Cauchy's Integral formula for Derivatives.

For $$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a}\, dz.$$

$$f'(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^2}\, dz$$

$$f''(a) = \frac{2!}{2\pi i} \int_c \frac{f(z)}{(z-a)^3}\, dz$$

$$f'''(a) = \frac{3!}{2\pi i} \int_c \frac{f(z)}{(z-a)^4}\, dz.$$

$\vdots$

$$f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

Note :

* $\int_c \frac{f(z)}{(z-a)}\, dz = 2\pi i\, f(a).$ if 'a' lies inside c

  $= 0$ if 'a' lies outside c.

* $\int_c \frac{f(z)}{(z-a)^2}\, dz = 2\pi i\, f'(a)$

* $\int_c \frac{f(z)}{(z-a)^3}\, dz = \frac{2\pi i}{2!} f''(a)$

* $f(z)$ is analytic inside and on c and

  $f(z)$ is not analytic outside c.

Problems :

Evaluate $\int_c \frac{1}{2z-3}\, dz$ if c is the circle $|z| = 1$.

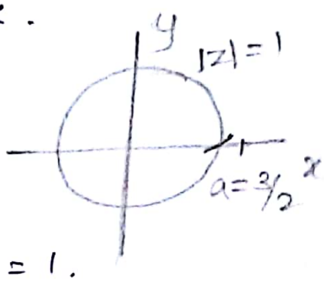
Soln cauchy's integral formula is

$$\int_c \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a), \text{ if a lies inside c}$$

$$= 0, \text{ if a lies outside c.}$$

Given, $\int_c \frac{1}{2z-3} dz = \int_c \frac{1}{2(z-3/2)} dz.$

$= \frac{1}{2}\int_c \frac{1}{(z-3/2)} dz.$



$\therefore f(z) = 1.$ and $a = 3/2 = 1.5$

$\Rightarrow$ a lies outside the circle $|z| = 1.$

By cauchy's integral formula,

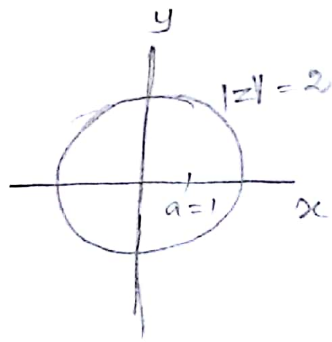
$$\int_c \frac{dz}{2z-3} = 0.$$

2) Evaluate $\int_c \frac{z}{(z-1)^3} dz$ where c is $|z| = 2$, using cauchy's integral formula.

soln :

W.K.T cauchy's integral formula is

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{n}(a)$$



Given, $\int_c \frac{z}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$

For $f(z) = z$, $a = 1 \; \because$ (lies inside $|z| = 2$)

$f(a) = a = 1$

$f'(a) = 0, \quad f''(a) = 0.$

$\therefore \int_c \frac{z}{(z-1)^3} dz = \frac{\cancel{2}\pi i}{\cancel{2}} (0). = \pi i(0)$

$\Rightarrow \int_c \frac{z}{(z-1)^3} dz = 0$

3) Evaluate $\int_c \frac{ze^z}{(z-a)^3} dz$, where $z=a$ lies inside the closed curve c using cauchy's integral formula.

Soln:

Given, $\int_c \frac{ze^z}{(z-a)^3} dz$.

$f(z) = ze^z$, The point $z=a$ lies inside c.

(ie) $\int_c \frac{f(z)\, dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{n}(a)$.

$\Rightarrow \int_c \frac{ze^z\, dz}{(z-a)^3} = \frac{2\pi i}{2!} f''(a)$.

where, $f(z) = ze^z$.

$f'(z) = ze^z + e^z$.

$f''(z) = ze^z + e^z + e^z = ze^z + 2e^z = e^z(z+2)$

$z=a \Rightarrow f''(a) = e^a(a+2)$.

$\Rightarrow \int_c \frac{ze^z\, dz}{(z-a)^3} = \frac{\cancel{2}\pi i}{\cancel{2}} e^a(a+2)$

$= \pi i\, e^a(a+2)$ //

---

4) Evaluate $\int_c \frac{z}{(z-1)^3} dz$, where c is $|z|=2$, using cauchy's integral formula.

Repeated. (2)

Soln:

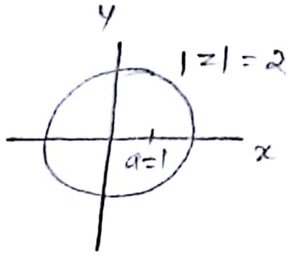
Given $\int_c \frac{z}{(z-1)^3} dz$,

Here $f(z) = z$, $a=1$ lies inside the circle $|z|=2$

Cauchy's integral formula,

$$\int_c \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)$$

$$\int_c \frac{z\,dz}{(z-a)^3} = \frac{2\pi i}{2!} f''(a)$$



$f(z) = z.$

$f'(z) = 1.$

$f''(z) = 0. \Rightarrow f''(a) = f''(1) = 0.$

$$\int_c \frac{z\,dz}{(z-1)^3} = \frac{2\pi i}{\cancel{2}} f''(1) = 0 //$$

---

5) Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where c is $|z| = 3$

Soln:

Given $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$.

Here, $f(z) = \sin \pi z^2 + \cos \pi z^2$

$a_1 = 1$ lies inside circle $|z| = 3$.

$a_2 = 2$ lies inside circle $|z| = 3$.

consider, $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$

$$= \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$\Rightarrow 1 = A(z-2) + B(z-1).$

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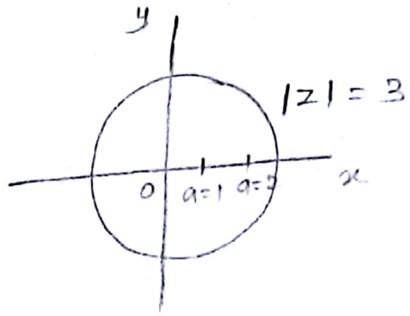
when $z = 2$,

$$A(0) + B(1) = 1.$$

$$\boxed{B = 1}$$



when $z = 1$,

$$A(1-2) + B(0) = 1.$$

$$-A = 1. \implies \boxed{A = -1}$$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}.$$

$$\therefore \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz + \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

By cauchy's integral formula,

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -(2\pi i\, f(1)) + (2\pi i\, f(2)).$$

$$f(z) = \sin \pi z^2 + \cos \pi z^2.$$

$$f(1) = \sin \pi + \cos \pi = 0 - 1 = -1.$$

$$f(2) = \sin 4\pi + \cos 4\pi = 0 + 1 = 1.$$

$$\therefore \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i(-1) + 2\pi i(1)$$

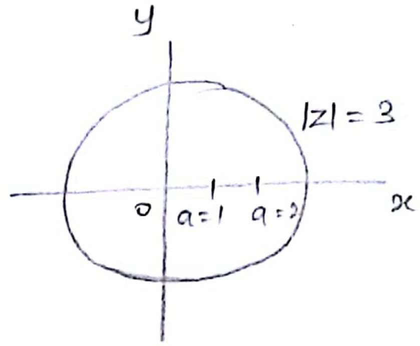
$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

6) using cauchy's integral formula, evaluate $\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ where c is $|z| = 3$.

Soln:

Given $\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz$.



$f(z) = \cos \pi z^2$.

$a_1 = 1$, lies inside $|z| = 3$

$a_2 = 2$, lies inside $|z| = 3$.

W.K.T $\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$ (previous problem)

$$\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -\int_c \frac{\cos \pi z^2}{z-1} dz + \int_c \frac{\cos \pi z^2}{z-2} dz$$

By cauchy's integral formula,

$$\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i f(1) + 2\pi i f(2).$$

$f(z) = \cos \pi z^2$.

$f(1) = \cos \pi = -1$

$f(2) = \cos 2\pi = 1$

$$\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i (-1) + 2\pi i (1)$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

$$\therefore \int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 4\pi i.$$

7) Evaluate $\int_c \frac{z-2}{z(z-1)} dz$, where c is the circle $|z| = 3$.

Soln:

Given, $\int_c \frac{z-2}{z(z-1)} dz$.

Here, $f(z) = z-2$

$a_1 = 0$ lies inside $c : |z| = 3$

$a_2 = 1$ lies inside $c : |z| = 3$

consider, $\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$

$$= \frac{A(z-1) + Bz}{z(z-1)}.$$

$\Rightarrow \quad 1 = A(z-1) + Bz$

when, $z = 1$.

$A(1-1) + B(1) = 1$

$\boxed{B = 1}$

when $z = 0$,

$A(0-1) + B(0) = 1$.

$-A = 1$

$\boxed{A = -1}$

$$\therefore \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}.$$

$$\int_c \frac{z-2}{z(z-1)} dz = -\int_c \frac{z-2}{z} dz + \int_c \frac{z-2}{z-1} dz$$

By cauchy's integral formula,

$$\int_c \frac{z-2}{z(z-1)} dz = -2\pi i f(0) + 2\pi i f(1)$$

Here, $f(z) = z-2$.

$f(0) = -2$.

$f(1) = -1$.

$$\int_c \frac{z-2}{z(z-1)} dz = -2\pi i f(0) + 2\pi i f(1)$$

$$= -2\pi i(-2) + 2\pi i(-1)$$

$$= 4\pi i - 2\pi i$$

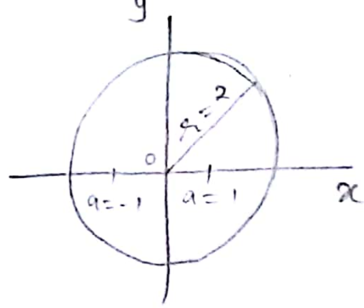
$$= 2\pi i \,//.$$

---

8) using cauchy's integral formula, evaluate $\int_c \frac{1}{z^2-1} dz$, where c is the circle with centre at $z=0$ and radius 2.

Soln:

$$\int_c \frac{1}{z^2-1} dz = \int_c \frac{1}{(z-1)(z+1)} dz$$



Here $f(z) = 1$

$a_1 = 1$ lies inside c,

$a_2 = -1$ lies inside c.

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$1 = A(z+1) + B(z-1)$$

Put $z = 1$

$1 = 2A$, $\boxed{A = 1/2}$

Put $z = -1$

$1 = -2B$, $\boxed{B = -1/2}$

$$\frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$$

$$\int_c \frac{1}{z^2-1}\,dz = \frac{1}{2}\int_c \frac{1}{z-1}\,dz - \frac{1}{2}\int_c \frac{1}{z+1}\,dz.$$

Here, $f(z) = 1$

$f(1) = 1$

$f(-1) = 1$

$$\int_c \frac{1}{z^2-1}\,dz = \frac{1}{\cancel{2}}\left(\cancel{2}\pi i\, f(1)\right) - \frac{1}{\cancel{2}}\left(\cancel{2}\pi i\, f(-1)\right)$$

$$= \pi i\,(1) - \pi i\,(1)$$

$$= 0.$$

$$\therefore \int_c \frac{1}{z^2-1}\,dz = 0.$$

---

9) using cauchy's integral formula, evaluate $\int_c \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$ where c is $|z| = 3/2$

Soln:

$$\int_c \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz.$$

Here, $f(z) = \cos \pi z^2$

$a_1 = 1$ lies inside $|z| = 3/2$.

