Chapter 2

FUNCTIONS OF SEVERAL VARIABLES

2. 1. INTRODUCTION :

This chapter is devoted to a study of functions depending on more than one independent variable. A real function $z = f(x, y)$ of two independent variables x and y, can be thought to represent a surface in the three-dimensional space referred to a set of co-ordinate axes X, Y, Z.

A simple example of a function of two independent variables x and y is $z = xy$, which represents the area of a rectangle whose sides are x and y.

◊ Continuity of a Function of Two Variables ◊

Definition : A function $z = f(x, y)$ is said to be continuous at the point (x_0, y_0) provided that a small change in the values of x and y produces a corresponding (small) change in the value of z. More precisely, if the value of $z = f(x, y)$ at (x_0, y_0) is z_0, then the continuity of the function at the point (x_0, y_0) means that

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0) = z_0$$

If a function is continuous at all points of some region R in the XY plane, then it is said to be continuous in the region R.

The definition of continuity of a function of more than two independent variables is similar.

◊ Partial Derivatives ◊

The analytical definition of the derivatives of a function $y = f(x)$ of single variable x is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Let Δz_x denote the increment in the function $z = f(x, y)$ where y is kept fixed and x is changed by an amount Δx,

(i. e) $\Delta z = f(x+\Delta x, y) - f(x, y)$. Then,

$\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$ is called the partial derivative of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$ or z_x.

Similarly, the partial derivative of z with respect y is defined and denoted as $\frac{\partial z}{\partial y}$ or z_y.

In general, if $z = f(x_n, x_n, ... x_n)$ is a function of n independent variables $x_1, x_2, x_3, ... x_n$ then $\frac{\partial z}{\partial x_i}$ denote the partial derivatives of z with respect to $x_i (i = 1, 2, ..., n)$, when the remaining variables are treated as constants.

Example 1

If $z = x^2 + y^2 + 3xy$, then $\frac{\partial z}{\partial x} = 2x + 3y$, $\frac{\partial z}{\partial y} = 2y + 3x$

Example 2

If $u = e^x \sin y \cos z$, then $\frac{\partial u}{\partial x} = e^x \sin y \cos z$ (when y and z are held constants) $\frac{\partial u}{\partial y} = e^x \cos y \cos z$ (Here x and z are treated as constants)

and $\frac{\partial u}{\partial z} = -e^x \sin y \sin z$ (both x and y are held constants).

Total Differential : If $z = f(x, y)$ is a function of two independent variables x and y, then the total differential of z is denoted as dz and defined as

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

In general, if $u = f(x_n, x_n, ... x_n)$, then the total differential of u is given by $du = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + ... + \frac{\partial f}{\partial x_n} dx_n$

It may thus be noted that the total differential is equal to the sum of the partial differentials.

Example 3

A metal box without a top has inside dimensions 6 ft, 4 ft and 2 ft. If the metal is 0.1 ft. thick, find the approximate volume by using the differential.

Let x, y, z be the dimensions of a metal box. Then its volume is $V = xyz$.

More its differential is given by

$$\begin{aligned} dV &= \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \\ &= yzdx + xzdy + xydz \\ &= 8 \times 0.2 + 12 \times 0.2 + 24 \times 0.1 \\ &= 1.6 + 2.4 + 2.4 \\ &= 6.4\, cu.ft \end{aligned}$$

Total Differentiation : Let $z = f(x, y)$ and let x and y be both functions of one independent variable t such that z becomes a function of this single independent variable and therefore, z may have a derivative with respect to t, called total derivative of z with respect to t, obtained via partial derivatives of f with respect to x and y.

(i. e) when $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, then $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

In general, if $z = f(x_1, x_2, ..., x_n)$ with $x_1 = g_1(t), x_2 = g_2(t), ..., x_n = g_n(t)$

then $\frac{dz}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + ... + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$.

Example 1

If $f(x,y) = x^2 + y^2$, where $x = r\cos\theta$ and $y = r\sin\theta$, find (i) $\frac{df}{dr}$ and $\frac{df}{d\theta}$ and also (ii) df.

Solution: Given $f(x,y) = x^2 + y^2$, where $x = r\cos\theta$ and $y = r\sin\theta$

(i) Now, $\frac{df}{dr} = \frac{\partial f}{\partial x}\frac{dx}{dr} + \frac{\partial f}{\partial y}\frac{dy}{dr} = 2x\cos\theta + 2y\sin\theta$

$= 2r(\cos^2\theta + \sin^2\theta) = 2r$

(ii) $\frac{df}{d\theta} = \frac{\partial f}{\partial x}\frac{dx}{d\theta} + \frac{\partial f}{\partial y}\frac{dy}{d\theta} = 2x(-r\sin\theta) + 2y(r\cos\theta)$

$= -2r^2\cos\theta\sin\theta + 2r^2\cos\theta\sin\theta = 0$

(iii) $df = 2rdr \quad \left(\because from (i) \frac{df}{dr} = 2r\right)$

$= 2xdx + 2ydy \ (\because r^2 = x^2 + y^2)$

Homogeneous Function : A function $f(x, y)$ is said to be a homogeneous function of degree 'n' if it can be expressed of the form $x^n\phi\left(\frac{y}{x}\right)$.

Euler's Theorem : If u is a homogeneous function of degree 'n',

then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$.

Example 2

If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x - y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Solution: Given $u = \sin^{-1}\left(\frac{x^2 + y^2}{x - y}\right)$

(i. e) $\sin u = \frac{x^2 + y^2}{x - y} = x^1\phi(y/x)$, '*sinu*' is a homogeneous function of degree 1. $\therefore$ Using Euler's theorem for the homogeneous function *sinu*, we have

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 1.\sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \sin u$$

$$(or)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u.$$

Total Differentiation : If $z = f(x, y)$, where x and y are functions of given two variables u and v, then $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$

Example 3

Find $\frac{dz}{dt}$, when $z = xy^2 + x^2y$, where $x = at^2$, $y = 2at$

Solution: We know that $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$

$$=2at(2at)^2+4at(at^2)(2at)+4a(at^2)(2at)+2a(at^2)^2$$

$$=8a^3t^3+8a^3t^4+8a^3t^3+2a^3t^4$$

$$=16a^3t^3+10a^3t^4$$

$$=2a^3t^3(8+5t)$$

Example 4

If $u=\sin(x/y)$, $x=e^t$, $y=t^2$, find $\frac{du}{dt}$.

Solution: Let $u=\sin(x/y)$, $x=e^t$, $y=t^2$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$=\cos(x/y)\left(\frac{1}{y}\right)(e^t)+\cos(x/y)\left(\frac{-x}{y^2}\right)(2t)$$

$$=\cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^2}\right)-\cos\left(\frac{e^t}{t^2}\right)2.\frac{e^t}{t^3}$$

$$=\frac{e^t}{t^2}\left[\cos\left(\frac{e^t}{t^2}\right)-\frac{2}{t}\right]$$

Example 5

If $u=x^2-y^2$, $v=2xy$, $f(x,y)=\varphi(u,v)$, show that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4(x^2+y^2)\left(\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}\right)$$

Solution: Let $\frac{\partial f}{\partial x}=\frac{\partial\varphi}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}.\frac{\partial v}{\partial x}=\frac{\partial\varphi}{\partial u}(2x)+\frac{\partial\varphi}{\partial v}(2y)$

$$\therefore\quad \frac{\partial^2 f}{\partial x^2}=\frac{\partial^2\varphi}{\partial x^2}(4x^2)+2\frac{\partial\varphi}{\partial x}(1)+\frac{\partial\varphi}{\partial y}(4y^2) \qquad (1)$$

$$\frac{\partial f}{\partial y}=\frac{\partial\varphi}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial\varphi}{\partial v}.\frac{\partial v}{\partial y}=\frac{\partial\varphi}{\partial u}(-2y)+\frac{\partial\varphi}{\partial v}(2x)$$

$$\therefore\quad \frac{\partial^2 f}{\partial y^2}=\frac{\partial^2\varphi}{\partial u^2}(4y^2)+\frac{\partial\varphi}{\partial u}(-2)+\frac{\partial^2\varphi}{\partial v^2}(4x^2) \qquad (2)$$

Equation (1) + (2) gives, $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\left(\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}\right)(4x^2+4y^2)$

$$=4(x^2+y^2)\left(\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}\right)$$

Example 6

If $u=f(r)$, $x=r\cos\theta$, $y=r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$.

Solution: Given $u=f(r)$ where $r=\sqrt{x^2+y^2}$

Now, $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}=f'(r).\frac{2x}{2\sqrt{x^2+y^2}}=f'(r)\frac{x}{r}$

$$\therefore\quad \frac{\partial^2 u}{\partial x^2}=\frac{x}{r}f''(r).\frac{\partial r}{\partial x}+f'(r)\frac{1}{r}+xf'(r)(-1)r^{-2}\frac{\partial r}{\partial x}$$

$$=\frac{x^2}{r^2}f''(r)+f'(r)\frac{1}{r}-\frac{x^2}{r^3}f'(r) \qquad (1)$$

Similarly, $\frac{\partial^2 u}{\partial y^2}=\frac{y^2}{r^2}f''(r)+f'(r)\frac{1}{r}-\frac{y^2}{r^3}f'(r)$ (2)

Equations (1) + (2) gives, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$.

