-1}x(t)}{dt^{M-1}} + \cdots + b_1\frac{dx(t)}{dt} + b_0 x(t) \quad \text{—— ③}$$

a_i & b_i → real constants & $a_N \neq 0$.

All practical systems have $M \leq N$.

- the general solution of eqn. ③ consists of Zero-input (natural) response and Zero-state (forced) response.

Total response = Zero input response + Zero state response.

Natural response:

- Natural response is the soln. of eqn. ③ with $x(t)=0$.

→ this soln. is also known as homogeneous soln. and is denoted by $y_h(t)$.

- equating the input terms in eqn. ④ to zero.

$$\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \dots + a_1\frac{dy(t)}{dt} + a_0 = 0 \quad \text{—— ④}$$

$$(\because a_N = 1)$$

Let us assume the soln. of eqn. ④ is of the form

$$y_h(t) = c e^{\lambda t}$$

$$\frac{dy_h(t)}{dt} = c.\lambda e^{\lambda t}$$

$$\frac{d^2 y_h(t)}{dt} = c.\lambda^2 e^{\lambda t}$$

$$\vdots$$

$$\frac{d^N y_h(t)}{dt^N} = c\lambda^N e^{\lambda t} \quad \text{—— ⑤}$$

Substituting these results in eqn. ⑤ gives

$$c.\lambda^N e^{\lambda t} + a_{N-1} c\lambda^{N-1} e^{\lambda t} + \dots + a_1 c\lambda e^{\lambda t} + a_0 c e^{\lambda t} = 0$$

$$\Rightarrow \lambda^N + a_{N-1}\lambda^{N-1} + \dots + a_1\lambda + a_0 = 0 \quad \text{—— ⑥}$$

→ this polynomial → characteristic eqn. of the S/m eqn. (6) $\Rightarrow$ represented in factorized form as

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_N)=0 \quad \text{—— (7)}$$

$\lambda_1, \lambda_2, \ldots \lambda_N \rightarrow$ roots of the characteristic eqn. called characteristic roots (or) eigen values (or) poles of the system.

- the nature of the response depends on the types of these roots.

Distinct roots

- If the roots $\lambda_1, \lambda_2, \ldots \lambda_N$ of eqn. (7) are distinct then the soln. has the terms

$c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, \ldots c_N e^{\lambda_N t}$. Therefore the soln. is of the form

$$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots c_N e^{\lambda_N t} \quad \text{—— (8)}$$

$c_1, c_2 \ldots c_N$ are arbitrary constants.

for ex. if the roots are $\lambda_1 = 2, \lambda_2 = 3$, then

$$y_h(t) = c_1 e^{2t} + c_2 e^{3t}$$

Repeated Roots:

- If a root λ_1 is repeated m times (root of multiplicity m) and remaining $(N-m)$ roots are distinct, then, the general soln. is of the form

$$y_h(t) = (c_1 + c_2 t + c_3 t^2 + \cdots + c_m t^{m-1}) e^{\lambda_1 t} + c_{m+1} e^{(\lambda_{m+1})t} + c_{m+2} e^{(\lambda_{m+2})t} + \cdots + c_N e^{\lambda_N t} \quad \text{--- (9)}$$

ex: if the roots of characteristic equation are $\lambda_1 = \lambda_2 = 3$ and $\lambda_3 = 2$ then

$$y_h(t) = (c_1 + c_2 t) e^{3t} + c_3 e^{2t}$$

Complex roots:

If the roots are complex say λ & λ^*

$\lambda_1 = a + jb$

$\lambda_2 = a - jb$ and the solution is

$$y_h(n) = c \cdot e^{at} [\cos(bt + \theta)]$$

Forced Response (zero state response)

→ response of the system when the initial conditions are zero.

→ Consists of 2 parts → homogeneous soln. & particular soln.

→ homogeneous soln can be obtained from the roots of characteristic equation.

→ particular soln. should satisfy the differential equation.

