Unit - 5

Markov Process

★ **Markov Chain**

$$P\{X_n = a_n | X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots X_0 = a_0\}$$
$$= P\{X_n = a_n | X_{n-1} = a_{n-1}\} \text{ for all } n$$

$\{X_n\}$ is called Markov chain

$a_1, a_2, \dots a_n$ are called states of Markov chain.

★ **One - step transition Probability**

$$P\{X_n = a_j | X_{n-1} = a_i\} = P_{ij}(n-1, n)$$

(one step from state a_i to a_j)

n - step

$$P\{X_n = a_j | X_0 = a_i\} = P_{ij}(n)$$

★ **Homogeneous Markov Chain**

→ one step transition pbty does not depend on step

$$P_{ij}(n-1, n) = P_{ij}(m-1, m)$$

★ **Transition Pbty Matrix (tpm)**

$P = (P_{ij})$ is tpm if (i) $P_{ij} \geq 0$ (ii) $\sum P_{ij} = 1 \;\forall\; i$

(sum of row element is 1)

Pbty distn → Row vector $P = (P_1, P_2, \dots P_n)$

[P_1 → pbty that the process is in state a_1]

Initial pbty $dist^n = P^0 = (P_1^0, P_2^0, \dots, P_n^0)$

→ Initial pbty for state 1

★ **Chapman - kolomogorov theorem**

If P is the tpm of a homogeneous markov chain then

$$P^{(n)} = P^n$$

Thus $P^{(1)} = P^{0} \times P$

$P^{(2)} = P^{(1)} \times P$

i.e., $P_{ij}^{(n)} = (P_{ij})^{n}$

Note: 1) $P_{ij}^{(2)} = P[X_2 = j \mid X_0 = i]$

$$= \sum_{k} P_{ik} P_{kj}$$

Similarly, $P_{ij}^{(3)} = \sum_{k} P_{ik} P_{kj}^{(2)}$

$P_{ij}^{(2)} \rightarrow$ transition from state i to state j in two steps.

2) If P is the tpm of regular chain and π (a row vector) is the steady state distⁿ then $\boxed{\pi P = P}$

Qu – Given tpm $P = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.11 \end{pmatrix}$

with initial Probabilities

$P_1^{(0)} = 0.4$, $P_2^{(1)} = 0.3$, $P_3^{(0)} = 0.3$

Find $P_1^{(1)}$, $P_2^{(1)}$, $P_3^{(1)}$

Soln – Given Initial pbty distⁿ,

$$P^{(0)} = (P_1^{(0)}, P_2^{(0)}, P_3^{(0)})$$

$$= (0.4 \quad 0.3 \quad 0.3)$$

$$P^{(1)} = P^{(0)} P$$

$$= (0.4 \quad 0.3 \quad 0.3) \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{pmatrix}$$

$$= (0.29 \quad 0.27 \quad 0.44)$$

$P_1^{(1)} = 0.29$, $P_2^{(1)} = 0.27$, $P_3^{(1)} = 0.44$

Qu – The tpm of a Markov chain with states 0, 1, 2 is

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

and $P(X_0 = i) = \frac{1}{3} \quad [i = 0, 1, 2]$

Find (i) $P(X_3 = 2 \mid X_2 = 1)$

(ii) $P(X_2 = 2, X_1 = 1, X_0 = 2)$

(iii) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$

(iv) $P(X_2 = 2)$

<u>soln.</u> $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$

(i) $P[X_3 = 2 \mid X_2 = 1] = P_{12}^{(1)} = 1/4 \quad \left[P[X_n = a_j \mid X_0 = a_i] = P_{ij}^{(n)} \right]$

(ii) $P(X_2 = 2, X_1 = 1, X_0 = 2) = P(X_2 = 2 \mid X_1 = 1, X_0 = 2)\, P(X_1 = 1, X_0 = 2)$

