Unit - V

A random variable (RV) is a rule (or function) that assigns a real number to every outcome of a random experiment, while a random process is a rule (or function) that assigns a time function to every outcome of a random experiment.

For example, Consider the random experiment of tossing a dice at $t=0$ and observing the number on the top face.

The sample space of this experiment consists of the outcomes $\{1, 2, 3, \dots 6\}$. For each outcome of the experiment, let us arbitrary assign a function of time t $(0 \le t < \infty)$ in the following manner.

outcome :	1	2	3	4	5	6
Function of time :	$x_1(t) = -4$	$x_2(t) = -2$	$x_3(t) = 2$	$x_4(t) = 4$	$x_5(t) = -t/2$	$x_6(t) = t/2$

The set of functions $\{x_1(t), x_2(t), \dots x_6(t)\}$ represents a random process.

Defn: A random process is a collection of R.V's $\{X(s,t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (parameter set or index set)).

The set of possible values of a any individual member of the random process is called state space

Any individual member itself is called a sample function of the process.

Markov Process: A random process in which the future value (next state) depends only on the present (current value) value and not on the past values, is called a Markov process.

Example: Consider the random experiment of tossing a fair coin many times.

There are two possible outcomes for each trial: head (denoted by 1) and

tail (denoted by 0)

each with probability $\frac{1}{2}$

Let X_n be the r.v denoting the outcome of the n^{th} toss.

Then $P(X_n=1)=\frac{1}{2}$ and $P(X_n=0)=\frac{1}{2}$ for $n=1,2,3,4,\ldots$

$X_1, X_2, X_3 \ldots$ are independent r.v's.

Now let S_n be a r.v. defined as follows

S_n = Total number of heads in the first n trials

$= X_1 + X_2 + \cdots + X_n$

S_n can take possible values $0, 1, 2, \ldots n$

let $S_n = k$

Then $S_{n+1} = \begin{cases} k, & \text{If the } (n+1)^{th} \text{ trial results in a tail} \\ k+1, & \text{If the } (n+1)^{th} \text{ trial results in a head} \end{cases}$

$\therefore P(S_{n+1} = k+1 / S_n = k) = \frac{1}{2}$

and $P(S_{n+1} = k / S_n = k) = \frac{1}{2}$

and these probabilities depend only on the value of S_n and not on the values of $S_1, S_2, \ldots S_{n-1}$.

This is a simple example of a Markov chain.

Note: Random processes $\{X(t)\}$ (with Markov property) which take discrete values, whether t is discrete or continuous, are called Markov chains.

Definition of a Markov chain

If, for all n, $P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots X_0 = a_0\}$

$= P\{X_n = a_n / X_{n-1} = a_{n-1}\}$,

then the process $\{X_n\}$, $n=0,1,\ldots$ is called a Markov chain. (M.C)

* $a_1, a_2, \ldots, a_n$ are called the states of the M.C.

* The conditional probability $P\{X_n = a_j / X_{n-1} = a_i\}$ is called the one-step transition probability from state a_i to state a_j at the nth step (trial) and is denoted by $p_{ij}(n-1, n)$.

* If the one step transition probability does not depend on the step, then the Markov chain is called a $\boxed{\text{homogeneous}}$ Markov chain.

* If the Markov chain is homogeneous, the one-step transition probability is denoted by p_{ij} and the matrix $P = (p_{ij})$ is called the one-step transition probability matrix (tpm).

Note: The tpm of a Markov chain is a stochastic matrix. i.e., (i) $p_{ij} \geq 0$ for all i, j

(ii) $\sum_j p_{ij} = 1$ for all i (i.e. sum of the elements of any row is 1).

* The Conditional probability that the process is in state a_j at step n, given that it was in state a_i at step 0, i.e $P\{X_n = a_j / X_0 = a_i\}$ is called the n-step transition probability and denoted by $p_{ij}^{(n)}$.

<u>Note:</u> $p_{ij}^{(1)} = p_{ij}$

<u>Definition:</u> If the probability that the process is in state a_i is p_i $(i = 1, 2, \ldots k)$ at any arbitrary step, then the row vector $p = (p_1, p_2, \ldots p_k)$ is called the probability distribution of the process at that time.

In particular, $p^{(0)} = \{p_1^{(0)}, p_2^{(0)}, \ldots p_k^{(0)}\}$ is the initial probability distribution, where $p_1^{(0)} = P[X_0 = 1]$, $p_2^{(0)} = P[X_0 = 2]$, ...

