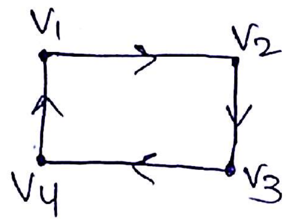
Unit-4

Graph Theory

* $G = (V, E)$
* A graph $G = (V, E)$ consists of non-empty set V called set of vertices and as such E of unordered or ordered pairs of elements called set of edges.

* Eg:-



$V = \{V_1, V_2, V_3, V_4\}$

$E = \{(V_1, V_2), (V_2, V_3), (V_3, V_4), (V_4, V_1)\}$

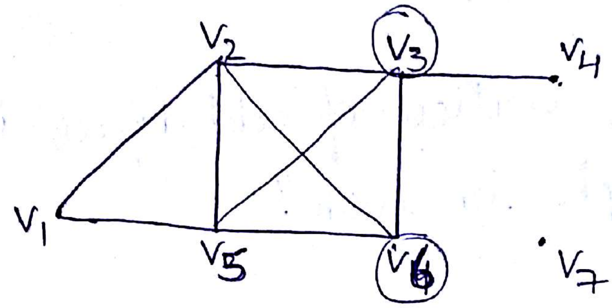
* An edge of a graph that joins vertex to itself is called a loop
* In a graph some pair of vertices are joined by more than one edge; such edges are called parallel edges.
* A graph in which there is only one edge b/w a pair of vertices is called a simple graph
* A graph in which loops & ||el edges are allowed is called pseudo graph.

Degree of a vertex :-

The degree of a vertex in an undirected graph is the no. of edges incident on it with the exception that a loop at a vertex ...

deg (V)

Eg:-



$deg(V_1) = 2$

$deg(V_2) = 4$

$deg(V_3) = 6 \ (4+2)$

$deg(V_4) = 1$

$deg(V_5) = 4$

$deg(V_6) = 5$

$deg(V_7) = 0$

** Hand shaking theorem:-

If $G = (V, E)$ is an undirected graph with e edges, then $\sum_i deg(v_i) = 2e$.

"The sum of degrees of all the vertices of an undirected graph is twice the no. of edges of the graph."

Proof:- Since every edge is incident with exactly 2 vertices every edge contributes 2 to the sum of the vertices.

All the e edges contributes 2e to the sum of the degree of the vertices.

$\sum_i deg(v_i) = 2e$

2) Theorem:-

"The no. of vertices of odd degree in an undirected graph is even"

Proof:-

Let $G = (V, E)$ be the undirected graph

Let V_1, V_2 be the set of vertices of G of even & odd degrees respectively.

Then by handshake theorem,

$$\sum_i \deg(v_i) = 2e.$$

$$\Rightarrow \sum_{v_i \in V_1} \deg(v_i) + \sum_{v_j \in V_2} \deg(v_j) = 2e$$

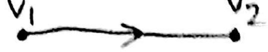
Since each $\deg(v_i)$ is even, $\sum_{v_i \in V_1} \deg(v_i)$ is even

$$\circledast \Rightarrow \sum_{v_j \in V_2} \deg(v_j) = 2e - \text{even} = \text{even}$$

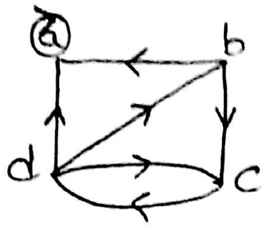
Since each $\deg(v_j)$ is even, the no. of terms contained in $\sum_{v_j \in V_2} \deg(v_j)$ is even.

* In a directed graph, direction will given for edges. No. of edges with v as their terminal vertex is called the <u>indegree of v</u> & is denoted by <u>$\deg^-(v)$</u>

No. of edges with v as their initial vertex is called the <u>outdegree of v</u> & is denoted by <u>$\deg^+(v)$</u>



Eg:-

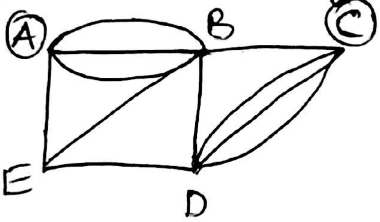


$\deg^-(a) = 3$	$\deg^+(a) = 1$
$\deg^-(b) = 1$	$\deg^+(b) = 2$
$\deg^-(c) = 2$	$\deg^+(c) = 1$
$\deg^-(d) = 1$	$\deg^+(c) = 3$