$= P(X_2 = 2 \mid X_1 = 1)\, P(X_1 = 1 \mid X_0 = 2)$

$= P(X_2 = 2 \mid X_1 = 1, X_0 = 2)\, P(X_1 = 1, X_0 = 2)$

$= P(X_2 = 2 \mid X_1 = 1)\, P(X_1 = 1 \mid X_0 = 2)\, P(X_0 = 2)$

$= P_{12}^{(1)} \times P_{21}^{(1)} \times P(X_0 = 2)$

$= \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{16}$

(OR)

$P(X_2 = 2, X_1 = 1, X_0 = 2) = P(X_0 = 2)\, P_{21}^{(1)}\, P_{12}^{(1)}$

$P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2) = P(X_0 = 2)\, P_{21}^{(1)}\, P_{12}^{(1)}\, P_{21}^{(1)}$

$= \frac{3}{64}$

$P(X_2 = 2) = \sum_{i=0}^{2} P(X_2 = 2 \mid X_0 = i)\, P(X_0 = i)$

$= P(X_2 = 2 \mid X_0 = 0)\, P(X_0 = 0) + P(X_2 = 2 \mid X_0 = 1)\, P(X_0 = 1)$
$+ P(X_2 = 2 \mid X_0 = 2)\, P(X_0 = 2)$

$= P_{02}^{(2)} \frac{1}{3} + P_{12}^{(2)} \left(\frac{1}{3}\right) + P_{22}^{(2)} \frac{1}{3} = \frac{1}{3}\left(\frac{1}{16} + \frac{3}{16} + \frac{4}{16}\right) = \frac{1}{6}$

$P^2 = P \cdot P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$

$$= \begin{pmatrix} \frac{10}{16} & \frac{5}{16} & \frac{1}{16} \\ 5/16 & 8/16 & 3/16 \\ 3/16 & 9/16 & 4/16 \end{pmatrix} = \begin{pmatrix} P_{00}^{(2)} & P_{01}^{(2)} & P_{02}^{(2)} \\ P_{10}^{(2)} & P_{11}^{(2)} & P_{12}^{(2)} \\ P_{20}^{(2)} & P_{21}^{(2)} & P_{22}^{(2)} \end{pmatrix}$$

Que – The tpm of $\{X_n\}$ having states 1, 2, 3 is

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

and the initial disturibution is $p^{(0)} = (0.7, 0.2, 0.1)$

Find (a) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

(b) $P(X_2 = 3)$

(a) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2) = P(X_0 = 2)\, P_{23}^{(1)} P_{33}^{(1)} P_{32}^{(1)}$

$= 0.2\,(0.2)(0.3)(0.4)$

$= 0.0048$

$p^{(0)} = (0.7, 0.2, 0.1)$

$= P(X_0 = 1, X_0 = 2, X_0 = 3)$

(b) $P(X_2 = 3) = \sum_{i=1}^{3} (P(X_2 = 3) \mid X_0 = i)\; P(X_0 = i)$

$= P(X_2 = 3 \mid X_0 = 1)\, P(X_0 = 1) + P(X_2 = 3 \mid X_0 = 2)\; P(X_0 = 2)$

$+ P(X_2 = 3 \mid X_0 = 3)\, P(X_0 = 3)$

$= P_{13}^{(2)} (0.7) + P_{23}^{(2)} (0.2) + P_{33}^{(2)} (0.1)$

$$P^2 = P.P = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

Que – suppose that pbty of a dry day following a rainy day is $\frac{1}{3}$ and the pbty of a rainy day following a dry day is $\frac{1}{2}$. Given May 1 is dry day. Find the pbty that

(i) May 3 is a dry day.

(ii) May 5 is a dry day.

