7.10.22

UNIT - 4 - FOURIER TRANSFORM

A fourier transform is a mathematical transform that decomposes functions depending on time into functions depending on frequency.

Fourier Transform pair

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

Inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(f(x))\,ds.$$

PROPERTIES.

1. Fourier transform is linear

$$F[af(x) + bg(x)] = aF(f(x)) + bF(g(x))$$

PROOF :

$$F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} (af(x) + bg(x))\,dx$$

Page No.:
Date: / /

$$\Rightarrow a\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx\right] + b\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} g(x)\,dx\right]$$

$$\Rightarrow a F(f(x)) + b F(g(x))$$

2. **SHIFTING THEOREM.**

If $F(f(x)) = F(s)$ then $F(f(x-a)) = e^{isa} F(s)$

$$F(f(x-a)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x-a)\,dx$$

$x - a = t$ $\quad\quad x = t + a$

$dx = dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(t+a)} f(t)\,dt$$

$$\Rightarrow e^{isa}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist} f(t)\,dt\right] = e^{isa} F(s)$$

't' $\longrightarrow$ dummy variable.

4. $F\left[e^{iax} f(x)\right] = F(s+a)$

$$F\left[e^{iax} \cdot f(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \cdot e^{iax} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx.$$

(5) MODULATION THEOREM

If $F(f(x)) = F(s)$ then

$F(f(x)\cos ax) =$

$$\frac{1}{2}\left[F(s-a) + F(s+a)\right]$$

$$F(f(x)\cos ax) = F\left(f(x)\left[\frac{e^{iax}+e^{-iax}}{2}\right]\right)$$

$$\Rightarrow \frac{1}{2}\left[F\left(e^{iax} f(x)\right) + F\left(e^{-iax} f(x)\right)\right]$$

By property (4)

$$\frac{1}{2}\left[F(s+a) + F(s-a)\right].$$

Date: / /

3. CHANGE OF SCALE PROPERTY

If $F(f(x)) = F(s)$ then

$$F\{f(ax)\} = \frac{1}{|a|} \cdot F\left(\frac{s}{a}\right)$$

Proof for ppt 2:

If $F(f(x)) = F(s)$ then $F(f(x-a)) = e^{isa} F(s)$

$$F(f(ax)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(ax)\,dx$$

$ax = t \quad x = \frac{t}{a}$

$a\,dx = dt$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(t/a)} f(t) \frac{dt}{a}$$

$$\Rightarrow \frac{1}{a}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(t/a)} f(t)\,dt\right.$$

$$\frac{1}{a}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)} f(t)\,dt\right.$$

Unit 2 & 3 - More 10 Q. only possible. _/_/___

Max 8 Q in Fourier transforms.

(6) If $F\{f(x)\} = F(s)$, then

Derivative of Fourier

$F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n}(F(s))$

(7) $F\left[\int_a^x f(x)\,dx\right] = \frac{F(s)}{-is}$

(8) $F[f'(x)] = (-is)\,F(s)$.

Q Find the FT of $f(x) = \begin{cases} x & \text{for } |x| \le a \\ 0 & \text{for } |x| > a. \end{cases}$

$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

10 M to integrate this.

$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x \cdot e^{isx}\, dx.$

$u = x \qquad dv = e^{isx}\,dx.$

$du = 1 \qquad v = \frac{e^{isx}}{is}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$

$v_1 = \frac{e^{isx}}{i^2 s^2} = \frac{-e^{isx}}{\cancel{i^2} s^2}$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$F[f(x)] = \frac{1}{\sqrt{2\pi}}\left[\frac{x e^{isx}}{is} + (1)\left(\frac{e^{isx}}{s^2}\right)\right]_{-a}^{a}$

$= \frac{1}{\sqrt{2\pi}}\left[\left[\frac{a e^{isa}}{is} - \frac{e^{isa}}{i^2 s^2}\right] - \left[\frac{(-a) e^{-isa}}{is} - \frac{e^{-isa}}{i^2 s^2}\right]\right]$

$\cancel{= \frac{1}{\sqrt{2\pi}}\left[ae\, e^{-isa}(a+1)\right.}$

$= \frac{1}{\sqrt{2\pi}}\left[\frac{a e^{isa}}{is} - \frac{e^{isa}}{i^2 s^2} + \frac{a e^{-isa}}{is} + \frac{e^{-isa}}{i^2 s^2}\right]$