$a_2 = 2$ lies outside $|z| = 3/2$.

$$\therefore \int_c \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz = \int_c \frac{\left(\frac{\cos \pi z^2}{z-2}\right)}{z-1}\,dz.$$

By cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\Rightarrow \int_c \frac{\left(\frac{\cos \pi z^2}{z-2}\right)}{z-1}\,dz = 2\pi i\, f(1)$$

Here, $f(z) = \dfrac{\cos \pi z^2}{z-2}$

$$f(1) = \frac{\cos \pi}{1-2} = \frac{-1}{-1} = 1.$$

(ie) $f(1) = 1.$

$$\therefore \int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i \, f(1)$$

$$= 2\pi i \, (1)$$

$$= 2\pi i$$

---

10) Evaluate $\displaystyle\int_c \frac{dz}{z^2 - 7z + 12}$, where c is the circle $|z| = 3.5$

soln:

Let $z^2 - 7z + 12 = 0$,

$(z-3)(z-4) = 0$.

$$\Rightarrow \int_c \frac{dz}{(z-3)(z-4)} \Leftarrow \int_c \frac{dz}{z^2 - 7z + 12}.$$

$a_1 = 3$ lies inside the circle $|z| = 3.5$

$a_2 = 4$ lies outside the circle $|z| = 3.5$

$$\int_c \frac{dz}{(z-3)(z-4)} = \int_c \frac{\left(\frac{dz}{z-4}\right)}{z-3}$$

By cauchy's integral formula,

$$\int_c \frac{f(z)\,dz}{z-a} = 2\pi i \, f(a)$$

$$\Rightarrow \int_c \frac{\left(\frac{dz}{z-4}\right)}{z-3} = 2\pi i \, f(3).$$

Here, $f(z) = \frac{1}{z-4}$

$$f(3) = \frac{1}{3-4} = -1.$$

$$\therefore \int_c \frac{dz}{z^2-7z+12} = 2\pi i(-1) = -2\pi i \,//.$$

---

11) using cauchy's integral formula and use it to evaluate $\int_c \frac{dz}{(z+1)^2(z-2)}$, where c is the circle $|z| = 3/2$.

Soln:

Given $\int_c \frac{dz}{(z+1)^2(z-2)}$

$a_1 = -1$ lies inside $|z| = 3/2$

$a_2 = 2$ lies outside $|z| = 3/2$

$$\int_c \frac{dz}{(z+1)^2(z-2)} = \int_c \frac{\left(\frac{dz}{z-2}\right)}{(z+1)^2}$$

By cauchy's integral formula,

$$\int_c \frac{dz}{(z+1)^2(z-2)} = 2\pi i\, f'(-1)$$

$$f(z) = \frac{1}{z-2}.$$

$$f'(z) = \frac{(z-2)(0) - 1(1)}{(z-2)^2} = \frac{-1}{(z-2)^2}.$$

$$f'(-1) = \frac{-1}{(-1-2)^2} = \frac{-1}{9}$$

$$\therefore \int_c \frac{dz}{(z+1)^2(z-2)} = \frac{-1}{9}(2\pi i).$$

---

12) Evaluate $\int_c \frac{z^2+1}{z^2-1}\,dz$, where c is $|z-1|=1$, using cauchy's integral formula.

Soln:

Given $|z-1|=1$ is a circle whose centre is 1 and radius 1.

$$\int_c \frac{z^2+1}{z^2-1}\,dz = \int_c \frac{z^2+1}{(z+1)(z-1)}\,dz.$$

$|z-1|=1$

$|x+iy-1|=1.$

$a_1 = 1$ lies inside $|z-1|=1$.

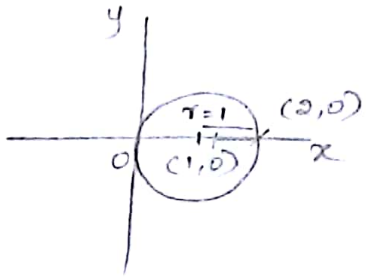
$a_2 = -1$ lies outside $|z-1|=1$.

$|(x-1)+iy|=1.$

$(x-1)^2+y^2=1$

$c(1,0),\ r=1.$

$$\therefore \int_c \frac{z^2+1}{z^2-1}\,dz = \int_c \frac{\left(\frac{z^2+1}{z+1}\right)}{(z-1)}\,dz.$$



By cauchy's integral formula is

$$\int_c \frac{z^2+1}{z^2-1}\,dz = 2\pi i\, f(1).$$

$$f(z) = \frac{z^2+1}{z+1}.$$

$$f(1) = \frac{1+1}{1+1} = \frac{2}{2} = 1.$$

$$\therefore \int_c \frac{z^2+1}{z^2-1}\,dz = 1.$$

---

13) using cauchy's integral formula evaluate $\int_c \frac{z+4}{z^2+2z+5}\,dz$, where c is the circle $|z+1-i|=2$

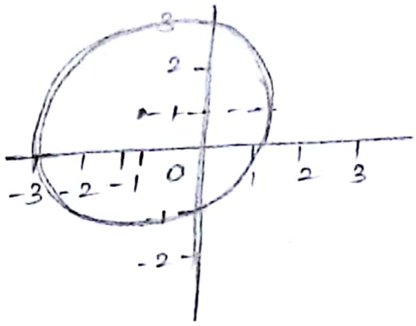
Soln:

Given, $|z+1-i|=2$

$|x+iy+1-i|=2. \Rightarrow |(x+1)+i(y-1)|=2$

$$(x+1)^2 + (y-1)^2 = 2^2.$$

is a circle whose centre is $(-1,1)$ and radius is 2.



[For $z^2 + 2z + 5 = 0$.

$$z = \frac{-2 \pm \sqrt{4-20}}{2}.$$

$$= \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

$z = -1 + 2i$, $\quad z = -1 - 2i$]

$$\int_c \frac{z+4}{z^2+2z+5}\, dz = \int_c \frac{z+4}{(z-(-1+2i))(z-(-1-2i))}\, dz.$$

$a_1 = -1+2i$ (ie) $(-1, 2)$ lies inside c.

$a_2 = -1-2i$ (ie) $(-1,-2)$ lies outside c.

$$= \int_c \frac{\left(\frac{z+4}{z-(-1-2i)}\right)}{(z-(-1+2i))}\, dz.$$

By cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

$$\int_c \frac{\left(\frac{z+4}{z-(-1-2i)}\right)}{(z-(-1+2i))}\, dz = 2\pi i\, f(-1+2i)$$

Here $f(z) = \dfrac{z+4}{z-(-1-2i)}$

$$f(-1+2i) = \frac{(-1+2i)+4}{-\cancel{1}+2i+\cancel{1}+2i} = \frac{3+2i}{4i}$$

$$\int_c \frac{z+4}{z^2+2z+5}\, dz = \cancel{2}\pi \cancel{i}\left(\frac{3+2i}{\cancel{4}_{2}\cancel{i}}\right) = \frac{\pi}{2}(3+2i) \text{ //}$$

Singularies - classification :

(i) **Zeros of an analytic function.**

If a function $f(z)$ is analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called a zero of $f(z)$.

Simple zero.

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $z = z_0$ is called a simple zero of $f(z)$ (or) a zero of the first order.

Zeros of order n.

If $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$ then z_0 is called zero of order n.

Singular points.

A point $z = z_0$ at which a function $f(z)$ fails to be analytic. is called a singular point (or) singularity of $f(z)$.

eg: Consider $f(z) = \frac{1}{z-3}$.

Here $z = 3$ is a singular point of $f(z)$

Types of singularity

a) **Isolated singularity**

A point $z = z_0$ is said to be isolated singularity of $f(z)$ if $f(z)$ is not analytic at $z = z_0$, there exists a neighbourhood of $z = z_0$ containing no other singularity.

eg: $f(z) = \frac{1}{z}$ is analytic everywhere except at $z=0$.

$z=0$ is an isolated singularity.

b) Removable singularity.

A singular point $z=z_0$ is called a removable singularity of $f(z)$, $\lim_{z\to z_0} f(z)$ exists finitely.

eg: $f(z) = \frac{\sin z}{z}$.

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \quad \text{(No -ve powers of } z\text{)}$$

$\therefore$ $z=0$ is a removable singularity of $f(z)$.

(ie) $\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z}{z} = 1$.

Poles :

An analytic function $f(z)$ with a singularity at $z=a$ if $\lim_{z\to a} f(z) = \infty$, then $z=a$ is a pole of $f(z)$.

Simple pole :

A pole of order one is called as simple pole.

eg: $f(z) = \frac{1}{(z-1)^4 (z-4)^2 (z+1)}$

Here, $z=-1$ is a simple pole

$z=1$ is a pole of order 4

$z=4$ is a pole of order 2.

Essential Singularity

If the principal part contains an infinite number of non-zero terms, then $z = z_0$ is known as essential singularity.

Integral function :

A function $f(z)$ which is analytic everywhere in the finite plane (except at infinitely) is called an integral function (or) an entire function.

eg: $\cos x$ is an entire function.

Meromorphic function :

A function $f(z)$ which is analytic everywhere in the finite plane except at finite number of poles is called a meromorphic function.

Problems :

1) Find the zeros of $f(z) = \dfrac{z^2+1}{1-z^2}$.

soln :

Zeros of $f(z)$ are given by $f(z) = 0$

$$f(z) = \frac{z^2+1}{1-z^2} = 0.$$

$$z^2 + 1 = 0.$$

$$z^2 = -1.$$

$$z = \pm\sqrt{-1} = \pm i.$$

$z = i, -i$ are simple zero.

2) Find the zeros of $f(z) = \sin\left(\frac{1}{z-a}\right)$

Soln:

$$f(z) = \sin\left(\frac{1}{z-a}\right)$$

The zeros of $f(z)$ are given by $f(z)=0$.

$$\therefore f(z) = \sin\left(\frac{1}{z-a}\right) = 0.$$

$$\sin\left(\frac{1}{z-a}\right) = 0$$

$$\frac{1}{z-a} = \sin^{-1}(0) = n\pi.$$

$$\Rightarrow z-a = \frac{1}{n\pi}.$$

$$\therefore z = a + \frac{1}{n\pi}, \quad n = \pm 1, \pm 2, \pm 3 \ldots$$

---

3) Find the isolated singularity of $f(z) = \frac{z+2}{z(z^2+1)}$

Soln:

$$f(z) = \frac{z+2}{z(z^2+1)}$$

Here $z(z^2+1) = 0.$

$z = 0, \quad z^2+1 = 0$

$z^2 = -1.$

$z = \pm\sqrt{-1} = \pm i,$

(ie) $z = 0, +i, -i$ are the singular points.

$\therefore z = 0, i, -i$ are isolated singularity.

---

4) what is the nature of the singularity $z=0$ of the function, $f(z) = \frac{\sin z - z}{z^3}$.

Soln: Given, $f(z) = \frac{\sin z - z}{z^3}$

The function $f(z)$ is not defined at $z=0$. $\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z - z}{z^3} = 0$

By L' Hospital's Rule,

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z - z}{z^3}$$

$$= \lim_{z\to 0} \frac{\cos z - 1}{3z^2} = \frac{0}{0}. \text{ (Indeterminate form)}$$

By L' Hospital's Rule.

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{-\sin z}{6z} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{z\to 0} \frac{-\cos z}{6} = \frac{-1}{6}. \text{ (finite)}.$$

$\therefore$ limit exists and is finite, the singularity at $z=0$ is a removable singularity.