Soln – let $D \to$ Dry day

$R \to$ Rainy day

$$P = \begin{array}{c c} & \begin{array}{cc} & X_n \\ D & R \end{array} \\ X_{n-1} \begin{array}{c} D \\ R \end{array} & \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} \end{array}$$

Initial pbty Distn

$$p^{(1)} = (1, 0) \qquad (p^{(1)} = (D, R))$$

$$p^{(2)} = p^{(1)} P = (1 \quad 0) \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$p^{(3)} = p^{(2)} P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$P(\text{May 3 is dry day}) = \frac{5}{12}$$

$$p^{(4)} = p^{(3)} P = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} \frac{29}{72} & \frac{43}{72} \end{pmatrix}$$

$$p^{(5)} = p^{(4)} P = \begin{pmatrix} \frac{173}{432} & \frac{259}{432} \end{pmatrix}$$

$$P(\text{May 5 is a dry day}) = \frac{173}{432}$$

Que – A fair die is tossed repeatedly. If X_n denotes the max. of numbers occurring in first n tosses. Find the tpm. Also find P^2 and $P(X_2 = 6)$

Soln. – Let $\pi = (\pi_1, \pi_2, \pi_3)$ be the stationary state distⁿ of Markov chain.

$$\pi P = \pi$$

$$(\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3)$$

$$\frac{3\pi_2}{4} + \frac{3}{4}\pi_3 = \pi_1 \quad \text{—— (1)}$$

$$\pi_1 + \frac{\pi_3}{4} = \pi_2 \quad \text{—— (2)}$$

$$\frac{\pi_2}{4} = \pi_3 \quad \text{—— (3)}$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \quad \text{—— (4)}$$

solving, $\pi_1 = \frac{15}{35}$, $\pi_2 = \frac{16}{35}$, $\pi_3 = \frac{4}{35}$

In the long run,

Pbty of buying A $= \frac{15}{35}$

Pbty of buying B $= \frac{16}{35}$

Pbty of buying C $= \frac{4}{35}$

Qu – A fair die is tossed repeatedly. If X_n denotes the max/– of numbers occuring in 1^{st} n tosses. Find the tpm. Also find P^2 and $P(X_2 = 6)$

soln. – $X_n \to$ Max/– of nos occuring in the nth trial

$X_{n+1} \to$ Max. of nos occuring in the (n+1) th trial.

$X_{n+1} = \max(X_n, \text{number in } (n+1)^{th} \text{ trial})$

$$P = X_n \begin{array}{c|cccccc} & & & X_{n+1} & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 2 & 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 3 & 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 4 & 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 5 & 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Let $X_n = 1$

$X_{n+1} = \max\{1, 1\} = 1$

Let $X_n = 2$

In $(n+1)^{th}$ trial

we get 1 $\to X_{n+1} = \max(2,1) = 2$

" 2 $\to X_{n+1} = \max(2,2) = 2$

" 3 $\to X_{n+1} = \max(2,3) = 3$

$$P^2 = P \cdot P = \begin{pmatrix} 1/36 & 3/36 & 5/36 & 7/36 & 9/36 & 11/36 \\ 0 & 4/36 & 5/36 & 7/36 & 9/36 & 11/36 \\ 0 & 0 & 9/36 & 7/36 & 9/36 & 11/36 \\ 0 & 0 & 0 & 16/36 & 9/36 & 11/36 \\ 0 & 0 & 0 & 0 & 25/36 & 11/36 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

$$= \frac{1}{36} \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}^2$$

Initial pbty Distn $P^0 = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

$$P(X_2=6) = \sum_{i=1}^{6} P(X_2=6 \mid X_0=i)\, P(X_0=i)$$

$$= P(X_2=6 \mid X_0=1) P(X_0=1) + P(X_2=6 \mid X_0=2) P(X_0=2) + \dots$$
$$+ P(X_2=6 \mid X_0=6)\, P(X_0=6)$$

$$= P_{16}^{(2)}\left(\frac{1}{6}\right) + P_{26}^{(2)}\left(\frac{1}{6}\right) + P_{36}^{(2)}\left(\frac{1}{6}\right) + P_{46}^{(2)}\left(\frac{1}{6}\right) + P_{56}^{(2)}\left(\frac{1}{6}\right)$$
$$+ P_{66}^{(2)}\left(\frac{1}{6}\right)$$

$$= \frac{1}{6}\left(\frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{36}{36}\right)$$

$$= \frac{91}{216}$$

☆ Classification of States

1) Accessible (or) reachable state → state j reached from state i if $P_{ij}^{(n)} \geq 0$

