UNIT – V

MULTIPLE INTEGRALS

Double Integration – cartesian and Polar co-ordinates – change of order of integration – change of variables between cartesian and polar co-ordinates – Area as a double integral – Triple integration – Volume as a triple integral.

Double Integration in Cartesian Co-ordinates:

Integrals of a function of two variables over a region in R^2 is called a double integral.

The double integral of a function $f(x,y)$ over a region D in R^2 is denoted by $\iint_D f(x,y)\,dx\,dy$.

(i) (ie) $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \int_c^d f(x,y)\,dx\,dy$,

if the limits are constants.

(ii) If the limits are variable, the check the given problem is in the correct form.

(ie) The limits for the inner integral are functions of x, then the first integral is with respect to y.

eg: $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dy\,dx \rightarrow$ [correct form].

Problem:

Evaluate $\int_0^1 \int_1^2 x(x+y)\,dy\,dx$.

Soln:

$$\int_0^1 \int_1^2 x(x+y)\,dy\,dx = \int_0^1 \left[\int_1^2 (x^2+xy)\,dy\right] dx.$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{2}\right]_{y=1}^{y=2} dx.$$

$$= \int_0^1 \left[(2x^2+2x) - \left(x^2+\frac{x}{2}\right)\right] dx.$$

$$= \int_0^1 \left(x^2 + \frac{3}{2}x\right) dx.$$

$$= \left[\frac{x^3}{3} + \frac{3}{2}\left(\frac{x^2}{2}\right)\right]_{x=0}^{x=1}.$$

$$= \left(\frac{1}{3} + \frac{3}{4}\right) + (0+0)$$

$$= \frac{13}{12} //.$$

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2) Evaluate $\int_1^2 \int_0^{x^2} x \, dy \, dx.$

$$\int_1^2 \left[\int_0^{x^2} x \, dy\right] dx = \int_1^2 \left[\int_0^{x^2} x \, dy\right] dx.$$

$$= \int_1^2 [xy]_0^{x^2} dx.$$

$$= \int_1^2 (x(x^2) - 0) dx.$$

$$= \int_1^2 x^3 dx.$$

$$= \left[\frac{x^4}{4}\right]_1^2$$

$$= \left[\frac{16}{4} - \frac{1}{4}\right]$$

$$= \frac{15}{4} //.$$

- Evaluate. $\int_{4}^{3}\int_{1}^{2} (x+y)^{-2}\, dx\, dy$.

$$\int_{4}^{3}\int_{1}^{2} (x+y)^{-2}\, dx\, dy = \int_{4}^{3}\left[\int_{1}^{2} (x+y)^{-2}\, dx\right] dy$$

$$= \int_{4}^{3}\left[\frac{(x+y)^{-1}}{-1}\right]_{x=1}^{x=2} dy$$

$$= \int_{4}^{3}\left[\frac{(2+y)^{-1}}{-1} - \frac{(1+y)^{-1}}{-1}\right] dy$$

$$= \int_{4}^{3}\left[\frac{-1}{2+y} + \frac{1}{1+y}\right] dy.$$

$$= \left[-\log(2+y) + \log(1+y)\right]_{4}^{3}$$

$$= \left[\log\left(\frac{1+y}{2+y}\right)\right]_{4}^{3}$$

$$= \left[\log\left(\frac{4}{5}\right) - \log\left(\frac{5}{6}\right)\right]$$

$$= \log\left[\frac{(4/5)}{(5/6)}\right]$$

$$= \log\left(\frac{24}{25}\right)$$

Evaluate $\int_{2}^{a}\int_{2}^{b} \frac{1}{xy}\, dx\, dy$.

$$\int_{2}^{a}\int_{2}^{b} \left(\frac{1}{x}\cdot\frac{1}{y}\right) dx\, dy = \int_{2}^{a}\left[\int_{2}^{b}\left(\frac{1}{x}\cdot\frac{1}{y}\right)dx\right] dy$$

$$= \int_{2}^{a}\left[\frac{1}{y}\log x\right]_{x=2}^{x=b} dy$$

$$= \int_2^a \frac{1}{y}\left[\log b - \log 2\right] dy.$$

$$= \int_2^a \frac{1}{y}\left(\log\left(\frac{b}{2}\right)\right) dy$$

$$= \log\left(\frac{b}{2}\right) \int_2^a \frac{1}{y}\, dy$$

$$= \log\left(\frac{b}{2}\right)\left[\log y\right]_2^a$$

$$= \log\left(\frac{b}{2}\right)\left[\log a - \log 2\right]$$

$$= \log\left(\frac{b}{2}\right)\left[\log\left(\frac{a}{2}\right)\right]$$

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5) Evaluate $\int_0^1 \int_0^{1-x} y\, dy\, dx.$ [correct form].

$$\int_0^1 \int_0^{1-x} y\, dy\, dx = \int_0^1 \left[\int_0^{1-x} y\, dy\right] dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx.$$

$$= \int_0^1 \left[\frac{(1-x)^2}{2} - 0\right] dx.$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\, dx.$$

$$= \frac{1}{2}\left[\frac{(1-x)^3}{3}(-1)\right]_0^1$$

$$= \frac{-1}{6}\left[0 - 1\right]$$

$$= \frac{1}{6} \,//.$$

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Shade the region of integration $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dx\,dy$

soln:

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dx\,dy = \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dy\,dx \quad \text{[correct form]}$$

x varies from $x=0$ to $x=a$

y varies from $y=\sqrt{ax-x^2}$ to $y=\sqrt{a^2-x^2}$

(ie) $y^2 = ax-x^2$ to $y^2=a^2-x^2$.

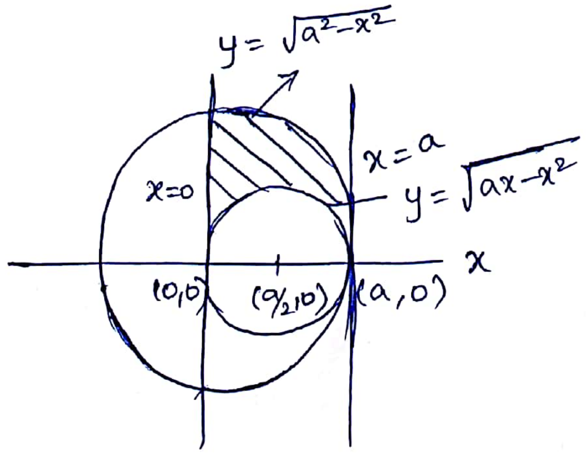
(ie) $x^2+y^2-ax=0$ to $x^2+y^2=a^2$.

$\left(x^2-ax+\frac{a^2}{4}\right)-\frac{a^2}{4}+y^2=0$ to $x^2+y^2=a^2$.

$\left(x-a/2\right)^2+y^2=\left(\frac{a}{2}\right)^2$ to $(x-0)^2+(y-0)^2=a^2$.

$\therefore \left(x-a/2\right)^2+y^2=\left(\frac{a}{2}\right)^2$ is a circle with centre $\left(\frac{a}{2},0\right)$ and radius $\left(\frac{a}{2}\right)$

$(x-0)^2+(y-0)^2=a^2$ is a circle with centre $(0,0)$ & $r=a$



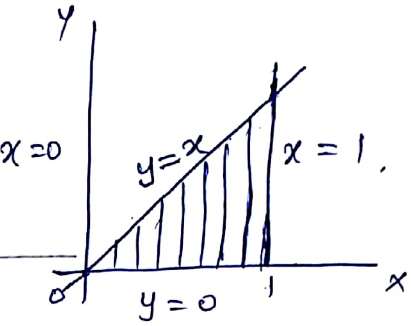
7) Sketch roughly the region of integration for

$$\int_0^1 \int_0^x f(x,y)\, dy\, dx$$

soln: Given, $\int_0^1 \int_0^x f(x,y)\, dy\, dx$.

x varies from $x=0$ to $x=1$.

y varies from $y=0$ to $y=x$.



8) Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1+x^2+y^2}$.

soln:

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx \quad [\text{correct form}]$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{y^2 + \left(\sqrt{(1+x^2)}\right)^2} dy \right] dx.$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right) \right]_{y=0}^{y=\sqrt{1+x^2}} dx.$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) - 0 \right] dx.$$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

$\tan^{-1}(1) = \pi/4$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx.$$

$\because \int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$

$$= \frac{\pi}{4} \left[\sinh^{-1}\left(\frac{x}{1}\right) \right]_0^1$$

$\because \sinh^{-1}\left(\frac{x}{a}\right) = \log \frac{x + \sqrt{x^2+a^2}}{a}$

$x=1, \quad a=1.$

$$= \frac{\pi}{4} \left[\sinh^{-1}(1) - 0 \right]$$

$$= \frac{\pi}{4} \left[\sinh^{-1}(1) \right]$$

$$= \frac{\pi}{4} \left[\log(1+\sqrt{2}) \right]$$

Double Integration in Polar co-ordinates.

Type :1 — Limits are constants :

Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r}{(r^2+a^2)^2}\, dr\, d\theta$.

soln: $\int_0^{\pi/2}\int_0^{\infty} \frac{r}{(r^2+a^2)^2}\, dr\, d\theta = \int_0^{\pi/2}\left[\int_0^{\infty} \frac{r}{(r^2+a^2)^2}\, dr\right] d\theta$.

$$= \int_0^{\pi/2}\left[\int_0^{\infty} \frac{(1/2)\, d(r^2)}{(r^2+a^2)^2}\right] d\theta$$

$r\,dr = \frac{1}{2}\, d(r^2)$

$$= \int_0^{\pi/2} \frac{1}{2}\left\{\int_0^{\infty}\left[(r^2+a^2)^{-2}(dr^2)\right]\right\} d\theta.$$

$$= +\frac{1}{2}\int_0^{\pi/2}\left[\frac{-1}{r^2+a^2}\right]_{r=0}^{r=\infty} d\theta.$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[0 - \frac{1}{a^2}\right] d\theta. = \frac{1}{2a^2}\int_0^{\pi/2} d\theta.$$

$$= \frac{1}{2a^2}\left[\theta\right]_0^{\pi/2} = \frac{1}{2a^2}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{4a^2}.$$

Type - 2 : Limits are Variables :

Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\, d\theta\, dr$.

soln: $\int_0^{\pi/2}\int_0^{\sin\theta} r\, d\theta\, dr = \int_0^{\pi/2}\int_0^{\sin\theta} r\, dr\, d\theta$ [correct form]

$$= \int_0^{\pi/2}\left[\int_0^{\sin\theta} r\, dr\right] d\theta.$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sin\theta} d\theta.$$

$$= \int_0^{\pi/2} \left[\frac{(\sin\theta)^2}{2} - 0 \right] d\theta .$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{1}{4} \int_0^{\pi/2} [1 - \cos 2\theta] \, d\theta .$$

$$= \frac{1}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} .$$

$$= \frac{1}{4} \left[\left(\frac{\pi}{2} - \frac{1}{2}(0) \right) - (0 - 0) \right]$$

$$= \frac{1}{4} \left(\frac{\pi}{2} \right)$$

$$= \frac{\pi}{8} \,//.$$

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3) Evaluate $\int_0^{\pi} \int_0^{\cos\theta} r \, dr \, d\theta$.

Soln: $\int_0^{\pi} \int_0^{\cos\theta} r \, dr \, d\theta = \int_0^{\pi} \left[\int_0^{\cos\theta} r \, dr \right] d\theta .$

$$= \int_0^{\pi} \left(\frac{r^2}{2} \right)_{r=0}^{r=\cos\theta} d\theta .$$

$$= \int_0^{\pi} \frac{1}{2} \left[\cos^2\theta - \cancel{\cos} \, 0 \right] d\theta .$$

$$= \frac{1}{2} \int_0^{\pi} \frac{1}{2} (1 + \cos 2\theta) \, d\theta .$$

$$= \frac{1}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi}$$

$$= \frac{1}{4} \left[(\pi + 0) - (0 + 0) \right]$$

$$= \pi/4 \,//.$$

Change of order of Integration :

Prove that $\int_0^3 \int_1^2 xy(x+y)\, dy\, dx = \int_1^2 \int_0^3 xy(x+y)\, dx\, dy$.

Soln:

$$LHS = \int_0^3 \int_1^2 xy(x+y)\, dy\, dx .$$

$$= \int_0^3 \int_1^2 (x^2y + xy^2)\, dy\, dx .$$

$$= \int_0^3 \left[x^2\left(\frac{y^2}{2}\right) + x\left(\frac{y^3}{3}\right)\right]_1^2 dx .$$

$$= \int_0^3 \left[2x^2 + \frac{8x}{3} - \frac{x^2}{2} - \frac{x}{3}\right] dx .$$

$$= \int_0^3 \left[\frac{3x^2}{2} + \frac{7x}{3}\right] dx .$$

$$= \left[\frac{\cancel{3}}{2}\left(\frac{x^3}{\cancel{3}}\right) + \frac{7}{3}\left(\frac{x^2}{2}\right)\right]_0^3$$

$$= \left[\frac{27}{2} + \frac{21}{2} - (0+0)\right] = \frac{48}{2} = 24 .$$

$$RHS = \int_1^2 \int_0^3 xy(x+y)\, dx\, dy$$

$$= \int_1^2 \int_0^3 (x^2y + xy^2)\, dx\, dy .$$

$$= \int_1^2 \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_0^3 dy$$

$$= \int_1^2 \left[9y + \frac{9y^2}{2}\right] dy$$

$$= \left[9\left(\frac{y^2}{2}\right) + \frac{9}{2}\left(\frac{y^3}{3}\right)\right]_1^2 = \left(18 + 12 \underbrace{- \frac{9}{2} - \frac{9^3}{\cancel{6}2}}_{-12/2}\right)$$

$$= 18 + 12 - 6 .$$

$$= 24 .$$

$\therefore$ LHS = RHS //

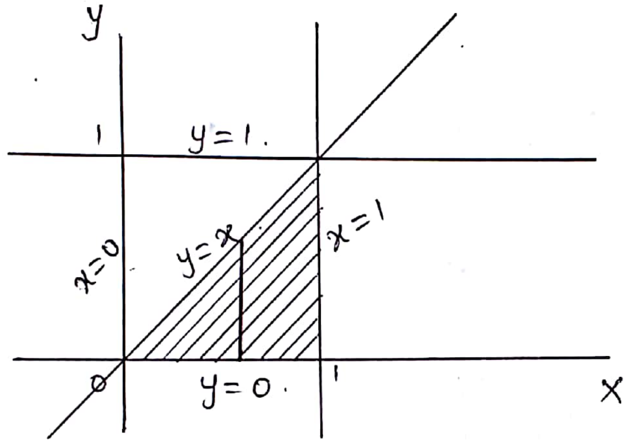
2) Change the order of integration and evaluate $\int_0^1 \int_0^x dy\, dx$.