$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{is}\left(e^{isa} + e^{-isa}\right) - \frac{1}{i^2 s^2}\left(e^{isa} - e^{-isa}\right)\right]$

__/__/__

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{is}(2\cos as) - \frac{1}{i^2 s^2}(2i\sin as)\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left(\frac{as\cos as - \sin as}{is^2}\right) \times \frac{i}{i} \rightarrow i \text{ can't be in Dr.} \quad \frac{1}{i} = -i$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{i}{s^2}\left(\sin as - as\cos as\right).$$

Q2. $f(x) = \begin{cases} 1 & \text{for } |x| \le a \\ 0 & \text{for } |x| > a. \end{cases}$ $\left[F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx}\,dx\right]$

Evaluate $\int_0^{\infty} \frac{\sin x}{x}\,dx.$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} (1) e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isa}}{is} - \frac{e^{-isa}}{is}\right].$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{is}\left(e^{isa} - e^{-isa}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{\cancel{i}s}\left(2\cancel{i}\sin as\right)$$

$$= \frac{2}{\sqrt{2\pi}}\frac{\sin as}{s} = \sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$$

Inversion Formula.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)] e^{-isx}\,ds$$

$$1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin as}{s} e^{-isx}\,ds.$$

$$\boxed{1 = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\sin as}{as} e^{-isx}\,ds}$$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s} (\cos sx - i \sin sx)\, ds$$

RP & IP

$$\boxed{1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx\, ds}$$

Hit & try. Set $x=0$

$$\boxed{\pi = \int_{-\infty}^{\infty} \frac{\sin as}{s}\, ds}$$

Sub $as = \theta$.

$a\,ds = d\theta$

$$\pi = \int_{-\infty}^{\infty} \frac{\sin\theta}{\theta/\cancel{a}} \frac{d\theta}{\cancel{a}}.$$

$$\pi = \int_{-\infty}^{\infty} \frac{\sin\theta}{\theta} d\theta$$

$\sin\theta$ odd fn.

$\sin\theta/\theta$ even fn.

So, $$\pi = 2\int_{0}^{\infty} \frac{\sin\theta}{\theta} d\theta$$

$$\boxed{\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin\theta}{\theta} d\theta}.$$

Q3) $f(x) = \begin{cases} 1-x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ & hence deduce $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2)\frac{e^{isx}}{is} - \frac{(-2x)\, e^{isx}}{i^2 s^2} + \frac{(-2)\, e^{isx}}{i^3 s^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(0 + \frac{2e^{is}}{i^2 s^2} - \frac{2e^{is}}{i^3 s^3}\right) - \left(0 - \frac{2e^{-is}}{i^2 s^2} - \frac{2e^{-is}}{i^3 s^3}\right) \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1}{i^2 s^2}\left(e^{is} + e^{-is}\right) - \frac{1}{i^3 s^3}\left(e^{is} - e^{-is}\right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\cos s}{i^2 s^2} - \frac{2 i \sin s}{i^3 s^3} \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$$

Inversion :

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(f(x)) \, e^{-isx} \, ds$$

$$1 - x^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3} \right) e^{-isx} \, ds.$$

$$1 - x^2 = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) e^{-isx} \, ds$$

$$1 - x^2 = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) (\cos sx - i\sin sx) \, ds.$$

$$1 - x^2 = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \cos sx \, ds.$$

Set $x = 1/2$

$\frac{3}{4}$

(Q) Show that the Fourier transform of $e^{-x^2/2}$ is $e^{-s^2/2}$ by finding the Fourier transform of $e^{-a^2x^2}$, $a > 0$.

$$F(e^{-a^2x^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2 + e^{isx}} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} \cdot e^{-s^2/4a^2} \, dx.$$

_/_/_

$$ax - \frac{is}{2a} = t.$$

$$\int_{-\infty}^{\infty} e^{-t^2} dx = \sqrt{\pi}.$$

$$a\,dx = dt.$$

$$\boxed{dx = \frac{dt}{a}}$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a} = \frac{1}{a} \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \times \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}a} e^{-s^2/4a^2}.$$

$$F(e^{-x^2/2}) = \frac{1}{\sqrt{2} \times 1/\sqrt{2}} \times e^{-s^2/4(1/2)} = e^{-s^2/2}.$$

Convolution thm:

The convolution of two fns $f(x)$ & $g(x)$ is defined by $f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$

The FT of convolution of $f(x)$ & $g(x)$ is a prod of their FT.

