Lecture 6 (21-03-2022)

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Changing the cartesian integrals into polar integrals:

$$\iint_R f(x,y)\,dx\,dy$$

$x = r\cos\theta,\quad y = r\sin\theta$

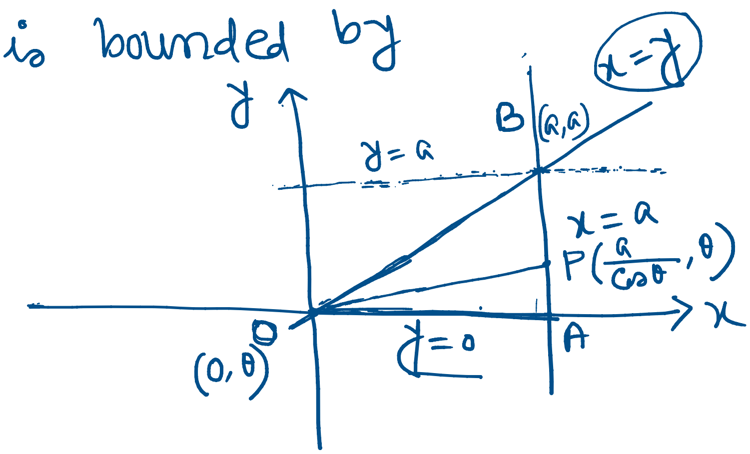
$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r,\theta)\, r\,dr\,d\theta$$

Example: change to polar coordinate and evaluate

$$\int_0^a\int_y^a x\,dx\,dy$$

sol^n: The region of integration is bounded by

$x=y,\ x=a,\ y=0,\ y=a$



$x = r\cos\theta,\quad y = r\sin\theta$

$x = a$

$r\cos\theta = a$

$r = \dfrac{a}{\cos\theta}$

The limit of r varies from 0 to $\dfrac{a}{\cos\theta}$

$x = y$

$r\cos\theta = r\sin\theta$

$r\cos\theta = r\sin\theta$

$\Rightarrow \tan\theta = 1 = \tan \pi/4$

$\Rightarrow \theta = \pi/4$

The limit of θ varies from 0 to $\pi/4$

the required integral is $\displaystyle\int_{\theta=0}^{\pi/4}\int_{r=0}^{\frac{a}{\cos\theta}} r\cos\theta \, r\,dr\,d\theta$

$$= \int_0^{\pi/4} \cos\theta \left[\frac{r^3}{3}\right]_0^{\frac{a}{\cos\theta}} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \cos\theta \cdot \frac{a^3}{\cos^3\theta}\, d\theta$$

$\int \sec^2\theta\, d\theta = \tan\theta$

$$= \frac{a^3}{3}\int_0^{\pi/4} \sec^2\theta\, d\theta$$

$$= \frac{a^3}{3}\Big[\tan\theta\Big]_0^{\pi/4}$$

$$= \frac{a^3}{3}(1-0) = \frac{a^3}{3}$$

$\boxed{\text{Example}}$ change to polar coordinates and evaluate

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dy\, dx \quad \Big| \quad e^{-r^2}$$

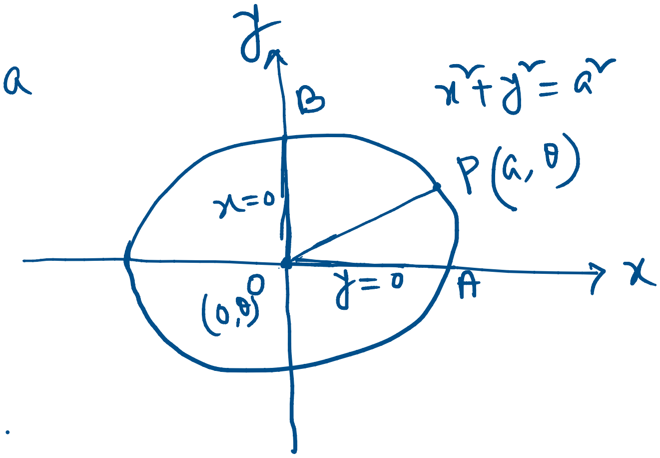
... integration is bounded by

Solⁿ: The region of integration is bounded by

$y=0,\ y=\sqrt{a^2-x^2},\ x=0,\ x=a$

$y^2=a^2-x^2$

$\Rightarrow x^2+y^2=a^2$



OAB is the region of integration.