5) Classify the singularity of $f(z) = \frac{e^{1/z}}{(z-a)^2}$

soln:

Given $f(z) = \frac{e^{1/z}}{(z-a)^2}$.

The pole of $f(z)$ are $(z-a)^2 = 0$.

$z = a$ is a pole of order 2.

$$\lim_{z\to 0} \frac{e^{1/z}}{(z-a)^2} = \frac{\infty}{a^2} = \infty \neq 0.$$

$e^{1/0} = e^{\infty} = \infty$

$\Rightarrow$ $z=0$ is removable singularity.

$\therefore$ $f(z)$ has no zeros.

6) classify the nature of singularities of the function $\frac{e^z}{z^2+4}$ and $e^{1/z}$.

Soln

(i) Let $f(z) = \frac{e^z}{z^2+4}$

The poles of $f(z)$ are obtained by equating the denominator to zero.

$$z^2+4 = 0.$$

$$z^2 = -4$$

$$z = \pm 2i$$

$$\lim_{z \to 0} \frac{e^z}{z^2+4} = \frac{1}{4} \neq 0.$$

$\therefore$ $z=0$ is a removable singularity.

$\therefore$ $f(z)$ has no zeros.

(ii) Let $f(z) = e^{1/z}$.

$\cancel{z=0}$.

Residues:

Formula for the evaluation of residues.

(i) Residue at a pole of order 'n'.

$$\text{Res}[f(z);\ z=a] = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]$$

(ii) Residue at a pole of order 1.

$$\text{Res}[f(z);\ z=a] = \lim_{z \to a} (z-a) f(z).$$

(iii) If $z = a$ is a simple pole of $f(z)$ and of

$f(z) = \dfrac{\phi(z)}{\psi(z)}$, then $\text{Res}[f(z) : z = a] = \dfrac{\phi(a)}{\psi'(a)}$

Problems :

1) Find the residue of $f(z) = \dfrac{z}{(z-1)^2}$ at its pole.

Soln:

$$f(z) = \frac{z}{(z-1)^2}$$

Here, $z = 1$ is a pole of order 2. , $\boxed{n = 2}$

$$\text{Res}[f(z) : z = a] = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]$$

$$\text{Res}[f(z) : z = 1] = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d^{2-1}}{dz^{2-1}} \left[\cancel{(z-1)^2} \left(\frac{z}{\cancel{(z-1)^2}}\right)\right]$$

$$= \frac{1}{1} \lim_{z \to 1} \underbrace{\frac{d}{dz}[z]}$$

$$= \lim_{z \to 1} (1) = 1.$$

$\therefore \text{Res}[f(z) : z = 1] = 1.$

---

2) Calculate the residue of $f(z) = \dfrac{1 - e^{2z}}{z^3}$

Soln:

$$f(z) = \frac{1 - e^{2z}}{z^3}.$$

Here $z = 0$ is a pole of order 3.

$\boxed{n = 3}$ & $\boxed{a = 0}$

$$\text{Res}[f(z) : z = 0] = \frac{1}{(3-1)!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[(z-0)^3 \frac{1 - e^{2z}}{z^3}\right]$$

$$= \frac{1}{2!} \lt_{z\to 0} \frac{d^2}{dz^2}\left[\cancel{z^3} \cdot \frac{(1-e^{2z})}{\cancel{z^3}}\right]$$

$$= \left(\frac{1}{2}\right) \lt_{z\to 0} \frac{d}{dz}(0 - 2e^{2z})$$

$$= \frac{1}{\cancel{2}} \lt_{z\to 0} (-\cancel{4}^{2} e^{2z})$$

$$= -2e^{2(0)}. \quad = -2.$$

$\therefore \quad Res[f(z) : z = 0] = -2.$

---

3) Test for singularity of $\frac{1}{z^2+1}$ and hence find the corresponding residues.

Soln.

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

$\because z^2+1 = 0.$
$z^2 = -1$
$z = \pm\sqrt{-1} = \pm i$
$z = i, -i$

Here, $z = i$ is a simple pole
$z = -i$ is a simple pole.

$$Res[f(z) : z=a] = \lt_{z\to a} (z-a) f(z).$$

For $z=i$,

$$= \lt_{z\to i} (\cancel{z-i}) \frac{1}{(\cancel{z-i})(z+i)}$$

$$= \lt_{z\to i} \frac{1}{(z+i)}$$

$$= \frac{1}{2i}.$$

$\therefore \quad Res[f(z) : z = i] = 1/2i$

For $z=-i$,

$$= \lt_{z\to -i} (\cancel{z+i}) \frac{1}{(z-i)(\cancel{z+i})}$$

$$= \lt_{z\to -i} \frac{1}{z-i} = -\frac{1}{2i}.$$

$Res[f(z) : z = -i] = -1/2i$ //.

4) calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Soln:

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$

Here, $z = -1$ is a pole of order 2.

$$Res[f(z) : z = a] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)]$$

$$= \frac{1}{1!} \lim_{z\to -1} \frac{d}{dz} \left[\cancel{(z+1)^2} \frac{e^{2z}}{\cancel{(z+1)^2}}\right]$$

$$= \lim_{z\to -1} \frac{d}{dz} [e^{2z}]$$

$$= \lim_{z\to -1} 2e^{2z}$$

$$= 2e^{-2} //$$

---

5) Find the residue at $z = 0$ of $\frac{1+e^z}{z\cos z + \sin z}$.

Soln:

$$f(z) = \frac{1+e^z}{z\cos z + \sin z}$$

$f(z)$ has a simple pole at $z = 0$.

Given, $f(z) = \frac{\phi(z)}{\psi(z)}$, then

$$Res[f(z) : z = a] = \frac{\phi(a)}{\psi'(a)}$$

$\phi(z) = 1+e^z$

$\phi(a) = 1+e^a$

$\psi(z) = z\cos z + \sin z$.

$\psi'(z) = -z\sin z + \cos z + \cos z$.

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(ie) $\phi'(z) = 2\cos z - z\sin z$.

$\phi'(a) = 2\cos a - a\sin a$.

when $z = 0$,

$$\phi(0) = 1+1 = 2$$

$$\phi'(0) = 2 - 0 = 2.$$

$$\therefore \quad \text{Res}[f(z) : z = 0] = \frac{\phi(0)}{\phi'(0)} = \frac{2}{2} = 1 //$$

6) Find the Residue of $\cot z$ at $z = 0$.

Soln:

$$f(z) = \cot z.$$

$$= \frac{\cos z}{\sin z}$$

Here, $\phi(z) = \cos z$

$\phi(0) = 1.$

At $z = 0$, $\psi(z) = \sin z$.

$\psi'(z) = \cos z$.

$\psi'(0) = 1$

$$\text{Res}[f(z) : z = 0] = \frac{\phi(a)}{\psi'(a)} = \frac{1}{1} = 1 //$$

7) Find the residues of $f(z) = \dfrac{1}{(z^2+a^2)^2}$ at its singularities.

Soln:

$$f(z) = \frac{1}{(z^2+a^2)^2}$$

$$= \frac{1}{(z-ai)^2(z+ai)^2}$$

$z^2+a^2 = 0$

$z^2 = -a^2$

$z = \pm\sqrt{-a^2}$

$z = \pm ai$

Here $z = ia$ is a pole of order 2.

$z = -ia$ is a pole of order 2.

$$Res[f(z) : z = ia] = \left(\frac{1}{1!}\right) \lim_{z \to ia} \frac{d}{dz}\left[\cancel{(z-ia)^2} \frac{1}{\cancel{(z-ia)^2}(z+ia)^2}\right]$$

$$= \lim_{z \to ia} \frac{d}{dz}\left[\frac{1}{(z+ia)^2}\right]$$

$$= \lim_{z \to ia} \frac{-2}{(z+ia)^3}$$

$$= \frac{-2}{(2ia)^3} = \frac{-2}{8 i^3 a^3} \qquad (i^2 = -1)$$

$$= \frac{-\cancel{2}}{-\cancel{8}_4 a^3 i} \times \frac{-i}{-i}$$

$$Res[f(z) : z = ia] = \frac{-i}{4a^3} \quad //$$

Now, $z = -ia$.

$$Res[f(z) : z = -ia] = \frac{1}{1!} \lim_{z \to -ia} \frac{d}{dz}\left[\cancel{(z+ia)^2} \frac{1}{(z-ia)^2 \cancel{(z+ia)^2}}\right]$$

$$= \lim_{z \to -ia} \left(\frac{-2}{(z-ia)^3}\right)$$

$$= \frac{-2}{(-2ia)^3} = \frac{\cancel{-}\cancel{2}}{-(\cancel{-8}_4 i a^3)} \times \frac{i}{i}$$

$$= \frac{i}{4a^3}$$

$$\therefore Res[f(z) : z = -ia] = \frac{i}{4a^3} \quad //.$$

8) calculate the residues of the fn/: $f(z) = +\frac{z\sin z}{(z-\pi)^3}$ at $z=\pi$.

soln:
$$f(z) = +\frac{z\sin z}{(z-\pi)^3}.$$

Here $z=\pi$ is a pole of order 3.

$$\text{Res}[f(z) : z=\pi] = \lim_{z\to\pi} \frac{1}{2!}\frac{d^2}{dz^2}\left[\cancel{(z-\pi)^3} \frac{(+z\sin z)}{\cancel{(z-\pi)^3}}\right]$$

$$= \lim_{z\to\pi}\left(\frac{1}{2}\right)\frac{d}{dz}\left[+(z\cos z + \sin z)\right]$$

$$= \frac{1}{2}\left[\lim_{z\to\pi}\left(+(z(-\sin z) + \cos z + \cos z)\right)\right]$$

$$= \frac{1}{2}\left[\lim_{z\to\pi}(-z\sin z + 2\cos z)\right]$$

$$= \frac{1}{2}\left[-\pi\sin\pi + 2\cos\pi\right]$$

$$= \frac{1}{2}(0 + 2(-1))$$

$$= \frac{1}{\cancel{2}}(-\cancel{2}) = -1$$

9) Find Residue of $f(z) = \frac{1}{z^2 e^z}$

$$f(z) = \frac{1}{z^2e^z} = \frac{e^{-z}}{z^2}.$$

$$\left[\begin{array}{l} e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \\ e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{array}\right.$$

$$= \frac{1}{z^2}\left[1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots\right]$$

$$= \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

$\text{Res}[f(z) : z=0]$ = coefficient of $(1/z)$ in Laurent expansion

$$\text{Res}[f(z) : z=0] = -1$$

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Cauchy's Residue Theorem :

If $f(z)$ be analytic at all point inside and on a simple closed curve c, except for a finite number of isolated singularity $z_1, z_2 \cdots z_n$ inside c.

$$\int_c f(z)dz = 2\pi i \left[\text{sum of Residues of } f(z) \text{ at } z_1, z_2 \cdots z_n\right]$$

Problems :

1) If c is the circle $|z| = 3$, then evaluate

$$\int_c \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz .$$

Soln:

$$f(z) = \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} .$$

Here $z = -1$ is a simple pole lies inside $|z| = 3$

$z = -2$ is a simple pole lies inside $|z| = 3$.

$$\text{Res}[f(z) : z = -1] = \lim_{z \to a} (z-a) f(z) .$$

$$= \lim_{z \to -1} \cancel{(z+1)} \frac{\cos \pi z^2 + \sin \pi z^2}{\cancel{(z+1)}(z+2)}$$

$$= \frac{\cos \pi (1) + \sin \pi (1)}{(-1+2)}$$

$$= \frac{-1+0}{1} = -1 .$$

$$\text{Res}[f(z) : z \to -1] = -1 .$$

$$\text{Res}[f(z) : z \to -2] = \lim_{z \to -2} \left[\frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)\cancel{(z+2)}}\right] \cancel{(z+2)}$$

$$= \frac{\cos 4\pi + i\sin \pi(4)}{-2+1}$$

$$= \frac{1+0}{-1} = -1.$$

$$Res[f(z) : z \to -2] = -1.$$

$\therefore$ By cauchy Residue theorem,

$$\int_c f(z)\,dz = 2\pi i\,[\text{sum of the Residues}]$$

$$= 2\pi i(-1-1)$$

$$= -4\pi i$$

---

2) Evaluate $\int_c \frac{dz}{\sin z}$, where c is $|z| = 4$.