Parseval's identity.

If $F(s)$ is FT $[F(f(x))]$ of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

For sine & cosine 0 to ∞ will be the range.

Q) $f(x) = \begin{cases} 1 & \text{for } |x| < a. \\ 0 & \text{if } |x| > a. \end{cases}$

& deduce $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2.$

—/—/—

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1 \cdot e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{is} \left[e^{isa} - e^{-isa} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{is} \left[2i \cdot \sin as \right].$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin as}{s} \right].$$

Using Parseval's Identity,

$$\int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{\sin as}{s} \right)^2 ds$$

$$= \int_{-a}^{a} 1^2\, dx.$$

$as = t$

$a\,ds = dt$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t/a} \right)^2 \frac{dt}{a} = \cancel{\varnothing}\; x \Big]_{-a}^{a}$$

$$\frac{2a}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = a + a$$

$$= \frac{2a \times \pi}{2a} = \pi.$$

$x, 1, 1-x^2, e^{-s^2/2}, 1Pi$

_/_/__

Q Find the FT of $f(x) = \begin{cases} 1-|x|, & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$ & deduce $\int_0^\infty \frac{\sin^4 t}{t^4}\,dt$.

1. $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\,dx$

2. $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(f(x)) \cdot e^{-isx}\,ds$.

3. $\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$.

$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^\infty f(x)\,dx$ $f(x)$ is even

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x)\cos sx\,dx \quad \text{[II term is odd it vanishes]}$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[0 - \frac{\cos s}{s^2}\right] - \left[0 - \frac{1}{s^2}\right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] = \sqrt{\frac{2}{\pi}} \left(\frac{2\sin^2(s/2)}{s^2} \right)$$

Parseval's identity : $\int_{-\infty}^{\infty} [f(x)]^2\,dx = \int_{-\infty}^{\infty} [F(s)]^2\,ds$.

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left[\frac{2\sin^2 s/2}{s^2} \right]^2 ds = \int_{-1}^{1} (1-|x|)^2\,dx$$

$$= 2\int_0^1 (1-x)^2\,dx = \frac{2}{3} \times \frac{\pi}{2} \times \frac{1}{4}$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin^2 s/2}{s^2} \right)^2 ds = \frac{2}{3} \times \frac{\pi}{2} \times \frac{1}{4}$$

Sub. $s/2 = t$, $ds = 2dt$.

$$\int_{-\infty}^{\infty} \frac{\sin^4 t \, 2dt}{(2t)^4} = \frac{1}{6}\sqrt{\frac{\pi}{2}}$$

$$2\int_{-\infty}^{\infty} \frac{\sin^4 t}{t^4} \times \frac{2}{2^4}\, dt = \frac{1}{6}\sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^{\infty} \frac{\sin^4 t}{t^4}\, dt = \frac{2^2}{6}\frac{\pi}{2} = \frac{2}{3} \times \frac{\pi}{2} = \frac{\pi}{3}$$

Fourier cosine & sine transform

The FCT pairs $\quad F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx.$

$$f(x) = \sqrt{\frac{2}{\pi}} = \int_0^{\infty} F_c(f(x)) \cos sx \, ds.$$

The FST pairs $\quad F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx.$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(f(x)) \sin sx \, ds$$

Properties of FCT & FST

→ Cosine & sine Ts are linear

→ $F_c[a f(x) + b g(x)] = a F_c(f(x)) + b F_c(g(x))$

→ FST $F_s[f(x) \sin ax] = \frac{1}{2}[F_c[s-a] - F_c[s+a]]$

→ $F_s[f(x) \cos ax] = \frac{1}{2}[F_s(s-a) + F_s(s+a)]$

→ $F_c[f(x) \sin ax] = \frac{1}{2}[F_s(a-s) + F_s(a+s)]$

→ $F_c[f(x) \cos ax] = \frac{1}{2}[F_c(s-a) + F_c(s+a)]$

→ $F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right)$

$\rightarrow F_s(f(ax)) = \frac{1}{a} F_s\left(\frac{s}{a}\right).$

$\rightarrow \int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds = \int_0^\infty |F_s(s)|^2 ds.$

$\rightarrow \int_0^\infty f(x) g(x)\, dx = \int_0^\infty F_c(s) \cdot G_c(s)\, ds.$

$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cdot \sin sx\, dx$

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(f(x)) \sin sx\, ds$

$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(f(x)) \cos sx\, ds$

$\int_0^\infty (f(x))^2 dx = \int_0^\infty [F_c(f(x))]^2 dx.$

$= \int_0^\infty (F_s(f(x)))^2 dx.$

$\int_0^\infty f(x) g(x)\, dx = \int_{-\infty}^{\infty} F_c(f(x))\, F_c(g(x))\, ds$

Q) Find FCT & FST of e^{-ax}, $a > 0$ & hence deduce the inversion formula.