$x^2+y^2=a^2 \qquad x=r\cos\theta,\ y=r\sin\theta$

$r^2\cos^2\theta+r^2\sin^2\theta=a^2$

$\Rightarrow r^2=a^2$

$\Rightarrow r=a$

The limits of r varies from 0 to a

The limits of θ varies from 0 to $\pi/2$

The required integral is $\displaystyle\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} e^{-r^2}\, r\,dr\,d\theta$

$-r^2=z$

$-2r\,dr=dz$

$\Rightarrow r\,dr=\frac{-dz}{2}$

r	0	a
z	0	$-a^2$

$$=\int_0^{\pi/2}\int_0^{-a^2} e^{z}\left(-\frac{dz}{2}\right)d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\int_{-a^2}^{0} e^{z}\,dz\,d\theta$$

$$- 2 \int_{0} \int_{-a^2}$$

$$= \frac{1}{2} \int_{0}^{\pi/2} \left[e^{z} \right]_{-a^2}^{0} d\theta$$

$$= \frac{1}{2} \int_{0}^{\pi/2} \left(1 - e^{-a^2} \right) d\theta$$

$$= \frac{1}{2} \left(1 - e^{-a^2} \right) \left[\theta \right]_{0}^{\pi/2}$$

$$= \frac{\pi}{4} \left(1 - e^{-a^2} \right)$$

$\boxed{\text{HW}}$ Evaluate $\int_{-a}^{a} \int_{0}^{\sqrt{a^2 - x^2}} (x^2 + y^2)\, dy\, dx$ by changing to polar coordinate.

Change of order of integration in double integral:

$$\int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Here the double integral with variable limits for y and constants limits for x.

Change the order of integration will change the limits

... c and d are constants.

Change the order of integration with ... constants.
of y from c to d where c and d are constants.
and the limits of x from $h_1(y)$ to $h_2(y)$.

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

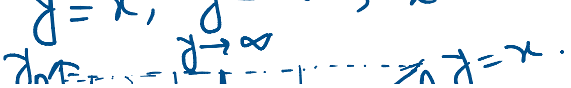
To evaluate this integral, we integrate 1st w.r.to x and then w.r.to y.

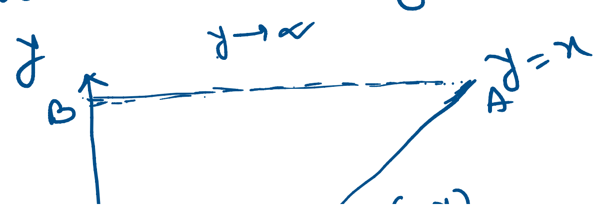
The process of converting a given double integral into its equivalent double integral by changing the order of integration is called the change of order of integration.

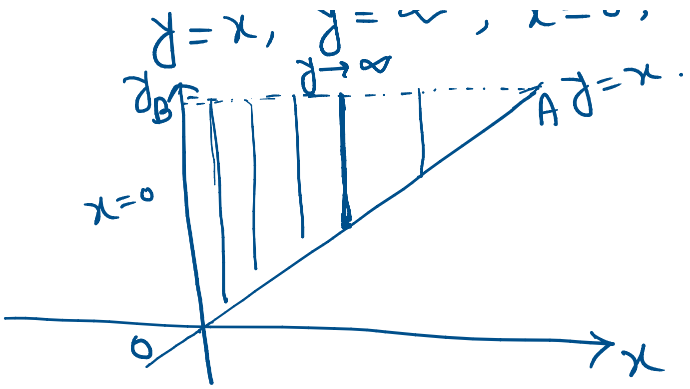
$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \longrightarrow \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

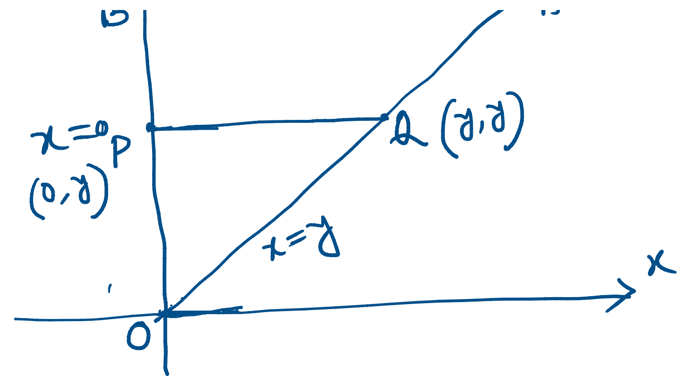
<u>Example:</u> Evaluate $\displaystyle\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ by changing the order of integration.