Soln:

$$f(z) = \frac{1}{\sin z}$$

$$\Rightarrow \sin z = 0.$$

$$z = \sin^{-1}(0) = n\pi.$$

$$z = 0, \pm\pi, \pm 2\pi \ldots$$

Given c is $|z| = 4$ Take $(\pi = 22/7)$

$z = 0, \pm\pi$ lies inside $c : |z| = 4$

$z = \pm 2\pi, \pm 3\pi \ldots$ lies outside $c : |z| = 4$.

$$Res[f(z) : z = 0] = \frac{\phi(0)}{\psi'(0)}.$$

$f(z) = \frac{1}{\sin z}$

$\phi(z) = 1$

$\psi(z) = \sin z$.

$\phi(z) = 1 \Rightarrow \phi(0) = 1.$

$\psi(z) = \sin z.$

$\psi'(z) = \cos z \Rightarrow \psi'(0) = 1.$

$$Res[f(z) : z = 0] = \frac{1}{1} = 1.$$

$$\text{Res}[f(z) : z = \pi] = \frac{\phi(\pi)}{\psi'(\pi)}$$

$$= \frac{1}{\cos\pi} = \frac{1}{-1} = -1.$$

$$\text{Res}[f(z) : z = -\pi] = \frac{\phi(-\pi)}{\psi'(-\pi)} = \frac{1}{\cos(-\pi)}$$

$$= \frac{1}{\cos\pi} = \frac{1}{-1} = -1.$$

$\therefore$ By cauchy's Residue theorem,

$$\int_c f(z)\,dz = 2\pi i \,(\text{sum of Residues})$$

$$= 2\pi i\,(\cancel{1} - \cancel{1} - 1)$$

$$= -2\pi i$$

---

3) Evaluate $\int_c \frac{z\sec z}{1-z^2}\,dz$, where c is the ellipse $4x^2+9y^2 = 9$.

soln:

$$f(z) = \frac{z\sec z}{1-z^2}.$$

$$f(z) = \frac{z}{\cos z\,(1+z)(1-z)}$$

Now, $\cos z\,(1+z)(1-z) = 0$.

$\cos z = 0$, $1+z = 0$, $1-z = 0$.

$z = \pm \pi/2, \pm 3\pi/2 \ldots$

$z = -1$

$z = 1$

$\cos z = 0$

$z = (2n+1)\pi/2$.

$\sin z = 0$

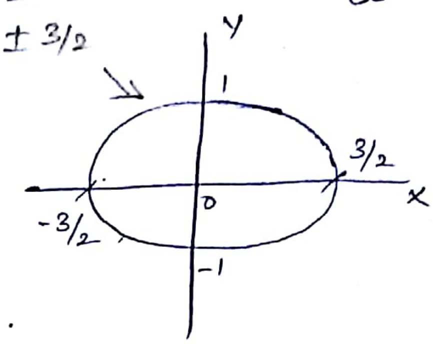
$z = n\pi$

$n = 0, \pm 1, \pm 2 \ldots$

Given, $4x^2 + 9y^2 = 9. \Rightarrow$ $x = 0,\ y = \pm 1$
$y = 0,\ x = \pm 3/2$

$\div$ by 9, $\frac{4}{9}x^2 + y^2 = 1.$

$$\frac{x^2}{(3/2)^2} + y^2 = 1.$$



The poles $z = \pm 1$ lies inside c.

The poles $z = \pm \pi/2,\ \pm 3\pi/2$ lies outside c.

$$\text{Res}\left[f(z) : z = 1\right] = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{z}{\cos z\,(1+z)\,\cancel{(1-z)}}$$

$$= \lim_{z \to 1} \frac{z}{\cos z\,(1+z)}$$

$$= \frac{1}{\cos 1\,(1+1)} = \frac{1}{2\cos 1}.$$

$$\text{Res}\left[f(z) : z \to -1\right] = \lim_{z \to -1} \cancel{(z+1)} \frac{z}{\cos z\,\cancel{(1+z)}\,(1-z)}$$

$$= \frac{-1}{\cos(-1)\,(1+1)} = \frac{-1}{2\cos 1}.$$

By cauchy's Residue theorem,

$$\int_c f(z)\,dz = 2\pi i\,(\text{Sum of the Residues})$$

$$= 2\pi i\left[\frac{1}{2\cancel{\cos 1}} - \frac{1}{2\cancel{\cos 1}}\right]$$

$$= 0 //.$$

---

4) If c is the circle $|z| = 3$, Evaluate $\int_c \tan z\,dz$

Soln:

$$f(z) = \tan z = \frac{\sin z}{\cos z}.$$

Take $\cos z = 0$, $z = (2n+1)\pi/2$ $\quad n = 0, \pm 1, \pm 2 \ldots$

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$z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2.$ $(\pi = 22/7)$
(1.57) 4.71 7.85

$\therefore$ $z = \pi/2$ lies inside $|z| = 3$.

$z = -\pi/2$ lies inside $|z| = 3$.

other points, $z = \pm 3\pi/2, \pm 5\pi/2 \ldots$ lies outside $|z| = 3$

$$\text{Res}[f(z): z = \pi/2] = \frac{\phi(a)}{\psi'(a)}.$$

$\phi(z) = \sin z$
$\psi(z) = \cos z$
$\psi'(z) = -\sin z$

$$= \frac{\sin \pi/2}{-\sin \pi/2} = \frac{1}{-1}$$

$$= -1.$$

$$\text{Res}[f(z): z = -\pi/2] = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = \frac{-\sin \pi/2}{\sin \pi/2}$$

$$= \frac{-1}{1} = -1.$$

By cauchy's Residue Theorem,

$$\int_c f(z)\,dz = 2\pi i [\text{sum of Residues}]$$

$$= 2\pi i[-1 - 1]$$

$$= -4\pi i$$

---

5) Evaluate $\int_c \frac{e^{z^2}}{\cos \pi z}\,dz$, where c is $|z| = 1$.

Soln:

$$f(z) = \frac{e^{z^2}}{\cos \pi z}.$$

$\cos \pi z = 0.$

$\pi z = \cos^{-1}(0) \Rightarrow \pi z = (2n+1)\pi/2.$

$n = 0, \pm 1, \pm 2 \ldots \Rightarrow z = \frac{(2n+1)}{2}.$

$z = \pm 1/2, \pm 3/2 \ldots$

$\therefore$ $z = 1/2$ lies inside $|z| = 1$. (simple pole)

$z = -1/2$ lies inside $|z| = 1$. (").

$$\text{Res}[f(z) : z = a] = \frac{\phi(a)}{\psi'(a)}$$

$\phi(z) = e^{z^2}$. $\Rightarrow$ $\phi(\pi/2) = e^{1/4}$.

$\phi(-1/2) = e^{1/4}$

$\psi(z) = \cos \pi z$.

$\psi'(z) = -\pi \sin \pi z$ $\Rightarrow$ $\psi'(1/2) = -\pi \sin(\pi/2) = -\pi$

$\psi'(-1/2) = -\pi \sin(-\pi/2)$

$= \pi \sin(\pi/2) = \pi$

$$\therefore \text{Res}[f(z) : z = 1/2] = \frac{e^{1/4}}{-\pi}$$

$$\text{Res}[f(z) : z = -1/2] = \frac{e^{1/4}}{\pi}.$$

By cauchy's Residue theorem,

$$\int_c f(z)\, dz = 2\pi i \,[\text{sum of Residues}]$$

$$= 2\pi i \left[\cancel{\frac{e^{1/4}}{-\pi}} + \cancel{\frac{e^{1/4}}{\pi}}\right]$$

$$= 2\pi i (0)$$

$$= 0 //$$

Contour Integration.

The complex integration along the zero curve used in evaluating the definite integral is called contour integration.

Type - I : $(0, 2\pi)$

Given, $\int_c f(\cos\theta, \sin\theta)\, d\theta \longrightarrow (1)$

The contour is chosen as the unit circle.

$c : |z| = 1.$

Now, $z = e^{i\theta}$, $0 \le \theta \le 2\pi$.

$dz = ie^{i\theta}\, d\theta. = iz\, d\theta.$

$$d\theta = \frac{dz}{iz}$$

$$z = e^{i\theta} = \cos\theta + i\sin\theta.$$

$$\cos\theta = \frac{1}{2}\left[\frac{z^2+1}{z}\right].$$

$$\sin\theta = \frac{1}{2i}\left[\frac{z^2-1}{z}\right]$$

when used in (1),

$$\int_c f(\cos\theta, \sin\theta)\, d\theta = \int_c f\left[\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right]\frac{dz}{iz},$$

$c : |z| = 1$

Problems :

1) Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$

soln:

Let $z = e^{i\theta}$

$dz = ie^{i\theta}\, d\theta = iz\, d\theta.$

$$d\theta = \frac{dz}{iz}.$$

$$\cos\theta = \frac{z^2+1}{2z}$$

Given $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \int_{c:|z|=1} \frac{1}{5+4\left[\frac{z^2+1}{2z}\right]} \left(\frac{dz}{iz}\right)$

$$= \frac{1}{i} \int_{c:|z|=1} \frac{1}{\left(\frac{10z+4z^2+4}{2\cancel{z}}\right)} \left(\frac{dz}{\cancel{z}}\right)$$

$$= \frac{1}{i} \int_{c:|z|=1} \frac{1}{2z^2+5z+2} \, dz.$$

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{1}{2i} \int_{c:|z|=1} \frac{1}{z^2+\frac{5}{2}z+1} \, dz \longrightarrow (1)$$

To find: $\int_{c:|z|=1} \frac{1}{z^2+\frac{5}{2}z+1} \, dz.$

Poles are $z^2+\frac{5}{2}z+1 = 0.$ $\qquad 2z^2+5z+2=0$

$\left(z+\frac{1}{2}\right)(z+2) = 0.$

$z = -2, \; -\frac{1}{2}.$

Here, $z = -\frac{1}{2}$ is a simple pole lies inside $|z|=1.$

$z = -2$ is a simple pole lies outside $|z|=1.$

$$\text{Res}\left[f(z) : z = -\tfrac{1}{2}\right] = \lim_{z\to -\frac{1}{2}} \left(z+\tfrac{1}{2}\right) \frac{1}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{1}{\left(-\frac{1}{2}+2\right)} = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}$$

By cauchy's Residue Theorem, $\int_c f(z)\,dz = 2\pi i\,[\text{sum of Residues}]$

$$= 2\pi i\left(\tfrac{2}{3}\right) \longrightarrow (2)$$

using (2) in (1), $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{1}{\cancel{2i}}\left[\frac{\cancel{4}^{2}\pi \cancel{i}}{3}\right] = \frac{2\pi}{3}$

2) Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ by contour integration.

soln:

Let $z = e^{i\theta}$.