Soln: Let $I = \int_0^1 \int_0^x dy\, dx$.

Here, y varies from $y=0$ to $y=x$.

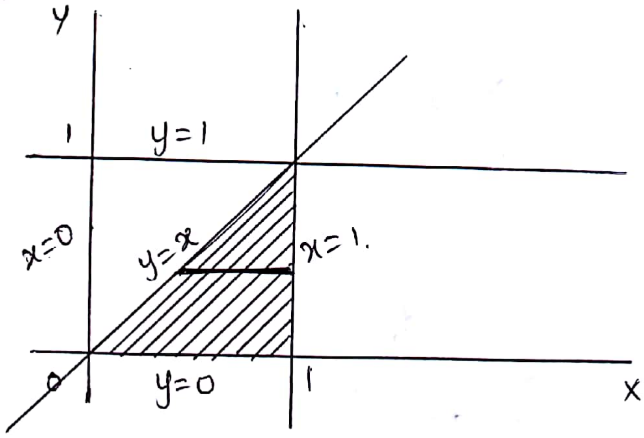
x varies from $x=0$ to $x=1$.

(ie) Before changing,



Changing the order means changing vertical into horizontal strip.

After changing,



Here, y varies from $y=0$ to $y=1$.

x varies from $x=y$ to $x=1$.

$$\therefore \quad I = \int_0^1 \int_y^1 dx\, dy$$

$$= \int_0^1 [x]_y^1\, dy \quad = \int_0^1 (1-y)\, dy$$

$$= \left[y - \frac{y^2}{2}\right]_0^1$$

$$= \left[1 - \tfrac{1}{2}\right]$$

$$I = \tfrac{1}{2}$$

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change the order of integration in $\int_0^a \int_x^a (x^2+y^2)\, dy\, dx$ and hence evaluate it.

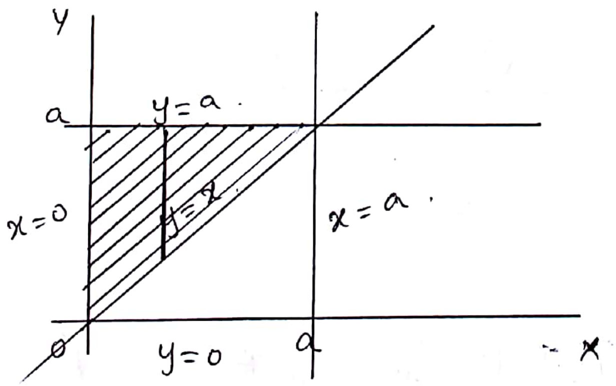
Soln:

Let $I = \int_0^a \int_x^a (x^2+y^2)\, dy\, dx.$

Here, y varies from $y = x$ to $y = a$.

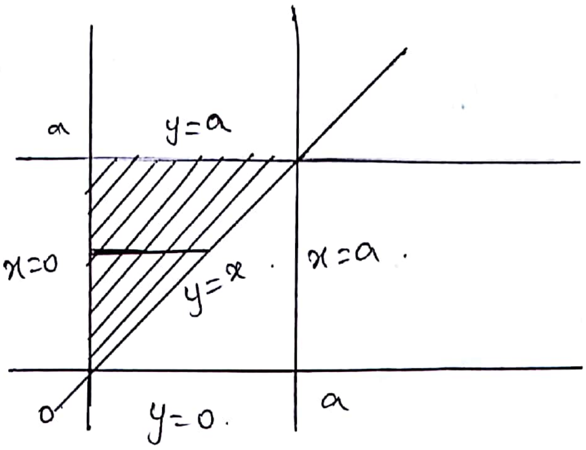
x varies from ~~y~~ $x = 0$ to $x = a$.

Before changing,



changing the order means changing vertical strip into horizontal strip.

After changing :



Here, x varies from $x=0$ to $x=y$.

y varies from $y=0$ to $y=a$.

$$\therefore \quad I = \int_0^a \int_0^y (x^2+y^2)\, dx\, dy.$$

$$= \int_0^a \left[\frac{x^3}{3} + y^2 x\right]_0^y dy$$

$$= \int_0^a \left(\frac{y^3}{3} + y^3\right) dy$$

$$= \left[\frac{1}{3}\left[\frac{y^4}{4}\right] + \frac{y^4}{4}\right]_0^a$$

$$= \frac{a^4}{12} + \frac{a^4}{4} = \frac{4a^4}{12}.$$

$$= \frac{a^4}{3} \;//$$

4) Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$ by changing the order of integration.

Soln :

Let $I = \int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$.

Here x varies from $x=1$ to $x=\sqrt{4-y}$

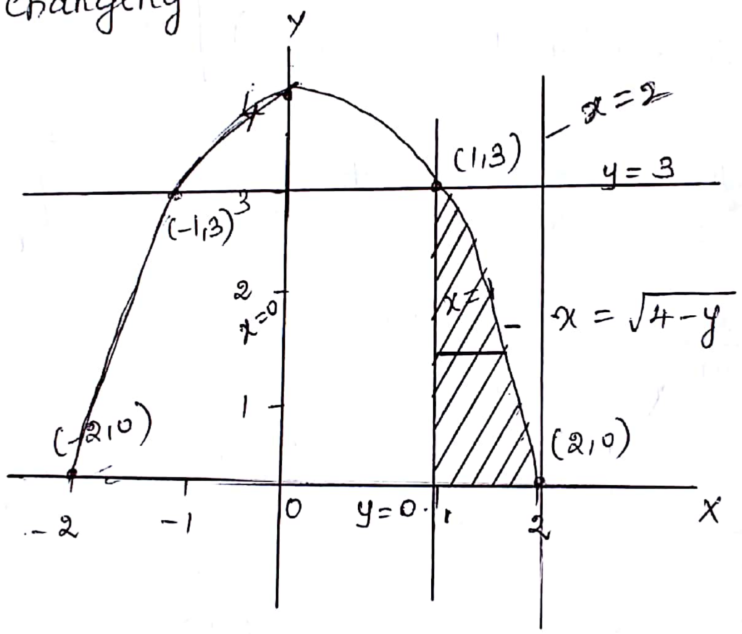
y varies from $y=0$ to $y=3$.

If $x=\sqrt{4-y} \Rightarrow x^2 = 4-y$

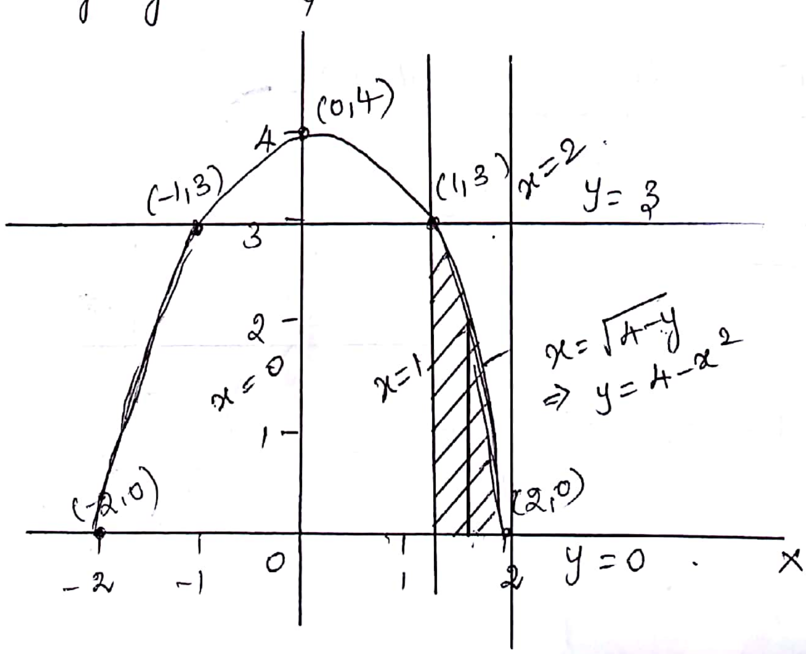
$\Rightarrow y = 4-x^2$.

x	0	1	-1	2	-2
y	4	3	3	0	0

Before changing



After changing



By changing the order,

$$I = \int_1^2 \int_0^{4-x^2} (x+y)\, dy\, dx.$$

$$= \int_1^2 \left[\int_0^{4-x^2} (x+y)\, dy \right] dx.$$

$$= \int_1^2 \left[xy + y^2/2 \right]_0^{4-x^2} dx.$$

$$= \int_1^2 \left[x(4-x^2) + \frac{1}{2}(4-x^2)^2 \right] dx.$$

$$= \int_1^2 4x - x^3 + \frac{1}{2}\left(16 + x^4 - 8x^2\right) \Big] dx$$

$$= \left[\frac{\cancel{4}^{2} x^2}{\cancel{2}} - \frac{x^4}{4} + 8x + \frac{x^5}{10} - \frac{4x^3}{3} \right]_1^2$$

$$= \left[\frac{x^5}{10} - \frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 + 8x \right]_1^2$$

$$= \left[\frac{\cancel{32}^{16}}{\cancel{10}_5} - \frac{\cancel{16}^{4}}{\cancel{4}} - \frac{32}{3} + 8 + 16 - \left(\frac{1}{10} - \frac{1}{4} - \frac{4}{3} + 2 + 8 \right) \right]$$

$$= \frac{16}{5} - \frac{28}{3} - \frac{1}{10} + \frac{1}{4} + 10$$

$$= \frac{192 - 560 - 6 + 15 + 600}{60}$$

$$= \frac{241}{60}$$

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5) change the order of $I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$, then evaluate.

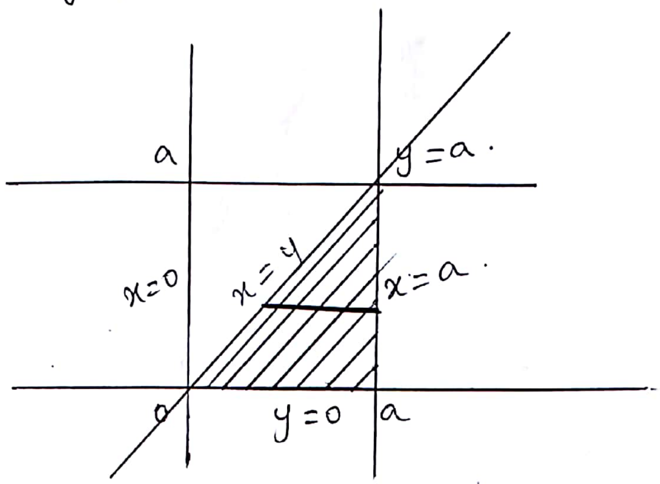
Soln:

Given, $I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy.$

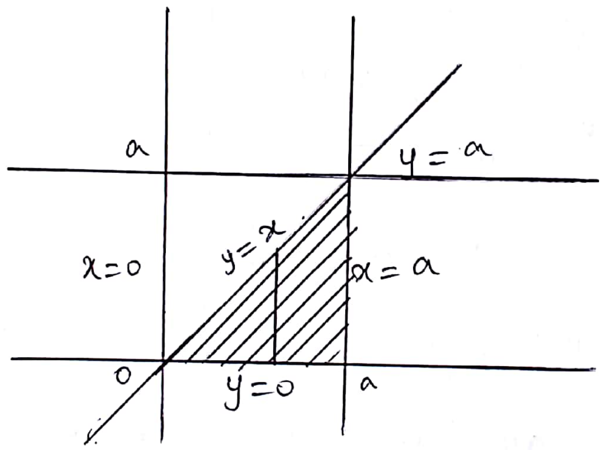
Here x varies from $x = y$ to $x = a$.

y varies from $y = 0$ to $y = a$.

Before changing,



After changing :



Here, x varies from $x = 0$ to $x = a$.

y varies from $y = 0$ to $y = x$.

By changing the order, we get,

$$I = \int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx.$$

$$= \int_0^a \left[\int_0^x x \left[\frac{1}{y^2+x^2} \right] dy \right] dx.$$

$$= \int_0^a \left[x \left(\frac{1}{x} \tan^{-1}\left(\frac{y}{x} \right) \right) \right]_0^x dx$$

$$= \int_0^a \cancel{x}\left(\frac{1}{\cancel{x}}\right)\left(\tan^{-1}1 - \tan^{-1}0\right) dx.$$

$$= \int_0^a \frac{\pi}{4}\, dx.$$

$$= \frac{\pi}{4}\left[x\right]_0^a = \frac{\pi}{4}(a-0)$$

$$= \frac{\pi a}{4} //.$$

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6) change the order of integration and hence evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$.