Differentiation of Implicit Functions :

Consider the implicit function $f(x, y) = 0$. Then, $\frac{dy}{dx}=-\frac{(\partial f/\partial x)}{(\partial f/\partial y)}$.

Example 7

Find $\frac{dy}{dx}$ if $xe^{-y}-2ye^x=1$.

Solution: Given $f(x,y)=xe^{-y}-2ye^{x}-1=0$

$$\frac{\partial f}{\partial x}=e^{-y}-2ye^{x} \;\&\; \frac{\partial f}{\partial y}=-xe^{-y}-2e^{x}$$

$$\frac{dy}{dx}=-\frac{(\partial f/\partial x)}{(\partial f/\partial y)}=-\frac{(e^{-y}-2ye^{x})}{(-xe^{-y}-2e^{x})}=\frac{e^{-y}-2ye^{x}}{xe^{-y}+2e^{x}}$$

Example 8

If $u=x\log(xy)$, where $x^3+y^3+3xy=1$, find du/dx.

Solution: Let $\dfrac{du}{dx}=\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\dfrac{dy}{dx}$ (1)

Now $f(x,y)=0$ is $x^3+y^3+3xy-1=0$

$$\frac{dy}{dx}=-\frac{(\partial f/\partial x)}{(\partial f/\partial y)}=-\frac{(3x^2+3y)}{(3y^2+3x)}=-\left(\frac{x^2+y}{y^2+x}\right) \tag{2}$$

Equation (1) becomes,

$$\frac{du}{dx}=x.\frac{1\times y}{xy}+\log(xy)+x.\frac{1\times x}{xy}\left(-\frac{x^2+y}{y^2+x}\right)$$

$$=1+\log(xy)-\frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

Example 9

Find dy/dx, if $(\cos x)^y=(\sin y)^x$

Solution: Given $(\cos x)^y=(\sin y)^x$

Taking *log* on both sides, we get, $y\log(\cos x)=x\log(\sin y)$

(i. e) $f(x,y)=y\log(\cos x)-x\log(\sin y)=0$

$$\frac{dy}{dx}=-\frac{(\partial f/\partial x)}{(\partial f/\partial y)}=-\frac{\left[y\frac{(-\sin x)}{\cos x}-\log(\sin y)\right]}{\left[\log(\cos x)-x\frac{\cos y}{\sin y}\right]}$$

$$\text{(i.e)}\quad \frac{dy}{dx}=\frac{y\tan x+\log(\sin y)}{\log(\cos x)-x\cot y}$$

Taylor's Theorem for Functions of Two Variables:

We know that by Taylor's theorem for a function $f(x)$ of a single variable x,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+....$$

Now let $f(x,y)$ be a function of two independent variables x and y. If y is kept constant, then by Taylor's theorem for a function of a single variable x, we have

$$f(x+h,y+k)=f(x,y+k)+h\frac{\partial}{\partial x}f(x,y+k)+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f(x,y+k)+\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}f(x,y+k)+...... \tag{1}$$

Now keeping x constant and applying Taylor's theorem for a function of single variable y, we have

$$f(x,y+k)=f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+...(2)$$

Using (2), we can write (1) as

$$f(x+h,y+k)=\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+.....\right]+h\frac{\partial}{\partial x}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+....\right]$$

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+.....\right]$$

$$+\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+.....\right]+...$$

$$=\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+......\right]$$

$$+\left[h\frac{\partial f}{\partial x}+hk\frac{\partial^2 f}{\partial x\partial y}+h\frac{k^2}{2!}\frac{\partial^3 f}{\partial x\partial y^2}+...\right]+\left[\frac{k^2}{2!}\frac{\partial^2 f}{\partial x^2}+...\right]+\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3}+...$$

$$=f(x,y)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\left(\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}+hk\frac{\partial^2 f}{\partial x\partial y}+\frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\right)$$

$$+\left(\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3}+\frac{h^2k}{2!}\frac{\partial^3 f}{\partial x^2\partial y}+\frac{hk^2}{2!}\frac{\partial^3 f}{\partial x\partial y^2}+\frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3}\right)+....$$

$$=f(x,y)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\frac{1}{2!}\left(\frac{h^2\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+\frac{k^2\partial^2 f}{\partial y^2}\right)$$

$$+\frac{1}{3!}\left(\frac{h^3\partial^3 f}{\partial x^3}+3h^2k\frac{\partial^3 f}{\partial x^2\partial y}+3hk^2\frac{\partial^3 f}{\partial x\partial y^2}+k^3\frac{\partial^3 f}{\partial y^3}\right)+...$$

$$=f(x,y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f+..........$$

Corollary: 1 Putting $x = a$ and $y = b$, we have

$$f(a+h,b+k)=f(a,b)+\left[hf_x(a,b)+kf_y(a,b)\right]$$
$$+\frac{1}{2!}\left[h^2 f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[h^3 f_{xxx}(a,b)+3h^2kf_{xxy}(a,b)+3hk^2 f_{xyy}(a,b)+k^3 f_{yyy}(a,b)\right]$$
$$+...$$

Corollary: 2 In corollary 1, put $a+h = x$ and $b+k = y$ so that $h = x\text{-}a$ and $k = y\text{-}b$, we have

$$f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$
$$+...$$

Corollary: 3 In corollary 2, put $a = 0, b = 0$, we have

$$f(x,y)=f(0,0)+\left[xf_x(0,0)+yf_y(0,0)\right]$$
$$+\frac{1}{2!}\left[x^2 f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2 f_{yy}(0,0)\right]+...$$

This is called **Maclaurin's theorem** for two variables.

Note : Corollary 3 is used to expand *f(x, y)* in powers of x and y ((i. e) near origin), whereas corollary 2 is used to expand *f(x, y)* in the neighbourhood of *(a, b)*.

Example1 $x^3y^3+3x^2y$

Expand $x^2y+3y-2$ in powers of *(x-1)* and *(y+2)* using Taylor's theorem upto terms of third degree.

Solution: The Taylor series expansion of *f(x, y)* in powers of *(x-a)* and *(y-b)* is given by

$$f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$
$$+...$$

Here $f(x,y)=x^2y+3y-2$, $a = 1$, $b = -2 \Rightarrow f(1,-2) = -10$

$f_x(x,y)=2xy,\qquad f_x(1,-2)=-4$

$f_y(x,y)=x^2+3\qquad f_{xy}(x,y)=2x$

$f_y(1,-2)=4\qquad f_{xy}(1,-2)=2$

$f_{xx}(x,y)=2y\qquad f_{yy}(x,y)=0$

$f_{xx}(1,-2)=-4\qquad f_{yy}(1,-2)=0$

$f_{xxx}(x,y)=0 \qquad f_{yyy}(x,y)=0$

$f_{xxy}(x,y)=2 \qquad f_{xyy}(x,y)=0$

From (1),

$$x^2y+3y-2=-10+\frac{1}{1!}[(x-1)(-4)+(y+2)(4)]$$
$$+\frac{1}{2!}\left[(x-1)^2(-4)+2(x-1)(y+2)(2)\right]$$
$$+\frac{1}{3!}\left[3(x-1)^2(y+2)(2)\right]+...$$

(or) $$x^2y+3y-2=-10-4(x-1)+4(y+2)-2(x-1)^2$$
$$+2(x-1)(y+2)+(x-1)^2(y+2)+....$$

Example 2

Expand $e^x\cos y$ in powers of x and y as far as the terms of the third degree.

Solution: Let $f(x,y)=e^x\cos y \Rightarrow f_x(x,y)=e^x\cos y \qquad f_x(0,0)=1$

$f_y(x,y)=-e^x\sin y \qquad f_{xxx}(x,y)=e^x\cos y$

$f_y(0,0)=0 \qquad f_{xxx}(0,0)=1$

$f_{xx}(x,y)=e^x\cos y \qquad f_{xxy}(x,y)=-e^x\sin y$

$f_{xx}(0,0)=1 \qquad f_{xxy}(0,0)=0$

$f_{xy}(x,y)=-e^x\sin y \qquad f_{xyy}(x,y)=-e^x\cos y$

$f_{xy}(0,0)=0 \qquad f_{xyy}(0,0)=-1$

$f_{yy}(x,y)=-e^x\cos y \qquad f_{yyy}(x,y)=e^x\sin y$

$f_{yy}(0,0)=-1 \qquad f_{yyy}(0,0)=0$

Taylor's series of $f(x, y)$ in powers of x and y is

$$f(x,y)=f(0,0)+\left[xf_x(0,0)+yf_y(0,0)\right]$$
$$+\frac{1}{2!}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]+...$$

$$\therefore e^x\cos y=1+\frac{1}{1!}[x.1+y.0]+\frac{1}{2!}[x^2.1+2xy.0+y^2(-1)]$$
$$+\frac{1}{3!}[x^3.1+3x^2y.0+3xy^2(-1)+y^3.0]+...$$
$$=1+\frac{x}{1!}+\frac{x^2-y^2}{2!}+\frac{x^3-3xy^2}{3!}+...$$

Example 3

Expand $\tan^{-1}\left(\frac{y}{x}\right)$ using Taylor's series near (1, 1) upto quadratic terms.

