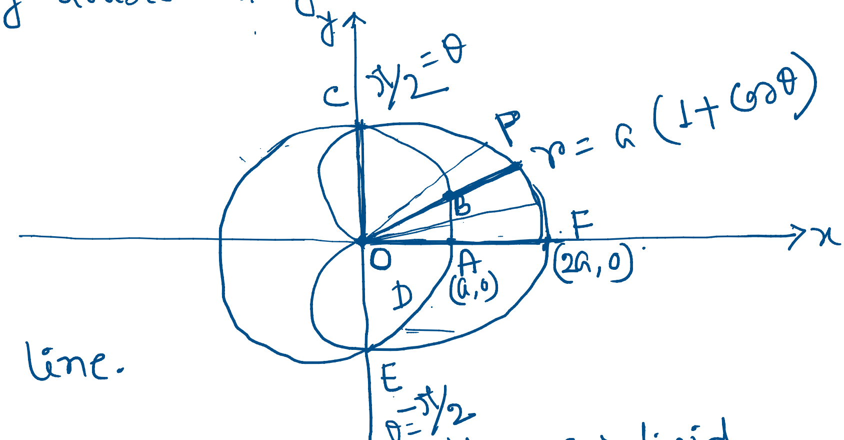
Lecture 5 (16-03-2022)

16 March 2022 14:20

Example: Find the area that lies inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$ by double integral:



Sol^n:

The cardioid $r = a(1+\cos\theta)$ is symmetric about the initial line.

The point of intersection of the line $\theta = 0$ with the cardioid is given by $r = 2a$ i.e $F = (2a, 0)$.

We perform the inner integration w.r.to r, we have to treat θ as constant temporarily and find the limit of r.

The limit of r: The variation of the r coordinate of any point on the line OP.

value of r at B is a and the value of r at P is $a(1+\cos\theta)$

The limit of r varies from a to $a(1+\cos\theta)$.

When we perform the outer integral, we have to find the limit of θ. We have to find the variation of line op. op varies from the line OF to OC

The limit of θ varies from 0 to $\pi/2$

The integral is $= 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} f(r,\theta)\, r\, dr\, d\theta$

The integral is $= 2\int_0 \int_a f(r,\theta)\, r\, dr\, d\theta$

If $f(r,\theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R.

$\therefore$ Required area is $2\times \int_0^{\pi/2}\int_a^{a(1+\cos\theta)} 1.\, r\, dr\, d\theta$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2}\left[a^2(1+\cos\theta)^2 - a^2\right] d\theta$$

$$= a^2\int_0^{\pi/2}\left(\cancel{1}+2\cos\theta+\frac{2\cos^2\theta}{2}-\cancel{1}\right) d\theta$$

$$= a^2\int_0^{\pi/2}\left(2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$$

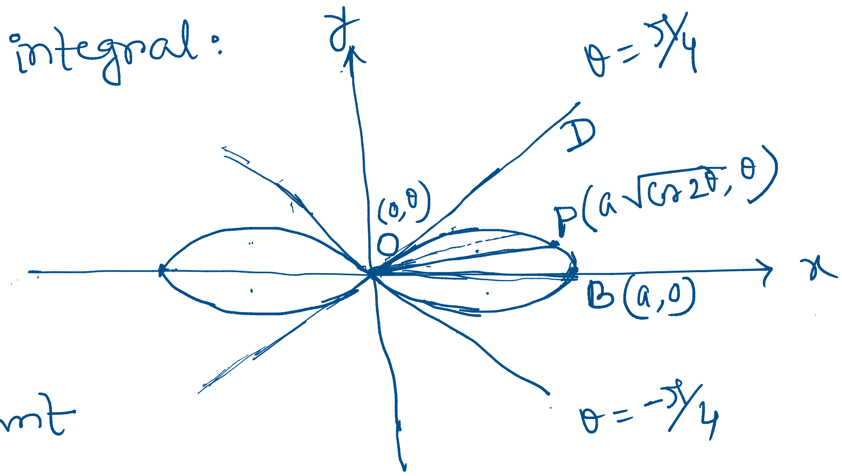
$$= a^2\left[2\sin\theta+\theta/2+\frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= a^2\left[\left(2+\frac{\pi}{4}+0\right)-(0+0+0)\right]$$

$$= a^2\left(\frac{8+\pi}{4}\right) = \frac{a^2}{4}(8+\pi)$$

<u>Example:</u> Find the area enclosed by the lemniscate by double integral: y $\theta = \pi/4$

Example: Find the area by double integral: $r^2 = a^2\cos 2\theta$,



Solⁿ:

We see that the total area is 4 times the first quadrant portion.

When we perform the inner integration w.r.to r, we have to consider θ as constant temporarily and find the limit of r.

The limit of r: The variation of r coordinate of any point on the OP.

The limit of r varies from 0 to $a\sqrt{\cos 2\theta}$

The limit of θ varies from 0 to $\pi/4$

$\therefore$ Required area is $4 \times \displaystyle\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r \, dr \, d\theta$

$$= 4\int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\int_0^{\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0$$

$$= a^2 (1-0) = a^2$$

$\boxed{\text{HW}}$ Find the area of a circle of radius a by double integral in polar coordinate

Ans: πa^2 sq. unit.