Soln :

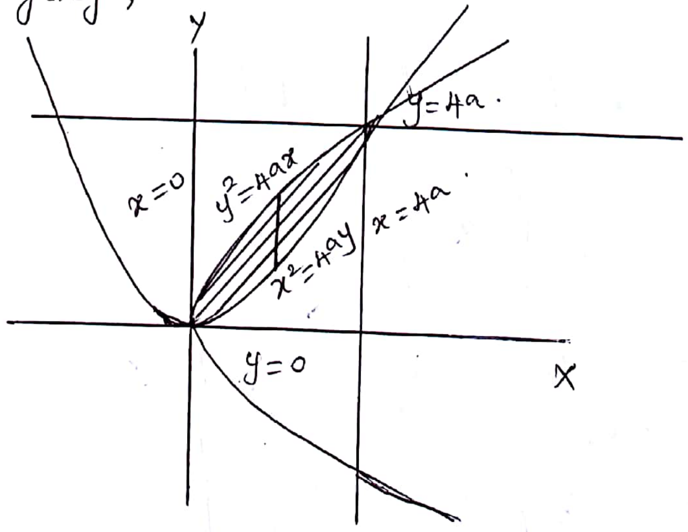
$$I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx.$$

Here y varies from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$.

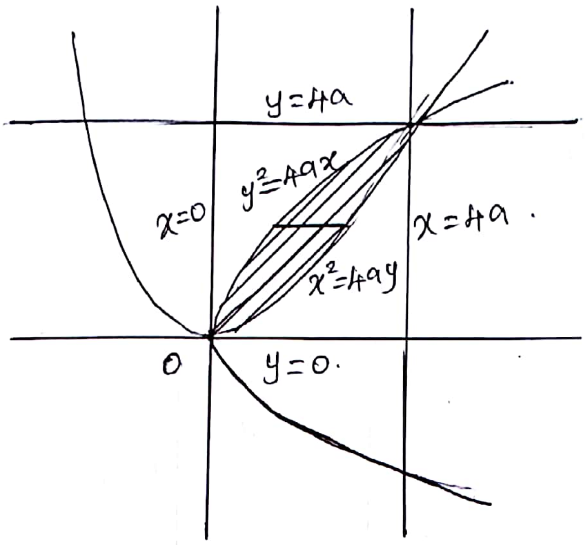
x varies from $x = 0$ to $x = 4a$.

From y, $x^2 = 4ay$ and $y^2 = 4ax$ represents the parabola.

Before changing,



After changing.



$y^2 = 4ax.$

$\Rightarrow x = y^2/4a$

$x^2 = 4ay$

$\Rightarrow x = 2\sqrt{ay}.$

Here, x Varies from $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}.$

y Varies from $y = 0$ to $y = 4a.$

By changing the order,

$$I = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} xy \, dx \, dy$$

$$= \int_0^{4a} \left[y\left(\frac{x^2}{2}\right) \right]_{y^2/4a}^{2\sqrt{ay}} dy.$$

$$= \int_0^{4a} y \left[\frac{4ay}{2} - \frac{y^4}{16a^2(2)} \right] dy$$

$$= \frac{1}{2} \int_0^{4a} \left[4ay^2 - \frac{y^5}{16a^2} \right] dy$$

$$= \frac{1}{2} \left[4a\left(\frac{y^3}{3}\right) - \frac{y^6}{6(16a^2)} \right]_0^{4a}.$$

$$= \frac{1}{2} \left[\frac{4a(64a^3)}{3} - \frac{4096\, a^6}{16 \times 6a^2} \right]$$

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$$= \frac{1}{2}\left[\frac{256a^4}{3} - \frac{256a^4}{6}\right]$$

$$= \frac{1}{\cancel{2}}(\overset{128}{\cancel{256}}\,a^4)\left[\frac{1}{3} - \frac{1}{6}\right]$$

$$= \overset{64}{\cancel{128}}\,a^4\left(\frac{1}{\underset{3}{\cancel{6}}}\right).$$

$$= \frac{64a^4}{3} //.$$

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7) Change the order of integration in $\int_0^\infty \int_0^y y e^{-y^2/x} \, dx \, dy$ and hence evaluate.

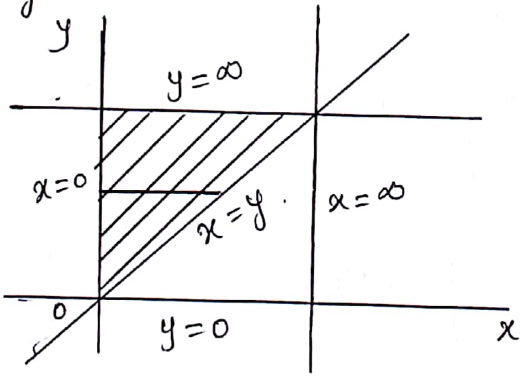
Soln:

Let $I = \int_0^\infty \int_0^y y e^{-y^2/x} \, dx \, dy$

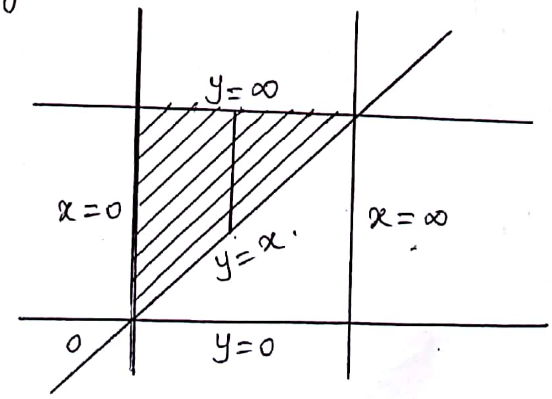
Here, x varies from $x = 0$ to $x = y$.

y varies from $y = 0$ to $y = \infty$.

Before changing.



After changing.



By changing the order,

Here, y varies from $y = x$ to $y = \infty$

x varies from $x = 0$ to $x = \infty$.

$$\therefore \quad I = \int_0^\infty \int_x^\infty y e^{-y^2/x} \, dy \, dx.$$

$$= \int_0^\infty \left[\cancel{y} \left(\frac{e^{-y^2/x}}{-\cancel{2y}/x} \right) \right.$$

$$= \int_0^\infty \left[\int_x^\infty \frac{e^{-y^2/x}}{2} \, d(y^2) \right] dx$$

$$= \frac{1}{2} \int_0^\infty \left[\int_x^\infty e^{-y^2/x} \, dy^2 \right] dx$$

$$= \frac{1}{2} \int_0^\infty \left[\frac{e^{-y^2/x}}{-1/x} \right]_{y=x}^{y=\infty} dx$$

$$= \frac{1}{2} \int_0^\infty \left[0 + x e^{-x} \right] dx \quad = \frac{1}{2} \int_0^\infty x e^{-x} \, dx.$$

$$= \frac{1}{2} \left[x(-e^{-x}) - 1(e^{-x}) \right]_{x=0}^{x=\infty}$$

$$= \frac{1}{2} \left[(0 - 0) - (0 - 1) \right]$$

$$= 1/2 \; //.$$

Change the order of integration and hence Evaluate

$$\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2 - y^2}} xy \, dx \, dy.$$

Soln:

$$\text{Let} \quad I = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2 - y^2}} xy \, dx \, dy.$$

Here, x Varies from $x=0$ to $x = \frac{a}{b}\sqrt{b^2-y^2}$.

y Varies from $y=0$ to $y=b$.

Take, $x = \frac{a}{b}\sqrt{b^2-y^2}$

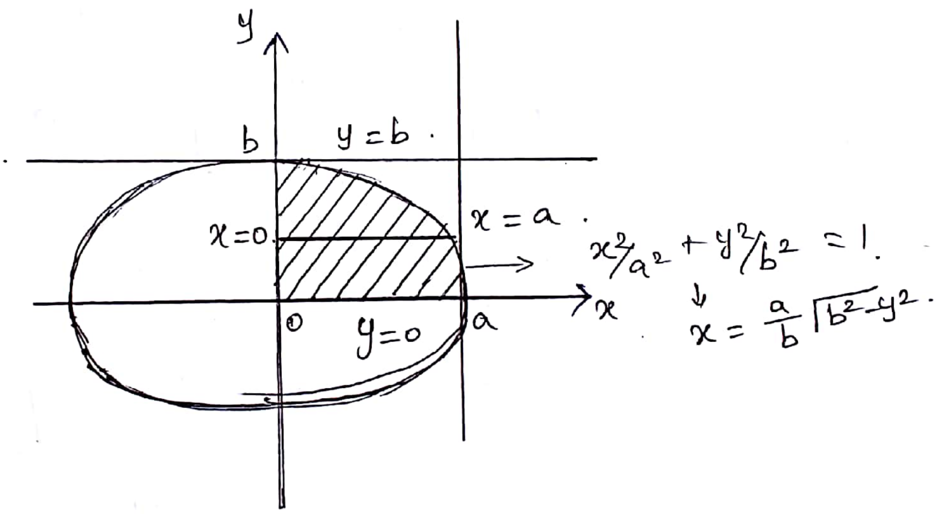
$$x^2 = \frac{a^2}{b^2}(b^2-y^2) = \frac{a^2b^2}{b^2} - \frac{a^2y^2}{b^2}.$$

$$x^2b^2 = a^2b^2 - a^2y^2.$$

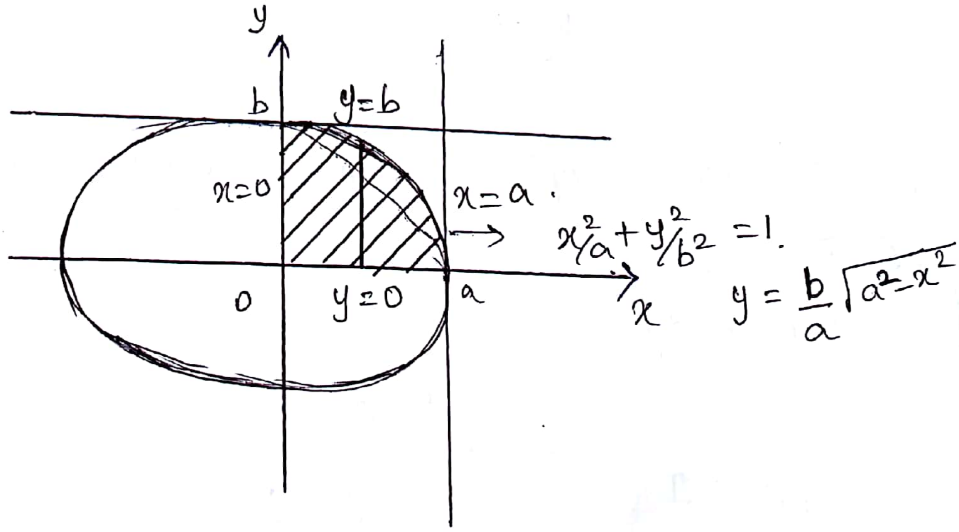
$$x^2b^2 + a^2y^2 = a^2b^2.$$

÷ by (a^2b^2) $\Rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$, represents ellipse.

Before changing,



After changing,



By changing the order,

$$I = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy \, dy \, dx .$$

(ie) y varies from $y=0$ to $y = \frac{b}{a}\sqrt{a^2-x^2}$.

x varies from $x=0$ to $x=a$.

$$= \int_0^a \left[\int_0^{b/a\sqrt{a^2-x^2}} xy \, dy \right] dx .$$

$$= \int_0^a \left[x\left(\frac{y^2}{2}\right) \right]_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} dx .$$

$$= \int_0^a \left[\frac{x}{2}\left(\frac{b^2}{a^2}(a^2-x^2) \right) \right] dx .$$

$$= \int_0^a \frac{b^2}{2a^2} \left[a^2x - x^3 \right] dx .$$

$$= \left[\frac{b^2}{2a^2} \left(a^2\left(\frac{x^2}{2}\right) - \frac{x^4}{4} \right) \right]_{x=0}^{x=a} .$$

$$= \frac{b^2}{2a^2} \left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - (0) \right]$$

$$= \frac{b^2a^2}{2}\left(\frac{1}{4}\right)$$

$$= \frac{b^2a^2}{8} //$$

Change the order of integration in $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y \, dx \, dy$ and then evaluate.

Soln:

Let $$I = \int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y \, dx \, dy .$$

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Here x varies from $x = a - y$ to $x = \sqrt{a^2 - y^2}$.

y varies from $y = 0$ to $y = a$.

Take, $x = \sqrt{a^2 - y^2}$.

$x^2 = a^2 - y^2$

$\Rightarrow x^2 + y^2 = a^2$, represents a circle.

Given, $x = a - y \Rightarrow y = a - x$.

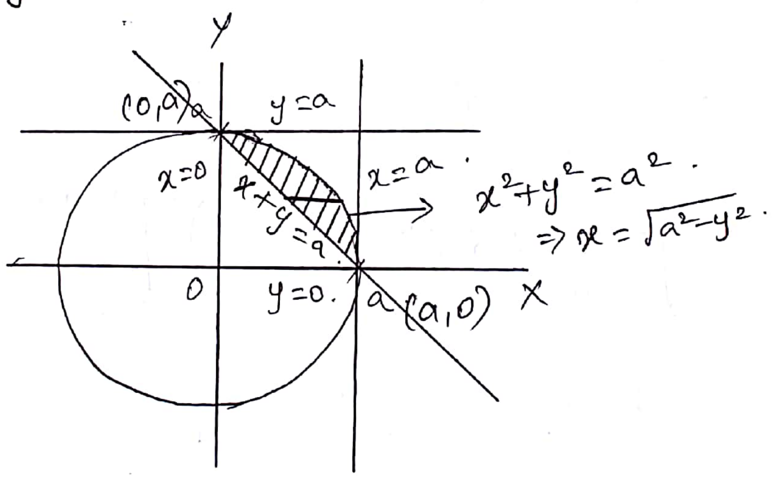
(ie) $x + y = a$.

