Lecture 2 (09-03-2022)

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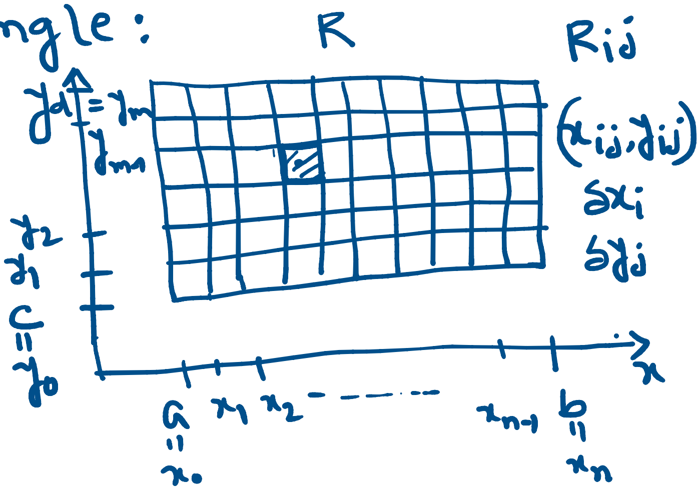
Double integral over a rectangle:



Suppose that $f(x,y)$ is defined on a rectangular region

$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

R divided into small rectangles of area

$$\delta A_{ij} = \delta x_i \, \delta y_j$$

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_{n-1} < x_n = b$$

$$c = y_0 < y_1 < y_2 < \cdots < y_{j-1} < y_j < \cdots < y_{m-1} < y_m = d$$

$$\delta A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$$

Volume of one piece of rectangle

$$= f(x_{ij}, y_{ij})\, \delta A_{ij}$$

Volume of all rectangles

$$V = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij})\, \delta A_{ij}$$

Now, take limit $\delta x_i \to 0$, $\delta y_j \to 0$, $\delta A_{ij} \to 0$

$$\iint_R f(x,y)\, dA = \lim_{\max(\delta A_{ij}) \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij})\, \delta A_{ij}$$

For equispaced points, $\sum^{n} \sum^{m} f(x_{ij}, y_{ij})\, \delta A$

For equispaced points,

$$\iint_R f(x,y)\, dA = \underset{\substack{m \to \infty \\ n \to \infty}}{\mathrm{Lt}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij})\, \delta A$$

Thus, the limit of sum is defined as the double integral of the function $f(x,y)$ over the region R and it is written as

$$\iint_R f(x,y)\, dA \quad \text{or} \quad \iint_R f(x,y)\, dx\, dy$$

$\boxed{\text{Note}}$ The continuity of f is a sufficient condition for the existance of the double integral, but not necessary. The above limit exist for many discontinuous function as well.

properties of double integrals:

1. Linearity:

$$\iint_R [f(x,y) \pm g(x,y)]\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$$

2. Constant comes outside:

$$\iint K f(x,y)\, dA = K \iint f(x,y)\, dA,$$ where K is any constant.

2. [illegible]

$$\iint_R K f(x,y)\, dA = K \iint_R f(x,y)\, dA,$$ [illegible] constant.

3. $\iint_R f(x,y)\, dA \geqslant 0$ if $f(x,y) \geqslant 0$ on R.

4. $\iint_R f(x,y)\, dA \geqslant \iint_R g(x,y)\, dA$ if $f(x,y) \geqslant g(x,y)$ on R.

5. If R is the union of two non-overlapping regions R_1 and R_2, then

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA.$$

Evaluting double integral:

Now we will find the value of the integral

$$\iint_R f(x,y)\, dA \text{ or } \iint_R f(x,y)\, dx\, dy$$

we will now evaluate double integral by calculating two successive single integrals.

The partial derivative of a function $f(x,y)$ are [illegible]

The partial derivative of a function $f(x,y)$ was calculated by holding one of the variable fixed and differentiating with respect to other variable.

Let us consider the reverse of this process, partial integral. The symbols

$$\int_a^b f(x,y)\,dx \quad \text{and} \quad \int_c^d f(x,y)\,dy$$

denotes the partial definite integral.

First integral, called the partial definite integral w.r.to x, is evaluated by considering y as fixed and integrating w.r.to x.

Similarly, the second integral, called the partial definite integral w.r.to y, is evaluated by considering x as fixed and integrating w.r.to y.

Example

$$\int_0^1 xy^2\,dx = y^2\int_0^1 x\,dx = y^2\left[\frac{x^2}{2}\right]_0^1 = \frac{y^2}{2}$$

$$\int_0^1 xy^2\,dy = x\int_0^1 y^2\,dy = x\left[\frac{y^3}{3}\right]_0^1 = \frac{x}{3}$$

$$\int_0^1 xy^2\,dy = x\int_0^1 y^2\,dy = x\left[\frac{y^3}{3}\right]_0^1 = x/3$$

A partial definite integral w.r.to x is a function of y and hence can be integrated w.r.to y.

Similarly, a partial definite integral w.r.to y is a function of x and hence can be integrated w.r.to x.

This two-stage integration process is called iterated (or repeated) integration. We introduce the following notation

$$\int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

<u>Example:</u> Evaluate

ⓐ $\int^{3}\int^{4} (40-2xy)\,dy\,dx$

(a) $\int_{1} \int_{2} (40 - 2xy)$

(b) $$\int_{2}^{4} \int_{1}^{3} (40 - 2xy)\, dx\, dy$$