$dz = ie^{i\theta} d\theta = iz\,d\theta$.

$d\theta = \frac{dz}{iz}$.

$\cos\theta = \frac{z^2+1}{2z}$.

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{c:|z|=1} \frac{1}{2+\left[\frac{z^2+1}{2z}\right]} \left(\frac{dz}{iz}\right)$$

$$= \int_{|z|=1} \frac{1}{\frac{4z+z^2+1}{2\cancel{z}}} \frac{dz}{i\cancel{z}}$$

$$= \frac{2}{i} \int_{|z|=1} \frac{1}{z^2+4z+1} dz. \longrightarrow (1)$$

To find $\int_{|z|=1} \frac{1}{z^2+4z+1} dz$.

$z^2+4z+1 = 0$.

$$z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2}.$$

$z = -2 \pm \sqrt{3}$

$$\left[\begin{array}{l} z = -2+\sqrt{3} = 0.268 \\ z = -2-\sqrt{3} = -3.732 \end{array}\right]$$

$\therefore z = -2+\sqrt{3}$ is a simple pole lies inside $|z|=1$

$z = -2-\sqrt{3}$ is a simple pole lies outside $|z|=1$

$$\text{Res}\left[f(z) : z = -2+\sqrt{3}\right] = \lim_{z \to -2+\sqrt{3}} (z+2-\sqrt{3}) \frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

$$= \lim_{z \to -2+\sqrt{3}} \frac{1}{(z+2+\sqrt{3})}$$

$$= \frac{1}{(-\cancel{2}+\sqrt{3}+\cancel{2}+\sqrt{3})}$$

$$= \frac{1}{2\sqrt{3}}$$

By cauchy's Residue theorem,

$$\int_c f(z)\,dz = \cancel{2}\pi i\left[\frac{1}{\cancel{2}\sqrt{3}}\right] = \frac{\pi i}{\sqrt{3}}. \longrightarrow (2)$$

using (2) in (1).

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{\cancel{i}}\left[\frac{\pi \cancel{i}}{\sqrt{3}}\right] = \frac{2\pi}{\sqrt{3}}$$

---

3) Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$, using contour integration.

Soln:

$$z = e^{i\theta}.$$

$$d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{z^2-1}{2iz}.$$

$$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \int_{c:|z|=1} \frac{1}{13+5\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{\cancel{2}iz}$$

$$= \int_{|z|=1} \frac{1}{\left(\frac{26iz+5z^2-5}{2i\cancel{z}}\right)} \frac{dz}{i\cancel{z}}$$

$$= 2\int_{|z|=1} \frac{1}{5z^2+26iz-5}\,dz.$$

$$\frac{-25}{1 \mid 25}$$

$$= \frac{2}{5}\int_{|z|=1} \frac{1}{z^2+\frac{26}{5}iz-1}\,dz \longrightarrow (1)$$

To find $\int_{|z|=1} \frac{1}{z^2+\frac{26}{5}iz-1}\,dz.$

$$z = \frac{-\frac{26}{5}i \pm \sqrt{\left(\frac{26}{5}i\right)^2 + 4}}{2}$$

$$= \frac{-\frac{26}{5}i \pm \sqrt{\frac{-676}{25} + 4}}{2} = -\frac{26}{5}i \pm \sqrt{\frac{-576}{25}}$$

$$z = \frac{-\frac{26}{5}i \pm \frac{24i}{5}}{2} = -5i, \ -\frac{i}{5}$$

$z = -5i$ is a simple pole lies outside $|z| = 1$

✓ $z = -\frac{i}{5}$ is a simple pole lies inside $|z| = 1$

$$\text{Res}\left[f(z) : z = -\frac{i}{5}\right] = \lim_{z \to -i/5} \cancel{\left(z + \frac{i}{5}\right)} \frac{1}{\cancel{\left(z + i/5\right)}(z + 5i)}$$

$$= \lim_{z \to -i/5} \frac{1}{(-i/5 + 5i)}$$

$$= \frac{5}{24i}$$

By cauchy's Residue theorem,

$$\int_c f(z)\, dz = \cancel{2}\pi \cancel{i} \left(\frac{5}{\cancel{24}_{12} \cancel{i}}\right) = \frac{5\pi}{12} \longrightarrow ②$$

using ② in ①

$$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta} = \frac{\cancel{2}}{\cancel{5}}\left(\frac{\cancel{5}\pi}{\cancel{12}_{6}}\right)$$

$$= \frac{\pi}{6} \,//$$

Ass

7) Using contour integration, Evaluate $\int_0^{2\pi} \frac{d\theta}{5-\sin\theta}$.

soln

$z = e^{i\theta}$.

$d\theta = \frac{dz}{iz}$.

$\sin\theta = \frac{z^2-1}{2iz}$

$\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta} = \frac{2\pi}{3}$.

$$\int_0^{2\pi} \frac{d\theta}{5-\sin\theta} = \int_{c:|z|=1} \frac{1}{5-\left[\frac{z^2-1}{2i\cancel{z}}\right]} \left(\frac{dz}{i\cancel{z}}\right)$$

$$= \int_{|z|=1} \frac{2}{10iz - z^2+1}\, dz.$$

$$= -2\int_{|z|=1} \frac{1}{z^2-10iz-1}\, dz. \longrightarrow (1).$$

To find $\int_{|z|=1} \frac{1}{z^2-10iz-1}\, dz$

$z^2 - 10iz - 1 = 0$.

$$z = \frac{10i \pm \sqrt{(10i)^2+4}}{2} = \frac{10i \pm \sqrt{-96}}{2}.$$

$$= \frac{10i \pm \sqrt{-16\times 6}}{2} = \frac{10i \pm 4\sqrt{6}i}{2}.$$

$$z = 5i \pm 2\sqrt{6}i$$

$\because$ $z = 5i + 2\sqrt{6}i = 9.898i$

$z = 5i - 2\sqrt{6}i = -0.101i$

$z = 5i + 2\sqrt{6}i$ is a simple pole outside $|z|=1$ $(\sqrt{6} = 2.449)$

$z = 5i - 2\sqrt{6}i$ is a simple pole inside $|z|=1$.

$$\text{Res}\left[f(z) : z = 5i - 2\sqrt{6}i\right] = \lim_{z \to 5i-2\sqrt{6}i} \cancel{(z-5i+2\sqrt{6}i)} \frac{1}{\cancel{(z-5i+2\sqrt{6}i)}(z-5i-2\sqrt{6}i)}$$

$$= \frac{1}{(\cancel{5i}-2\sqrt{6}i - \cancel{5i} - 2\sqrt{6}i)} = \frac{1}{-4\sqrt{6}i}$$

By cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left[\text{sum of the Residues}\right]$$

$$= \cancel{2}\pi \cancel{i}\left(\frac{-1}{\cancel{4}_{2}\sqrt{6}\cancel{i}}\right)$$

$$= \frac{-\pi}{2\sqrt{6}}. \longrightarrow (2)$$

using (2) in (1).

$$\int_0^{2\pi} \frac{d\theta}{5-\sin\theta} = -\cancel{2}\left(\frac{-\pi}{\cancel{2}\sqrt{6}}\right) = \frac{\pi}{\sqrt{6}} //$$

---

5) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$ $(0<a<1)$

soln:

$$z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}.$$

$$\cos\theta = \frac{z^2+1}{2z}.$$

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \int_{C:|z|=1} \frac{1}{1-\cancel{2}a\left(\frac{z^2+1}{\cancel{2}\cancel{z}}\right)+a^2}\left(\frac{dz}{i\cancel{z}}\right)$$

$$= \int_{|z|=1} \frac{1}{1-a\left(\frac{z^2+1}{z}\right)+a^2}\,\frac{dz}{i}$$

$$= \frac{1}{i}\int_{|z|=1} \frac{1}{z-az^2-a+a^2z}\,dz$$

$$= \frac{1}{i}\int_{|z|=1} \frac{1}{-a\left[-\frac{z}{a}+z^2+1-az\right]}\,dz$$

$$= \frac{1}{-ai} \int_{|z|=1} \frac{1}{z^2 - (a+\frac{1}{a})z + 1} \, dz$$

$$= \frac{1}{-ai} \int_{|z|=1} \frac{1}{z^2 - \left(\frac{a^2+1}{a}\right)z + 1} \, dz \longrightarrow (1)$$

Poles are $z^2 - \left(\frac{1+a^2}{a}\right)z + 1 = 0$.

$$z = \frac{\left(\frac{1+a^2}{a}\right) \pm \sqrt{\frac{(a^2+1)^2}{a^2} - 4}}{2}$$

$$= \frac{\left[\left(\frac{a^2+1}{a}\right) \pm \sqrt{\frac{a^4+2a^2+1-4a^2}{a^2}}\right]}{2}$$

$$= \frac{\left[\left(\frac{a^2+1}{a}\right) \pm \sqrt{\frac{(a^2-1)^2}{a^2}}\right]}{2}$$

$$z = \frac{(a^2+1) \pm (a^2-1)}{2a}$$

$\therefore z = a$ is a simple pole lies inside $|z| = 1$.

$z = 1/a$ is a simple pole lies outside $|z| = 1$.

[since, $0 < a < 1$, for $a = 1/2 \Rightarrow z = \frac{1}{a} = 2$]

$$Res[f(z) : z = a] = \lim_{z \to a} \cancel{(z-a)} \frac{1}{\cancel{(z-a)}(z-1/a)}$$

$$= \lim_{z \to a} \frac{1}{z - 1/a} = \frac{1}{a - 1/a}$$

$$= \frac{a}{a^2-1} \longrightarrow (2)$$

using ② in ①.