Solution: The Taylor series expansion for $f(x, y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)\right] \quad (1)$$
$$+...$$

Here $f(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$, $a=1=b \Rightarrow f(a,b)=f(1,1)=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4}$

$$f_x(x,y)=\frac{1}{1+(y^2/x^2)}\left(\frac{-y}{x^2}\right)=\frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)=\frac{-y}{x^2+y^2} \Rightarrow f_x(1,1)=\frac{-1}{2}$$

$$f_y(x,y)=\frac{1}{1+\frac{y^2}{x^2}}\times\frac{1}{x}=\frac{x}{x^2+y^2} \Rightarrow f_y(1,1)=\frac{1}{2}$$

$$f_{xx}(x,y)=(-y)(x^2+y^2)^{-2}2x=\frac{-2xy}{(x^2+y^2)^2} \quad \therefore f_{xx}(1,1)=\frac{-2}{4}=\frac{-1}{2}$$

$$f_{xy}(x,y)=\frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow f_{xy}(1,1)=0$$

$$f_{yy}(x,y)=(x)(x^2+y^2)^{-2}2y=\frac{2xy}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1)=\frac{1}{2}.$$

Using these values in (1) we get,

$$\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{4}+\frac{1}{1!}\left[(x-1)\left(-\frac{1}{2}\right)+(y-1)\frac{1}{2}\right]$$

$$+\frac{1}{2!}\left[(x-1)^2\left(-\frac{1}{2}\right)+(x-1)(y-1)\times 0\right.$$

$$\left.+(y-1)^2\left(\frac{1}{2}\right)\right]+\ldots\ldots$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{4}-\left(\frac{x-1}{2}\right)+\left(\frac{y-1}{2}\right)-\frac{(x-1)^2}{4}+\frac{(y-1)^2}{4}+\ldots$$

Example 4

Using Taylor's series expand $e^x\log(1+y)$ up to terms of the third degree in the neighborhood of origin.

Solution: The Taylor's series expansion of $f(x, y)$ near $(0, 0)$ is

$$f(x,y)=f(0,0)+\frac{1}{1!}\left[(x-0)f_x(0,0)+(y-0)f_y(0,0)\right]$$

$$+\frac{1}{2!}\left[(x-0)^2f_{xx}(0,0)+2(x-0)(y-0)f_{xy}(0,0)+(y-0)^2f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[\begin{array}{l}(x-0)^3f_{xxx}(0,0)+3(x-0)^2(y-0)f_{xxy}(0,0)+3(x-0)(y-0)^2\\ f_{xyy}(0,0)+(y-0)^3f_{yyy}(0,0)+\ldots\end{array}\right] \quad (1)$$

Now $f(x,y)=e^x\log(1+y)\Rightarrow f(0,0)=0$ as $log1 = 0$

$f_x(x,y)=e^x\log(1+y)$; $\quad f_{xxx}(x,y)=e^x\log(1+y)$

$f_y(x,y)=\dfrac{e^x}{1+y}$; $\quad f_{xxy}(x,y)=\dfrac{e^x}{1+y}$

$f_{xx}(x,y)=e^x\log(1+y)$ $\quad f_{xyy}(x,y)=-e^x(1+y)^{-2}$

$f_{xy}(x,y)=\dfrac{e^x}{1+y}$ $\quad f_{yyy}(x,y)=2e^x(1+y)^{-3}$

$f_{yy}(x,y)=-e^x(1+y)^{-2}$

$f_x(0,0)=0;\quad f_{xx}(0,0)=0;\quad f_{xxx}(0,0)=0$

$f_y(0,0)=0;\quad f_{xy}(0,0)=1;\quad f_{xxy}(0,0)=1$

$f_{yy}(0,0)=-1;\quad f_{xyy}(0,0)=-1;\quad f_{yyy}(0,0)=2$

Using these values in (1), we get

$$e^x\log(1+y)=0+\frac{1}{1!}[(x-0)\times 0+(y-0)(1)]$$

$$+\frac{1}{2!}\left[(x-0)^2\times 0+2(x-0)(y-0)(1)+(y-0)^2(-1)\right]$$

$$+\frac{1}{3!}\left[(x-0)^3\times 0+3x^2y(1)+3xy^2(-1)+(y-0)^3(2)\right]+\ldots$$

$$\Rightarrow e^x\log(1+y)=y+xy-\frac{y^2}{2}+\frac{x^2y}{2}-\frac{xy^2}{2}+\frac{y^3}{3}+\ldots$$

Example 5

Expand $xy^2+2x-3y$ in powers of $(x+2)$ and $(y-1)$ up to third degree terms.

Solution: The Taylor series expansion of $f(x, y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)\right] \quad (1)$$

$+\ldots\ldots$

Here $f(x, y) = xy^2+2x-3y$; $a = -2$, $b = 1$.

$\therefore f(-2,1) = (-2)(1)+2(-2)-3(1) = -9$

$f_x(x,y)=y^2+2 \quad f_{xx}(x,y)=0$

$f_x(-2,1)=3 \quad f_{xy}(x,y)=2y$

$f_y(x,y)=2xy-3 \quad f_{xy}(-2,1)=2$

$f_y(-2,1)=-7 \quad f_{yy}(x,y)=2x$

$f_{xxx}(x,y)=0$ $\qquad$ $f_y(-2,1)=-4$

$f_{xyy}(x,y)=2$ $\qquad$ $f_{yyy}(x,y)=0$

$f_{xxy}(x,y)=0$

Substituting these values in (1), we get

$$xy^2+2x-3y=-9+3(x+2)-7(y-1)+2(x+2)(y-1)-2(y-1)^2$$
$$+(x+2)(y-1)^2+\dots$$

Example 6

Find the Taylor series expansion of e^{xy} at (1, 1) up to third degree terms.

Solution: Taylor series expansion of *f(x, y)* at (1, 1) is given by

$$f(x,y)=f(1,1)+\left[(x-1)f_x(1,1)+(y-1)f_y(1,1)\right]+\frac{1}{2!}\left[(x-1)^2f_{xx}(1,1)\right.$$
$$\left.+2(x-1)(y-1)f_{xy}(1,1)+(y-1)^2f_{yy}(1,1)\right]+\dots\dots\dots \quad (1)$$

Here $f(x,y)=e^{xy}$; $a=1=b$ $\quad \therefore f(1,1)=e^{1\times1}=e$

$f_x(x,y)=e^{xy}.y \Rightarrow f_x(1,1)=e^{1\times1}\times1=e$

$f_y(x,y)=e^{xy}x \Rightarrow f_y(1,1)=e^{1\times1}\times1=e$

$f_{xx}(x,y)=y.e^{xy}.y \Rightarrow f_{xx}(1,1)=1^2\times e^{1\times1}=e$

$f_{xy}(x,y)=e^{xy}.1+xe^{xy}y \Rightarrow f_{xy}(1,1)=2e$

$f_{yy}(x,y)=x^2e^{xy} \Rightarrow f_{yy}(1,1)=1^2\times e^{1\times1}=e$

$f_{xxx}(x,y)=y^3.e^{xy} \Rightarrow f_{xxx}(1,1)=e$

$f_{xxy}(x,y)=e^{xy}y+y(e^{xy}.1+xye^{xy}) \Rightarrow f_{xxy}(1,1)=3e$

$f_{xyy}(x,y)=x^2e^{xy}y+e^{xy}2x \Rightarrow f_{xyy}(1,1)=3e$

$f_{yyy}(x,y)=x^3e^{xy} \Rightarrow f_{yyy}(1,1)=e$

Using these values in (1), we get

$$e^{xy}=e+\frac{1}{1!}((x-1)e+(y-1)e)+\frac{1}{2!}\left[(x-1)^2e+2(x-1)(y-1)2e+(y-1)^2e\right]$$
$$+\frac{1}{3!}\left[(x-1)^3e+3(x-1)^2(y-1)3e+3(x-1)(y-1)^23e+(y-1)^3e\right]+\dots$$

$$\Rightarrow \quad e^{xy}=e\left[1+(x-1)+(y-1)+\frac{(x-1)^2}{2}+2(x-1)(y-1)+(y-1)^2+\frac{(x-1)^3}{6}\right.$$
$$\left.+\frac{3}{2}(x-1)^2(y-1)+\frac{3}{2}(x-1)(y-1)^2+(y-1)^3+\dots\right]$$

Example 7

Using Taylor's series, verify that

$$\cos(x+y)=1-\frac{(x+y)^2}{2!}+\frac{(x+y)^4}{4!}-\dots$$

Solution: The required expansion is possible if we obtain the expansion in powers of x and y

