Lecture 8 (23-03-2022)

23 March 2022 14:20

Example: Evaluate $\displaystyle\int_0^1\int_0^{1-z}\int_0^{1-y-z} xyz \, dx\, dy\, dz$

Sol^n:

$$\int_0^1\int_0^{1-z} yz \left[\frac{x^2}{2}\right]_0^{1-y-z} dy\, dz$$

$(1-z-y)^2$

$$= \frac{1}{2}\int_0^1\int_0^{1-z} yz\,(1-y-z)^2\, dy\, dz$$

$$= \frac{1}{2}\int_0^1\int_0^{1-z} yz\left\{(1-z)^2 - 2(1-z)y + y^2\right\} dy\, dz$$

$$= \frac{1}{2}\int_0^1 z\left[(1-z)^2\frac{y^2}{2} - 2(1-z)\frac{y^3}{3} + \frac{y^4}{4}\right]_0^{1-z} dz$$

$$= \frac{1}{2}\int_0^1 \left[\frac{z(1-z)^4}{2} - \frac{2}{3}z(1-z)^4 + \frac{z}{4}(1-z)^4\right] dz$$

$\frac{6-8+3}{12} = \frac{1}{12}$

$$= \frac{1}{2}\int_0^1 z(1-z)^4\left\{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right\} dz$$

$z = 1-(1-z)$

$$= \frac{1}{24}\int_0^1 \{1-(1-z)\}(1-z)^4\, dz$$

$$= \frac{1}{24}\int_0^1 \left\{(1-z)^4 - (1-z)^5\right\} dz$$

$(1-z)^5 \quad (1-z)^6 \Big]^1$

$$= \frac{1}{24}\left[\frac{(1-z)^5}{-5} - \frac{(1-z)^6}{-6}\right]_0^1$$

$$= \frac{1}{24}\left[-\frac{1}{5}(0-1) + \frac{1}{6}(0-1)\right]$$

$$= \frac{1}{24}\left[\frac{1}{5} - \frac{1}{6}\right] = \frac{1}{24}\,\frac{6-5}{30} = \frac{1}{24}\cdot\frac{1}{30} = \frac{1}{720}$$

Example: Evaluate $\displaystyle\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$ $\quad\Big|\; e^{x}\cdot e^{y}\cdot e^{z}$.

Solution: $\displaystyle\int_0^{\log 2}\int_0^{x} e^{x+y}\cdot\left[e^{z}\right]_0^{x+y} dy\,dx$ $\qquad e^{x+y}\cdot e^{x+y} = e^{2x+2y}$

$$= \int_0^{\log 2}\int_0^{x} e^{x+y}\left[e^{x+y} - 1\right] dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x}\left[e^{2x+2y} - e^{x+y}\right] dy\,dx$$

$$= \int_0^{\log 2}\left\{e^{2x}\cdot\left[\frac{e^{2y}}{2}\right]_0^{x} - e^{x}\cdot\left[e^{y}\right]_0^{x}\right\} dx$$

$$= \int_0^{\log 2}\left\{\frac{1}{2}e^{2x}\left(e^{2x}-1\right) - e^{x}\left(e^{x}-1\right)\right\} dx$$

$\left(-\frac{1}{2}-1\right)e^{2x}$

$$= \int_0 \Big(\quad \quad$$

$$= \int_0^{\log 2} \left(\frac{e^{4x}}{2} - \frac{1}{2} e^{2x} - e^{2x} + e^{x} \right) dx \qquad \left(-\frac{1}{2} - 1\right) e^{2x}$$

$$- \frac{3}{2} e^{2x}$$

$$= \int_0^{\log 2} \left[\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right] dx \qquad \Bigg| \qquad e^{b\log a} = e^{\log a^b} = a^b$$

$$\frac{1-6+8}{8}$$

$$= \left[\frac{e^{4x}}{8} - \frac{3}{2} \cdot \frac{e^{2x}}{2} + e^{x} \right]_0^{\log 2}$$

$$= \left(\frac{e^{4\log 2}}{8} - \frac{3}{4} e^{2\log 2} + e^{\log 2} \right) - \left(\frac{1}{8} - \frac{3}{4} + 1 \right)$$

$$= \left(\frac{16}{8} - \frac{3}{4} \times 4 + 2 \right) - \frac{3}{8}$$

$$= (2 - 3 + 2) - \frac{3}{8} = 1 - \frac{3}{8} = \frac{5}{8}$$

Example: Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{(x+y)^2} dz\, dy\, dx$

Solution: $\displaystyle\int_0^1 \int_0^{1-x} [z]_0^{(x+y)^2} dy\, dx$

$$= \int_0^1 \int_0^{1-x} (x+y)^2 \, dy\, dx$$

0 0

$$= \int_0^1 \left[\frac{(x+y)^3}{3}\right]_0^{1-x} dx$$

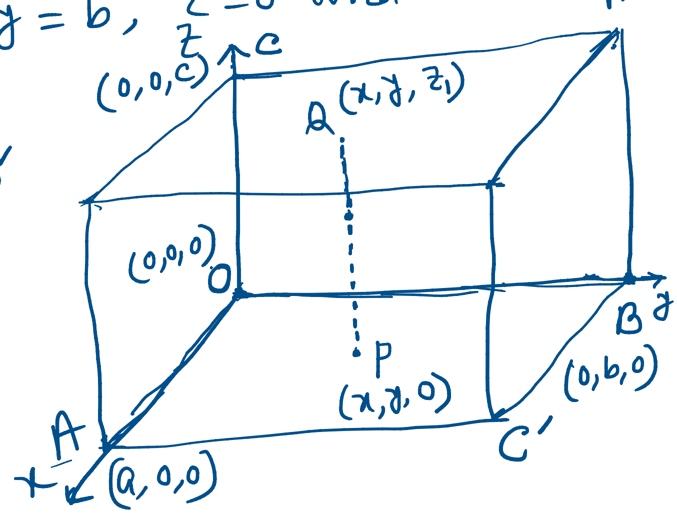
$$= \int_0^1 \frac{1}{3}\left(1-x^3\right) dx$$

$$= \frac{1}{3}\left[x - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3}\left[\left(1-\frac{1}{4}\right) - (0-0)\right] = \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4}$$

$\boxed{\text{HW}}$ Evaluate $\displaystyle\int_0^a\int_0^{a-x}\int_0^{a-x-y} x^2\,dz\,dy\,dx$ Answer $\rightarrow \dfrac{a^5}{60}$

Example: Evaluate $\displaystyle\iiint_V (x+y+z)\,dx\,dy\,dz$, where V is the volume of the rectangular parallelopiped bounded by $x=0$, $x=a$, $y=0$, $y=b$, $z=0$ and $z=c$



Soln:

$$I = \iiint (x+y+z)\,dz\,dy\,dx$$

The limits of z varies from

The limits of z varies from
0 to c.

The line PQ covers the entire volume of parallelopiped. Hence, double integral got after innermost integration is to be evaluated over the plane region $OAC'B$.

The limits of y varies from 0 to b
and " " of x " " 0 to a

$$\int_0^a \int_0^b \int_0^c (x+y+z)\, dz\, dy\, dx$$

Ans: $\frac{abc}{2}(a+b+c)$