If

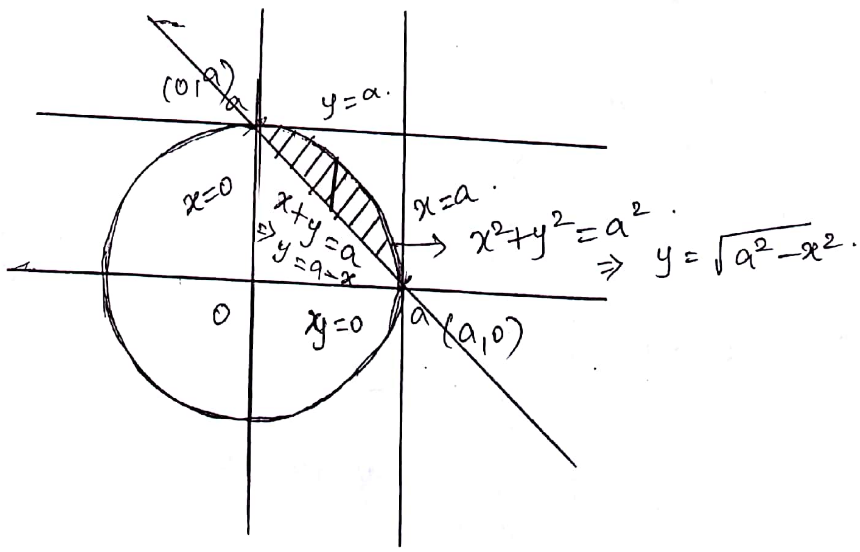
x	0	a
y	a	0.

$\Rightarrow (0, a)$ & $(a, 0)$.

Before changing,



After changing,



Here, x varies from $x=0$ to $x=a$.

y varies from $y=a-x$ to $y=\sqrt{a^2-x^2}$.

By changing the order,

$$I = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y \, dy \, dx.$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_{y=a-x}^{y=\sqrt{a^2-x^2}} dx.$$

$$= \int_0^a \frac{1}{2}\left[(\cancel{a^2}-x^2) - (\cancel{a^2}+x^2-2ax)\right] dx.$$

$$= \frac{1}{2}\int_0^a (2ax - 2x^2)\, dx. = \int_0^a (ax - x^2)\, dx.$$

$$= \left[a\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_0^a$$

$$= \left(\frac{a^3}{2} - \frac{a^3}{3}\right) = a^3\left(\frac{3-2}{6}\right)$$

$$I = \frac{a^3}{6} //$$

Change the order of integration in $\int_0^1 \int_y^{2-y} xy \, dx \, dy$ and hence evaluate it.

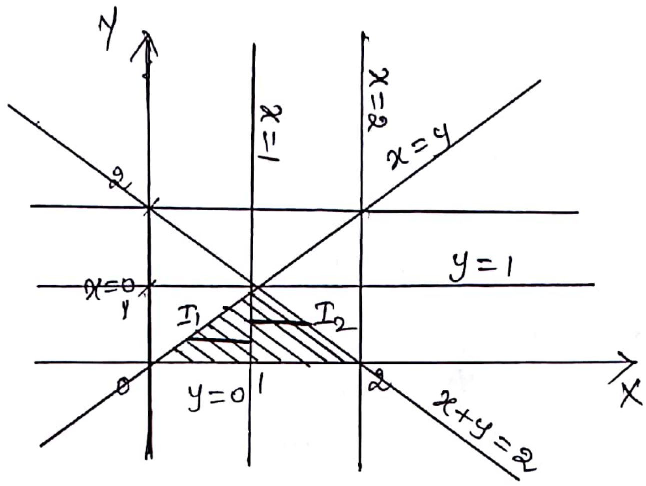
Soln:

Let $I = \int_0^1 \int_y^{2-y} xy \, dx \, dy$

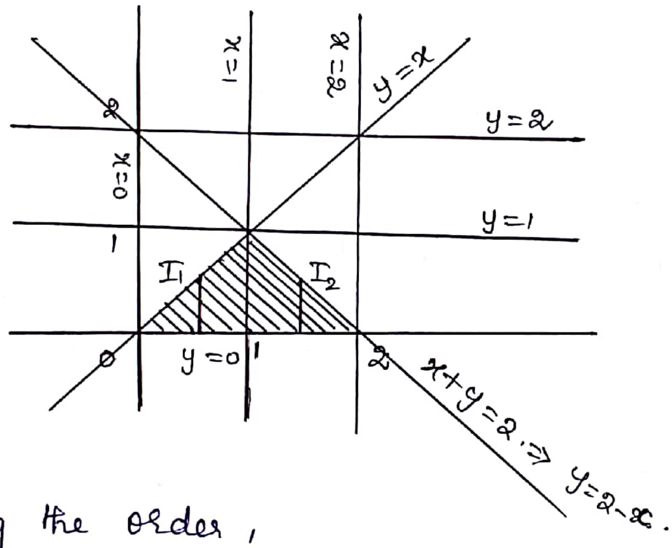
Here, x varies from $x=y$ to $x=2-y$

y varies from $y=0$ to $y=1$.

Before changing,



After changing,



By changing the order,

Given, $I = \int_0^1 \int_y^{2-y} xy \, dx \, dy$

$= \int_0^1 \int_y^1 xy \, dx \, dy + \int_0^1 \int_y^{2-y} xy \, dx \, dy$

$= I_1 + I_2.$

After changing,

In I_1 $\Rightarrow$ x Varies from $x=0$ to $x=1$.

y Varies from. $y=0$ to $y=x$.

In I_2 $\Rightarrow$ x varies from $x=1$ to $x=2$.
y varies from $y=0$ to $y=2-x$.

$$\underline{I} = I_1 + I_2$$

$$= \int_0^1 \int_0^x xy\, dy\, dx + \int_1^2 \int_0^{2-x} xy\, dy\, dx$$

$$= \int_0^1 \left[x\left(\frac{y^2}{2}\right) \right]_{y=0}^{y=x} dx + \int_1^2 \left[x\left(\frac{y^2}{2}\right) \right]_{y=0}^{y=2-x} dx.$$

$$= \int_0^1 \left[\frac{x^3}{2} \right] dx + \int_1^2 \frac{x}{2}(4+x^2-4x)\, dx$$

$$= \frac{1}{2}\int_0^1 x^3\, dx + \frac{1}{2}\int_1^2 (x^3+4x-4x^2)\, dx.$$

$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[\frac{x^4}{4} + \cancel{4}^{2}\left(\frac{x^2}{\cancel{2}}\right) - 4\left(\frac{x^3}{3}\right)\right]_1^2$$

$$= \left(\frac{1}{8}+0\right) + \frac{1}{2}\left[\frac{16}{4} + 2(4) - 4\left(\frac{8}{3}\right) - \left(\frac{1}{4}+2-\frac{4}{3}\right)\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[4 + 8 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[10 - \frac{1}{4} - \frac{28}{3}\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{120-3-112}{12}\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{5}{12}\right]$$

$$= \frac{1}{8} + \frac{5}{24}.$$

$$= \frac{3+5}{24}$$

$$= \frac{8}{24}$$

$$I = \frac{1}{3} //$$

11) change the order of integration in $\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy$ and hence evaluate.

Soln: Let $I = \int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy$

$= \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ (correct form)

Here, y varies from $y = \frac{x^2}{a}$ to $y = 2a - x$.

x varies from $x = 0$ to $x = a$.

(ie) $y = \frac{x^2}{a}$

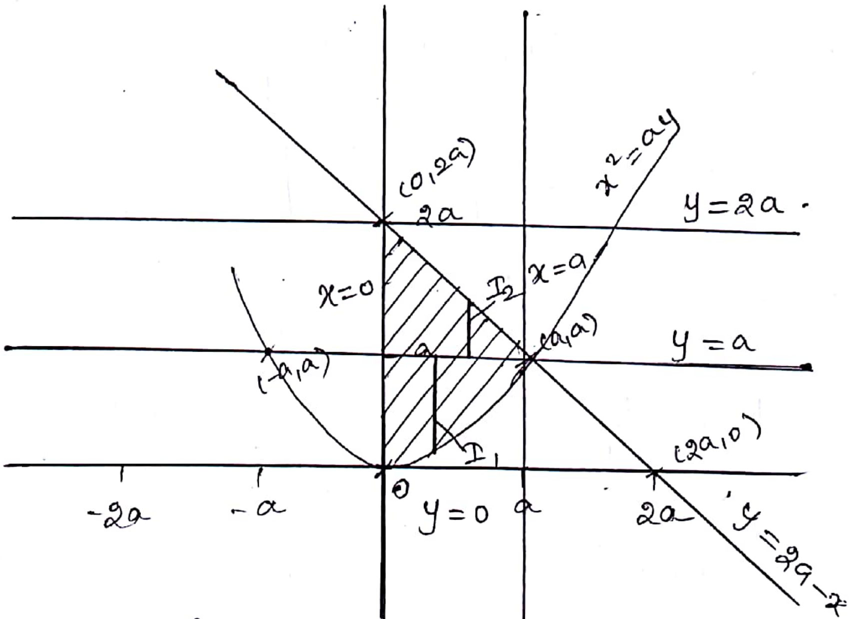
$x^2 = ay$

$x =$	$-a$	0	a
y	a	0	a

$y = 2a - x$

$x =$	0	a	$2a$
y	$2a$	a	0

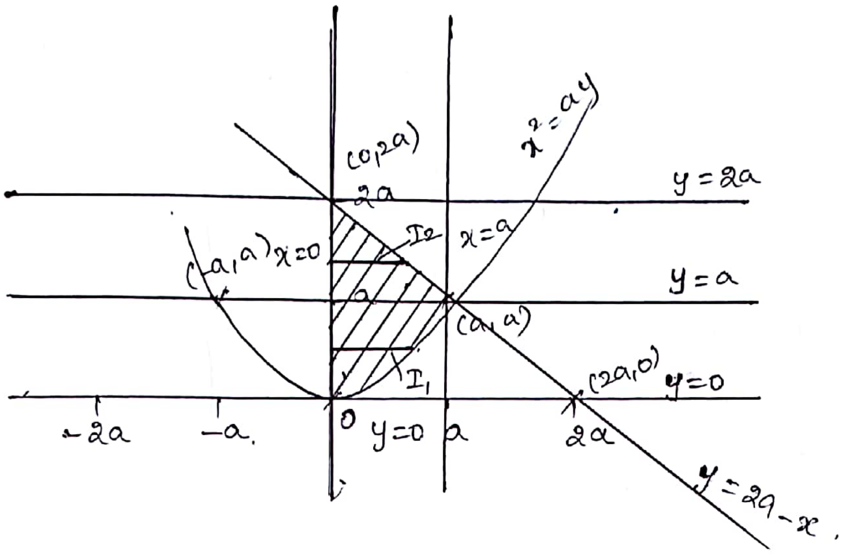
Before changing,



$\therefore I = \int_0^a \int_{x^2/a}^{a} xy\,dy\,dx + \int_0^a \int_a^{2a-x} xy\,dy\,dx.$

$= I_1 + I_2$

After changing,



By changing the order,

$I_1 \rightarrow$ y varies from $y = 0$ to $y = a$.

x varies from $x = 0$ to $x = \sqrt{ay}$

$I_2 \rightarrow$ y varies from $y = a$ to $y = 2a$.

x varies from $x = 0$ to $x = 2a - y$

$$I = I_1 + I_2$$

$$= \int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy + \int_a^{2a} \int_0^{2a-y} xy \, dx \, dy$$

$$= \int_0^a \left[y\left(\frac{x^2}{2}\right) \right]_0^{\sqrt{ay}} dy + \int_0^{2a} \left[y\left(\frac{x^2}{2}\right) \right]_0^{2a-y} dy.$$

$$= \frac{1}{2} \int_0^a y(ay) \, dy + \frac{1}{2} \int_0^{2a} y\,(4a^2 + y^2 - 4ay) \, dy$$

$$= \frac{a}{2} \int_0^a y^2 \, dy + \frac{1}{2} \int_0^{2a} (4a^2 y + y^3 - 4ay^2) \, dy$$

$$= \frac{a}{2} \left[\frac{y^3}{3} \right]_0^a + \frac{1}{2} \left[\frac{4a^2 y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3} \right]_0^{2a}.$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[8a^4 + 4a^4 - \frac{4a^4}{3} - \left(2a^4 + \frac{a^4}{4} - \frac{4a^4}{3}\right)\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[12a^4 - \frac{34a^4}{3} - 2a^4 - \frac{a^4}{4} + \frac{4a^4}{3}\right]$$

$$= \frac{a^4}{6} + \frac{a^4}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$$= \frac{a^4}{6} + \frac{a^4}{2}\left[\frac{120 - 112 - 3}{12}\right]$$

$$= \frac{a^4}{6} + \frac{5a^4}{24}$$

$$= \frac{9a^4}{24}$$

$$I = \frac{3a^4}{8} \,//$$

Area as a double integral (cartesian form).

$$\text{Area} = \iint_R dx\,dy = \iint_R dy\,dx$$

Problems :

1) Evaluate $\iint_R xy\,dx\,dy$, where R is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Soln : $\boxed{x = 2a}$ and $x^2 = 4ay$.