PROBLEMS:-

1) Verify handshaking theorem,

(i)



$|V| = 5$

$e = |E| = 13 \cancel{+2}$

$\deg(A) = 4+2 = 6$

$\deg(B) = 4+2 = 6$

$\deg(C) = 4+2 = 6$

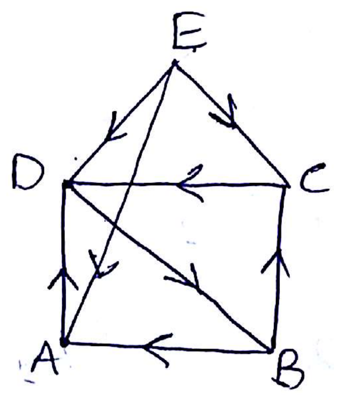
$\deg(D) = 3+2 = 5$

$\deg(E) = 3$

sum of degrees $= 26 = 2 \times 13$

$\therefore \quad \sum_i \deg(v_i) = 2e$

2) Verify $\sum deg^-(V) = \sum deg^+(V) = e$



$e = |E| = 8$

(in)	(out)
$deg^-(A) = 2$	$deg^+(A) = 1$
$deg^-(B) = 1$	$deg^+(B) = 2$
$deg^-(C) = 2$	$deg^+(C) = 1$
$deg^-(D) = 3$	$deg^+(D) = 1$
$deg^-(E) = 0$	$deg^+(E) = 3$
8	8

Special Graphs :-

(1) Complete graph:-

A simple graph is which there is exactly one edge between each pair of vertices is call a complete graph.

The complete graph with n vertices is denoted by K_n

K_1 K_2 K_3 K_4

Note:-

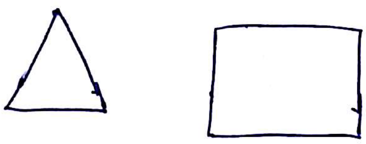
The no. of edges in K_n is ${}^{n}C_2 = \frac{n(n-1)}{2}$

$({}^{n}C_2)$

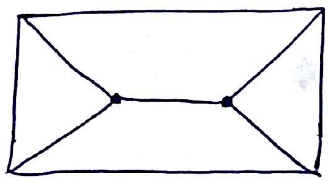
(complete graph are regular)

(2) Regular graph:-

If every vertex of a simple graph has the same degree then the graph is called regular graph



2-regular graphs



3-regular graph

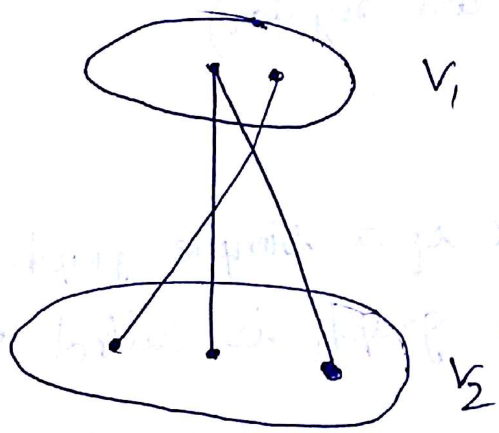
Note:-

Complete graphs are regular but not conversely true.

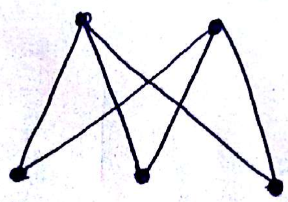
3) Bipartite graph:-

If vertex edge V of a simple graph can be partitioned into two sets V_1, V_2 such that every edge of G connects a vertex in V_1 & a vertex in V_2 then G is called bipartite graph.

If each vertex of V_1 is connected with every vertex of V_2 then G is called complete bipartite graph and is denoted by $K_{m,n}$ where $m = |V_1|$ & $n = |V_2|$ (number of elements)



Bipartite graph



$K_{2,3}$

(1.) Prove that the no. of edges in a bipartite graph with n vertices is atmost $\frac{n^2}{4}$.

Sol:- Let the vertex set be partitioned into V_1, V_2.

Let V_1 contain x vertices. Then V_2 contains $(n-x)$ vertices.