$f(x,y)=\cos(x+y) \Rightarrow f(0,0)=\cos 0=1$

$f_x(x,y)=-\sin(x+y)(1) \Rightarrow f_x(0,0)=\sin 0=0$

$f_y(x,y)=-\sin(x+y)(1) \Rightarrow f_y(0,0)=0$

$f_{xx}(x,y)=-\cos(x+y)(1) \Rightarrow f_{xx}(0,0)=-\cos 0=-1$

$f_{xy}(x,y)=-\cos(x+y)(1) \Rightarrow f_{xy}(0,0)=-\cos 0=-1$

$f_{yy}(x,y)=-\cos(x+y)(1) \Rightarrow f_{yy}(0,0)=-1$

$f_{xxx}(x,y)=\sin(x+y) \Rightarrow f_{xxx}(0,0)=\sin 0=0$

$f_{xxy}(x,y)=\sin(x+y) \Rightarrow f_{xxy}(0,0)=\sin 0=0$

$f_{xyy}(x,y)=\sin(x+y) \Rightarrow f_{xyy}(0,0)=\sin 0=0$

$f_{yyy}(x,y)=\sin(x+y) \Rightarrow f_{yyy}(0,0)=\sin 0=0$

$f_{xxxx}(x,y)=\cos(x+y) \Rightarrow f_{xxxx}(0,0)=\cos 0=1$

$f_{xxxy}(x,y)=\cos(x+y) \Rightarrow f_{xxxy}(0,0)=\cos 0=1$

Similarly other terms.

$$\therefore \cos(x+y) = 1 + \frac{1}{2!}\left[(x-0)^2(-1) + 2(x-0)(y-0)(-1) + (y-0)^2(-1)\right]$$
$$+\frac{1}{4!}\left[(x-0)^4(1)+\ldots\right]$$

$$\Rightarrow \quad \cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \ldots$$

Example 8 ....

Expand $\frac{(x+h)(y+k)}{x+h+y+k}$ in a series of powers of h and k up to the second degree terms.

Solution: Let $f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$ and $f(x,y) = \frac{xy}{x+h}$.

Taylor's series of $f(x+h, y+k)$ in powers of h and k is

$$f(x+h,y+k) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(\frac{h^2\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + \frac{k^2\partial^2 f}{\partial y^2}\right) + \frac{1}{3!}\left(\frac{h^3\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \ldots\ldots \quad (1)$$

$$f_x(x,y) = y\left[\frac{(x+y)(1) - x(1)}{(x+y)^2}\right] = \frac{y^2}{(x+y)^2}$$

By symmetry $f_y(x,y) = \frac{x^2}{(x+y)^2}$, $f_{xx}(x,y) = \frac{-2y^2}{(x+y)^3}$,

$f_{yy}(x,y) = \frac{-2x^2}{(x+y)^3}$, $f_{xy}(x,y) = \frac{(x+y)^2(2x) - x^2 2(x+y)}{(x+y)^4} = \frac{2xy}{(x+y)^3}$

Using in (1), we get

$$\frac{(x+h)(y+k)}{x+h+y+k} = \frac{xy}{x+y} + \frac{1}{1!}\left(\frac{hy^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2}\right)$$
$$+\frac{1}{2!}\left(h^2\left(\frac{-2y^2}{(x+y)^3}\right) + 2hk\frac{2xy}{(x+y)^3} + k^2\left(\frac{-2x^2}{(x+y)^3}\right)\right)$$

Example 9 ....

Expand $(x^2y + \sin y + e^x)$ in powers of $(x-1)$ and $(y-p)$.

Solution: The Taylor's series expansion of $f(x, y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ldots \quad (1)$$

Here $f(x,y) = x^2y + \sin y + e^x; a = 1, b = \pi$ and

$$f(1,\pi) = 1^2\pi + \sin\pi + e = \pi + e \text{ as } \sin\pi = 0$$

$$f_x(x,y) = 2xy + e^x \Rightarrow f_x(1,\pi) = 2\pi + e$$
$$f_y(x,y) = x^2 + \cos y \Rightarrow f_y(1,\pi) = 1 + \cos\pi = 0$$

$$f_{xx}(x,y) = 2y + e^x \Rightarrow f_{xx}(1,\pi) = 2\pi + e$$
$$f_{xy}(x,y) = 2x \Rightarrow f_{xy}(1,\pi) = 2$$

$$f_{yy}(x,y) = -\sin y \Rightarrow f_{yy}(1,\pi) = 0$$

Using in (1), we get

$$x^2y + \sin y + e^x = \pi + e + (x-1)(2\pi + e) + \frac{1}{2}(x-1)^2(2\pi+e) \cdot + (x-1)(y-\pi) + \ldots$$

Example 10 ....

Find the expansion for $\cos x \cos y$ in powers of x and y up to terms of 3rd degree.

Solution: Let $f(x,y) = \cos x\cos y \Rightarrow f(0,0) = \cos 0\cos 0 = 1$

$$f_x(x,y) = -\sin x\cos y \quad\Rightarrow\quad f_x(0,0) = 0;$$
$$f_y(x,y) = -\cos x\sin y \quad\Rightarrow\quad f_y(0,0) = 0$$
$$f_{xx}(x,y) = -\cos x\cos y \quad\Rightarrow\quad f_{xx}(0,0) = -1;$$
$$f_{xy}(x,y) = \sin x\sin y \quad\Rightarrow\quad f_{xy}(0,0) = 0$$
$$f_{yy}(x,y) = -\cos x\cos y \Rightarrow f_{yy}(0,0) = -1;$$
$$f_{xxx}(x,y) = \sin x\cos y \Rightarrow f_{xxx}(0,0) = 0$$
$$f_{xxy}(x,y) = \cos x\sin y \Rightarrow f_{xxy}(0,0) = 0;$$
$$f_{xyy}(x,y) = \sin x\cos y \Rightarrow f_{xyy}(0,0) = 0$$

and $f_{yyy}(x,y) = \cos x\sin y \Rightarrow f_{yyy}(0,0) = 0$.

The required expansion is $\cos x\cos y = 1 - \frac{x^2}{2} - \frac{y^2}{2} - \ldots$

Example 11 ....

Find the Taylor's series expansion of $e^x \sin y$ near the point $\left(-1, \frac{\pi}{4}\right)$ up to the 3rd degree terms.

Solution: Taylor's series of $f(x, y)$ near the point $\left(-1, \frac{\pi}{4}\right)$ is

$$f(x,y) = f\left(-1,\frac{\pi}{4}\right) + \frac{1}{1!}\left\{(x+1)f_x\left(-1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)f_y\left(-1,\frac{\pi}{4}\right)\right\}$$
$$+\frac{1}{2!}\left[\begin{matrix}(x+1)^2 f_{xx}\left(-1,\frac{\pi}{4}\right) + 2(x+1)\left(y-\frac{\pi}{4}\right)f_{xy}\left(-1,\frac{\pi}{4}\right) \\ +\left(y-\frac{\pi}{4}\right)^2 f_{yy}\left(-1,\frac{\pi}{4}\right)\end{matrix}\right] + \ldots \quad (1)$$

$$f(x,y) = e^x \sin y;\ f_x = e^x \sin y;\ f_y = e^x \cos y$$
$$f_{xx} = e^x \sin y;\ f_{xy} = e^x \cos y;\ f_{yy} = -e^x \sin y$$
$$f_{xxx} = e^x \sin y;\ f_{xxy} = e^x \cos y;\ f_{xyy} = -e^x \sin y$$
$$f_{yyy} = -e^x \cos y$$

$$f\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_x\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_y\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}}$$
$$f_{xx}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_{xy}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_{yy}\left(-1,\frac{\pi}{4}\right) = -\frac{1}{e\sqrt{2}}$$
$$f_{xxx}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_{xxy}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_{xyy}\left(-1,\frac{\pi}{4}\right) = -\frac{1}{e\sqrt{2}}$$
$$f_{yyy}\left(-1,\frac{\pi}{4}\right) = -\frac{1}{e\sqrt{2}}$$

Using these values in (1), we get

$$e^x \sin y = \frac{1}{e\sqrt{2}}\left\{1 + \frac{1}{1!}\left[(x+1) + \left(y-\frac{\pi}{4}\right)\right]\right\}$$
$$+\frac{1}{2!}\left\{(x+1)^2 + 2(x+1)\left(y-\frac{\pi}{4}\right) - \left(y-\frac{\pi}{4}\right)^2\right\}$$
$$+\frac{1}{3!}\left\{(x+1)^3 + 3(x+1)^2\left(y-\frac{\pi}{4}\right) - 3(x+1)\left(y-\frac{\pi}{4}\right)^2 - \left(y-\frac{\pi}{4}\right)^3\right\} + \ldots$$