$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$

$$\left[\begin{aligned} \int e^{ax} \sin bx\, dx &= \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx] \\ \int e^{ax} \cos bx\, dx &= \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx] \end{aligned}\right]$$

$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a \sin sx - s \cos sx) \right]_0^\infty$

$= \sqrt{\frac{2}{\pi}} \left[0 - \left[\frac{1}{a^2+s^2} [0 - s] \right] \right]$

formula: $\boxed{F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \left(\frac{s}{a^2+s^2} \right)}$ —— (1)

_/_/_

Don't do

<u>Inversion formula</u> (~~do~~ integration for inversion formula).

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx \, ds.$$

$$e^{-ax} \times \frac{\pi}{2} = \int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds. \quad \text{—} (2)$$

If $a = \alpha$,

$$\int_0^\infty \frac{s \sin sx \, ds}{s^2+\alpha^2} = e^{-\alpha x} \frac{\pi}{2}$$

Nr – will have s. (sine)

~~Nr~~ – No s in Nr (cosine).

<u>FCT</u>

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} \left[-a \cos sx + s \sin sx \right] \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2} (-a) \right] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \quad \text{—} (3)$$

<u>Inversion formula</u>

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \cos sx \, dx.$$

$$e^{-ax} \times \frac{\pi}{2a} = \int_0^\infty \frac{1}{s^2+a^2} \cos sx \, dx. \quad \text{—} (4)$$

If $a = \alpha$, $\displaystyle\int_0^\infty \frac{\cos sx \, ds}{s^2+\alpha^2} = e^{-\alpha x} \frac{\pi}{2}$

__/__/__

Q2 Find FST of $\frac{x}{x^2+a^2}$ & FCT of $\left(\frac{1}{a^2+x^2}\right)$

FST:

$$F_s\left(x/x^2+a^2\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{x^2+a^2}\sin sx\,dx.$$

$$\cancel{u = \frac{x}{x^2+a^2}} \qquad \cancel{dv = \sin sx\,dx}.$$

Refer previous Q eqn ②,

$$= \sqrt{\frac{2}{\pi}} \times e^{-as} \times \frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}\cdot e^{-as}, \quad a>0.$$

$$\frac{\pi}{\sqrt{\pi}} = \sqrt{\pi}.$$

$$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

FCT: Refer Q1 eqn ④,

$$F_c\left(\frac{1}{a^2+x^2}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{a^2+x^2}\cos sx\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\left(e^{-as}\times\frac{\pi}{2a}\right)$$

$$= \frac{1}{a}\sqrt{\frac{\pi}{2}} \times e^{-as}$$

Q3 Using Parseval's identity, evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ & $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}dx$, $a>0$

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty [F_c(f(x))]^2 ds.$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty [F_c(e^{-ax})]^2 ds.$$

$$= \int_0^\infty \frac{2}{\pi}\left(\frac{a}{s^2+a^2}\right)^2 ds. \quad \text{(use Q1 eqn ③)}$$

$$\Rightarrow \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2a^2}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds.$$

$$\left[0+\frac{1}{2a}\right]\times \pi/2a^2 = \int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds.$$

$$\frac{\pi}{4a^3} = \int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds$$

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a^3}$$

$$\int_0^\infty (e^{-ax})^2\,dx = \int_0^\infty \left(F_s[e^{-ax}]\right)^2 ds.$$

$$= \int_0^\infty \frac{2}{\pi}\left(\frac{s}{s^2+a^2}\right)^2 ds. \text{ [using Q1 eqn 1]}$$

$$\Rightarrow \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds.$$

$$\left[0+\frac{1}{2a}\right]\times\frac{\pi}{2} = \int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds.$$

$$\frac{\pi}{4a^3} = \int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds.$$

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a^3}.$$

Q4) $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ using transform methods.