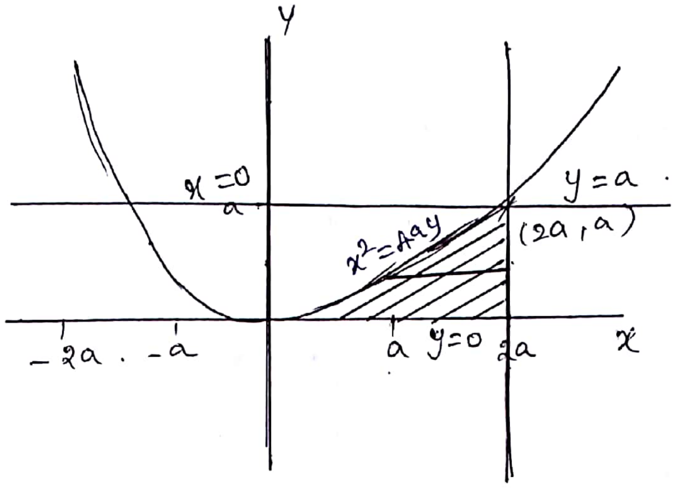
Sub $x = 2a$ in $x^2 = 4ay$

$$\cancel{4}a^{\cancel{2}} = \cancel{4a}y$$

$$a = y$$

(ie) $\boxed{y = a}$

$\therefore$ Point of intersection (x, y) is $(2a, a)$.



$$I = \iint_R xy \, dx \, dy$$

$$= \int_0^a \int_{2\sqrt{ay}}^{2a} xy \, dx \, dy.$$

$$= \int_0^a \left[y \left(\frac{x^2}{2} \right) \right]_{2\sqrt{ay}}^{2a} dy$$

$$= \int_0^a \frac{y}{2} (4a^2 - 4ay) \, dy$$

$$= \frac{1}{2} \int_0^a (4a^2 y - 4ay^2) \, dy$$

$$= \frac{1}{2} \left[{}^2\cancel{4} a^2 \left(\frac{y^2}{\cancel{2}} \right) - 4a \left(\frac{y^3}{3} \right) \right]_0^a$$

$$= \frac{1}{2} \left[2a^4 - \frac{4a^4}{3} \right]$$

$$= \frac{1}{2} \left[\frac{6a^4 - 4a^4}{3} \right] = \frac{1}{\cancel{2}} \left(\frac{\cancel{2}a^4}{3} \right)$$

$$I = \frac{a^4}{3}$$

2) using double integration. Find the area enclosed by the curves $y = 2x^2$ and $y^2 = 4x$.

Soln:

Given, $y = 2x^2$ and $y^2 = 4x$.

Sub $y = 2x^2$ in $y^2 = 4x$, we get,

$$\cancel{4}x^4 = \cancel{4}x$$

$$x^4 - x = 0.$$

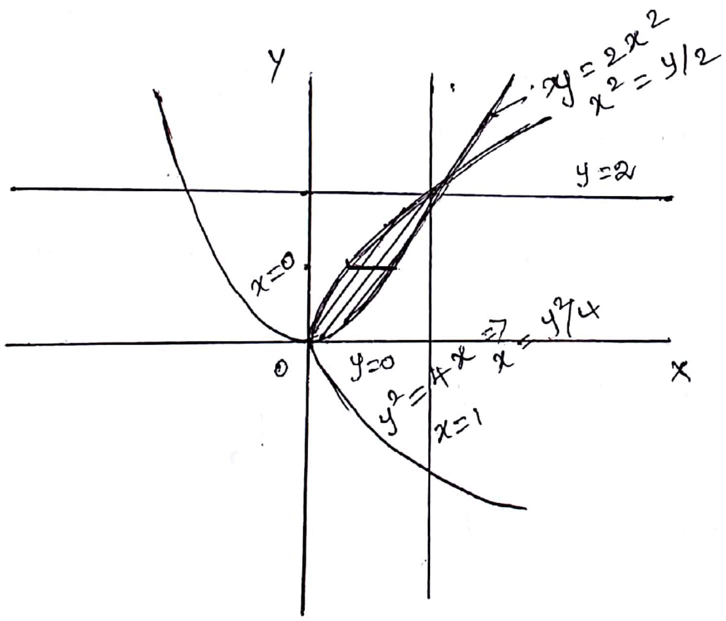
$$x(x^3 - 1) = 0.$$

$$x = 0, \quad x^3 - 1 = 0, \quad x = 1.$$

when $x = 0 \Rightarrow y = 0.$

$x = 1 \Rightarrow y = 2.$

Point of intersection is $(0, 0)$ & $(1, 2)$.



Here, y Varies from $y = 0$ to $y = 2$

x Varies from $x = \frac{y^2}{4}$ to $x = \sqrt{\frac{y}{2}}$

$\therefore$ Area $= \iint_R dx\, dy$

$= \int_0^2 \int_{y^2/4}^{\sqrt{y/2}} dx\, dy$

$= \int_0^2 [x]_{y^2/4}^{\sqrt{y/2}} dy$

$= \int_0^2 \left(\sqrt{\frac{y}{2}} - \frac{y^2}{4}\right) dy$

$= \left[\frac{1}{\sqrt{2}} \left(\frac{y^{3/2}}{3/2}\right) - \frac{1}{4}\left(\frac{y^3}{3}\right)\right]_0^2$

$= \left[\frac{\cancel{2}}{\cancel{\sqrt{2}}} \frac{(\cancel{2})^{3/2}}{3} - \frac{1}{\cancel{4}}\left(\frac{\cancel{8}^{\,2}}{3}\right) - (0-0)\right]$ $(2\sqrt{2})$

$= \frac{4}{3} - \frac{2}{3}$

$= \frac{2}{3}$ //.

Using double integration, find the area bounded by $y = x$ and $y = x^2$.

Soln: Given $y = x$ and $y = x^2$.

$x^2 - x = 0$

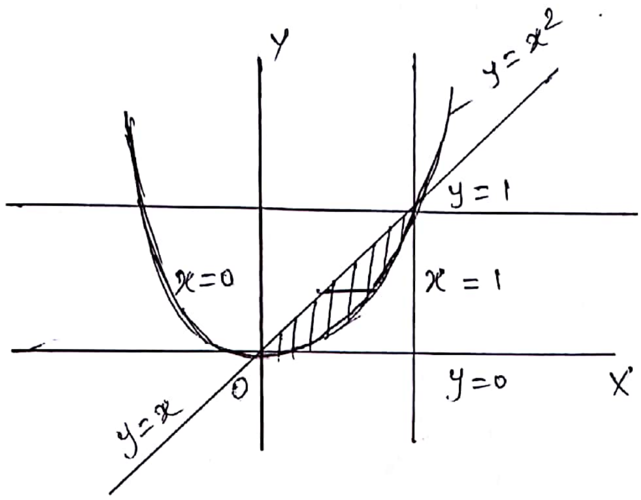
$x(x-1) = 0.$

$x = 0$ and $x = 1$.

when $x = 0 \Rightarrow y = 0.$

$x = 1 \Rightarrow y = 1$

$\therefore$ Point of intersection is $(0,0)$ & $(1,1)$



Here x varies from $x=y$ to $x=\sqrt{y}$

y varies from $y=0$ to $y=1$

$$\text{Area} = \iint_R dx\,dy$$

$$= \int_0^1 \int_y^{\sqrt{y}} dx\,dy$$

$$= \int_0^1 [x]_y^{\sqrt{y}}\,dy$$

$$= \int_0^1 (\sqrt{y} - y)\,dy$$

$$= \left[\frac{y^{3/2}}{3/2} - \frac{y^2}{2}\right]_0^1$$

$$= \left[\frac{1}{3}(2) - \frac{1}{2} - (0-0)\right]$$

$$= \frac{4-3}{6}$$

$$= \frac{1}{6}$$

$$\boxed{\text{Area} = 1/6}$$

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4) Find the area bounded by the parabola $y^2 = 4-x$ and $y^2 = 4-4x$ as a double integral and evaluate.

Soln :

Given, $y^2 = 4-x$ and $y^2 = 4-4x$.

$$\cancel{4} - x = \cancel{4} - 4x.$$

$$4x = x$$

$$4x - x = 0$$

$$3x = 0 \Rightarrow x = 0.$$

when $x = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

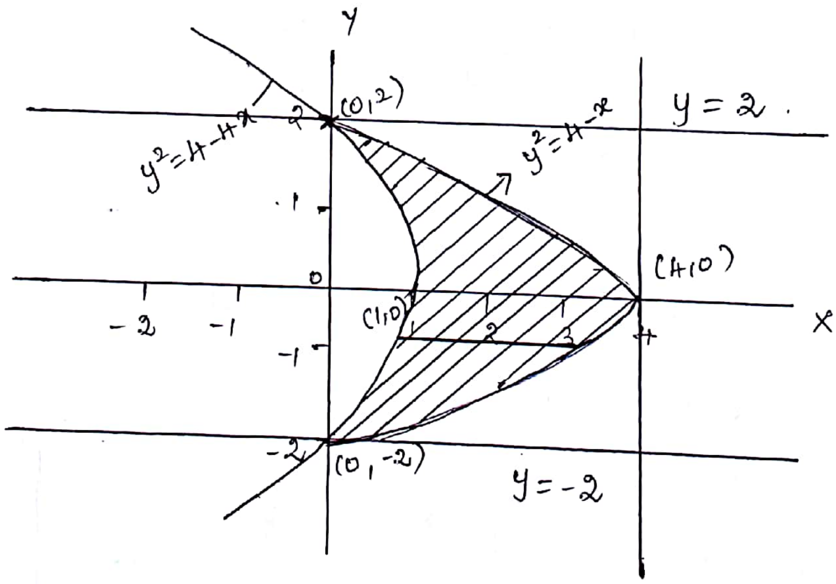
Point of intersection :

$y^2 = 4-x$

x	0	4
y	± 2	0

$y^2 = 4-4x$.

x	0	1
y	± 2	0



Here, x varies from $x = \frac{1}{4}(4-y^2)$ to $x = 4-y^2$

y varies from $y = -2$ to $y = 2$.

$y^2 = 4-x$.

$\Rightarrow x = 4-y^2$

$y^2 = 4-4x$

$\Rightarrow x = \frac{1}{4}(4-y^2)$

$$\text{Area} = \iint_R dx\,dy$$

$$= \int_{-2}^{2} \int_{\frac{1}{4}(4-y^2)}^{4-y^2} dx\,dy$$

$$= \int_{-2}^{2} \Big[x \Big]_{\frac{1}{4}(4-y^2)}^{4-y^2} dy$$

$$= \int_{-2}^{2} \Big[1(4-y^2) - \frac{1}{4}(4-y^2) \Big] dy$$

$$= \int_{-2}^{2} \frac{3}{4}(4-y^2)\, dy$$

$$= \frac{3}{4}\Big[4y - \frac{y^3}{3} \Big]_{-2}^{2}$$

$$= \frac{3}{4}\Big[8\cancel{y} - \frac{8}{3} - \Big[(-8) + \frac{8}{3}\Big]\Big]$$

$$= \frac{3}{4}\Big[16 - \frac{16}{3} \Big]$$

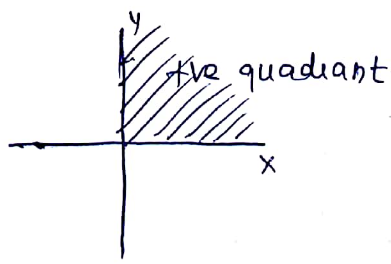
$$= \frac{\cancel{3}}{4}\Big[\frac{48-16}{\cancel{3}} \Big] = \frac{32}{4}$$

$$= 8 //.$$

---

5) Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$.

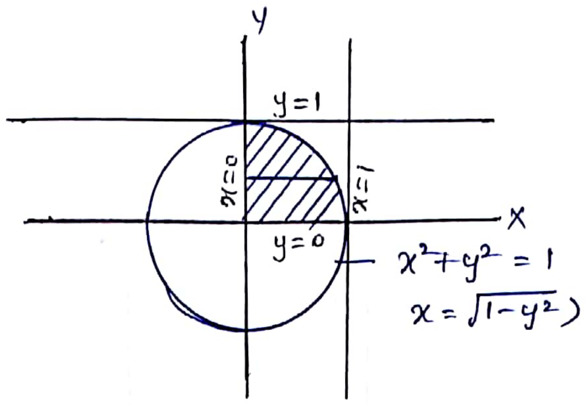
<u>soln</u>:



Given $x^2+y^2=1$.

$$x^2 = 1-y^2$$

$$x = \sqrt{1-y^2}.$$



Here x varies from $x=0$ to $x=\sqrt{1-y^2}$

y varies from $y=0$ to $y=1$.

$$\text{Area} = \iint_R xy \, dx \, dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx \, dy$$

$$= \int_0^1 \left[y\left(\frac{x^2}{2}\right) \right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \left[\frac{y(1-y^2)}{2} - 0 \right] dy$$

$$= \int_0^1 \frac{1}{2}(y - y^3) \, dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= \frac{1}{8} \,//$$

Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln:

Area of ellipse = 4 (Area of 1^{st} quadrant)

(ie) $= 4\left[\iint_R dx \, dy \right]$

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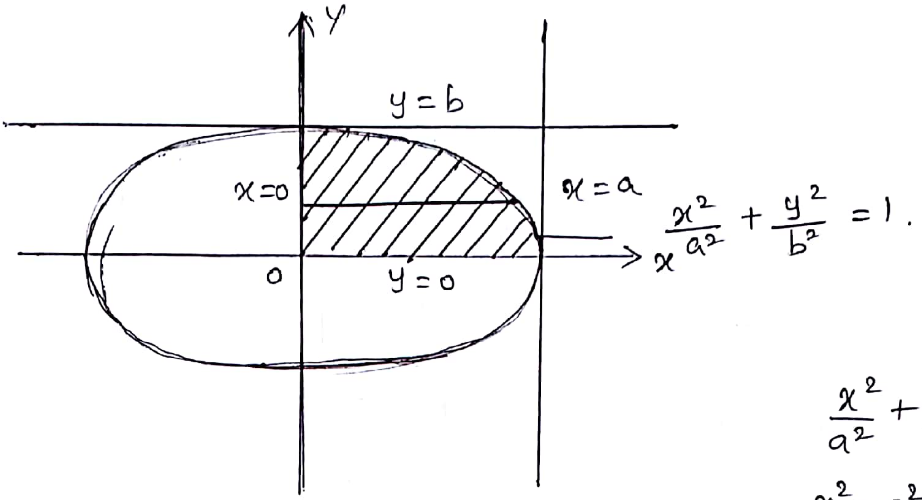


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 = a^2\left[1 - \frac{y^2}{b^2}\right]$$

$$x^2 = \frac{a^2}{b^2}\left[b^2 - y^2\right]$$

$$x = \frac{a}{b}\left(\sqrt{b^2-y^2}\right)$$

Here, x Varies from $x = 0$ to $x = \frac{a}{b}\left[\sqrt{b^2-y^2}\right]$

y Varies from $y = 0$ to $y = b$

Area of ellipse $= 4 \iint\limits_R dx\, dy.$

$$= 4 \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx\, dy$$

$$= 4 \int_0^b \left[x\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= 4 \int_0^b \left[\frac{a}{b}\sqrt{b^2-y^2}\right] dy.$$

$$= \frac{4a}{b}\left[\int_0^b \sqrt{b^2-y^2}\, dy\right]$$

W.k.T $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$= \frac{4a}{b}\left[\frac{y\sqrt{b^2-y^2}}{2} + \frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right)\right]_0^b$$

$$= \frac{4a}{\cancel{b}}\left[\frac{b^{\cancel{2}}}{\cancel{2}} \times \frac{\pi}{\cancel{2}}\right] = \pi ab_{//} \quad \left(\because \sin^{-1}(1) = \pi/2\right)$$

S.T the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

soln

Given, $y^2 = 4ax$

$x^2 = 4ay \Rightarrow y = \frac{x^2}{4a}$

Equating, $\left(\frac{x^2}{4a}\right)^2 = 4ax$.