■ EXERCISE ■

1. Find the Taylor's series expansion of x^y near the point (1, 1) up to the second degree terms. [Ans: $x^y = 1 + (x-1) + (x-1)(y-1)$].
2. Using Taylor's series, verify that
$$\log(1+x+y) = (x+y) - \frac{1}{2}(x+y)^2 + \frac{1}{3}(x+y)^3 - \ldots$$
3. Using Taylor's series, verify that
$$\tan^{-1}(x+y) = (x+y) + \frac{1}{3}(x+y)^3 + \ldots + \infty.$$
4. Expand $e^x \sin y$ in powers of x and y as far as terms of 3rd degree. [Ans: $e^x \sin y = y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \ldots$]

5. Expand $f(x, y) = sin(xy)$ in powers of $(x-1)$ and $\left(y-\frac{\pi}{2}\right)$.

[Ans: $1-\frac{1}{8}\pi^2(x-1)^2-\frac{1}{2}\pi(x-1)\left(y-\frac{\pi}{2}\right)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2+....$]

6. Expand $xy^2+\sin(xy)$ at the point $\left(1,\frac{\pi}{2}\right)$ up to terms of second degree.

$$\left[\begin{array}{l} Ans: 1+\pi^2+\frac{\pi^2}{4}(x-1)+\pi\left(y-\frac{\pi}{2}\right) \\ +\frac{1}{2!}\left[-\frac{\pi^2}{4}(x-1)^2+\pi(x-1)\left(y-\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2\right]+.... \end{array}\right]$$

7. Find the Taylor's for the function $f(x,y)=\frac{1}{xy}$, through terms of degree two for (2,-1).

$$\left[\begin{array}{l} Ans: -\frac{1}{2}+\frac{1}{4}(x-2)-\frac{1}{2}(y+1)-\frac{1}{8}(x-2)^2 \\ +\frac{1}{4}(x-2)(y+1)-\frac{1}{2}(y+1)^2+... \end{array}\right]$$

8. Expand $\frac{y^2}{x^3}$ at (1,-1). [Ans:

$1-3(x-1)-2(y+1)+6(x+1)^2+6(x-1)(y+1)+(y+1)^2+....$].

9. If $f(x,y)=\tan^{-1}(xy)$, compute an approximate value of $f(0.9,-1.2)$. [Ans: -0.823].Hint: expand near (1,-1); take $h = -0.1, k = -0.2$]

10. Expand $f(x,y)=21+x-20y+4x^2+xy+6y^2$ in Taylor series of maximum order about the point (-1,2).

$$\left[\begin{array}{l} Ans: f(x,y)=6-5(x+1)+3(y-2)+4(x+1)^2 \\ +(x+1)(y-2)+6(y-2)^2+... \end{array}\right]$$

11. Expand $e^{ax}\ sinby$ at (0, 0) using Taylor's series.

$\left[Ans: by+abxy+\frac{1}{3!}(3a^2bx^2y-b^3y^3)+...\right]$

12. Expand e^{xy} at (1, 1) up to 3 terms.

$\left[Ans: e\left(1+(x-1)+(y-1)+\frac{1}{2!}\left[(x-1)^2+4(x-1)(y-1)+(y-1)^2\right]\right)\right]$

13. Expand $e^x log(1+y)$ as the Taylor's series in the neighborhood of (0, 0). $\left[Ans: y+xy-\frac{y^2}{2}+\frac{x^2y}{2}-\frac{xy^2}{2}+...\right]$

◊ Maxima and Minima of Functions of Two Variables ◊

A function $f(x, y)$ is said to have a maximum value at $x = a, y = b$ if $f(a, b) > f(a+h, b+k)$, for small and independent values of h and k, positive or negative. A function $f(x, y)$ is said to have a minimum value at $x = a, y = b$ if $f(a, b) < f(a+h, b+k)$, for small and independent values of h and k positive or negative. Thus, $f(x, y)$ has a maximum or minimum value at a point (a, b) according as $\Delta f = f(a+h,b+k)-f(a,b) < or > 0$. A minimum or a maximum value of a function is called its extreme value.

◊ Conditions for f(x, y) to be Maximum or Minimum ◊

By Taylor's theorem, we have

$$\Delta f = f(a+h,b+k)-f(a,b)$$

$$=\left[hf_x(a,b)+kf_y(a,b)\right]+\frac{1}{2!}\left[h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b)\right] \quad (1)$$

$+$.....................................

For small values of h and k, the second and higher order terms are still smaller and may be neglected.

Thus sign of $\Delta f = sign\ of\ [h_x f(a,b)]+[kf_y(a,b)]$.

Taking $h = 0$, the sign of Δf changes with sign of k. Similarly taking $k = 0$, the sign of Δf changes with the sign of h. Since Δf changes sign with h and k, $f(x, y)$ can not have a maximum (or) minimum value at (a, b) unless $f_x(a,b) = 0 = f_y(a,b)$.

Hence the necessary conditions for $f(x, y)$ to have a maximum (or) a minimum value at (a, b) are $f_x(a,b) = 0,\ f_y(a,b) = 0$.

If these conditions are satisfied, then for small values of h and k, we have from (1).

$$\Delta f = \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$

$$= \frac{1}{2!}\left[h^2 r + 2hks + k^2 t\right] \text{ where } r = f_{xx}(a,b), s = f_{xy}(a,b), t = f_{yy}(a,b)$$

$$\Delta f = \frac{1}{2r}\left[h^2 r^2 + 2hkrs + k^2 rt\right]$$

$$= \frac{1}{2r}\left[(h^2 r^2 + 2hkrs + k^2 s^2) + k^2 rt - k^2 s^2\right] + \ldots$$

$$= \frac{1}{2r}\left[(hr + ks)^2 + k^2(rt - s^2)\right] + \ldots \qquad (2)$$

Now $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$.

In this case, Δf will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then $f(x, y)$ has a maximum or minimum at (a, b) according as $r < 0$ or $r > 0$.

If $rt - s^2 < 0$, then Δf changes sign with h and k. Hence there is neither a maximum nor a minimum value at (a, b). The point (a, b) is a saddle point in this case.

If $rt - s^2 = 0$, no conclusion can be drawn about a maximum or minimum value at (a, b) and hence further investigation is required.

Note : The point (a, b) is called a stationary point if $f_x(a,b) = 0$, $f_y(a,b) = 0$. The value $f(a, b)$ is called a stationary value. Thus every value is a stationary value but the converse may not be true.

◊ Rule to find the Extreme Value of a function z=f(x, y) ◊

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(i). Solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, similarly. Let (a, b); (c, d),.... be the solutions of these equations.

(ii). For each solution in step (ii) find $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$

(iii). a) If $rt - s^2 > 0$ and $r<0$ for a particular solution (a, b) of step then z has a maximum value at (a, b).

b) If $rt - s^2 > 0$ and $r > 0$ for a particular solution (a, b) of step (ii), then z has a minimum value at (a, b).

c) If $rt - s^2 < 0$ for a particular solution (a, b) of step (ii) then z has no extreme value at (a, b).

d) If $rt - s^2 = 0$, this case is doubtful and requires further investigation.

Example 1 ....

Examine for extreme values of $x^2 + y^2 + 6x + 12$.

Solution: Let $f(x, y) = x^2 + y^2 + 6x + 12$ and $f_x = 2x + 6;\ f_y = 2y$.

For maximum or minimum, $f_x = 0, f_y = 0 \Rightarrow 2x+6 = 0;\ 2y = 0$

Solving we get $x = -3$, $y = 0$. Stationary point is (-3, 0)

$r = f_{xx} = 2$ at (-3,0), $rt - s^2$ gives 4 (i.e) $rt - s^2 > 0$, $t = f_{xy} = 0$,

$s = f_{xy} = 2$. As r is 0, $rt - s^2 > 0 \Rightarrow$ (-3, 0) is a minimum point.

and $f(-3,0) = 9-18+12 = 3$.

$\therefore$ Minimum value = 3.

Example 2

Examine for extreme values of $xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$.

Solution: Let $z = f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$

$$f_x = y - \frac{a^3}{x^2};\ f_y = x - \frac{a^3}{y^2}$$

Equation f_x and f_y to zero

$$y - \frac{a^3}{x^2} = 0 \qquad (1)$$

$$x - \frac{a^3}{y^2} = 0 \qquad (2)$$

From (1), $y = \dfrac{a^3}{x^2}$. Put this value in (2), we get $x - \dfrac{a^3}{\left(\dfrac{a^6}{x^4}\right)} = 0$

$$(i.e)\ x - \frac{a^3 x^4}{a^6} = 0\ \ (i.e)\ x\left(1 - \frac{a^3 x^3}{a^6}\right) = 0\ \ (i.e)\ x\left(1 - \frac{x^3}{a^3}\right) = 0$$

(i. e) $x = 0, a$.

When $x = 0 \Rightarrow y = \infty$, when $x = a \Rightarrow y = a$.