⇒ the forced response of the system is obtained by adding particular soln. and homogeneous soln. and then finding the coeffs. of the homogeneous soln. so that the combined response $y_h(t) + y_p(t)$ satisfies the zero initial conditions.

S.No.	Input $x(t)$	particular soln. $y_p(t)$
1.	A	K
2.	e^{at}, $a \neq \lambda_i$ $(i = 1, 2, \ldots n)$	$A \cdot e^{at}$
3.	e^{at}, $a = \lambda_i$	$A \cdot t \cdot e^{at}$
4.	$\cos(\omega t + \theta)$	$A \cos(\omega t + \phi)$

Table: The form of particular soln. for diff. I/ps.

Total response

→ obtained by adding the natural response and forced response.

$$y(t) = y_n(t) + y_f(t).$$

⇒ If we are not interested in 2 separate solns, then it is possible to obtain directly the total response in the same way as forced response by using actual initial conditions.

1. An LTIC system is specified by equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

The input $x(t) = e^{-t}u(t)$.

Find i) Natural response for initial conditions

$$y(0^+) = 3, \quad \frac{dy(0^+)}{dt} = 0.$$

ii) forced response

iii) Total response.

Soln:

Given $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$ ——— ①

Natural response can be obtained by equating the input terms in diff. equation to zero.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 0.$$

- Characteristic equation can be obtained by substituting $y_h(t) = c.e^{\lambda t}$ in the above eqn.

$$\lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda+2)(\lambda+3) = 0$$

$$\lambda_1 = -2, \quad \lambda_2 = -3.$$

The homogeneous soln. is

$$y_h(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

$$y(0^+) = c_1 + c_2$$

$$\frac{dy(0^+)}{dt} = -2c_1 e^{-2t} + (-3)c_2 e^{-3t} \Big|_{t=0}.$$

$$= -2c_1 - 3c_2.$$

from the initial conditions,

$$y(0^+) = 3, \quad \frac{dy(0^+)}{dt} = 0.$$

from the above eqns.

$$c_1 + c_2 = 3$$

$$2c_1 + 3c_2 = 0$$

$$\Rightarrow c_2 = -6$$

$$c_1 = 9.$$

$$\boxed{y_n(t) = 9e^{-2t} - 6e^{-3t}}$$

Forced Response:

The homogeneous soln. is given by

$$y_h(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

for input $x(t) = e^{-t}$, the particular solution is of the form,

$$\boxed{y_p(t) = k \cdot e^{-t}}$$

from which $\frac{dy_p(t)}{dt} = -ke^{-t}$

$$\frac{d^2 y_p(t)}{dt^2} = ke^{-t}$$

The particular solution should satisfy the diff. equation.

∴ substitute the values in eqn ①

$$\frac{d^2 y_p(t)}{dt^2} + 5\frac{dy_p(t)}{dt} + 6 y_p(t) = \frac{dx(t)}{dt} + 4x(t)$$

$$k \cdot e^{-t} - 5k \cdot e^{-t} + 6k \cdot e^{-t} = -e^{-t} + 4e^{-t}$$

$$2k \cdot e^{-t} = 3e^{-t}$$

$$2k = 3$$

$$\boxed{k = 1.5}$$

$$\boxed{y_p(t) = 1.5e^{-t}}$$

Forced response is the sum of homogeneous and particular solution.

$$y_f(t) = c_1 e^{-2t} + c_2 e^{-3t} + 1.5 e^{-t} \quad \text{—— (A)}$$

$$y(0^+) = c_1 + c_2 + 1.5$$

$$\frac{dy}{dt}(0^+) = -2c_1 e^{-2t} - 3c_2 e^{-3t} - 1.5 e^{-t} \Big|_{t=0}$$

$$\frac{dy}{dt}(0^+) = -2c_1 - 3c_2 - 1.5$$

– to obtain forced response equate initial conditions to zero.

$$c_1 + c_2 = -1.5$$

$$2c_1 + 3c_2 = -1.5$$

Solving for c_1 and c_2,

$$c_2 = 1.5$$

$$c_1 = -3$$

$$\boxed{y_f(t) = -3e^{-2t} + 1.5e^{-3t} + 1.5e^{-t}}$$

Total response

$$y(t) = y_n(t) + y_f(t)$$

$$= (9 e^{-2t} - 6e^{-3t}) + (-3e^{-2t} + 1.5e^{-3t} + 1.5e^{-t})$$

$$\boxed{y(t) = 6e^{-2t} - 4.5e^{-3t} + 1.5e^{-t}}$$

Total response can also be found without finding natural response and forced response separately. For this proceed upto A like in forced response.