$\frac{x^4}{16a^2} = 4ax$.

$x^4 = 64a^3x$.

$x^4 - 64a^3x = 0$.

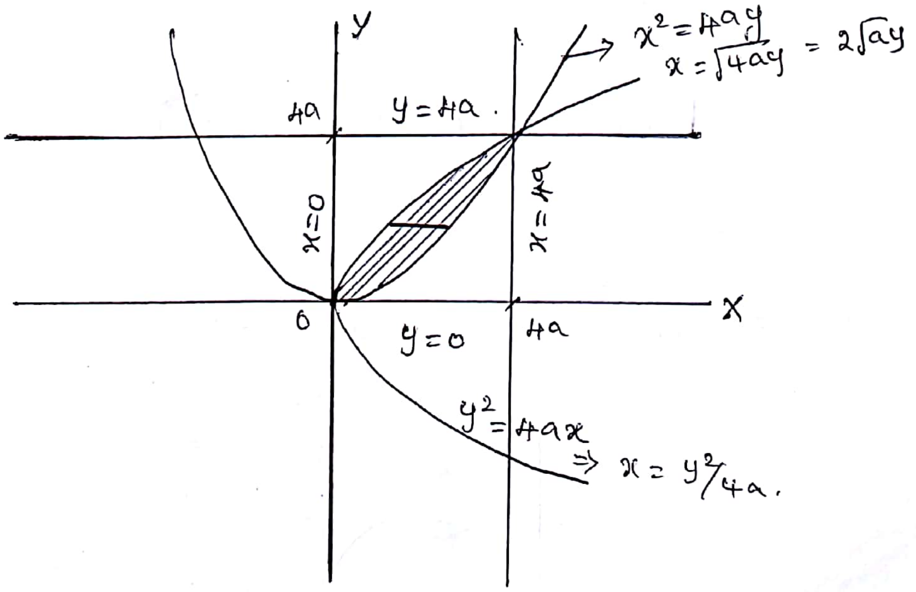
$x(x^3 - 64a^3) = 0$.

$x = 0$ and $x^3 = 64a^3$.

$x = 4a$.

when $x = 0 \Rightarrow y = 0$.
$x = 4a \Rightarrow y = 4a$ } (in $y = x^2/4a$)

Point of intersection is $(0,0)$ & $(4a, 4a)$.



Here, x varies from $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$

y varies from $y=0$ to $y = 4a$.

$$\text{Area} = \iint_R dx\, dy$$

$$= \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx\, dy$$

$$= \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy$$

$$= \left[2\sqrt{a}\left(\frac{y^{3/2}}{3/2}\right) - \frac{1}{4a}\left(\frac{y^3}{3}\right) \right]_0^{4a}$$

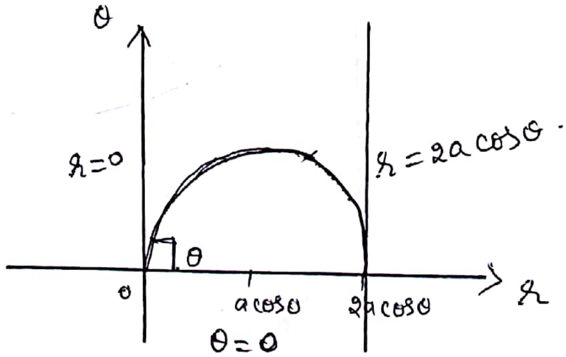
$$= \left[\frac{4\sqrt{a}}{3}(4a\sqrt{4a}) - \frac{1}{\cancel{12}_3 \cancel{a}}(\cancel{64}^{16} a^{\cancel{3}}) - (0-0) \right]$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{(32-16)a^2}{3} //$$

$$= \frac{16}{3} a^2 //.$$

Area as a double integral (In Polar co-ordinates)

1) Evaluate $\iint r^2 \sin\theta \, dr\, d\theta$, where R is the semi circle $r = 2a\cos\theta$ about the initial line.



Limit for r is $r=0$ to $r=2a\cos\theta$.

Limit for θ is $\theta=0$ to $\theta=\pi/2$.

Let $I = \iint r^2 \sin\theta \, dr\, d\theta$.

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta.$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} \sin\theta \, d\theta.$$

$$= \int_0^{\pi/2} \frac{8a^3\cos^3\theta}{3} \cdot \sin\theta \, d\theta.$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta.$$

W.K.T $\boxed{\int_0^{\pi/2} \sin^n x \cos^m x \, dx = \frac{[(m-1)(m-3)\cdots]\cdot[(n-1)(n-3)\cdots]}{(m+n)(m+n-2)(m+n-4)\cdots} \times [\pi/2, \text{only if } m,n\text{-even}]}$

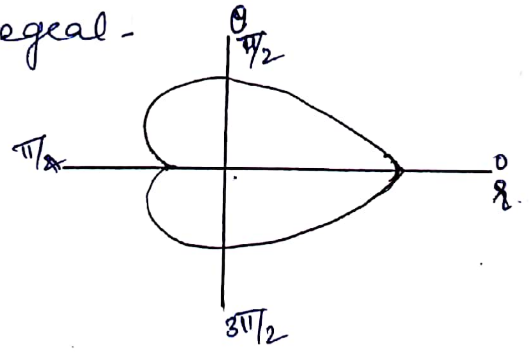
Here, $m=3$, $n=1$

$$= \frac{8a^3}{3}\left[\frac{(3-1)}{(3+1)(3+1-2)}\right]$$

$$= \frac{2\cancel{8}a^3\,(\cancel{2})}{3\times\cancel{4}\times\cancel{2}}$$

$$= \frac{2a^3}{3} //.$$

Find the area of the cardioid, $r = a(1+\cos\theta)$, using a double integral.



$\theta=0$, $r=2a$

$\theta=\pi/2$, $r=a$

$\theta=\pi$, $r=0$.

$\theta=3\pi/2$, $r=a$.

$\theta=2\pi$, $r=2a$.

Limit for r is $r=0$ to $r=a(1+\cos\theta)$

Limit for θ is $\theta=0$ to $\theta=2\pi$.

$$\text{Area} = \iint_R r\,dr\,d\theta .$$

$$= \int_0^{2\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta .$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta .$$

$$= \int_0^{2\pi} \frac{1}{2}\left[a^2(1+\cos\theta)^2 - 0\right] d\theta .$$

$$= \frac{a^2}{2}\int_0^{2\pi} (1+\cos^2\theta + 2\cos\theta)\,d\theta .$$

$$= \frac{a^2}{2}\left[\int_0^{2\pi} d\theta + \int_0^{2\pi}\cos^2\theta\,d\theta + 2\int_0^{2\pi}\cos\theta\,d\theta\right].$$

$$= \frac{a^2}{2}\left[[\theta]_0^{2\pi} + \underbrace{4\int_0^{\pi/2}\cos^2\theta\,d\theta} + 2[\sin\theta]_0^{2\pi}\right]$$

$$= \frac{a^2}{2}\left[(2\pi) + \cancel{4}^{2}\left[\frac{(2-1)}{\cancel{2}}\cdot\frac{\pi}{2}\right] + 2[0]\right]$$

$$= \frac{a^2}{2}\left[\frac{\cancel{2}\pi}{\cancel{2}} + 2\pi\right]$$

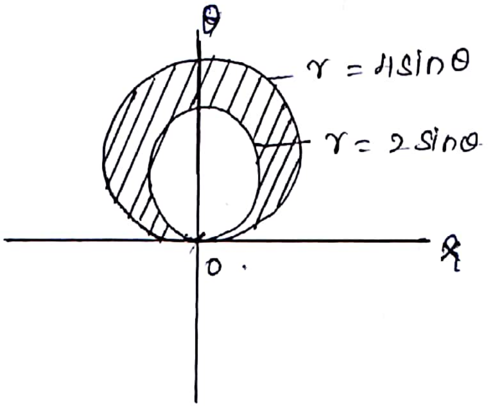
$$= \frac{a^2}{2}[3\pi]$$

$$= \frac{3\pi a^2}{2} \,//.$$

$$\because \int_0^{\pi/2}\cos^n x\,dx = \frac{(n-1)(n-3)\cdots}{n(n-2)\cdots} \times \frac{\pi}{2} \text{ (if } n\text{-even)}$$

calculate $\iint r^3 \, dr \, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

soln: Given $r = 2\sin\theta$, $r = 4\sin\theta$.



Let $I = \iint_R r^3 \, dr \, d\theta$.

$$= \int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr \, d\theta.$$

$$= \int_0^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta.$$

$$= \int_0^{\pi} \frac{1}{4}\left[256.\sin^4\theta - 16.\sin^4\theta\right] d\theta.$$

$$= \int_0^{\pi} \frac{1}{\cancel{4}}\left[\overset{60}{\cancel{240}} \sin^4\theta\right] d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta.$$

$$= 60 \left[2\left(\int_0^{\pi/2} \sin^4\theta \, d\theta\right)\right]$$

$$= 120 \int_0^{\pi/2} \sin^4\theta \, d\theta \Big]$$

W.K.T $\boxed{\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\cdots}{n(n-2)\cdots} \times \left[\pi/2 \text{ only if } n\text{-even}\right]}$

$$= 120 \cdot \left[\frac{(4-1)(4-3)}{4(4-2)} \cdot \frac{\pi}{2} \right]$$

$$= \cancel{120}\,\cancel{60}\,15 \left[\frac{3 \times 1}{\cancel{4} \times \cancel{2}} \times \frac{\pi}{2} \right]$$

$$= \left(\frac{15 \times 3}{2} \right) \pi$$

$$= \frac{45\pi}{2} \text{ //.}$$

Triple Integration (cartesian co-ordinates)

1) Evaluate $\int_0^a \int_0^b \int_0^c xyz \; \underline{dz\, dy\, dx}$. (correct form)

$$I = \int_0^a \int_0^b \left[\int_0^c xyz \right] dz\, dy\, dx.$$

$$= \int_0^a \int_0^b \left[xy \left(\frac{z^2}{2} \right) \right]_0^c dy\, dx.$$

$$= \frac{1}{2} \int_0^a \int_0^b xyc^2 \, dy\, dx$$

$$= \frac{c^2}{2} \int_0^a \left[\frac{xy^2}{2} \right]_0^b dx$$

$$= \frac{c^2}{4} \int_0^a (x(b^2) - 0)\, dx.$$

$$= \frac{b^2c^2}{4} \left[\frac{x^2}{2} \right]_0^a$$

$$= \frac{a^2b^2c^2}{8} \text{ //.}$$

Evaluate $\int_0^a \int_0^b \int_0^c e^{x+y+z}\, dz\, dy\, dx$.

$$I = \int_0^a \int_0^b \int_0^c (e^x \cdot e^y \cdot e^z)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b \left[e^x \cdot e^y \left(e^z\right)_0^c \right] dy\, dx.$$

$$= \int_0^a \int_0^b e^x\, e^y \left(e^c - e^0\right) dy\, dx.$$

$$= (e^c - 1) \int_0^a \left[e^x \cdot \left(e^y\right)_0^b \right] dx.$$

$$= (e^c - 1) \int_0^a \left[e^x \left(e^b - e^0\right) \right] dx.$$

$$= (e^c - 1)(e^b - 1) \left[e^x \right]_0^a$$

$$= (e^a - 1)(e^b - 1)(e^a - e^0)$$

(ie) $I = (e^a - 1)(e^b - 1)(e^c - 1)$

Evaluate $\int_0^2 \int_1^3 \int_1^2 x y^2 z\, dz\, dy\, dx$.

$$I = \int_0^2 \int_1^3 \int_1^2 x y^2 (z)\, dz\, dy\, dx.$$

$$= \int_0^2 \int_1^3 \left[x y^2 \left(\frac{z^2}{2} \right) \right]_1^2 dy\, dx.$$

$$= \int_0^2 \int_1^3 x y^2 \left(\frac{4}{2} - \frac{1}{2} \right) dy\, dx.$$

$$= \frac{\cancel{3}}{2} \int_0^2 x \left(\frac{y^3}{\cancel{3}} \right)_1^3 dx$$

$$= \frac{1}{2}\int_0^2 x(27-1)\,dx.$$

$$= \frac{26}{2}\int_0^2 x\,dx.$$

$$= 13\left[\frac{x^2}{2}\right]_0^2$$

$$= 13(2-0)$$

$$= 26//.$$

---

4) Evaluate $\int_0^1\int_0^{1-x}\int_0^{x+y} e^z \,dx\,dy\,dz$

$$I = \int_0^1\int_0^{1-x}\int_0^{x+y} [e^z]_{\cancel{0}}^{\cancel{x+y}} \underbrace{dz\,dy\,dx}$$ (correct form)

$$= \int_0^1\int_0^{1-x} \left[e^z\right]_0^{x+y} dy\,dx$$

$$= \int_0^1\int_0^{1-x} (e^{x+y} - e^0)\,dy\,dx.$$

$$= \int_0^1\int_0^{1-x} (e^x \cdot e^y - 1)\,dy\,dx.$$

$$= \int_0^1 \left[e^x e^y - y\right]_0^{1-x} dx.$$

$$= \int_0^1 \left[\underbrace{e^{\cancel{x}} e^{1-\cancel{x}}} - (1-x) - (e^x e^0 - 0)\right] dx.$$

$$= \int_0^1 (e^1 - (1-x) - e^x)\,dx.$$

$$= \left[x e^1 - x + \frac{x^2}{2} - e^x\right]_0^1$$

$$= e^{1} - 1 + \tfrac{1}{2} - [e^{1}] - [(e^{0})]$$

$$= \cancel{e^{1}} - \tfrac{1}{2} - \cancel{e^{1}} + 1$$

$$I = \tfrac{1}{2} \,//.$$

Evaluate $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$

Soln:

$$I = \int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^b \left[\frac{x^3}{3} + (y^2+z^2)\,x\right]_0^c dy\,dz$$

$$= \int_0^a\int_0^b \left[\frac{c^3}{3} + c(y^2+z^2)\right] dy\,dz.$$

$$= \int_0^a \left[\frac{c^3}{3}(y) + c\left(\frac{y^3}{3}\right) + cz^2y\right]_0^b dz$$

$$= \int_0^a \left(\frac{c^3b}{3} + \frac{cb^3}{3} + cbz^2\right) dz.$$

$$= \left[\frac{bc^3}{3}z + \frac{b^3c}{3}z + cb\left(\frac{z^3}{3}\right)\right]_0^a$$

$$= \frac{bc^3a}{3} + \frac{b^3ca}{3} + \frac{cba^3}{3}$$

$$I = \frac{a^3bc}{3} + \frac{ab^3c}{3} + \frac{abc^3}{3} \,//.$$

$$= \frac{1}{3}(abc)\left[a^2+b^2+c^2\right] //$$

Volume as a triple integral :

1) Evaluate $\iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region of integration bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$.