Omit $(0, \infty)$; $\Rightarrow$ stationary point is (a, a)

$$r = 2\frac{a^3}{x^3};\ t = 2\frac{a^3}{y^3};\ s = 1 \text{ at } (a, a)$$

$$\Rightarrow r = \frac{2a^3}{a^3} = 2;\ t = 2;\ s = 1.\ \therefore rt - s^2 \textit{ gives } 4 - 1 = 3 > 0$$

As $rt - s^2 > 0$, $r > 0 \Rightarrow$ the point (a, a) is a minimum point.

$$f(a,a) = a^2 + \frac{a^3}{a} + \frac{a^3}{a}$$

(i. e) $3a^2$ is the minimum value.

Example 3

Examine the function $x^3 + y^3 - 12x - 3y + 20$ for extreme values.

Solution: Let $f(x,y) = x^3 + y^3 - 12x - 3y + 20$

$$f_x = 3x^2 - 12;\ f_y = 3y^2 - 3\quad f_{xx} = 6x;\ f_{yy} = 6y;\ f_{xy} = 0$$

The stationary points are given by $f_x = 0, f_y = 0$

$$\Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$$

Also $3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

The critical points are (2, 1), (2, -1), (-2, 1), (-2, -1).

Now at (2, 1); $r = f_{xx} = 12,\ t = f_{yy} = 6,\ s = 0$

$$\therefore \Delta = rt - s^2 \Rightarrow (12 \times 6) - 0^2 = 72 > 0$$

(i. e) (2,1) is a minimum point as $r > 0,\ \Delta > 0$

$\therefore f(2,1) = 8+1-24-3+20 = 2$

$\therefore$ Minimum value = 2

At $(2,-1)$: $r = 12,\ t = -6,\ s = 0$

$\Rightarrow \Delta < 0\ \Rightarrow$ (2,-1) is a saddle point.

Similarly (-2, 1) is also a saddle point.

At $(-2,-1)$: $r = -12,\ t = -6,\ s = 0$

$$\Rightarrow \Delta = rt - s^2 = 72 - 0$$

$\Rightarrow \Delta > 0,\ r < 0$ at (-2,-1)

$\therefore$ (-2,-1) is a maximum point.

$\therefore f(-2,-1) = -8-1+24+3+20 = 38$

$\therefore$ At (-2,-1); the maximum value = 38.

Example 4

Identify the saddle point and extreme points of $f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$.

Solution: Let $f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$.

$f_x = 4x^3 - 4x$, $f_y = -4y^3 + 4y$,

$f_{xx} = 12x^2 - 4$; $f_{xy} = 0$; $f_{yy} = -12y^2 + 4$

The stationary points are given by $f_x = 0, f_y = 0$

$\therefore 4x^3 - 4x = 0; \quad 4y - 4y^3 = 0$

$x^3 - x = 0; \quad y - y^3 = 0$

$x(x^2-1) = 0; \quad y(1-y^2) = 0$

$x = 0, \pm 1, \quad y = 0, \pm 1$

The stationary points are (0, 0), $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm 1)$.

At (0, 0): r = -4, t = 4, s = 0

$\therefore \Delta = rt - s^2 = -16 < 0 \Rightarrow$ *(0, 0)* is a saddle point.

At *(0, 1)*: r = -4, s = 0, t = -8

$\therefore \Delta = rt - s^2 > 0$

As $rt - s^2 > 0$, $r < 0$, *(0, 1)* is a maximum point.

$f(0,1) = -1+2 = 1$

Similarly (0,-1) is a maximum point.

At $(\pm 1, 0)$: r = 12-4 = 8, s = 0, t = 4

$\therefore \Delta = rt - s^2 = 32 > 0$ as $\Delta > 0, r > 0 \Rightarrow (\pm 1, 0)$ is a minimum point.

$f(\pm 1, 0) = 1-2 = -1$

The points $(\pm 1, \pm 1)$ are saddle points.

Example 5

Examine $f(x,y) = x^3 + y^3 - 3axy$ for maxima and minima.

Solution: Given $f(x,y) = x^3 + y^3 - 3axy$

Now, $f_x = 3x^2 - 3ay$; $\quad f_y = 3y^2 - 3ax$

$f_{xx} = 6x$; $\quad f_{yy} = 6y$

$f_{xy} = -3a$

The stationary points are obtained from $f_x = \frac{\partial f}{\partial x} = 0$ & $f_y = \frac{\partial f}{\partial y} = 0$

where $3x^2 - 3ay = 0$ and $3y^2 - 3ax = 0$.

(i. e) $x^2 - ay = 0$ and $y^2 - ax = 0$

Using these two equations, we get the stationary points as *(0, 0)* and *(a, a)*.

Now, at *(0, 0)*: $r = f_{xx} = 0$, $t = f_{yy} = 0$ and $s = f_{xy} = -3a$

$\therefore rt - s^2 = 0 - 9a^2 < 0$.

The point *(0, 0)* is neither a maximum nor a minimum point.

At *(a, a)*: $r = f_{xx} = 6a$, $t = f_{yy} = 6a$ and $s = f_{xy} = -3a$ such that

$rt - s^2 = 36a^2 - 9a^2 > 0$...

Also, $r = f_{xx}$ at $(a, a) = 6a$

$\Rightarrow$ is positive when a is positive and r is negative when a is negative.

(i. e) The point *(a, a)* is a minimum if $a > 0$ and *(a, a)* is a maximum if $a<0$.

Example 6

Find the maximum or minimum value of $\sin x + \sin y + \sin(x+y)$.

Solution: Given, $f(x,y) = \sin x + \sin y + \sin(x+y)$

$f_x = \cos x + \cos(x+y)$; $f_y = \cos y + \cos(x+y)$

$f_{xx} = -\sin x - \sin(x+y)$, $f_{yy} = -\sin y - \sin(x+y)$

$f_{xy} = -\sin(x+y)$

The stationary points are obtained by equating f_x to 0 and f_y to 0.

(i.e) $f_x = 0 \Rightarrow \cos x + \cos(x+y) = 0$ (1)

$f_y = 0 \Rightarrow \cos y + \cos(x+y) = 0$ (2)

From (1), $\cos x = -\cos(x+y) = \cos(\pi - (x+y)) \Rightarrow x = \pi - (x+y)$

(i.e) $2x + y = \pi$ (3)

Solving (3) and (4), we get, $x = \frac{\pi}{3}$, $y = \frac{\pi}{3}$.

Then the stationary point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$: $r = f_{xx} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$, $s = f_{xy} = -\frac{\sqrt{3}}{2}$,

$t = f_{yy} = -\sqrt{3}$ $\therefore rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$

Also, $r < 0$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$\therefore$ The point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a maximum point.

Hence the maximum value of the given function is

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3}$$

$$= \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\left(\pi - \frac{\pi}{3}\right)$$

$$= \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{\pi}{3} = 3\sin\frac{\pi}{3} = 3\frac{\sqrt{3}}{2}.$$

Example 7

Determine the maxima or minima of the function $x^3y^2(12 - x - y)$ where $x > 0$, $y > 0$. Also find the extreme value.

Solution: Given $f(x,y) = x^3y^2(12 - x - y) = 12x^3y^2 - x^4y^2 - x^3y^3$

Now, $f_x = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$ *and* $f_y = 24x^3y - 2x^4y - 3x^3y^2$

$f_x = 0$ & $f_y = 0 \Rightarrow 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow x^2y^2(36 - 4x - 3y) = 0 \Rightarrow 36 - 4x - 3y = 0$ (since $x > 0$, $y > 0$)

(i. e) $4x + 3y = 36$ (1)

and $24x^3y - 2x^4y - 3x^3y^2 = 0$

$x^3y(24 - 2x - 3y) = 0$

$\Rightarrow 24 - 2x - 3y = 0$ (since $x > 0$, $y > 0$)

(i. e) $2x + 3y = 24$ (2)

Solving (1) and (2), we get, $y = 4$ and $x = 6$.

Now, $f_{xx} = 72xy^2 - 12x^2y^2 - 6xy^3$, $f_{xy} = 72x^2y - 8x^3y - 9x^2y^2$

and $f_{yy} = 24x^3 - 2x^4 - 6x^3y$

Then at $(6, 4)$, $r = f_{xx} = -2204$, $s = -1728$, $t = -2592$

$\therefore rt - s^2 > 0$. Also $r < 0$.

Then point (6, 4) is a maximum point.

The maximum value of the given function is 6912.

◊ Lagrange's Method of Multipliers ◊

This method is to find the maximum or minimum value of a function of three or more variables, given the constraints. Let $f(x, y, z)$ be a function of three variables which is to be tested for maximum or minimum value.

Let the variables x, y, z be connected by a relation

$$\varphi(x, y, z) = 0 \quad (1)$$

The conditions for $f(x, y, z)$ to have a maximum point, or a minimum point are

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0, \ \frac{\partial f}{\partial z} = 0$$

By total differentials, we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \quad (2)$$

Similarly from (1), we have that

$$\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = 0 \quad (3)$$

Equation (2) + λ.equation (3), ultimately gives the following:

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = 0, \ \frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y} = 0 \ \ and \ \ \frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z} = 0$$

(Here 'λ' is called the Lagrange's multiplier).

Solving the above equations along with the given relation, we will get the values of x, y, z and the Lagrange's multiplier λ.