Let $f(x) = x(n-x) = nx - x^2$

$f'(x) = n - 2x$

$f'(x) = 0$

$\Rightarrow n - 2x = 0$

$\Rightarrow n = 2x$

$\Rightarrow x = n/2$

$f''(x) = -2 < 0.$

Max value of $f(x) = f(n/2) = n\left(\frac{n}{2}\right) - \left(\frac{n}{2}\right)^2$

$= \frac{n^2}{2} - \frac{n^2}{4}$

$= \frac{n^2}{4}$

Matrix representation:-

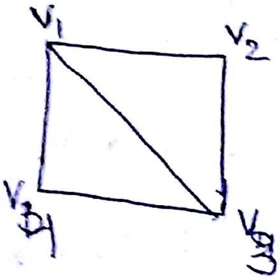
Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$. Then $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

is called the adjacency matrix of G.

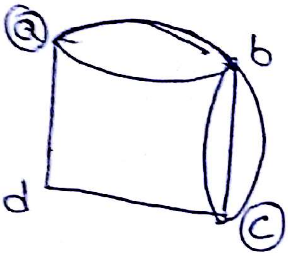
Eg:-

(1.)



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(2.)

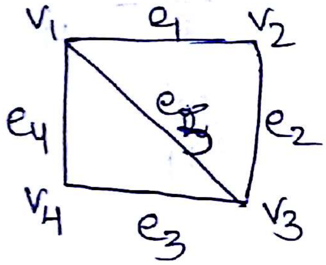


$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

2.) If $G = (V, E)$ is an undirected graph with $V = \{v_1, v_2, v_3, \dots, v_n\}$ & $E = \{e_1, e_2, e_3, \dots, e_m\}$ then $B = (b_{ij})_{n \times m}$ where $b_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident on } v_i \\ 0 & \text{otherwise} \end{cases}$

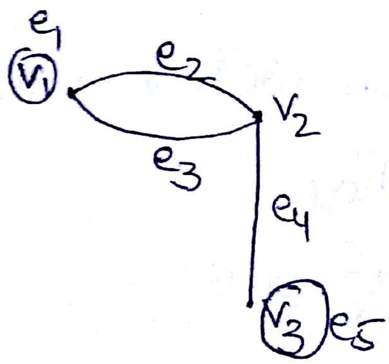
is called incident matrix of G.

Eg:- (1.)



$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

(2.)



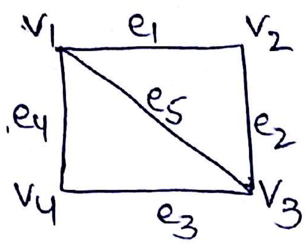
$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Paths, cycles, connectivity:-

A path in a graph is a finite alternating sequence of vertices & edges, beginning & ending with vertices.

- The no. of edges in a path is called length of path
- If initial & final vertices of a path are same then the path is called a cycle (or) a circuit.

Eg:-



$v_1\, e_1\, v_2\, e_2\, v_3$ – path of length 2

$v_1\, e_5\, v_3\, e_3\, v_4\, e_4\, v_1$ – ~~Path~~ cycle path of length 3

Shortest path algorithm:-

The shortest path b/w two vertices in a weighted graph is a path of least weight.

Warshall's algorithm:-

The weight matrix $W = (w_{ij})$

where $w_{ij} = \begin{cases} w(v_i v_j) & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$

Let $|V| = n$

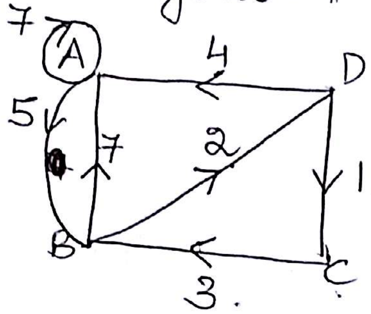
Initial matrix L_0 is the same as the weight matrix W except that each non-diagonal 0 in W is replace by ∞.