Then substitute actual initial conditions.

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t} + 1.5 e^{-t}$$

$$y(0) = C_1 + C_2 + 1.5 = 3$$

$$\frac{dy(0^+)}{dt} = -2C_1 - 3C_2 - 1.5 = 0$$

$$C_1 + C_2 = 1.5$$

$$2C_1 + 3C_2 = -1.5$$

Solving for C_1 and C_2,

$$C_1 = 6, \quad C_2 = -4.5$$

$$\boxed{y(t) = 6e^{-2t} - 4.5e^{-3t} + 1.5e^{-t}}$$

2. Using Classical method, solve

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + x(t)$$

if the initial conditions are

$$y(0^+) = \frac{9}{4}, \quad \frac{dy(0^+)}{dt} = 5.$$

and if the input is $e^{-3t}u(t)$.

Soln:

The characteristic equ. can be obtained by substituting $y_h(t) = Ce^{\lambda t}$ in the diff. equation.

The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda+2)^2 = 0.$$

The homogeneous soln. is

$$y_h(t) = (C_1 + t \cdot C_2)e^{-2t}.$$

For input $x(t) = e^{-3t}u(t)$, the particular soln. is of the form

$$y_p(t) = K \cdot e^{-3t}u(t)$$

- Substituting particular soln. $y_p(t)$ and input $x(t)$ in the diff. eqn,

$$\frac{d^2y_p(t)}{dt^2} + 4\cdot\frac{dy_p(t)}{dt} + 4y_p(t) = \frac{d}{dt}x(t) + x(t)$$

$$\frac{d^2}{dt^2}K\cdot e^{-3t} + 4\frac{d}{dt}(Ke^{-3t}) + 4(K\cdot e^{-3t}) = \frac{d}{dt}(e^{-3t}) + e^{-3t}$$

$$K(-3)(-3)e^{-3t} + 4K(-3)e^{-3t} + 4K\cdot e^{-3t} = (-3)e^{-3t} + e^{-3t}$$

$$9Ke^{-3t} - 12Ke^{-3t} + 4Ke^{-3t} = -2e^{-3t}$$

$$Ke^{-3t} = -2e^{-3t}$$

$$\boxed{K = -2}$$

$$\therefore y_p(t) = -2e^{-3t}u(t)$$

The total response

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = (c_1 + tc_2)e^{-2t} - 2e^{-3t}$$

from the total response

$$y(0) = c_1 - 2$$

$$\frac{dy(t)}{dt} = -2c_1e^{-2t} + c_2(e^{-2t} - 2te^{-2t}) + 6e^{-3t}$$

$$\frac{dy(0^+)}{dt} = -2c_1 + c_2 + 6$$

On substituting initial conditions,

$$y(0^+) = \frac{9}{4}$$

$$\frac{dy(0^+)}{dt} = 5$$

$$c_1 - 2 = \frac{9}{4}$$

$$c_1 = \frac{17}{4}$$

$$-2c_1 + c_2 + 6 = 5 \qquad c_1 = \frac{17}{4}$$

$$-2\left(\frac{17}{4}\right) + c_2 + 6 = 5 \qquad c_2 = \frac{15}{2}$$

$$c_2 = -1 + \frac{17}{2} = \frac{15}{2}$$

$$\therefore \boxed{y(t) = \left(\frac{17}{4} + \frac{15}{2}t\right)e^{-2t} - 2 \cdot e^{-3t}}$$