These values give finally the required maximum or minimum value of the function $f(x, y, z)$.

Example 1

A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions if the total surface area is minimum.

Solution: Given volume, $\varphi(x, y, z) = xyz - 32 = 0$ (1)

The required function is the total surface area

$S = f(x, y, z) = xy + 2xz + 2yz$ (2)

At the critical points, we have

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = 0 \Rightarrow y + 2z + \lambda yz = 0 \quad (3)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y} = 0 \Rightarrow x + 2z + \lambda xy = 0 \quad (4)$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z} = 0 \Rightarrow 2x + 2y + \lambda xy = 0 \quad (5)$$

(3) $\times x$ – (4) $\times y$ gives, $2(zx - zy) = 0, \ z \neq 0 \Rightarrow x - y = 0$

$x = y$ (6)

(4) $\times$ y – (5)) $\times$ z gives, $xy - 2xz = 0$

$y^2 - 2yz = 0$ (using (6)) $\Rightarrow y(y-2z) = 0 \Rightarrow y - 2z = 0 \ (y \neq 0)$

$z = y/2$ (7)

Using (6) and (7) in (1), we get,

$x.x.x/2 = 32 \Rightarrow x^3 = 64 \Rightarrow x = 4, \therefore y = 4$ and $z = 2$.

The dimensions are 4 cm, 4 cm and 2 cm.

Example 2

Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution: The given ellipsoid is $\varphi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ (1)

The required function is the volume of the parallelopiped, given by

$V = 8xyz = f(x, y, z)$, (2)

where the dimensions are $2x$, $2y$ and $2z$.

At the critical points we have

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = 0 \Rightarrow 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \quad (3)$$

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \Rightarrow 8xz + \lambda\left(\frac{2y}{b^2}\right) = 0$ (4)

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0 \Rightarrow 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0$ (5)

Equation (3) $\times x$ + (4)) $\times y$ + (5)) $\times z$ gives

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

(i. e) $2\lambda = -24xyz$ (using (1))

$\therefore \lambda = -12xyz$ (6)

Using (6) in (5), we get

$$8xy + (-12xyz)\left(\frac{2z}{c^2}\right) = 0 \Rightarrow 8xy\left(1 - \frac{3z^2}{c^2}\right) = 0$$

$$\Rightarrow \frac{3z^2}{c^2} = 1 \text{ (or) } z = \frac{c}{\sqrt{3}} \quad (\because x \neq 0, y \neq 0)$$

Similarly by using (6) in (4) and (3), we get

$$y = \frac{b}{\sqrt{3}} \quad and \quad x = \frac{a}{\sqrt{3}}.$$

The maximum volume of rectangular parallelepiped is

$$V = 8xyz = \frac{abc}{3\sqrt{3}} \text{ cubic units.}$$

Example 3 ....

Find the dimensions of the rectangular box, open at the top of maximum capacity whose surface is 432 sq.cm.

Solution: Let x, y, z be the dimensions of the rectangular box, open at the top. Given its surface area

$\varphi(x, y, z) = xy + 2yz + 2zx - 432 = 0$ (1)

The required function is its volume $V = xyz = f(x, y, z)$ (2)

At the critical point we get

$yz + \lambda(y + 2z) = 0$ (3)

$xz + \lambda(x + 2z) = 0$ (4)

$xy + \lambda(2y + 2x) = 0$ (5)

Equation (3) $\times x$ - (4) $\times y$ gives, $2\lambda z(x - y) = 0 \Rightarrow x = y \ (z \neq 0, \lambda \neq 0)$ (6)

Equation (3)$\times x$ - (5)$\times z$ gives,

$\lambda y(x - 2z) = 0 \Rightarrow z = \frac{x}{2} \ (y \neq 0, \lambda \neq 0)$ (7)

Using (6) and (7) in (1), we get,

$$x^2 + x^2 + x^2 = 432 \Rightarrow 3x^2 = 432 \Rightarrow x^2 = 144 \therefore x = 12$$

Hence $y = 12$ and $z = 6$.

Thus, the dimensions of the rectangular box open at the top of maximum capacity are 12 cm, 12 cm and 6 cm.

Example 4 ....

Find the maximum and minimum distance of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

Solution: Given $\varphi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ (1)

The required equation is $f(x, y, z)$ = square of the distance from the point (3, 4, 12) to the sphere $= (x-3)^2 + (y-4)^2 + (z-12)^2$ (2)

At the critical points we have,

$2(x-3) + 2\lambda x = 0$ (3)

$2(y-4) + 2\lambda y = 0$ (4)

$2(z-12) + 2\lambda z = 0$ (5)

From (3), (4) and (5), we get,

$x = \frac{3}{1+\lambda}, y = \frac{4}{1+\lambda}$ and $z = \frac{12}{1+\lambda}$ (6)

Using these values in (1), we get,

$$\frac{9}{(1+\lambda)^2}+\frac{16}{(1+\lambda)^2}+\frac{144}{(1+\lambda)^2}=1$$

$$(i.e)\ (1+\lambda)^2=169 \Rightarrow 1+\lambda=\pm 13 \qquad (7)$$

Using (7) in (6) we get the points as:

$$\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right) \text{ and } \left(\frac{-3}{13},\frac{-4}{13},\frac{-12}{13}\right)$$

$\therefore$ The distances are:

$$\sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2}=12 \text{ and}$$

$$\sqrt{\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2}=14$$

Thus, the maximum distance is 14 and the minimum distance is 12.

Example 5 ....

If $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$, find the values of x, y, z which make $x+y+z$ is minimum.

Solution: Given $\varphi(x,y,z)\equiv\frac{3}{x}+\frac{4}{y}+\frac{5}{z}-6=0 \qquad (1)$

The required function is $f(x,y,z)=x+y+z \qquad (2)$

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial\varphi}{\partial x}=0\Rightarrow\left(1-\frac{3\lambda}{x^2}\right)=0 \qquad (3)$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial\varphi}{\partial y}=0\Rightarrow\left(1-\frac{4\lambda}{y^2}\right)=0 \qquad (4)$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial\varphi}{\partial z}=0\Rightarrow\left(1-\frac{5\lambda}{z^2}\right)=0 \qquad (5)$$

From (3), (4) and (5), we get

$$3y^2=4x^2 \qquad (6)$$

$$4z^2=5y^2 \qquad (7)$$

$$(i.e)\ y=\pm\frac{2}{\sqrt{3}}x \qquad (8)$$

$$\text{and } y=\pm\frac{2}{\sqrt{5}}z \qquad (9)$$

Using (8) and (9) in (1), we get,

$$\frac{3}{x}+\frac{2\sqrt{3}}{x}+\frac{5}{\sqrt{5}\left(\frac{x}{\sqrt{3}}\right)}=6 \Rightarrow \frac{3}{x}+\frac{2\sqrt{3}}{x}+\frac{\sqrt{15}}{x}=6$$

$$\sqrt{3}(\sqrt{3}+\sqrt{5}+2)=6x \Rightarrow 3+2\sqrt{3}+\sqrt{15}=6x$$

$$\Rightarrow x=\frac{\sqrt{3}}{6}(\sqrt{3}+\sqrt{5}+2),\quad y=\frac{1}{3}(\sqrt{3}+\sqrt{5}+2)$$

$$\text{and } z=\frac{\sqrt{5}}{6}(\sqrt{3}+\sqrt{5}+2).$$

■ EXERCISE ■

1. If $u=\sin\left(\frac{x}{y}\right)$, $x=e^t, y=t^2$, find $\frac{du}{dt}$.

   $\left[Ans: e^t\cos\left(\frac{e^t}{t^2}\right)\left(\frac{t-2}{t^3}\right)\right]$

2. If $x=u^2-v^2$, $y=2uv$, find $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

   $\left[Ans: \frac{\partial u}{\partial x}=\frac{u}{2(u^2+v^2)}, \frac{\partial u}{\partial y}=\frac{v}{2(u^2+v^2)}, \frac{\partial v}{\partial x}=\frac{-v}{2(u^2+v^2)}, \frac{\partial v}{\partial y}=\frac{u}{2(u^2+\right.$

11. Find the minimum value of $x^2 + y^2 + z^2$ respect to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. *[Ans: The function is maximum at (3, 3, 3) and the maximum value is 27]*

12. The temperature at any point *(x, y, z)* in space is $T = 400xyz^2$. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = 1$. *[Ans: 50]*

13. Find the maximum value of $x^m y^n z^p$ given that $x + y + z = a$.

[Ans: The maximum value is $\left[a^{m+n+p} \cdot \frac{m^m n^n p^p}{(m+n+p)^{m+n+p}} \right]$

14. If $\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6$, find the values of *x, y, z* which makes $x + y + z$ is minimum.

$$\left[Ans: x = \frac{\sqrt{3}}{6}(\sqrt{3} + 2 + \sqrt{5}), y = \frac{2}{6}(\sqrt{3} + 2 + \sqrt{5}) \,\&\, z = \frac{\sqrt{5}}{6}(\sqrt{3} + 2 + \sqrt{5}) \right]$$

16. Find the dimensions of the rectangular box without a top of maximum capacity, whose surface is 108 sq.cm. *[Ans: 6 cm, 6 cm and 3 cm]*.