$L_r = \{ l_r(i,j)$ where

Note :-

If G is a directed pseudo graph (with loops & ∥el edges), to get the initial matrix L_0 replace all 0's by ∞ in $W = (w_{ij})$ by ∞

1. Find the shortest distance matrix using Warshall's algorithm.



Sol:-

$$W = \begin{array}{c|cccc} & A & B & C & D \\ A & 7 & 5 & 0 & 0 \\ B & 7 & 0 & 0 & 2 \\ C & 0 & 3 & 0 & 0 \\ D & 4 & 0 & 1 & 0 \end{array}$$

Initial matrix $L_0 = \begin{pmatrix} 7 & 5 & \infty & \infty \\ 7 & \infty & \infty & 2 \\ \infty & 3 & \infty & \infty \\ 4 & \infty & 1 & \infty \end{pmatrix}$ (replace all zero's by ∞)

Iteration 1 ($K=1$) To find $L_1(l_1(i,j))$

$l_1(i,j) = \min\{l_0(i,j),\ l_0(i,K) + l_0(K,j)\}$
$\quad = \min\{l_0(i,j),\ l_0(i,1) + l_0(1,j)\}$

$l_1(1,1) = \min\{l_0(1,1), l_0(1,1) + l_0(1,1)\} = \min\{7, 7+7\} = 7$
$l_1(1,2) = \min\{l_0(1,2), l_0(1,1) + l_0(1,2)\} = \min\{5, 7+5\} = 5$
$l_1(1,3) = \min\{l_0(1,3), l_0(1,1) + l_0(1,3)\} = \min\{\infty, 7+\infty\} = \infty$
$l_1(1,4) = \min\{l_0(1,4), l_0(1,1) + l_0(1,4)\} = \min\{\infty, 7+\infty\} = \infty$

Scanned by CamScanner

$l_1(2,1) = \min\{l_1(2,1),\ l_1(2,1) + l_1(1,1)\} = \min(7, 7+7) = 7$

$l_1(2,2) = \min\{\infty,\ 7+5\} = 12$

$l_1(2,3) = \min\{\infty,\ 7+\infty\} = \infty$

$l_1(2,4) = \min\{2,\ 7+\infty\} = 2$

$l_1(3,1) = \min\{\infty,\ \infty+7\} = \infty$

$l_1(3,2) = \min\{3,\ \infty+5\} = 3$

$l_1(3,3) = \min(\infty,\ \infty+\infty) = \infty$

$l_1(3,4) = \min(\infty,\ \infty+\infty) = \infty$

$l_1(4,1) = \min(4,\ 4+7) = 4$

$l_1(4,2) = \min(\infty,\ 4+5) = 9$

$l_1(4,3) = \min(1,\ 4+\infty) = 1$

$l_1(4,4) = \min(\infty,\ 4+\infty) = \infty$

$$L = \begin{pmatrix} 7 & 5 & \infty & \infty \\ 7 & 12 & \infty & 2 \\ \infty & 3 & \infty & \infty \\ 4 & 9 & 1 & \infty \end{pmatrix}$$

Iteration 2 $(k = r = 2)$

$l_2(1,1) = \min\{l_1(1,1),\ l_1(1,2) + l_1(2,1)\} = \min(7, 5+7) = 7$

$l_2(1,2) = \min\{l_1(1,2),\ l_1(1,2) + l_1(2,2)\} = \min(5, \infty) = 5$

$l_2(1,3) = \min\{l_1(1,3),\ l_1(1,2) + l_1(2,3)\} = \min(\infty, 5+\infty) = \infty$

$l_2(1,4) = \min\{l_1(1,4),\ l_1(1,2) + l_1(2,4)\} = \min(\infty, 5+2) = 7$

$l_2(2,1) = \min(7,\ \infty+7) = 7$

$l_2(2,2) = \min(12,\ 12+12) = 12$

$l_2(2,3) = \min(\infty,\ \infty+\infty) = \infty$

$l_2(2,4) = \min(2,\ \infty+2) = 2$

$l_2(3,1)=10$	$l_2(4,1)=4$
$l_2(3,2)=3$	$l_2(4,2)=9$
$l_2(3,3)=\infty$	$l_2(4,3)=1$
$l_2(3,4)=5$	$l_2(4,4)=11$

$$\therefore\ L_2=\begin{pmatrix}7 & 5 & \infty & 7\\ 7 & 12 & \infty & 2\\ 10 & 3 & \infty & 5\\ 4 & 9 & 1 & 11\end{pmatrix}$$