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} = \frac{-1}{\cancel{d}\cancel{i}}\left[2\pi \cancel{i}\left(\frac{\alpha}{a^2-1}\right)\right]$$

$$= \frac{-2\pi}{a^2-1}, \quad 0 < a < 1$$

---

6) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\, d\theta$ $\quad \int_{-\pi}^{\pi} = 2\int_0^{\pi} = \int_0^{2\pi}$

Soln:

Let $z = e^{i\theta}$

$d\theta = \frac{dz}{iz}$ $\qquad$ ||| ly $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{6}$

$\cos\theta = \frac{z^2+1}{2z}$

$z^2 = (e^{i\theta})^2$

$z^2 = e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

$z^2 = \text{RP of } \cos 2\theta$.

$$\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\, d\theta = \int_{c:|z|=1} \frac{\text{RP of } z^2}{5 - \cancel{4}^{2}\left[\frac{z^2+1}{\cancel{2z}}\right]} \left(\frac{dz}{i\cancel{z}}\right)$$

$$= RP\left(\frac{1}{i}\right) \int_{c:|z|=1} \frac{z^2}{5z - 2z^2 - 2}\, \frac{dz}{\cancel{z}}$$

$$= RP\left(\frac{1}{-i}\right) \int_{|z|=1} \frac{z^2}{2z^2-5z+2}\, dz \longrightarrow ①$$

$$= RP\left(\frac{-1}{i}\right) \int_{|z|=1} \frac{z^2}{(2z-1)(z-2)}\, dz.$$

Here, $z=2$ is a simple pole. lies outside $|z|=1$

✓ $z = 1/2$ is a simple pole lies inside $|z|=1$

$$\text{Res}\left[f(z) : z = 1/2\right] = \lim_{z \to 1/2} \cancel{(z - 1/2)} \cdot \frac{z^2}{\cancel{(z-1/2)}(z-2)}$$

$$= \lim_{z \to 1/2} \frac{z^2}{z-2} = \frac{1/4}{1/2 - 2}.$$

$$= \frac{1/4}{-3/2} = \frac{1}{\cancel{4}_2}\left(\frac{-\cancel{2}}{3}\right)$$

$$= -\frac{1}{6}$$

By cauchy's Residue theorem,

$$\int_c f(z)\,dz = 2\pi i \left[\text{sum of the Residues}\right]$$

$$= \cancel{2}\pi i\left[-\frac{1}{\cancel{6}_3}\right]$$

$$= -\frac{\pi i}{3}. \longrightarrow (2)$$

using (2) in (1),

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\,d\theta = RP\left(-\frac{1}{i}\right) \int_{|z|=1} \frac{z^2}{(2z-1)(z-2)}\,dz$$

$$= RP\left(\frac{-1}{2i}\right) \int_{|z|=1} \frac{z^2}{(z-1/2)(z-2)}\,dz$$

$$= RP\left(\frac{-1}{2\cancel{i}}\right)\left(-\frac{\pi \cancel{i}}{3}\right)$$

$$= RP\left(\frac{\pi}{6}\right)$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\,d\theta = \frac{\pi}{6} \,//$$

---

7) using contour integration, Evaluate the real integral

$$\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\,d\theta.$$

Soln:

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta.$$

$$= \cancel{RP}\left(\tfrac{1}{2}\right)\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta.$$

$z = e^{i\theta}.$

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}.$$

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = RP\left(\frac{1}{2}\right)\int_{c:|z|=1} \frac{1+2z}{5+\cancel{4}\left(\frac{z^2+1}{\cancel{2z}}\right)}\left(\frac{dz}{i\cancel{z}}\right)$$

$$= RP\left(\frac{1}{2i}\right)\int_{|z|=1} \frac{1+2z}{5z+2z^2+2}\,dz$$

$$= RP\left(\frac{1}{2i}\right)\int_{|z|=1} \frac{1+2z}{2z^2+5z+2}\,dz.$$

$$= RP\left(\frac{1}{2i}\right)\int_{|z|=1} \frac{1+2z}{(2z+1)(z+2)}\,dz.$$

$$= RP\left(\frac{1}{4i}\right)\int_{|z|=1} \frac{1+2z}{(z+1/2)(z+2)}\,dz. \longrightarrow (1)$$

To find $\displaystyle\int_{|z|=1} \frac{1+2z}{(z+1/2)(z+2)}\,dz.$

Here, $z=-1/2$ lies inside $|z|=1$ (simple pole)

$z=-2$ is a simple pole lies outside $|z|=1$

$$\text{Res}\left[f(z) : z=-1/2\right] = \lim_{z\to -1/2} \cancel{(z+1/2)}\,\frac{(1+2z)}{\cancel{(z+1/2)}(z+2)}$$

$$= \frac{1+\cancel{2}(-1/\cancel{2})}{(-1/2+2)} = \frac{1-1}{(-3/2)} = 0.$$

By cauchy's Residue theorem,

$$\int_c f(z)\,dz = 2\pi i(0) = 0. \longrightarrow (2)$$

using (2) in (1),

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = RP\left(\frac{1}{4i}\right)\int_{|z|=1}\frac{1+8z}{(z+\frac{1}{2})(z+2)}\,dz$$

$$= RP\left(\frac{1}{4i}\right)(0)$$

$$= 0$$

---

8) Solve $\int_{-\pi}^{\pi}\frac{a\cos\theta}{a+\cos\theta}\,d\theta = 2\pi a\left[1-\frac{a}{\sqrt{a^2-1}}\right]$, $a>1$.

Soln:

$$\int_{-\pi}^{\pi}\frac{a\cos\theta}{a+\cos\theta}\,d\theta = 2\int_0^{\pi}\frac{a\cos\theta}{a+\cos\theta}\,d\theta = \int_0^{2\pi}\frac{a\cos\theta}{a+\cos\theta}\,d\theta$$

S.T $\int_0^{2\pi}\frac{\cos\theta}{3+\sin\theta}\,d\theta = 0$

Let $z = e^{i\theta}$

$d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$.

$$\int_0^{2\pi}\frac{a\cos\theta}{a+\cos\theta}\,d\theta = RP\int_{c:|z|=1}\frac{az}{a+\left(\frac{z^2+1}{2z}\right)}\left(\frac{dz}{iz}\right)$$

$$= RP\int_{|z|=1}\frac{az}{\left(\frac{2az+z^2+1}{2\cancel{z}}\right)}\frac{dz}{i\cancel{z}}$$

$$= RP\left(\frac{2a}{i}\right)\int_{|z|=1}\frac{z}{z^2+2az+1}\,dz. \longrightarrow (1)$$

To find $\int_{|z|=1}\frac{z}{z^2+2az+1}\,dz$.

$z^2+2az+1 = 0.$

$$z = \frac{-2a\pm\sqrt{4a^2-4}}{2} = -a\pm\sqrt{a^2-1}.$$

$z = -a+\sqrt{a^2-1}$, $z = -a-\sqrt{a^2-1}$.

Given $a > 1$, Take $a = 2$.

$z = -2 + \sqrt{4-1} = -2 + \sqrt{3} = -0.268$.

$z = -2 - \sqrt{4-1} = -2 - \sqrt{3} = -3.732$.

From our assumption,

$z = -a + \sqrt{a^2-1}$ is a simple pole lies inside $|z| = 1$.

$z = -a - \sqrt{a^2-1}$ is a simple pole lies outside $|z| = 1$

$$\text{Res}\left[f(z) : z = -a + \sqrt{a^2-1}\right] = \lim_{z \to -a+\sqrt{a^2-1}} \frac{(z + \cancel{a - \sqrt{a^2-1}}) \times z}{(\cancel{z + a - \sqrt{a^2-1}})(z - (-a - \sqrt{a^2-1}))}$$

$$= \lim_{z \to -a+\sqrt{a^2-1}} \frac{z}{z - (-a - \sqrt{a^2-1})}$$

$$= \frac{(-a + \sqrt{a^2-1})}{(-\cancel{a} + \sqrt{a^2-1}) + \cancel{a} + \sqrt{a^2-1})}$$

$$= \frac{-a + \sqrt{a^2-1}}{+2\sqrt{a^2-1}}.$$

using Cauchy's Residue theorem,

$$\int_c f(z)\,dz = \cancel{2}\pi i \left[\frac{-a + \sqrt{a^2-1}}{\cancel{2}\sqrt{a^2-1}}\right]$$

$$= \pi i \left[1 - \frac{a}{\sqrt{a^2-1}}\right] \longrightarrow (2)$$

using (2) in (1),

$$\int_{-\pi}^{\pi} \frac{a \cos\theta}{a + \cos\theta}\,d\theta = RP\left[\frac{2a}{\cancel{i}} \times \pi \cancel{i} \left(1 - \frac{a}{\sqrt{a^2-1}}\right)\right]$$

$$= 2\pi a \left(1 - \frac{a}{\sqrt{a^2-1}}\right), \quad a > 1.$$

Type - II

The integral of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$, where $P(x)$ and $Q(x)$ are polynomials in x such that the degree of Q exceeds that of P at least by two .and $Q(x)$ does not vanish for any x.

Problem :

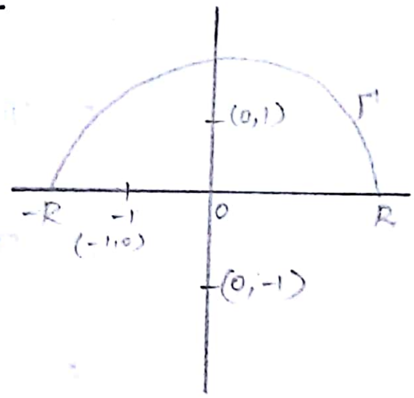
1) using contour integration evaluate $\int_{-\infty}^{\infty} \frac{x\, dx}{(x+1)(x^2+1)}$

soln:

consider $\int_c f(z)\, dz = \int_c \frac{z\, dz}{(z+1)(z^2+1)} \longrightarrow (1)$

where c consists of the semi circle $\Gamma : |z| = R$ and the bounding diameter $[-R, R]$ c is the upper half of the semicircle Γ with bounding dia. $[-R, R]$

$$\int_c f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz \longrightarrow (2)$$



To find (2),

$$\int_c \frac{z\, dz}{(z+1)(z^2+1)}$$

$(z+1)\ (z^2+1) = 0.$

$z = -1, \qquad z = \pm i$

$z = -1$ is a simple pole which lies on the real axis.

$z = i$ is a simple pole which lies inside Γ.

$z = -i$ is a simple pole which lies outside Γ.

$$\operatorname{Res}[f(z) : z = -1] = \lim_{z \to -1} \cancel{(z+1)} \frac{z}{\cancel{(z+1)}(z+i)(z-i)}$$

$$= \frac{-1}{(-1+i)(-1-i)} = \frac{-1}{(-1)^2 - (i)^2}$$

$$= \frac{-1}{1-(-1)} = \frac{-1}{1+1} = \frac{-1}{2}$$

(ie) $\text{Res}[f(z) : z \to -1] = \frac{-1}{2}$.

$$\text{Res}[f(z) : z = i] = \lim_{z \to i} (\cancel{z}-i) \frac{z}{(z+1)(z+i)(z \cancel{-i})}$$

$$= \frac{\cancel{i}}{(i+1)(2\cancel{i})} = \frac{1}{2(i+1)}$$

By cauchy's Residue theorem.

$$\int_c f(z)\,dz = 2\pi i\,(\text{Sum of the residues}) + \pi i\,(\text{sum of Residues of the real axis})$$

$$= \cancel{2}\pi i\left(\frac{-1}{\cancel{2}}\right) + \cancel{2}\pi i\left(\frac{1}{\cancel{2}(i+1)}\right)$$

$$= \frac{\pi i}{i+1} - \frac{\pi i}{2}.$$

$$= \pi i\left[\frac{2-(i+1)}{2(i+1)}\right] = \pi i\left(\frac{2-i-1}{2(i+1)}\right)$$

$$= \pi i\left(\frac{1-i}{2(1+i)}\right) \times \left(\frac{1-i}{1-i}\right)$$

$$= \frac{\pi i\,(1-i)^2}{2[1-(-1)]} = \frac{\pi i\,(\cancel{1}-2i-\cancel{1})}{2(2)}$$

$$= \frac{-\cancel{2}\pi i^2}{\cancel{4}\,2} = \frac{-\pi(-1)}{2}$$

$$\therefore \int_c f(z)\,dz = \frac{\pi \cancel{i}}{2} \longrightarrow (3)$$

(2) $\Rightarrow$ $$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = \frac{\pi i}{2}$$

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If $R \to \infty$, then $\int_{\Gamma} f(z)dz \to 0$.

$$\therefore \int_{-R}^{R} f(x)\,dx = \frac{\pi}{2}.$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2}$$

(ie) $$\int_{-\infty}^{\infty} \frac{x\,dx}{(x+1)(x^2+1)} = \frac{\pi}{2} //.$$

---

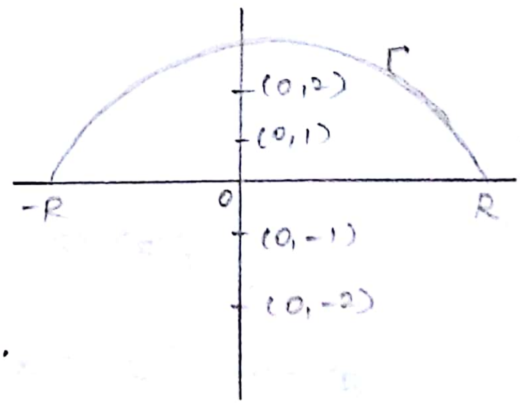
2) using contour integration, evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

soln:

consider, $$\int_c f(z)\,dz = \int_c \frac{z^2}{(z^2+4)(z^2+1)} \longrightarrow (1)$$

where c consists of the semicircle $\Gamma : |z| = R$ and bounding diameter $[-R, R]$

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz \longrightarrow (2)$$



To find $\int_c f(z)\,dz$.

$$\int_c \frac{z^2}{(z^2+1)(z^2+4)}\,dz.$$

Poles are $(z^2+1)(z^2+4) = 0$.