$$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_c(f(x))\,F_c(g(x))\,dx.$$

$$f(x) = e^{-ax};\ g(x) = e^{-bx};\ a,b>0.$$

$$\int_0^\infty e^{-ax}e^{-bx}\,dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right)\left(\sqrt{\frac{2}{\pi}}\frac{b}{b^2+s^2}\right) ds$$

$$\int_0^\infty e^{-(a+b)x}\,dx = \frac{2}{\pi}ab\int_0^\infty \frac{1}{(a^2+s^2)(b^2+s^2)}\,ds.$$

$$\left.\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty \times \frac{\pi}{2ab} = \int_0^\infty \frac{1}{(a^2+s^2)(b^2+s^2)}\,ds.$$

$$\frac{\pi}{2ab(a+b)} = \int_0^\infty \frac{1}{(a^2+x^2)(b^2+x^2)}\,dx.$$

(Q) Find FST of $1/x$.

$$F_s(1/x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx.$$

$sx = \theta$. ~~$dx = \sin sx\,dx$~~.

$s\,dx = d\theta$.

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\theta}{\theta/s} \times \frac{d\theta}{s}.$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} = \sqrt{\pi/2}$$

(Q) Find FST & FCT of x^{n-1}.

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, \quad n>0.$$

Replace x with ax.

$$\int_0^\infty e^{-ax}(ax)^{n-1}a\,dx = \Gamma(n).$$

$$a^n\int_0^\infty e^{-ax}x^{n-1}\,dx = \Gamma(n)$$

$\cos\theta - i\sin\theta = e^{-i\theta}$.

$\cos\pi/2 - i\sin\frac{\pi}{2} = e^{-i\pi/2}$

$-i = e^{-i\pi/2}$.

Let $a = is$.

$$(is)^n \int_0^\infty e^{-isx}x^{n-1}\,dx = \Gamma(n)$$

$$\int_0^\infty (\cos sx - i\sin sx)x^{n-1}\,dx = \frac{\Gamma n}{i^n s^n} = \frac{(-i)^n\,\Gamma(n)}{s^n}$$

$$= \left(e^{-i\pi/2}\right)^n \frac{\Gamma(n)}{s^n}$$

$$= \cos\frac{n\pi}{2} \times \frac{\Gamma(n)}{s^n} - i\sin\frac{n\pi}{2}\frac{\Gamma(n)}{s^n}.$$

Comparing LHS & RHS,

$$\int_0^\infty x^{n-1}\cos sx\,dx = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2}$$

$$\Rightarrow F_c(x^{n-1}) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2}.$$

$$\int_0^\infty x^{n-1}\sin sx\,dx = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}$$

$$F_s(x^{n-1}) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}$$

$$\boxed{F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2} \quad \Bigg| \quad F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\frac{s}{a^2+s^2}}$$

Q ST $F_s(xf(x)) = \frac{-d}{ds}\left(F_c(f(x))\right)$. ~~the thing is~~

$$F_c(xf(x)) = \frac{d}{ds}\left(F_s(f(x))\right)$$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$\frac{d}{ds}(F_c(f(x))) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)(-\sin sx).x.dx$$

$$\frac{-d}{ds}\left[F_c(f(x))\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty (xf(x))\sin sx\,dx = F_s(xf(x))$$

_/_/__

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx.$$

$$\frac{d}{ds}(F_s(f(x))) = \sqrt{\frac{2}{\pi}} \int_0^\infty (x f(x)) \cos sx \, dx = F_c(x f(x)).$$

Q) Find $F_c(e^{-a^2x^2})$ & hence find $F_s(xe^{-a^2x^2})$.

$$F_c(e^{-a^2x^2}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx$$

$e^{i\theta} = \underset{RP}{\cos\theta} + \underset{IP}{i\sin\theta}$

$$= R.P \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} e^{-isx} \, dx.$$

RP of this is $\cos\theta$

$$= \frac{1}{a\sqrt{2}} e^{-s^2/4a^2} \quad \text{(Refer Q4)}$$

$$F_s(xe^{-a^2x^2}) = \frac{-d}{ds}(F_c(e^{-a^2x^2}))$$

$$= \frac{-1}{a\sqrt{2}} e^{-s^2/4a} \cdot \frac{-2s}{4a^2}$$

$$= \frac{2s}{4\sqrt{2}a^3} e^{-s^2/4a^2}.$$

Fourier Integral Thm.

If $f(x)$ is a piecewise continuously differentiable & absolutely integrable in the interval $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) e^{i(x-t)s} \, dt \, ds.$$