Iteration 3 $(k=3)$

$l_3(1,1) = \min(7, \infty+10) = 7$

$l_3(1,2) = \min(5, \infty+\ \ ) = 5$

$l_3(1,3) = \min(\infty, \infty+\ \ ) = \infty$

$l_3(1,4) = \min(7, \infty+\ \ ) = 7$

$l_3(2,1) = \min(7, \infty+\ \ ) = 7$

$l_3(2,2) = \min(12, \infty+\ \ ) = 12$

$l_3(2,3) = \min(\infty, \infty+\ \ ) = \infty$

$l_3(2,4) = \min(2, \infty+\ \ ) = 2$

$l_3(3,1) = \min(10, \infty+\ \ ) = 10$

$l_3(3,2) = \min(3, \infty+\ \ ) = 3$

$l_3(3,3) = \min(\infty, \infty+\ \ ) = \infty$

$l_3(3,4) = \min(5, \infty+\ \ ) = 5$

$l_3(4,1) = \min(4, 1+10) = 4$

$l_3(4,2) = \min(9, 1+3) = 4$

$l_3(4,3) = \min(1, 1+\infty) = 1$

$l_3(4,4) = \min(11, 1+5) = 6$

$$L_3 = \begin{pmatrix} 7 & 5 & \infty & 7 \\ 7 & 12 & \infty & 2 \\ 10 & 3 & \infty & 5 \\ 4 & 4 & 1 & 6 \end{pmatrix}$$

Iteration-4 $(K=4)$

$$L_4 = \begin{pmatrix} 7 & 5 & 8 & 7 \\ 7 & 6 & 3 & 2 \\ 9 & 3 & 6 & 5 \\ 4 & 4 & 1 & 6 \end{pmatrix}$$

$\therefore$ Shortest path matrix is

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 7 & AB & ABDC & ABD \\ BA & BDCB & BDC & BD \\ CBDA & CB & CBDC & CBD \\ DA & DCB & DC & DCBD \end{bmatrix} \end{array}$$

Eulerian & Hamiltonian graph:-

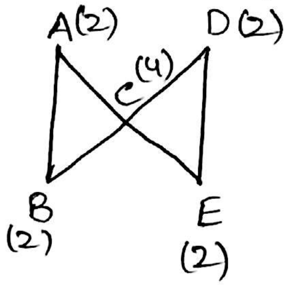
- A path of a graph G is called an eulerian path if it includes each edge exactly once.
- A circuit of a graph G is called an eulerian circuit if it includes each edge of G exactly once.
- A graph containing an eulerian circuit is called an eulerian graph.
- A path of a graph G is called a hamiltonian path if it includes each vertex of G exactly once
- A circuit of a graph G is called a hamiltonian circuit if it includes each vertex of G exactly once, except the starting & end vertices

• A graph having a hamiltonian circuit is called a hamiltonian graph.

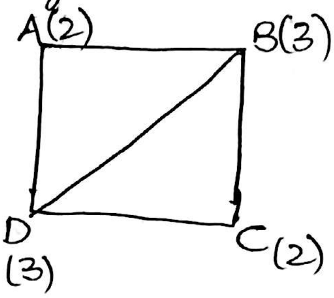
Note:-

→ A connected graph is eulerian graph if ~~Eulerian~~ all the vertices are of even degree.

Eg:- (1)



(2)



(*)

• ABCDECA – Eulerian circuit

• Eulerian graph

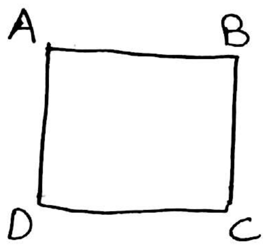
• not hamiltonian circuit(graph)

(*)

• Not eulerian graph

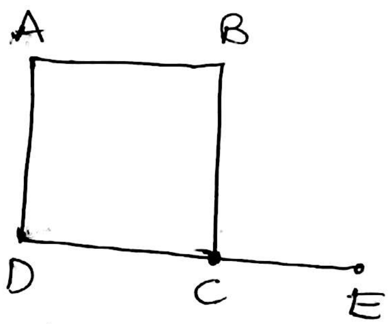
• ABCDA – Hamiltonian circuit (graph)

(3)



• Both eulerian & hamiltonian graphs

(4)



• Neither eulerian nor hamiltonian graph.

Trees:-

A connected graph without any circuit is called a "tree".

Eg:-

Scanned by CamScanner

Properties :-

1. An undirected graph is a tree if & only if there is a unique simple path between every pair of vertices.

2. A tree with n vertices has $(n-1)$ edges.

Spanning tree :-

Let G be a connected graph. A sub-graph of G containing all the vertices of G.