$z^2+1 = 0$ | $z^2+4 = 0$

$z = \pm i$ | $z \pm 2i$

✓ $z = i$ is a simple pole lies inside Γ.

$z = -i$ is a simple pole lies outside Γ.

✓ $z = 2i$ is a simple pole lies inside Γ

$z = -2i$ is a simple pole lies outside Γ.

$$\text{Res}[f(z) : z = i] = \lim_{z \to i} \cancel{(z-i)} \frac{z^2}{(z+i)\cancel{(z-i)}(z+2i)(z-2i)}$$

$$= \frac{i^2}{(i+i)(i+2i)(i-2i)}$$

$$= \frac{i^2}{(2i)(3i)(-i)} = \frac{i^2}{-6i^3} = \frac{-1}{+6i}$$

$$\text{Res}[f(z) : z = 2i] = \lim_{z \to 2i} \cancel{(z-2i)} \frac{z^2}{(z+i)(z-i)(z+2i)\cancel{(z-2i)}}$$

$$= \frac{(2i)^2}{(3i)(i)(4i)}$$

$$= \frac{-4}{12 i^3} = \frac{\cancel{-4}}{\cancel{-12}_3 i} = \frac{1}{3i}$$

Cauchys Residue theorem,

$$\int_c f(z)\,dz = 2\pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right]$$

$$= \frac{2\pi \cancel{i}}{\cancel{i}}\left[\frac{1}{3} - \frac{1}{6}\right] = 2\pi\left(\frac{2-1}{6}\right)$$

$$= \cancel{2}\pi\left(\frac{1}{\cancel{6}_3}\right)$$

$$\int_c f(z)\,dz = \frac{\pi}{3}.$$

$$② \Rightarrow \int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$\int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = \frac{\pi}{3}$$

If $R \to \infty$, then $\int_\Gamma f(z)\,dz \to 0$.

$$\int_{-R}^{R} f(x)\,dx = \frac{\pi}{3} \quad \text{(ie)} \quad \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$$

3) P.T $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{a+b}.$

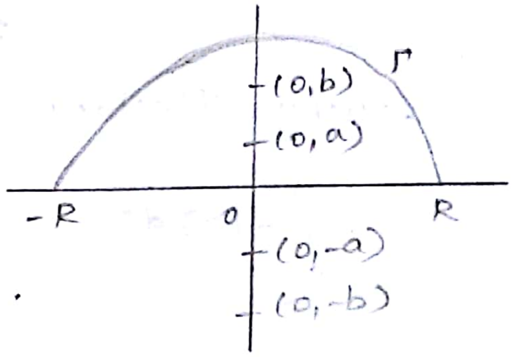
soln:

consider, $\displaystyle\int_c f(z)\,dz = \int_c \frac{z^2}{(z^2+a^2)(z^2+b^2)}\, dz. \longrightarrow (1)$

where, c consists of the semicircle $\Gamma : |z| = R$ and bounding diameter $[-R, R]$

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz \longrightarrow (2)$$

To find, $\displaystyle\int_c f(z)\,dz.$



$$\int_c \frac{z^2}{(z^2+a^2)(z^2+b^2)}\, dz$$

Poles are $(z^2+a^2)(z^2+b^2) = 0.$

$z^2+a^2 = 0$ | $z^2+b^2 = 0.$

$z = \pm ai$ | $z = \pm bi$

$z = ai$ is a simple pole lies inside Γ.

$z = -ai$ is a simple pole lies outside Γ.

$z = bi$ is a simple pole lies inside Γ.

$z = -bi$ is a simple pole lies outside Γ.

$$Res[f(z) : z = ai] = \lim_{z\to ai} \cancel{(z-ai)} \frac{z^2}{(z+ai)\cancel{(z-ai)}(z+bi)(z-bi)}$$

$$= \frac{-a^2}{(2ai)(ai+bi)(ai-bi)}$$

$$= \frac{-a^2}{2ai(a^2i^2 - b^2i^2)} = \frac{-\cancel{a}^2}{2\cancel{a}i(-a^2+b^2)}$$

$$= \frac{-a}{2i(+b^2-a^2)} = \frac{\cancel{+}a}{\cancel{+}2i(a^2-b^2)} = \frac{a}{2i(a^2-b^2)}$$

$$\text{Res}[f(z) : z = bi] = \lim_{z \to bi} \cancel{(z-bi)} \frac{z^2}{(z+ai)(z-ai)(z+bi)\cancel{(z-bi)}}$$

$$= \frac{-b^2}{(bi+ai)(bi-ai)(2bi)}$$

$$= \frac{-b^2}{(b^2i^2 - a^2i^2)(2bi)} = \frac{-\cancel{b^2}}{(-b^2+a^2)(2\cancel{b}i)}$$

$$= \frac{-b}{2i(a^2-b^2)}$$

cauchys Residue theorem,

$$\int_c f(z)\, dz = 2\pi i \left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)} \right]$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{2i}(a^2-b^2)} (a-b)$$

$$= \frac{\pi\, \cancel{(a-b)}}{\cancel{(a-b)}(a+b)} = \frac{\pi}{a+b} \rightarrow (3)$$

using (3) in (2),

$$\int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = \frac{\pi}{a+b}.$$

If $R \to \infty$, then $\int_{\Gamma} f(z)\, dz \to 0$.

$$\therefore \int_{-R}^{R} f(x)\, dx = \frac{\pi}{a+b}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi}{a+b}.$$

(ie) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{a+b}$ //

4) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $(a,b>0)$

soln:

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

consider $\int_c f(z)\,dz = \int_c \frac{dz}{(z^2+a^2)(z^2+b^2)} \longrightarrow ①$

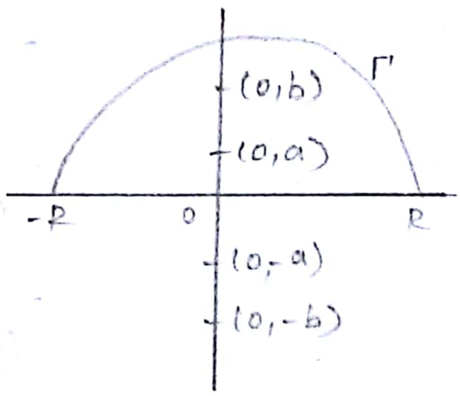
where c is the upper half of the semicircle Γ with boun--ding diameter $[-R, R]$.

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

If $R\to\infty$, $\int_\Gamma f(z)\,dz \longrightarrow 0$,

Then, $\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx. \longrightarrow ②$

To find $\int_c \frac{dz}{(z^2+a^2)(z^2+b^2)}$



$$(z^2+a^2)(z^2+b^2) = 0$$

$z^2+a^2=0$	$z^2+b^2=0.$
$z=\pm ai$	$z=\pm bi$

✓ $z = ai$ is a simple pole lies inside Γ.

$z=-ai$ is a simple pole lies outside Γ.

✓ $z= bi$ is a simple pole lies inside Γ.

$z=-bi$ is a simple pole lies outside Γ.

$$\text{Res}[f(z) : z=ai] = \lim_{z\to ai} \cancel{(z-ai)} \times \frac{1}{\cancel{(z-ai)}(z+bi)(z+ai)(z-bi)}$$

$$= \frac{1}{(2ai)(ai+bi)(ai-bi)} = \frac{1}{2ai(-a^2+b^2)}$$

(ie) $\text{Res}[f(z) : z = ai] = \dfrac{-1}{2ai(a^2-b^2)}$

$$\text{Res}[f(z) : z=bi] = \lim_{z \to bi} \cancel{(z-bi)} \times \frac{1}{(z-ai)(z+ai)(z+bi)\cancel{(z-bi)}}$$

$$= \frac{1}{(bi-ai)(bi+ai)(2bi)}$$

$$= \frac{1}{2bi(a^2-b^2)}$$

cauchys Residue theorem,

$$\int_c f(z)\,dz = 2\pi i \,(\text{sum of the Residues})$$

$$= 2\pi i \left[\frac{-1}{2ai(a^2-b^2)} + \frac{1}{2bi(a^2-b^2)}\right]$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{2}\cancel{i}} \left[\frac{-1}{a(a^2-b^2)} + \frac{1}{b(a^2-b^2)}\right]$$

$$= \frac{\pi}{(a^2-b^2)} \left[-\frac{1}{a} + \frac{1}{b}\right]$$

$$= \frac{\pi}{(a^2-b^2)} \left[\frac{-b+a}{ab}\right]$$

$$= \frac{\pi}{(a+b)\cancel{(a-b)}} \left[\frac{\cancel{(a-b)}}{ab}\right]$$

$$\int_c f(z)\,dz = \frac{\pi}{ab(a+b)}$$

$$\therefore (2) \Rightarrow \int_{-R}^{R} f(x)\,dx = \frac{\pi}{ab(a+b)}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

5) Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx$ by using contour integration

Soln:

Consider $\int_c f(z)\,dz = \int_c \frac{z^2-z+2}{z^4+10z^2+9}\,dz \longrightarrow$ (1).

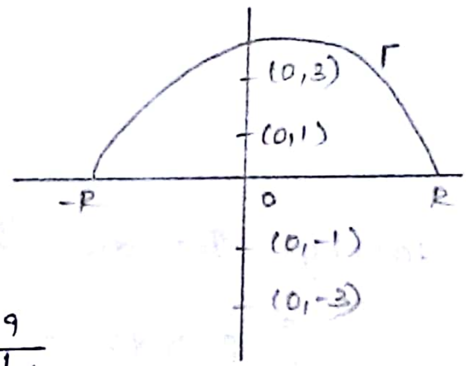
where c is the semicircle Γ and its bounding diameter $[-R, R]$.

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

If $R \to \infty$, then. $\int_\Gamma f(z)\,dz$.

Now, $\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx \longrightarrow$ (2)

To find $\int_c \frac{z^2-z+2}{z^4+10z^2+9}\,dz$.



$z^4 + 10z^2 + 9 = 0.$

Consider $r = z^2$.

$r^2 + 10r + 9 = 0.$

$(r+9)(r+1) = 0.$

$(z^2+9)(z^2+1) = 0.$

$z = \pm 3i, \quad z = \pm i$

$z = 3i$ is a simple pole. which lies inside Γ.

$z = -3i$ is a simple pole which lies outside Γ.

$z = i$ is a simple pole which lies inside Γ.