2) communicating state → Two states i & j are reachable to each other

3) Irreducible chain → If every state is reachable from every other state i.e., $P_{ij}^{(n)} > 0$ for every i and j

4) Return state

$$P_{ii}^{(n)} > 0$$

5) Absorbing state

no other state is accessible from i

$$P_{ii} = 1$$
$$P_{ij} = 0 \text{ for } i \neq j$$

6) period (return state)

$$d_i = \gcd\{m / P_{ii}^{(m)} > 0\}$$

state i is Periodic with Period d_i if $d_i > 1$

state i is aperiodic if $d_i = 1$

7) first return pbty $\longrightarrow f_{ii}^{(n)}$ ($i \to i$ for first time at n^{th} step)

8) If $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$ then the state is Recurrent (or) persistent state

< 1 " " " non (or) transient state

9) $\mu_i = \sum n f_{ii}^{(n)}$
(mean)

If $\mu_i \to$ finite (non null persistent)

If $\mu_i = \infty$ (null Persistent)

10) Ergodic state

A non null persistent and aperiodic state are called ergodic.

Theorem 1

If a markov chain is irreducible then all the states are of same type.

Theorem 2

If a markov chain is finite and irreducible then the states are non null persistent.

Theorem 3

A markov chain which is irreducible and aperiodic is ergodic.

Qu – Three boys A, B, C are throwing a ball. A always throw a ball to B. B always throw a ball to C but C is just as likely to throw the ball to B as to A. Find the tpm and classify the states.

$$P = \begin{array}{c c} & \begin{array}{ccc} & X_n & \\ A & B & C \end{array} \\ X_{n-1} \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}$$

$$P^2 = P \cdot P = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$$P^3 = P^2 P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$P_{11}^{(3)} > 0$, $P_{13}^{(2)} > 0$, $P_{21}^{(2)} > 0$, $P_{22}^{(2)} > 0$, $P_{33}^{(2)} > 0$ and all other $P_{ij}^{(1)} > 0$

$\therefore$ the chain is irreducible.

<u>For state B</u>

$$P_{ii}^{(2)} > 0 \;,\; P_{ii}^{(3)} > 0 \;, \ldots$$

$$\text{period} = \gcd\{2,3\} = 1$$

B is a periodic

Hence, A and C are also periodic. States are finite

All states A, B, C are non null persistent.

$\therefore$ All the states are ergodic.

$$d_i = \gcd\{ m \mid P_{ij}^{(m)} > 0 \}$$

<u>Que</u> – Find the nature of the states with
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

<u>Soln.</u> – $$P^2 = P \cdot P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$P_{00}^{(2)} > 0$, $P_{01}^{(1)} > 0$, $P_{02}^{(2)} > 0$

$P_{10}^{(1)} > 0$, $P_{11}^{(2)} > 0$, $P_{12}^{(1)} > 0$

$P_{20}^{(2)} > 0$, $P_{21}^{(1)} > 0$, $P_{22}^{(2)} > 0$

All states are communicate to each other.

$\therefore$ It is irreducible.

$$P^3 = P^2 \cdot P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} = P \quad \left[d_i = \gcd\left(m : P_{ii}^{(m)} > 0\right) \right]$$

$P^4 = P^2$ and so on.

$P^{2n} = P^2$, $P^{2n+1} = P$

For state 0, $d_1 = \gcd(2, 4, 6, ---) = 2$

for state 1, $d_2 = \gcd(2, 4, 6, \ldots) = 2$

For state 2, $d_3 = \gcd(2, 4, 6, \ldots) = 2$

All the states are periodic with period 2.

Since the states are finite and irreducible, all the states are non null persistent. All the states are not ergodic.

<u>Que</u> – Given

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

(i) Is the chain ergodic?

(ii) Find the invariant pbty. ($\pi P = \pi$)