<u>Sol</u>: The region of integration is bounded by
$y = x,\ y = \infty,\ x = 0,\ x = \infty$









The limits of x varies from 0 to y

and " " of y " " 0 to ∞

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot [x]_0^y \, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot y \, dy$$

$$= \int_0^\infty e^{-y} \, dy$$

$$= -\left[e^{-y}\right]_0^\infty = -(0-1) = 1$$

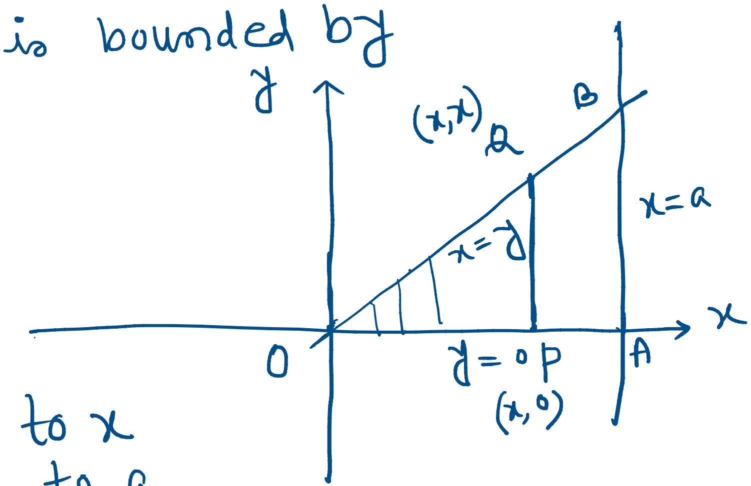
<u>Example:</u> change the order of integration

$$\int_0^a \int_y^a \frac{x}{x^2+y^2} \, dx \, dy$$ and hence evaluate it

$0 \quad y$

<u>Sol^n:</u> The region of integration is bounded by

$x = y, \; x = a, \; y = 0, \; y = a$



OAB is the bounded region.

The limits of y varies from 0 to x

The limits of x " " 0 to a

$$\int \frac{dy}{x^2+y^2} = \frac{1}{x} \cdot \tan^{-1}(y/x)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}(x/a)$$

$$\int_0^a \int_0^x \frac{x}{x^2+y^2} \, dy \, dx$$

$$= \int_0^a \cancel{x} \cdot \frac{1}{\cancel{x}} \left[\tan^{-1}\left(y/x\right)\right]_0^x dx$$

$$\tan^{-1} 1 = \tan^{-1} \tan \pi/4 = \frac{\pi}{4}$$

$$\tan^{-1} 0 = \tan^{-1} \tan 0 = 0$$

$$= \int_0^a \left(\tan^{-1} 1 - \tan^{-1} 0\right) dx$$

$$= \int_0^a \frac{\pi}{4} \, dx = \frac{\pi}{4} \left[x\right]_0^a = \frac{\pi a}{4}$$

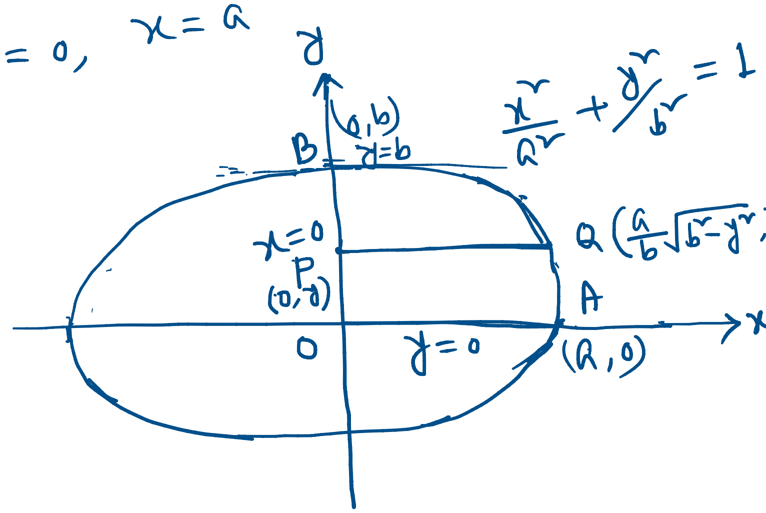
<u>Example:</u> Change the orders of integration in