A sub-graph T of G is a tree having all the vertices of G is called a spanning tree of G.

A spanning tree with smallest total weight is called a <u>minimum spanning tree</u>

Kruskal's algorithm :-

Let $|V| = n$

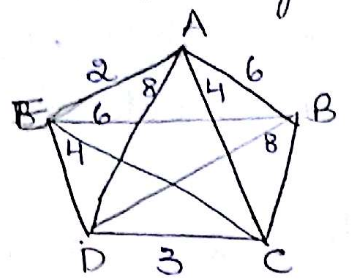
Step-1: Arrange the edges of G in the ascending order of their weights.

Step-2: An edge with minimum weight is selected.

Step-3: Edges with minimum weight that do not form circuit with already selected edges are successivel added.

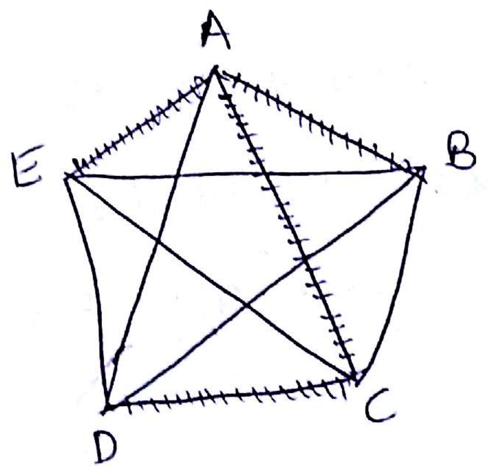
Step-4: Stop this procedure if $(n-1)$ edges have been selected.

1.) Find a minimum spanning tree for the following graph:

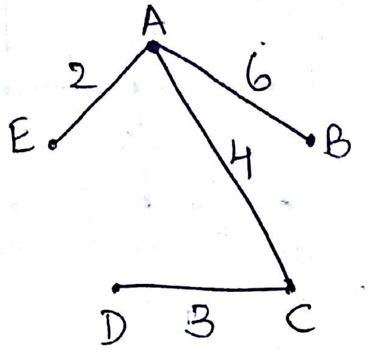


Sol:-

Edge	Weight	Included in the spanning tree or not	if not circuit formed
AE	2	Yes	—
CD	3	Yes	—
AC	4	Yes	—
EC	4	No	AECA
AB	6	Yes	—
BC	6	—	
EB	6	—	
ED	7	—	
BD	8	—	
AD	8	—	

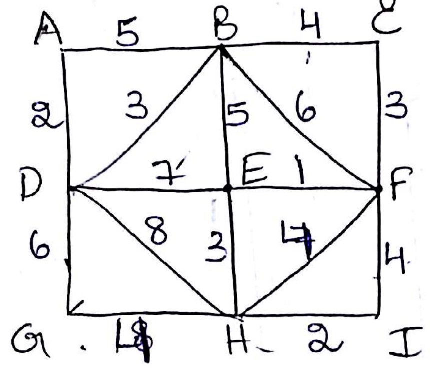


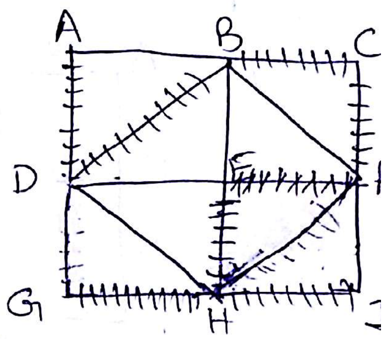
Minimum Spanning tree



length $= 2+3+4+6 = 15$

Find the minimum spanning tree for following graph.

