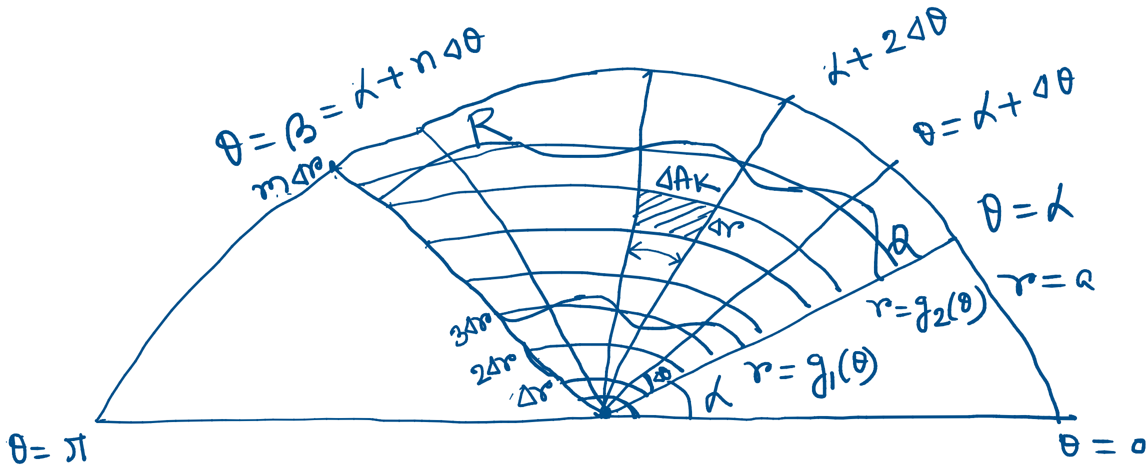
Double integral in polar form:

When we defined the double integral of a function over a region R in the xy-plane, we began by cutting the region R into rectangles whose sides were parallel to the coordinate axes. These were the natural shape to use because their sides have either constant x-values or constant y-values.

In polar coordinate, the natural shape is a "polar rectangle" whose sides have constant r and θ values.



Suppose that a function $f(r,\theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$.

Suppose also that $0 \le g_1(\theta) \le g_2(\theta) \le a$ for every value of θ between α and β.

The arcs are cut from circles centred at the origin with raddi $\Delta r, 2\Delta r, \text{------}, m\Delta r$. Where $\Delta r = a/m$.

given by $\theta = \alpha, \alpha + \Delta\theta, \alpha + 2\Delta\theta,$ ------

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The rays are given by $\theta = \alpha, \alpha + \Delta\theta, \alpha + 2\Delta\theta$, -----

$\alpha + n\Delta\theta = \beta$ $\quad \Delta\theta = (\beta - \alpha)/n$.

The arcs and rays patition R into small patches called "polar rectanges".

We number the polar rectangles that lie inside R, calling their areas $\Delta A_1, \Delta A_2$, ------------, ΔA_n.

let (r_k, θ_k) be the center of the polar rectangle whose area is ΔA_k.

The area of whole region

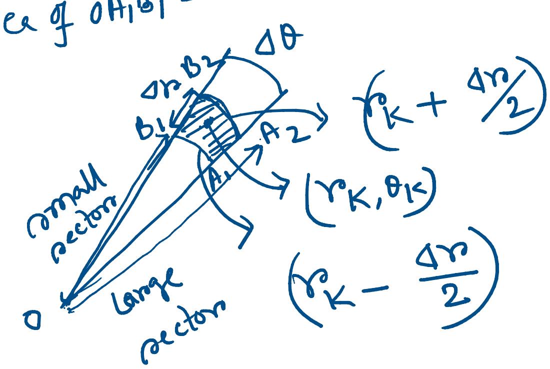
$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, \Delta A_k \quad \text{------} \quad (1)$$

If f is continuous throughout R, this sum will approach a limit as we refine the grid to make Δr and $\Delta\theta$ go to zero.

The limit is called double integral of f over R. In symbols.

$$\lim_{n \to \infty} S_n = \iint_R f(r, \theta)\, dA.$$

Area of OA_2B_2 − Area of OA_1B_1 = ΔA_k



To evaluate this limit, we first have to write the sum S_n in a way that expresses ΔA_k in

have to ... ... in
a way that expresses ΔA_K in
terms of Δr and $\Delta\theta$.

The radius of inner arc bounding ΔA_K is $\left(r_K - \frac{\Delta r}{2}\right)$
The " " outer " " " is $\left(r_K + \frac{\Delta r}{2}\right)$

The area of the circular sector subtended by
these arcs at the origin are

For inner radius : $\frac{1}{2}\left(r_K - \frac{\Delta r}{2}\right)^2 \Delta\theta$

For outer radius : $\frac{1}{2}\left(r_K + \frac{\Delta r}{2}\right)^2 \Delta\theta$

ΔA_K = area of large sector − area of small sector

$$= \frac{1}{2}\Delta\theta \left\{\left(r_K + \frac{\Delta r}{2}\right)^2 - \left(r_K - \frac{\Delta r}{2}\right)^2\right\}$$

$\Delta A_K = \Delta x \, \Delta y$

$$\Delta A_K = \frac{\Delta\theta}{2} \, 2 \, r_K \, \Delta r = r_K \, \Delta r \, \Delta\theta$$

combining this result with Eq (1), we get

$$S_n = \sum_{K=1}^{n} f(r_K, \theta_K) \, r_K \, \Delta r \, \Delta\theta$$

A version of Fubini's theorem now says that the
limit approached by these sums can be evaluated
... integrations w.r.to r and θ

limit approached by ...
by repeated single integrations w.r.to r and θ

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

$\boxed{\text{Example}}$ Evaluate $\displaystyle\int_0^{\pi} \int_0^{a\sin\theta} r\, dr\, d\theta$

Soln:

$$I = \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi} a^2 \sin^2\theta\, d\theta$$

$$= \frac{a^2}{2}\, \frac{1}{2} \int_0^{\pi} (1 - \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{4} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4} \left[\pi - \frac{\sin 2\pi}{2}\right]$$

$$= \frac{\pi a^2}{4}$$

H.W $\boxed{\text{Ex}}$ $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta$ Ans: $\frac{32}{9}$

$-\frac{5}{2}$ 0