<u>Chapman-kolmogorov Theorem:</u>

If P is the tpm of a homogeneous Markov chain, then the n-step tpm $P^{(n)}$ is equal to P^n.

i.e, $[p_{ij}^{(n)}] = [p_{ij}]^n$

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Regular Matrix: A stochastic matrix P is said to be regular matrix if all the entries P^m are positive (for some positive integer m).

Regular Markov chain: A homogeneous Markov chain is regular if its tpm is regular.

Theorems: (i) If $p = \{p_i\}$ is the state prob. distribution of the process at an arbitrary time, then that the prob. distribution after $\boxed{\text{one step is } pP}$ (where P is tpm) and that after $\boxed{n \text{ steps is } pP^n}$.

(ii) If a homogeneous M.C is $\boxed{\text{regular}}$, then every sequence of state prob. distributions approac a unique fixed prob. dist. called the stationary (state) distribution or steady state distribution of the Markov chain.

ie, $\lim_{n\to\infty} \{p^{(n)}\} = \pi$, where the state prob. dist. at step n.

$p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots, p_k^{(n)})$ and the stationary distribution $\pi = (\pi_1, \pi_2, \ldots \pi_k)$ are row vectors.

(iii) If P is the tpm of the regular chain, then $\boxed{\pi P = \pi}$ (π is a row vector).

Limiting distribution

Suppose a person chooses between four places for dinner: Home, Restaurants A, B & C.

If he eats at one place on a day, the next day he goes to any of the four places with various probabilities.

The t.p.m. is given below

$$P = \begin{array}{c} \\ H \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} H & A & B & C \end{array} \\ \begin{bmatrix} 0.25 & 0.20 & 0.25 & 0.30 \\ 0.20 & 0.30 & 0.20 & 0.30 \\ 0.25 & 0.25 & 0.40 & 0.10 \\ 0.30 & 0.30 & 0.10 & 0.30 \end{bmatrix} \end{array}$$

If he eats at home on day 1, then the initial distribution is

$$p^{(1)} = [1 \; 0 \; 0 \; 0]$$

The distribution for day 2 is

$$p^{(2)} = p^{(1)} P = [0.25 \quad 0.20 \quad 0.25 \quad 0.30]$$

$$p^{(3)} = p^{(2)} P = [0.255 \quad 0.2625 \quad 0.2325 \quad 0.25]$$

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Proceeding, we find

$p^{(16)} = p^{(20)} = p^{(30)} = p^{(50)} = [0.2495 \;\; 0.2634 \;\; 0.2339 \;\; 0.2532]$

ie, the days go on, the state vector approaches a fixed vector. This distribution is called the <u>steady state dist.</u> or <u>stationary dist.</u> of the Markov chain.

<u>Remark:</u> Every Markov chain need not have a stationary distribution.

For example, Consider a M.C. with t.p.m $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and initial distribution $p^{(0)} = [1 \;\; 0]$

Then $p^{(1)} = p^{(0)} P = [0 \;\; 1]$, $p^{(2)} = p^{(1)} P = [1 \;\; 0]$, $p^{(3)} = p^{(2)} P = [0 \;\; 1]$, $p^{(4)} = p^{(3)} P = [1 \;\; 0]$, ...

ie, $p^{(2n)} = [1 \;\; 0]$, and $p^{(2n+1)} = [0 \;\; 1]$

ie, the system ossillates and does not have a steady state distribution.

Classification of states of a Markov chain

If $p_{ij}^{(n)} > 0$ for some n and for all i and j, then every state can be reached from every other state, and the M.C is said to be irreducible.

Otherwise, the chain is said to be reducible or non irreducible.

State i of a M.C is called a return state, If $p_{ii}^{(n)} > 0$ for some $n > 1$.

The period d_i of a return state i is the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$ ie, $d_i = GCD(m : p_{ii}^{(m)} > 0)$. state i is periodic with period d_i If $d_i > 1$ and aperiodic If $d_i = 1$.

The prob. that the chain returns to state i, having started from state i, for the first time at the n^{th} step is denoted by $f_{ii}^{(n)}$ and is called the first return time probability.

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, the return to state i is certain and the state i is said to be persistent or recurrent. The state i is said to be transient If the return to state i is uncertain ie, If $F_{ii} < 1$.

$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ is called the mean recurrence time of the state i. The state i is said to be nonnull persistent if its mean recurrence time μ_{ii} is finite and null persistent, if $\mu_{ii} = \infty$.

A nonnull persistent and aperiodic state is called ergodic.

Theorem:

(1) If a M.C is irreducible, all its states are of the same type. They are either all transient or all null persistent or all non-null persistent.

All the states are either aperiodic or periodic with same period.

(2) If a M.C is finite and irreducible, then all its states are non-null persistent.