Soln :

Given,

$x+y+z=1 \Rightarrow z = 1-x-y$.

$x+y = 1 \Rightarrow y = 1-x$.

$\therefore$ x varies from $x=0$ to $x=1$.

y varies from $y=0$ to $y=1-x$

z varies from $z=0$ to $z=1-x-y$.

$$\therefore I = \iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx.$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[(\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}+1)^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[(2)^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

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$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4}(1-x) + (\cancel{x}+1-\cancel{x}+1)^{-1} - (0+(x+1)^{-1})\right] dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4}(1-x) + (2)^{-1} \cancel{+} - (x+1)^{-1}\right] dx.$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4} - \frac{x}{4} + \frac{1}{2} \cancel{+} - (x+1)^{-1}\right] dx$$

$$= -\frac{1}{2}\left[\frac{1}{4}x - \frac{1}{4}\left(\frac{x^2}{2}\right) + \frac{1}{2}(x) \cancel{+} - \log\frac{(x+1)^{\cancel{1}}}{\cancel{1}}\right]_0^1$$

$$= -\frac{1}{2}\left[\left(\frac{1}{4} - \frac{1}{8} + \frac{1}{2} \cancel{+} - \log(2)\right) - 0\right]$$

$$= -\frac{1}{2}\left[\frac{2-1+4 \cancel{+} - 8\log 2}{8}\right]$$

$$= -\frac{1}{2}\left[\frac{5 \cancel{+} - 8\log 2}{8}\right]$$

$$= \frac{8\log 2 - 5}{16} \quad //.$$

Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $x=0$.

Soln:

Given $y+z=4 \Rightarrow z=4-y$.

$x^2+y^2=4 \Rightarrow y^2 = 4-x^2$.

$y = \pm\sqrt{4-x^2}$.

And $x^2 = 4 \Rightarrow x = \pm 2$

Here, x varies from $x=-2$ to $x=2$

y varies from $y=-\sqrt{4-x^2}$ to $y=+\sqrt{4-x^2}$

z varies from $z=0$ to $z=4-y$.

$$\text{Volume} = \iiint_V dz\, dy\, dx.$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-y} dz\, dy\, dx.$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (z)_0^{4-y}\, dy\, dx.$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx.$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4\, dy - y\, dy)\, dx$$

$$= \int_{-2}^{2} \left(4y - \frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

$$= \int_{-2}^{2} \left[4\sqrt{4-x^2} - \left(\frac{4-x^2}{2}\right) - \left(-4\sqrt{4-x^2} - \frac{(4-x^2)}{2}\right)\right] dx$$

$$= \int_{-2}^{2} \left(4\sqrt{4-x^2} - \cancel{\frac{(4-x^2)}{2}} + 4\sqrt{4-x^2} + \cancel{\frac{(4-(x^2))}{2}}\right) dx$$

$$= 8 \int_{-2}^{2} \sqrt{4-x^2}\, dx.$$

W.K.T $\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)}$

$$= 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2} \sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}.$$

$$= 8\left[0 + 2\sin^{-1}(1) - (0 + 2\sin^{-1}(-1))\right]$$

$$= 8\left[\cancel{2}(\pi/\cancel{2}) - (-2\sin^{-1}(1))\right]$$

$$= 8\left[\pi + \cancel{2}(\pi/\cancel{2})\right] = 8(2\pi)$$

$$= 16\pi$$

Find the volume of the sphere. $x^2+y^2+z^2=a^2$ without using transformation.

soln:

Volume of a sphere = 8 × Volume in an octant.

Given, $x^2+y^2+z^2=a^2 \Rightarrow z = \sqrt{a^2-x^2-y^2}$.

$x^2+y^2=a^2 \Rightarrow y = \sqrt{a^2-\cancel{y^2}x^2}$

$x^2=a^2 \Rightarrow x = a$.

Volume of a sphere $= 8\iiint_V dz\,dy\,dx$.

$$= 8\int_0^a \int_0^{\sqrt{a^2-\cancel{y^2}x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-\cancel{y^2}x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}}\,dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-\cancel{y^2}x^2}} \left(\sqrt{a^2-x^2-y^2}\right) dy\,dx.$$

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\,dy\right) dx.$$

$$= 8\int_0^a \left[\frac{y\sqrt{a^2-x^2-y^2}}{2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left[0 + \frac{a^2-x^2}{x}\left(\frac{\pi}{2}\right)\right]dx.$$

$$= 2\pi \int_0^a (a^2-x^2)\,dx.$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[a^3 - \frac{a^3}{3}\right]$$

$$= 2\pi\left(\frac{2a^3}{3}\right)$$

$$= \frac{4\pi a^3}{3} \;//.$$

---

4) Evaluate $\iiint_V \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the first octant of the sphere $x^2+y^2+z^2=a^2$.

soln:

z varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$

y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

x varies from $x=0$ to $x=a$.

$$\therefore\ I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}.$$ (correct form)

W.K.T $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{(a^2-x^2-y^2)-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx.$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{\pi}{2}\right] dy\, dx.$$

$$= \frac{\pi}{2} \int_0^a \left(y\right)_0^{\sqrt{a^2-x^2}} dx.$$

$$= \frac{\pi}{2} \int_0^a \left(\sqrt{a^2-x^2}\right) dx.$$

$$= \frac{\pi}{2} \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{\pi}{2} \left[0 + \frac{a^2}{2} \sin^{-1}(1)\right]$$

$$= \frac{\pi}{2} \left[\frac{a^2}{2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{a^2 \pi^2}{8} \ //.$$

---

5) Find the Volume of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Soln:

Volume of an ellipsoid = 8 × Volume of an octant

Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\frac{z^2}{c^2} = 1 - \left(\frac{x^2}{a^2}\right) - \left(\frac{y^2}{b^2}\right)$$

$$z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

And $\frac{x^2}{a} + \frac{y^2}{b^2} = 1.$

$$\Rightarrow \therefore \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

$$= b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}}.$$

Also, $\frac{x^2}{a^2} = 1. \Rightarrow x^2 = a^2.$

$$\Rightarrow x = a.$$

$\therefore$ x Varies from $x=0$ to $x=a$.

y Varies from $y=0$ to $y = b\sqrt{1 - x^2/a^2}$

z Varies from $z=0$ to $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

$$\therefore \text{Volume} = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\, dy\, dx.$$

$$= 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} [z]_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dy\, dx.$$

$$= 8c\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left(\sqrt{\left(1-\frac{x^2}{a^2}\right) - \frac{y^2}{b^2}}\right) dy\, dx.$$

$(x^{1y}/_{(\cdot/\cdot)})$ by b^2 for the first term (inside root)

$$= 8c \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right) - y^2}{b^2}}\; dy\; dx.$$

$$= \frac{8c}{b} \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right) - y^2}\; dy\; dx.$$

Take $b\sqrt{1-\frac{x^2}{a^2}} = t. \Rightarrow t^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$$= \frac{8c}{b} \int_0^a \int_0^t \sqrt{t^2-y^2}\; dy\; dx.$$

$$= \frac{8c}{b} \int_0^a \left[\frac{y\sqrt{t^2-y^2}}{2} + \frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right)\right]_0^t dx$$

$$= \frac{8c}{b} \int_0^a \left(0 + \frac{t^2}{2}\sin^{-1}(1) - 0\right) d\cancel{t}x.$$

$$= \frac{8c}{b} \int_0^a \frac{t^2}{2}\cdot\frac{\pi}{2}\; dx.$$

$$= 2\frac{\cancel{8}c}{\cancel{b}} \int_0^a \frac{\cancel{b^2}\left(1-\frac{x^2}{a^2}\right)}{\cancel{2}}\, \frac{\pi}{\cancel{2}}\; dx.$$

$$= \frac{2c\pi}{\cancel{b}}(\cancel{b^2}) \int_0^a \left(1 - x^2/a^2\right) dx.$$

$$= 2c\pi(b)\left[x - \frac{1}{a^2}\left(\frac{x^3}{3}\right)\right]_0^a$$

$$= 2c\pi(b)\left[a - \frac{a}{3} - 0\right]$$

$$= 2c\pi(b)\left(\frac{2a}{3}\right)$$

$$V = \frac{4\pi abc}{3} \;//$$

6) Evaluate the integration $\iiint_V xyz\, dz\, dy\, dx$, taken throughout the volume for which $x, y, z \geq 0$ and $x^2+y^2+z^2 \leq 9$.

Soln: We use spherical polar co-ordinates,

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta.$$

$$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi.$$

Given,

$$\iiint_V xyz\, dz\, dy\, dx = \iiint_V r^5 \sin^3\theta\cos\theta\sin\phi\cos\phi\, dr\, d\theta\, d\phi.$$

Limits are $0 \leq r \leq 3$

$0 \leq \theta \leq \pi/2$.

$0 \leq \phi \leq \pi/2$.

$$\therefore \Rightarrow \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 r^5 \sin^3\theta\cos\theta\sin\phi\cos\phi\, dr\, d\theta\, d\phi.$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^3 \sin^3\theta\cos\theta\left(\frac{\sin 2\phi}{2}\right) d\theta\, d\phi.$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{729}{6}\sin^3\theta\cos\theta\,\frac{\sin 2\phi}{2}\, d\theta\, d\phi.$$

$$= \frac{729}{12}\int_0^{\pi/2}\left[\sin^3\theta\cos\theta\, d\theta\right]\int_0^{\pi/2}\sin 2\phi\, d\phi$$

Take, $t = \sin\theta$.

$dt = \cos\theta \, d\theta$.

when $\theta = 0 \Rightarrow t = 0$.

$\theta = \pi/2 \Rightarrow t = 1$

$$V = \frac{729}{12}\left[\int_0^1 t^3 \, dt\right]\left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2}.$$

$$= \frac{729}{12}\left[\frac{t^4}{4}\right]_0^1 \left[-\frac{1}{2}(\cos 2\phi)\right]_0^{\pi/2}$$

$$= \frac{729}{12}\left[\frac{1}{4}\right]\left(\frac{-1}{2}\right)(-1-1)$$

$$= \frac{\overset{243}{\cancel{729}}(-1)(-\cancel{2})}{4\,\cancel{12} \times 4 \times \cancel{2}} = \frac{243}{16}.$$

$$V = \frac{243}{16} \,//.$$

(Or)

<u>Soln</u> :

$x^2+y^2+z^2 \leq 9, \quad x, y, z \geq 0.$

Given. $x^2+y^2+z^2 \leq 9 \Rightarrow z^2 = 9 - x^2 - y^2.$

$z = \sqrt{9-x^2-y^2}$

$x^2+y^2 \leq 9 \Rightarrow y = \sqrt{9-x^2}.$

$x^2 \leq 9 \Rightarrow \cancel{y}\; x = 3.$

Here, x varies from $x = 0$ to $x = 3$.