JACOBIANS :

Definition : If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u, v with respect to x, y and is denoted by the symbol $J\left(\frac{u, v}{x, y}\right)$ *or* $\frac{\partial(u, v)}{\partial(x, y)}$

Similarly if u, v, w be functions of x, y, z then the Jacobian of u, v, w with respect to x, y, z is $J\left(\frac{u,v,w}{x,y,z}\right)$ or $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

◊ Properties of Jacobians ◊

Property 1 If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v then $J_1 J_2 = 1$.

(i. e) $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$.

Proof. Let $u = u(x, y)$ and $v = v(x, y)$, so that u and v are functions of x, y.

Differentiating partially w. r. to u and v, we get

$$1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} = u_x x_u + u_y y_u, \qquad 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = u_x x_v + u_y y_v$$

$$0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} = v_x x_u + v_y y_u, \qquad 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = v_x x_v + v_y y_v$$

Now $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ (Interchanging rows and columns in the second determinant)

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$$

$$= \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \text{ using } (1).$$

Property 2 If u, v are functions of r, s where r, s are functions of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$.

Proof. Since u and v are composite functions of x and y.

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = u_r r_x + u_s s_x \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = u_r r_y + u_s s_y \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} = v_r r_x + v_s s_y \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} = v_r r_y + v_s s_y \end{aligned}\right\} \quad (1)$$

Now $\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$ (Interchanging rows and columns in the second determinant.)

$$= \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix} = \begin{vmatrix} u_r r_x + u_s s_y & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} \text{ using (1)}$$

Note: If the Jacobian value is zero then u and v are functionally related.

Example 1 ....

If $x = u^2 - v^2$ and $y = 2uv$ find the Jacobian of x and y with respect to u and v.

Solution: Let $J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = 4(u^2 + v^2)$

Example 2

If $x = rcos?, y = rsin?$, verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Solution: Let $x = rcos?, y = rsin?$. Then

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

Now $r^2 = x^2 + y^2$ *and* $\theta = \tan^{-1}\frac{y}{x}$. Then

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$. Also $\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$.

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2+y^2}{r^3} = \frac{1}{r}.$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r.\frac{1}{r} = 1.$$

Example 3

If $u = yz/x,\ v = zx/y,\ w = xy/z$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

Solution: Let

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1)$$

$= 4$ (Expanding through 1^{st} row.)

(Taking yz, xz, xy as common factors from 1^{st}, 2^{nd} and 3^{rd} columns respectively)

Example 4

If $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$$

Solution: Let

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Taking out common factors (r from second column and rsin? from third column)

$$= r^2 \sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

(Expanding by third row)

$$= r^2 \sin\theta \left\{ \cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix} \right\}$$

$$= r^2 \sin\theta[\cos\theta(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi)$$

$$+\sin\theta(\sin\theta\cos^2\phi + sin\theta\sin^2\phi)]$$

$$= r^2 \sin\theta(\cos^2\theta + \sin^2\theta)$$

$$= r^2 \sin\theta$$

Example 5

If $x = r\cos\theta,\ y = r\sin\theta,\ z = z,$ evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Solution: Let $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

Example 6

If $u = xyz,\ v = xy + yz + zx$ and $w = x + y + z,$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Solutions: Let $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

Using $C_1 \to C_2 - C_1, C_2 \to C_3 - C_2$

We get $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} z(x-y) & x(y-z) & xy \\ x-y & y-z & y+x \\ 0 & 0 & 1 \end{vmatrix}$

Taking $(x-y)$, $(y-z)$ as common factors from column 1 and column 2, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (x-y)(y-z)\begin{vmatrix} z & x & xy \\ 1 & 1 & y+x \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (x-y)(y-z)(z-x).$$

Example 7

Are the functions $u = \frac{x+y}{1-xy}$ *and* $v = \tan^{-1}x + \tan^{-1}y$ functionally dependent? If so, find the relation between them.

Solution: Let $u = \frac{x+y}{1-xy}$. Differentiate w. r. to x partially we get,

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

Similarly $\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$ (since u is symmetric)

$v = \tan^{-1} x + \tan^{-1} y$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}; \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

$\Rightarrow$ u and v are functionally dependent.

Now $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left\{\frac{x+y}{1-xy}\right\}$

$\Rightarrow v = \tan^{-1} u \quad \Rightarrow u = \tan v$.

Example 8 ....

If $u = y+z; v = x+2z^2; w = x-4yz-2y^2$, find the Jacobian of u, v, w with respect to x, y, z. Comment on the result.

Solution: Let $u = y+z;\ v = x+2z^2;\ w = x-4yz-2y^2$. Differentiate them partially w. r. to x, y, z, we get

$$\frac{\partial u}{\partial x} = 0; \frac{\partial u}{\partial y} = 1; \frac{\partial u}{\partial z} = 1, \quad \frac{\partial v}{\partial x} = 1; \frac{\partial v}{\partial y} = 0; \frac{\partial v}{\partial z} = 4z$$

$$\frac{\partial w}{\partial x} = 1; \frac{\partial w}{\partial y} = -4z-4y; \frac{\partial w}{\partial z} = -4y.$$

Now $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4y-4z & -4y \end{vmatrix}$$

$$= -1(-4y-4z)+(-4y-4z) = 0$$

$\therefore$ u, v and w are functionally dependent.

Now $v - w = 2z^2 + 4yz + 2y^2 = 2(y+z)^2 = 2u^2$

Hence the functional relationship is $2u^2 = v - w$.

Example 9 ....

If $u = x+y+z,\ uv = y+z,\ uvw = z,$ show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.

Solution: Given $z = uvw \Rightarrow \frac{\partial z}{\partial u} = vw, \quad \frac{\partial z}{\partial v} = uw, \quad \frac{\partial z}{\partial w} = uv$

and $y + z = uv$.

$\Rightarrow y + uvw = uv \Rightarrow y = uv - uvw$

$$\Rightarrow \frac{\partial y}{\partial u} = v - vw, \quad \frac{\partial y}{\partial v} = u - uw, \quad \frac{\partial y}{\partial w} = -uv.$$

Also $u = x+y+z \Rightarrow x = u - y - z = u - uv + uvw - uvw$

$$\Rightarrow x = u - uv \Rightarrow \frac{\partial x}{\partial u} = 1 - v; \frac{\partial x}{\partial v} = -u; \frac{\partial x}{\partial w} = 0.$$

Now $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix} = \begin{vmatrix} 1-vw & -uw & -uv \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}.$$

(Using $R_1 \to R_1 + R_2,\ R_2 \to R_2 + R_3$)

From $R_1 \to R_1 + R_3$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = u^2 v \text{ (By expanding through 1}^{\text{st}} \text{ row)}$$

Example 10

If $u=x^2-2y,\ v=x+y+z,\ w=x-2y+3z$, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Solution: Let $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}2x & -2 & 0\\ 1 & 1 & 1\\ 1 & -2 & 3\end{vmatrix}$

$= 2x(3+2)+2(3-1)$ (expanding through 1st row)

$= 10x+4.$

EXERCISE

1. If $u=x^2-2y,\ v=x+y$, prove that $\dfrac{\partial(u,v)}{\partial(x,y)}=2x+2$.
2. If $u=x(1-y),\ v=xy$, prove that $JJ'=1$.
3. If $u=\dfrac{x}{y-z},\ v=\dfrac{y}{z-x},\ w=\dfrac{z}{x-y}$, show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=0$.
4. Determine whether $u=\sin^{-1}x+\sin^{-1}y$ *and* $v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$ are functional dependent. [Ans: $v=\sin u$]
5. If $u=xy+yz+zx,\ v=x^2+y^2+z^2$ *and* $w=x+y+z$. Determine whether there is a functional relationship between u,v,w and if so, find it. [Ans: $w^2-v-2u=0$]
6. For the transformation $x=e^u\cos v,\ y=e^u\sin v$, prove that $\dfrac{\partial(x,y)}{\partial(u,v)}\dfrac{\partial(u,v)}{\partial(x,y)}=1$.
7. Find the value of $\dfrac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$, if $y_1=(1-x_1),\ y_2=x_1(1-x_2),\ y_3=x_1x_2(1-x_3)$. *[Ans: $-x_1^2x_2$]*

8. If $u=x+y+z,\ u^2v=y+z,\ u^3w=z$, show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=u^{-5}$.
9. Test whether $u=\dfrac{x+y}{x-y},\ v=\dfrac{xy}{(x-y)^2}$ are functionally dependent. If so, state the relation between them. *[Ans: $u^2-4v=1$]*
10. Find the value of the Jacobian $\dfrac{\partial(u,v)}{\partial(r,\theta)}$, where $u=x^2-y^2,\ v=2xy$ and $r=$ *a cos?*, $y=r$ *sin?*. *[Ans: $4r^3$]*.