$\boxed{\text{Ex}}$ Change the order of integration in

$$\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y \, dx \, dy$$ and then evaluate it.

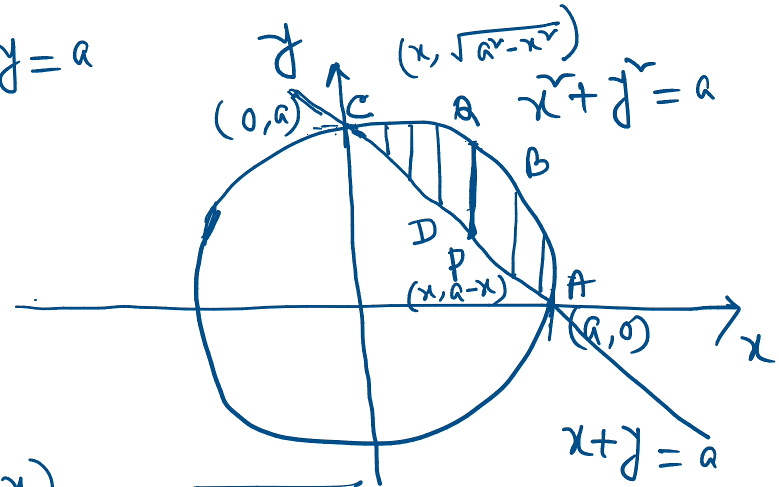
Ans: $a^3/6$

Sol^n: The region of integration is bounded by

$x = a-y, \quad x = \sqrt{a^2-y^2}, \quad y=0, \quad y=a$

$x+y=a \qquad\qquad x^2 = a^2-y^2$

$\Rightarrow \frac{x}{a} + \frac{y}{a} = 1 \qquad x^2+y^2=a^2$



The limits of y varies from $(a-x)$ to $\sqrt{a^2-x^2}$

The limits of x varies from 0 to a

$$\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y \, dy \, dx$$

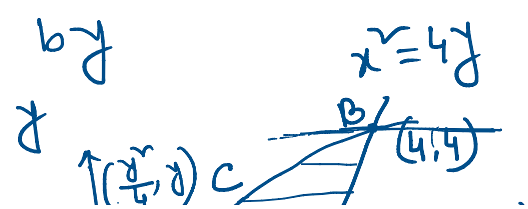
$\boxed{\text{Ex}}$ Change the order of integration in

$$\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy \, dx$$ and then evaluate it.

Ans: $\frac{16}{3}$

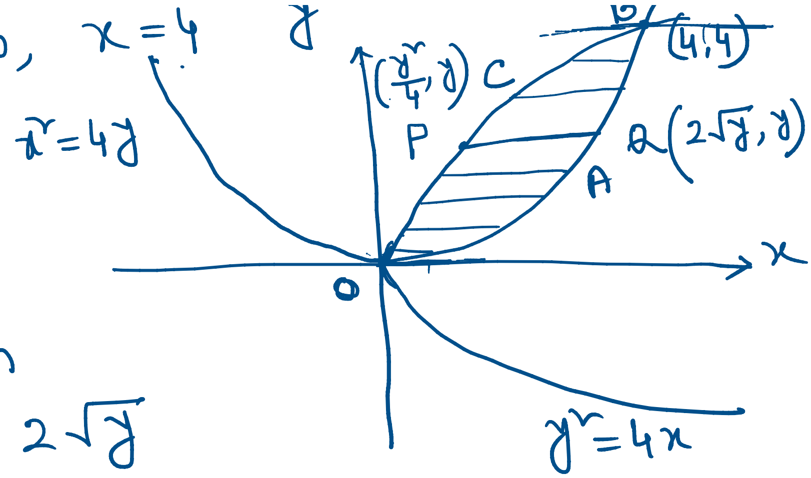
Sol^n: The given integral is bounded by

$y = x^2/4, \quad y = 2\sqrt{x}, \quad x=0, \quad x=4$



$y = x^2/4$, $y = 2\sqrt{x}$, $x = 0$, $x = 4$

$x^2 = 4y$ $\quad y^2 = 4x$



The limits of x varies from $y^2/4$ to $2\sqrt{y}$

The limits of y varies from 0 to 4

$$\int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\, dy$$

Triple integral:

The triple integral is denoted by

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)\, dx\, dy\, dz$$

$$\Rightarrow \int_{z_0}^{z_1}\left[\int_{y_0}^{y_1}\left[\int_{x_0}^{x_1} f(x,y,z)\, dx\right] dy\right] dz$$

<u>Example:</u> $\displaystyle\int_0^2\int_1^3\int_1^2 xy^2z\, dz\, dy\, dx$

$$\int_0^1 \int_1^1 \int_1^1$$

$$= \int_0^2 \int_1^3 xy^2 \left[\frac{z^2}{2}\right]_1^2 dy\,dx$$

$$= \frac{1}{2} \int_0^2 \int_1^3 xy^2 (4-1)\, dy\,dx$$

$$= \frac{3}{2} \int_0^2 x \cdot \left[\frac{y^3}{3}\right]_1^3 dx$$

$$= \frac{3}{2} \cdot \frac{1}{3} \int_0^2 x\,[27-1]\, dx$$

$$= \frac{1}{2} \cdot 26 \left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{13}{2}\,[4-0] = 26$$

$\boxed{\text{Example 2}}$ Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\, dz\,dy\,dx$

Sol^n: $\int_0^1 \int_0^{1-x} \left[e^z\right]_0^{x+y} dy\,dx$

$e^x \cdot e^y$

$$= \int_0^1 \int_0^{1-x} \left(e^{x+y} - 1\right) dy\,dx$$

$$\int_0^1 \left[\quad y \quad\quad y \right]^{1-x} dx$$

$$= \int_0^1 \left[e^x \cdot e^y - y \right]_0^{1-x} dx$$

$$= \int_0^1 \left[e^x \left(e^{1-x} - 1 \right) - (1-x) \right] dx$$

$$= \int_0^1 \left(e - e^x - 1 + x \right) dx$$

$$= \int_0^1 \left[(e-1) - e^x + x \right] dx$$

$$= (e-1)\left[x\right]_0^1 - \left[e^x\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1$$

$$= (e-1) - (e-1) + \frac{1}{2}(1-0)$$

$$= \frac{1}{2}$$

$\boxed{\text{Ex}}$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \, dz \, dy \, dx$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right).$$

$$1-x^2-y^2 = \left(\sqrt{1-x^2-y^2}\right)^2$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right]_0^{\sqrt{1-x^2-y^2}} dy \, dx$$

$\sin^{-1} 1 =$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} \, dy \, dx$$

$$= \int_0 \int_0 \frac{\pi}{2} \cdot dy\, dx$$

$$= \frac{\pi}{2} \int_0^1 \left[y \right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \frac{1}{2} \left\{ x\sqrt{1-x^2} + \sin^{-1}(x) \right]_0^1$$

$$= \frac{\pi}{4} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi^2}{8}$$

$$\int \sqrt{a^2-x^2}\, dx$$

$$= \frac{1}{2} \left\{ x\sqrt{a^2-x^2} + a^2 \sin^{-1}(x/a) \right\}$$

$\boxed{\text{HW}}$ Evaluate $\int_0^a \int_0^b \int_0^c e^{x+y+z}\, dz\, dy\, dx$

Answer: $(e^a-1)(e^b-1)(e^c-1)$