y varies from $y = 0$ to $y = \sqrt{9-x^2}$

z varies from $z = 0$ to $z = \sqrt{9-x^2-y^2}$

$$V = \iiint_V xyz \, dz \, dy \, dx.$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xyz \, dz \, dy \, dx.$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{9-x^2-y^2}} dy \, dx.$$

$$= \frac{1}{2} \int_0^3 \int_0^{\sqrt{9-x^2}} xy \left[\left(\sqrt{9-x^2-y^2}\right)^2 - 0\right] dy \, dx.$$

$$= \frac{1}{2} \int_0^3 \int_0^{\sqrt{9-x^2}} x \left[y\left(9-x^2-y^2\right)\right] dy \, dx.$$

$$= \frac{1}{2} \int_0^3 \int_0^{\sqrt{9-x^2}} x \left[9y - x^2 y - y^3\right] dy \, dx.$$

$$= \frac{1}{2} \int_0^3 \left[x\left(\frac{9y^2}{2} - x^2\left(\frac{y^2}{2}\right) - \frac{y^4}{4}\right)\right]_0^{\sqrt{9-x^2}} dx.$$

$$= \frac{1}{2} \int_0^3 x\left[\frac{9}{2}\left(\sqrt{9-x^2}\right)^2 - \frac{x^2}{2}\left(\sqrt{9-x^2}\right)^2 - \frac{1}{4}\left(\left(\sqrt{9-x^2}\right)\right)^4\right] dx$$

$$= \frac{1}{2} \int_0^3 x\left[\frac{9}{2}\left(9-x^2\right) - \frac{x^2}{2}\left(9-x^2\right) - \frac{1}{4}\left(9-x^2\right)^2\right] dx$$

$$= \frac{1}{2} \int_0^3 \frac{9}{2}\left[9x - x^3\right] - \frac{1}{2}\left[9x^3 - x^5\right] - \frac{x}{4}\left[81 + x^4 - 18x^2\right] dx$$

$$= \frac{1}{2}\int_0^3\left[\frac{9}{2}\left(9x - x^3\right) - \frac{1}{2}\left(9x^3 - x^5\right) - \frac{1}{4}\left(81x + x^5 - 18x^3\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{9}{2}\left(\frac{9x^2}{2} - \frac{x^4}{4}\right) - \frac{1}{2}\left(\frac{9x^4}{4} - \frac{x^6}{6}\right) - \frac{1}{4}\left[\frac{81x^2}{2} + \frac{x^6}{6} - \frac{\cancel{18}^{9}x^4}{\cancel{4}_2}\right]\right)_0^3$$

$$= \frac{1}{2}\left[\frac{9}{2}\left(\frac{81}{2} - \frac{81}{4}\right) - \frac{1}{2}\left(\frac{729}{4} - \frac{729}{6}\right) - \frac{1}{4}\left[\cancel{\frac{729}{2}} + \frac{729}{6} - \cancel{\frac{729}{2}}\right] - (0)\right]$$

$$= \frac{1}{2}\left[\frac{9(81)}{2}\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{2}(729)\left(\frac{1}{4} - \frac{1}{6}\right) - \frac{1}{4}\left(\frac{729}{6}\right)\right]$$

$$= \frac{1}{2}\left[\frac{729}{2}\left(\frac{1}{4}\right) - \frac{729}{\cancel{2}}\left(\frac{\cancel{2}}{24}\right) - \frac{729}{24}\right]$$

$$= \frac{1}{2}\left[\frac{729}{8} - \underbrace{\frac{729}{24} - \frac{729}{24}}\right]$$

$$= \frac{1}{2}\left[\frac{729}{8} - \cancel{2}\left(\frac{729}{\cancel{24}_{12}}\right)\right]$$

$$= \frac{1}{2}\left[729\left(\frac{1}{8} - \frac{1}{12}\right)\right]$$

$$= \frac{1}{2}\left[\cancel{729}^{243}\left(\frac{3-2}{\cancel{24}_{8}}\right)\right] = \frac{1}{2}\left[243\left(\frac{1}{8}\right)\right]$$

$$= \frac{243}{16} \;//$$

7) Evaluate : $I = \int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z} \, dz \, dy \, dx$.

Soln:

$$I = \int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^x . e^y . e^z \, dz \, dy \, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^x . e^y \, e^z \right]_{z=0}^{z=x+\log y} dy \, dx.$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^x e^y \left(e^{x+\log y} - \underbrace{e^0}_{1} \right) \right] dy \, dx.$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^x e^y . e^x . e^{\log y} - e^x e^y \right] dy \, dx.$$

$e^{\log y} = y$

$$= \int_0^{\log 2} \int_0^{x} \left(e^{2x} \left(e^y . y \right) - e^x e^y \right) dy \, dx.$$

$$= \int_0^{\log 2} \int_0^{x} \left(e^{2x} \left(y e^y \right) - e^x e^y \right) dy \, dx.$$

$(\because \int uv \, dx = uv_1 - u'v_2 + \cdots)$

$$= \int_0^{\log 2} \left[e^{2x} \left(y e^y - e^y \right) - e^x e^y \right]_{y=0}^{y=x} dx.$$

$$= \int_0^{\log 2} \left[e^{2x} \left(x e^x - e^x \right) - e^x e^x - \left(e^{2x}(0-1) - e^x \right) \right] dx$$

$$= \int_0^{\log 2} \left[e^{2x} e^x (x-1) - \cancel{e^{2x}} + \cancel{e^{2x}} + e^x \right] dx.$$

$$= \int_0^{\log 2} \left[\underbrace{e^{3x}}_{v}\,\underbrace{(x-1)}_{u} + e^x \right] dx.$$

$\left[\because \int uv\,dx = uv_1 - u'v_2 + \cdots \text{ (Bernoulli's formula)}\right]$

$$= \left[(x-1)\left(\frac{e^{3x}}{3}\right) - 1\left(\frac{e^{3x}}{3^2}\right) + e^x \right]_0^{\log 2}.$$

$$= \left[(\log 2 - 1)\frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} + e^{\log 2} - \left((0-1)\frac{1}{3} - \frac{1}{9} + 1\right) \right]$$

$$= \left[(\log 2 - 1)\frac{\cancel{e}^{\cancel{\log} 2^3}}{3} - \frac{\cancel{e}^{\cancel{\log} 2^3}}{9} + \cancel{e}^{\cancel{\log} 2} - \left(-\frac{1}{3} - \frac{1}{9} + 1\right) \right]$$

$$= \left[(\log 2 - 1)\frac{8}{3} - \frac{8}{9} + 2 - \left(\frac{-3-1+9}{9}\right) \right]$$

$$= \left(\frac{8}{3}\log 2\right) - \frac{8}{3} - \frac{8}{9} + 2 - \left(\frac{5}{9}\right)$$

$$= \frac{8}{3}\log 2 + \left(\frac{-24-8+18}{9}\right) - \frac{5}{9}$$

$$= \frac{8}{3}\log 2 - \frac{14}{9} - \frac{5}{9}$$

$$= \frac{8}{3}\log 2 - \frac{19}{9} \;//$$

8) Find the Volume of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, $x = 0$, $y = 0$, $z = 0$.

Soln

To find $\iiint_V dz\,dy\,dx$.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

$$\Rightarrow z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right).$$

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \Rightarrow \quad y = b\left(1 - \frac{x}{a}\right).$$

$$\frac{x}{a} = 1 \quad \Rightarrow \quad x = a.$$

$\therefore$ x Varies from $x = 0$ to $x = a$.

y Varies from $y = 0$ to $y = b\left(1 - \frac{x}{a}\right)$

z Varies from $z = 0$ to $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$.

$\therefore$ Let $I = \iiint\limits_V dz\, dy\, dx.$

$$= \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dz\, dy\, dx$$

$$= \int_0^a \int_0^{b(1-x/a)} \left[z \right]_0^{c(1-x/a-y/b)} dy\, dx.$$

$$= \int_0^a \int_0^{b(1-x/a)} \left(c\left(1 - x/a - y/b\right) - 0 \right) dy\, dx.$$

$$= c\int_0^a \left[y - \frac{x}{a}(y) - \frac{1}{b}\left(\frac{y^2}{2}\right) \right]_0^{b(1-x/a)} dx.$$

$$= c\int_0^a \left[b(1-x/a) - \frac{x}{a}\left(b(1-x/a)\right) - \frac{\cancel{b}^2}{2\cancel{b}}(1-x/a)^2 \right] dx$$

$$= bc\int_0^a \underbrace{(1-x/a) - \frac{x}{a}(1-x/a)} - \frac{1}{2}(1-x/a)^2 \Big] dx.$$

$$= bc\int_0^a (1-x/a)\left[1-x/a\right] - \tfrac{1}{2}\left[1-x/a\right]^2 dx.$$

$$= bc\int_0^a \left[(1-x/a)^2 - \tfrac{1}{2}(1-x/a)^2\right]dx$$

$$= bc\int_0^a \left[1-x/a\right]^2 (1-\tfrac{1}{2})\, dx.$$

$$= \frac{bc}{2}\int_0^a (1-x/a)^2\, dx.$$

$$= \frac{bc}{2}\left[\frac{(1-x/a)^3}{3(-1/a)}\right]_0^a$$

$$= \frac{bc}{2\times 3}\left[(-a)(1-x/a)^3\right]_0^a$$

$$= \frac{bc}{6}\left[-a[(\cancel{1-1})^3 - (1)^3]\right]$$

$$= \frac{bc}{6}(a)$$

$$= \frac{abc}{6} //$$

Area as a double integral :

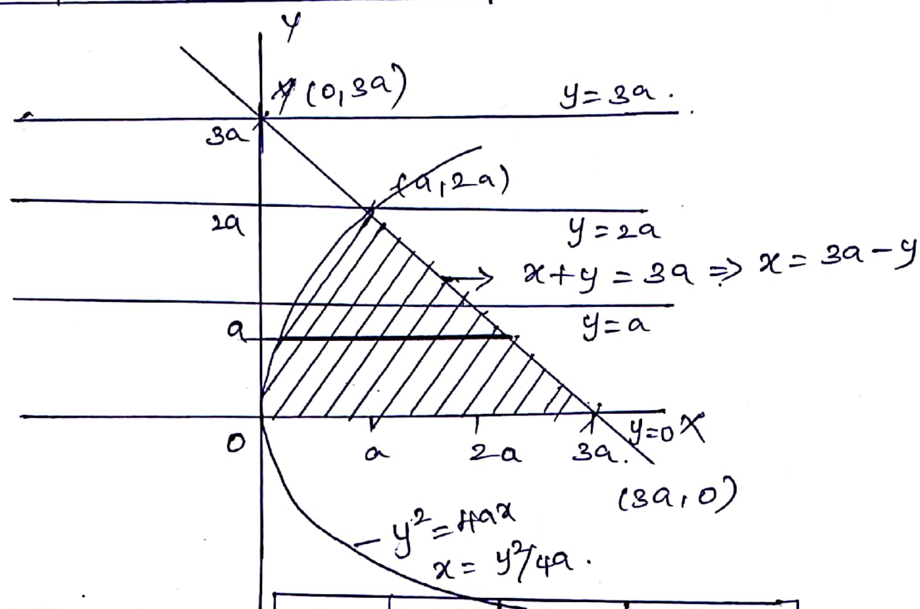
1) Find the area enclosed by the curve $y^2 = 4ax$ and the lines $x+y=3a$, $y=0$.

Soln: Given $y^2 = 4ax$.

$x+y = 3a$, $y=0$.

Take $x + y = 3a \Rightarrow y = 3a - x$.

x	0	a	$3a$.
y	$3a$	$2a$	0.



$y^2 = 4ax \Rightarrow$

x	0	a	$-a$
y	0	$2a$	$-2a$.

$$\text{Area} = \iint_R dx\,dy$$

$$= \int_0^{2a} \int_{y^2/4a}^{3a-y} dx\,dy.$$

$$= \int_0^{2a} [x]_{y^2/4a}^{3a-y}\,dy.$$

$$= \int_0^{2a} \left[3a - y - \frac{y^2}{4a}\right] dy.$$

$$= \left[3ay - \frac{y^2}{2} - \frac{1}{4a}\left(\frac{y^3}{3}\right)\right]_0^{2a}$$

$$= \left(6a^2 - \frac{4a^2}{2} - \frac{\cancel{8}^{2}a^{\cancel{3}\,2}}{\cancel{12}_{3}\cancel{a}}\right) - (0)$$

$$= 6a^2 - 2a^2 - \frac{2}{3}a^2$$

$$= 4a^2 - \frac{2}{3}a^2$$

$$= a^2\left(4 - \frac{2}{3}\right) = a^2\left(\frac{12-2}{3}\right)$$

$$= \frac{10a^2}{3}$$

<u>Triple Integral.</u>

Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{(a^2-x^2-y^2)-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\Bigg]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

i.e. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2}\,dy\,dx$$

$$= \frac{\pi}{2} \int_0^a y\Big]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{\pi}{2}\left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0\right]$$

$$= \frac{\pi}{2} \cdot \frac{a^2}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2 a^2}{8}.$$

2) Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}.$

In above problem, put $a=1$,

we get, $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}.$

3) Evaluate $\displaystyle\iiint_V dx\,dy\,dz$ where V is the finite region of space (tetrahedron) formed by planes $x=0$, $y=0$, $z=0$ and $2x+3y+4z=12$.

Let $I = \displaystyle\iiint_V dz\,dy\,dx$.

Gn $2x+3y+4z=12$ & $x=0$, $y=0$, $z=0$.

Limit of z : $z=0$ to $z=\frac{1}{4}(12-2x-3y)$

Limit of y : $y=0$ to $y=\frac{1}{3}(12-2x)$.

Limit of x : $x=0$ to $x=6$.

$$I = \int_0^6 \int_0^{\frac{1}{3}(12-2x)} \int_0^{\frac{1}{4}(12-2x-3y)} dz\, dy\, dx$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} z \Big]_0^{\frac{1}{4}(12-2x-3y)} dy\, dx.$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} \frac{1}{4}(12-2x-3y)\, dy \Big]\, dx$$

$$= \frac{1}{4}\int_0^6 \int_0^{\frac{1}{3}(12-2x)} (12-2x) - 3y \Big]\, dy\, dx.$$

$$= \frac{1}{4}\int_0^6 \Big[(12-2x)y - 3\left(\frac{y^2}{2}\right)\Big]_0^{\frac{1}{3}(12-2x)} dx.$$

$$= \frac{1}{4}\int_0^6 \left[\frac{(12-2x)^2}{3} - \frac{3}{2}\left[\frac{12-2x}{3}\right]^2\right] dx.$$

$$= \frac{1}{4}\int_0^6 \left[\frac{(12-2x)^2}{3} - \frac{(12-2x)^2}{6}\right] dx.$$

$$= \frac{1}{4}\int_0^6 \frac{2(12-2x)^2 - (12-2x)^2}{6} \Big]\, dx.$$

$$= \frac{1}{24}\int_0^6 (12-2x)^2\, dx.$$

$$= \frac{1}{24}\left[\frac{(12-2x)^3}{-6}\right]$$

$$= \frac{1}{24}\left[\frac{(12)^3}{6}\right]$$

$$= 12.$$