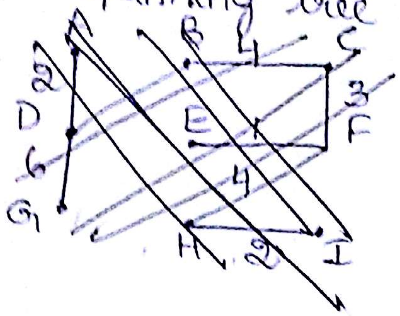


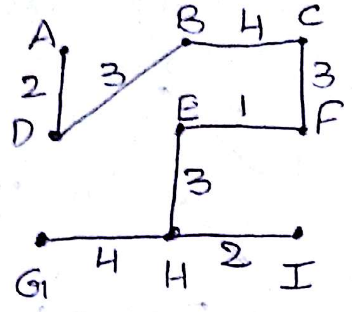


Sol:-

Edge	Weight	Included in the spanning tree or not	if not circuit formed
EF	1	Yes	—
AD	2	Yes	—
HI	2	Yes	—
BD	3	Yes	—
CF	3	Yes	—
EH	3	Yes	—
BC	4	Yes	—
HF	4	~~Yes~~ No	EFHE
FI	4	No	
GH	5	~~No~~	
AB	5	~~No~~	ADBA
BE	5	~~No~~	BCFEB
BF	6	~~No~~	BCFB
DG	6	~~Yes~~	—
DE	7	—	
DH	8	—	

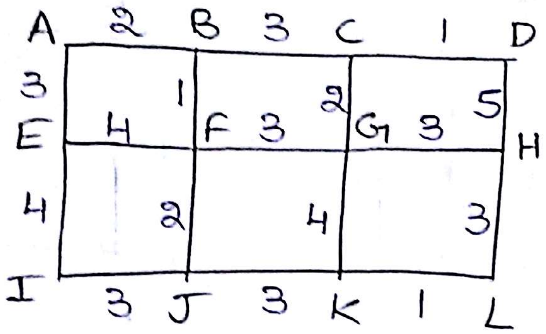
...nimum spanning tree is





length = 1+2+2+3+3+3+4+4
= 22

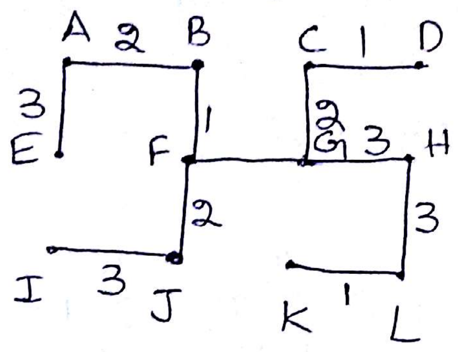
***3) Find minimum spanning tree for following graph



Sol:-

Edge	Weight	included in spanning tree or not	if not circuit formed
CD	1	Yes	—
BF	1	Yes	—
KL	1	Yes	—
AB	2	Yes	—
CG	2	Yes	—
FJ	2	Yes	—
FG	3	Yes	—
BC	3	No	BCGFB
AE	3	Yes	—
GH	3	Yes	—
HL	3	Yes	—
IJ	3	Yes	—
JK	3	~~No~~	~~JFGHLKJ~~
EF	4	—	—
EI	4	—	—
GK	4	—	—
DH	[illegible]		

Minimum Spanning tree is



length $= (1\times 3)+(2\times 3)+(3\times 5)$
$= 3+6+15$
$= 24$

4) The maximum no. of edges in a simple disconnected graph G with n vertices & k components is $\frac{(n-k)(n-k+1)}{2}$

Sol:- Proof:-

Let no. of vertices in the i^{th} component of G be n_i $(n_i \geq 1)$

then $n_1 + n_2 + \cdots + n_k = n$

$$\Rightarrow \sum_{i=1}^{k} n_i = n$$

$$\Rightarrow \sum_{i=1}^{k} n_i - k = n-k$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1) = n-k$$

$$\Rightarrow \left\{ \sum_{i=1}^{k} (n_i-1) \right\}^2 = (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i-1)^2 + 2\cdot \sum_{\substack{i,j=1 \\ \& i\neq j}}^{k} (n_i-1)(n_j-1) = n^2 - 2nk + k$$

Scanned by CamScanner

$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 - 2nk + k^2$; since $n_i \geq 1$

$\Rightarrow \sum_{i=1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$

$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq (n^2 - 2nk + k^2) + 2\sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$

$\Rightarrow \sum_{i=1}^{k} n_i^2 = n^2 - 2nk + k^2 + 2n - k$ —— (1)

The max. no. of edges in the ith component of G $= \frac{n_i(n_i-1)}{2}$

Max no. of sides of G $= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$

$= \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k) - \frac{n}{2}$

$= \frac{1}{2}(n^2 - 2nk + k^2 + n - k)$

$= \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$

$= \frac{(n-k)(n-k+1)}{2}$