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 \, dy \, dx$$ and then integrate it.

of integration bounded by

$\underline{\text{sol}^n}$: The region of integration bounded by

$y=0,\quad y=\frac{b}{a}\sqrt{a^2-x^2},\quad x=0,\quad x=a$



$$y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$\Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \frac{x^2}{a^2}=1-\frac{y^2}{b^2}$$

$$\Rightarrow x^2=\frac{a^2}{b^2}\left(b^2-y^2\right)$$

$$\Rightarrow x=\frac{a}{b}\sqrt{b^2-y^2}$$

The limits of x varies from 0 to $\frac{a}{b}\sqrt{b^2-y^2}$

and " " of y " " 0 to b

After changing the order of integration, the integral becomes

$$\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} x^2\,dx\,dy$$

$$\int_0^b\left[\frac{x^3}{3}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \int_0^b \left[\frac{x^3}{3}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$y = b\sin\theta$$

$$dy = b\cos\theta\, d\theta$$

y	0	b
θ	0	$\pi/2$

$$= \frac{1}{3}\int_0^b \frac{a^3}{b^3}\left(b^2-y^2\right)^{3/2} dy$$

$$= \frac{a^3}{3b^3}\int_0^{\pi/2} \left(b^2 - b^2\sin^2\theta\right)^{3/2} b\cos\theta\, d\theta$$

$$= \frac{a^3 b}{3}\int_0^{\pi/2} \cos^3\theta\,\cos\theta\, d\theta$$

$$= \frac{a^3 b}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{a^3 b}{3}\int_0^{\pi/2} \cos^2\theta\cdot\left(1-\sin^2\theta\right) d\theta$$

$$2\cos^2\theta = 1+\cos 2\theta$$

$$2\sin^2\theta = 1-\cos 2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{a^3 b}{3}\int_0^{\pi/2} \left(\cos^2\theta - \sin^2\theta\cos^2\theta\right) d\theta$$

$$= \frac{a^3 b}{3}\int_0^{\pi/2} \left[\frac{1}{2}\left(1+\cos 2\theta\right) - \frac{1}{4}\left(2\sin\theta\cos\theta\right)^2\right] d\theta$$

$$2\sin^2\theta = 1-\cos 2\theta$$

$$2\sin^2 2\theta = 1-\cos 4\theta$$

$$= \frac{a^3 b}{3}\int_0^{\pi/2} \left[\frac{1}{2}\left(1+\cos 2\theta\right) - \frac{1}{4}\sin^2 2\theta\right] d\theta$$

$$= \frac{a^3 b}{3}\int_0^{\pi/2} \left[\frac{1}{2}\left(1+\cos 2\theta\right) - \frac{1}{8}\left(1-\cos 4\theta\right)\right] d\theta$$

$$= \frac{a^3 b}{3}\int \left[\frac{1}{2}\right.$$

$$= \frac{a^3 b}{3} \int_0^{\pi/2} \left[\frac{1}{2}\left(1 + \cos 2\theta\right) - \frac{1}{8}\left(\cdots\right) \right]$$

$$= \frac{a^3 b}{3} \left\{ \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} - \frac{1}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} \right\}$$

$$= \frac{a^3 b}{3} \left\{ \frac{1}{2}\left(\pi/2 + 0\right) - \frac{1}{8}\left[\pi/2 - 0\right] \right\}$$

$$= \frac{a^3 b}{3} \left\{ \frac{\pi}{4} - \frac{\pi}{16} \right\} = \frac{a^3 b}{3} \; \frac{4\pi - \pi}{16} = \frac{a^3 b}{3} \cdot \frac{3\pi}{16}$$

$$= \frac{\pi a^3 b}{16}$$

$\boxed{\text{HW}}$ ① Change the order of integration in

$$\int_0^a \int_{a-y}^{\sqrt{a^2 - y^2}} y \, dx \, dy$$ and evaluate it.

<u>Answer</u>: $a^3/16$

② Change the order of integration in

$$\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy \, dx$$ and then evaluate it.

Answer: $\frac{16}{3}$