$z = -i$ is a simple pole which lies outside Γ.

$$\operatorname{Res}[f(z) : z = i] = \lim_{z \to i} \frac{\cancel{(z-i)}\ (z^2 - z + 2)}{\cancel{(z-i)}(z+i)\ (z-3i)\ (z+3i)}$$

$$= \frac{i^2 - i + 2}{(2i)(-2i)(4i)} = \frac{-1 - i + 2}{-16\, i^3}.$$

$$= \frac{1-i}{16i}$$

$$\operatorname{Res}[f(z) : z = 3i] = \lim_{z \to 3i} \frac{\cancel{(z-3i)} \times (z^2 - z + 2)}{(z-i)(z+i)\cancel{(z-3i)}(z+3i)}$$

$$= \frac{9i^2 - 3i + 2}{(2i)(4i)(6i)}$$

$$= \frac{-9 - 3i + 2}{48 i^3} = \frac{\cancel{-}(7+3i)}{\cancel{-}48i}$$

$$= \frac{7+3i}{48i}$$

Cauchy's Residue theorem,

$$\int_c f(z)\, dz = 2\pi i \left[\frac{1-i}{16i} + \frac{7+3i}{48i} \right]$$

$$= 2\pi i \left[\frac{3(1-i) + 7+3i}{48i} \right]$$

$$= \frac{2\pi i}{48i} \left[3 - \cancel{3i} + 7 + \cancel{3i} \right]$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{48}\cancel{i}} (\cancel{10}) \quad 24 \; 12 \quad 5 = \frac{5\pi}{12}$$

$$\therefore \int_{-R}^{R} f(x)\, dx = \frac{5\pi}{12}.$$

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx = \frac{5\pi}{12}$$

6) S.T $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

Soln:

$$\int_c f(z)\,dz = \int_c \frac{dz}{(z^2+1)^3} \longrightarrow (1)$$

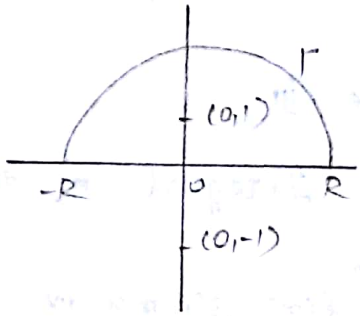
where c is the upper half of the semicircle and bounding diameter $[-R, R]$.

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

If $R \to \infty$, then $\int_\Gamma f(z)dz \to 0$.

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx \longrightarrow (2).$$

To find, $\displaystyle\int_c \frac{dz}{(z^2+1)^3}$



$(z^2+1)^3 = 0$.

$z^2+1 = 0$

$z = \pm i$ $(n = 3)$.

✓ $z = i$ is a ~~simple~~ pole of order 3 lies inside Γ.

$z = -i$ is a pole of order 3 lies outside Γ.

$$\text{Res}[f(z) : z = i] = \lim_{z \to i} \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} (z-i)^3 \cdot \frac{1}{(z-i)^3(z+i)^3}$$

$$= \lim_{z \to i} \frac{1}{2!} \frac{d^2}{dz^2} \cdot \frac{1}{(z+i)^3}$$

$$= \frac{1}{2} \lim_{z \to i} \frac{d}{dz}\left[\frac{-3}{(z+i)^4}\right]$$

$$= \frac{1}{2} \lim_{z \to i} \left[\frac{12}{(z+i)^5}\right]$$

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$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{12}^{6}}{(2i)^5}\right] = \frac{\cancel{6}\,3}{\cancel{32}_{16}\, i^5}$$

$$= \frac{3}{16i}$$

Cauchys Residue theorem,

$$\int_c f(z)\,dz = \cancel{2}\pi\cancel{i}\left[\frac{3}{\cancel{16}_{8}\cancel{i}}\right]$$

$$= \frac{3\pi}{8}.$$

$$\int_{-R}^{R} f(x)\,dx = \frac{3\pi}{8}.$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8} \,//$$

Type – III

Integral of the form $\int_{-\infty}^{\infty} f(x)\cos mx\,dx$ (or)

$\int_{-\infty}^{\infty} f(x)\sin mx\,dx$, where $f(x) \to 0$ as $x \to \infty$

Problems:

1) Evaluate $\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx$, $a > 0$.

soln:

$$\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx \longrightarrow (*)$$

To find, $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx$.

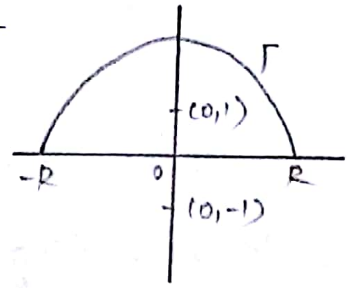
consider, $\int_c f(z)\,dz = \int_c \frac{\cos az}{z^2+1}\,dz \longrightarrow (1)$

where c is the upper half of the semicircle Γ and bounding diameter $[-R, R]$

- Let $e^{i\theta} = \cos\theta + i\sin\theta$.

$$e^{iaz} = \cos az + i\sin az$$

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$



If $R \to \infty$, $\int_\Gamma f(z)\,dz \to 0$.

$$\therefore \int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx \longrightarrow (2)$$

$$\int_c \frac{\cos az}{z^2+1}\,dz = RP \underbrace{\int_c \frac{e^{iaz}}{z^2+1}\,dz} \longrightarrow (3)$$

- Let $z^2+1 = 0$.

$$z = \pm i.$$

$z = i$ is a simple pole lies inside Γ.

$z = -i$ is a simple pole lies outside Γ.

$$\text{Res}\,[f(z) : z = i] = \lim_{z \to i} \cancel{(z-i)} \frac{e^{iaz}}{\cancel{(z-i)}(z+i)}$$

$$= \frac{e^{-a}}{2i}$$

cauchys Residue theorem,

$$\int_c f(z)\,dz = 2\pi \cancel{i} \left(\frac{e^{-a}}{2\cancel{i}}\right)$$

$$= \pi e^{-a}$$

(*) becomes, $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}$ //

2) S.T $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{\pi}{\cancel{2}}\left[\frac{\cancel{2} - e^{-ma}}{a^2}\right]$

soln

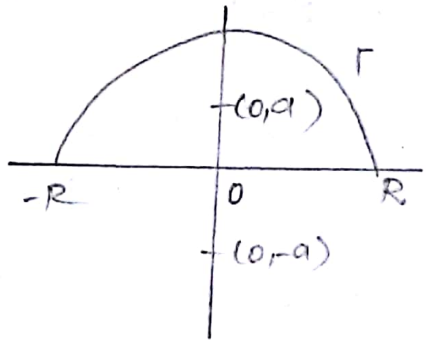
$$\int_c f(z)\,dz = \int_c \frac{\sin mz}{z(z^2+a^2)}\,dz = I.P \int_c \frac{e^{imz}}{z(z^2+a^2)}\,dz. \longrightarrow ①$$

W.K.T $$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz \longrightarrow ②$$

where c is the upper half of the semicircle Γ with its bounding diameter $[-R,R]$

If $R\to\infty$, $\int_\Gamma f(z)\,dz$.

②, $$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx.$$



To find $I.P \displaystyle\int_c \frac{e^{imz}}{z(z^2+a^2)}\,dz.$

$$z(z^2+a^2) = 0.$$

$$z = 0, \quad z = \pm ai.$$

$z=0$ is a simple pole lies on the real axis

$z = ai$ is a simple pole lies inside Γ.

$z = -ai$ is a simple pole lies outside Γ.

$$\text{Res}\,[f(z) : z=0] = \lim_{z\to 0} (z-0)\frac{e^{imz}}{\cancel{z}(z+ai)(z-ai)}$$

$$= \frac{e^0}{(0+a^2)} = \frac{1}{a^2}.$$

$$\text{Res}\left[f(z) : z = ai\right] = \lim_{z \to ai} \cancel{(z-ai)} \frac{e^{imz}}{z(z+ai)\cancel{(z-ai)}}$$

$$= \frac{e^{im(ai)}}{(ai)(2ai)} = \frac{e^{-ma}}{-2a^2}$$

cauchys Residue theorem,

$$\int_C f(z)\,dz = 2\pi i \,(\text{sum of Residues}) + \pi i \,(\text{sum of Res/: on the real axis})$$

$$= \cancel{2}\pi i \left(\frac{e^{-ma}}{-\cancel{2}a^2}\right) + \pi i \left(\frac{1}{a^2}\right)$$

$$= \pi i \left[\frac{1 - e^{-ma}}{a^2}\right]$$

$$\text{IP} \int_C \frac{e^{imz}}{z(z^2+a^2)}\,dz = \frac{\pi}{a^2}\left(1 - e^{-ma}\right)$$

Taylor and Laurent Expansions.

Taylor Series :

A function $f(z)$, analytic inside a circle c with centre at 'a', can be expanded in the series.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$
$$\cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots \infty$$

which is convergent at every point inside c.

Note :

i) Taking $a = 0$, Taylor's series reduces to

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \cdots \text{ to } \infty$$

which is known as Maclaurins series.

2) There is no negative powers of $(z-a)$.

Laurent's Series :

Let c_1 and c_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

Let $f(z)$ be analytic on c_1 and c_2 and in the annular region R between them. Then, for any point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where,
$$a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

Note :

1) If $f(z)$ is analytic inside c_2, then the Laurent's series residues to the Taylor's series of $f(z)$ with centre 'a', since in this case all the coefficients of negative powers in Laurent's are zero.

2) The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ consisting of positive integral powers of $(z-a)$ is called the analytic part of Laurent's series.

The part $\sum_{n=1}^{\infty} b_n (z-a)^n$ consisting of negative integral powers of $(z-a)$ is called the principal part of Laurent's series.

Some important Results :

1) $(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$ when $|z| < 1$

2) $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$ when $|z| < 1$

3) $(1-z)^{-2} = 1 + 2z + 3z^2 + \cdots$ when $|z| < \infty$

4) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$ when $|z| < \infty$

5) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$

6) $e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$

Problem:

1) Expand $f(z) = \log(1+z)$ as a Taylor's series about $z=0$ if $|z| < 1$.

soln: Given $f(z) = \log(1+z)$

Function	Value at $z=0$.
$f(z) = \log(1+z)$	$f(0) = \log(1) = 0$.
$f'(z) = \frac{1}{1+z}$	$f'(0) = \frac{1}{1} = 1$
$f''(z) = \frac{-1}{(1+z)^2}$	$f''(0) = \frac{-1}{1} = -1$
$f'''(z) = \frac{2}{(1+z)^3}$	$f'''(0) = \frac{2}{1} = 2$

Taylor's series about $z=0$ is,

$$f(z) = f(0) + f'(0)\frac{z}{1!} + f''(0)\frac{z^2}{2!} + f'''(0)\frac{z^3}{3!} + \cdots$$

$$= 0 + 1(z) + (-1)\frac{z^2}{2!} + 2\left(\frac{z^3}{3!}\right)$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

2) Expand $f(z) = \cos z$ about $z = \frac{\pi}{3}$ in Taylor's series

Soln:

Given $f(z) = \cos z$ about $z = \frac{\pi}{3}$

Function	Value at $z = \frac{\pi}{3}$
$f(z) = \cos z$	$f(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \cancel{\frac{\sqrt{3}}{2}}\ \frac{1}{2}$
$f'(z) = -\sin z$	$f'(\frac{\pi}{3}) = -\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$
$f''(z) = -\cos z$	$f''(\frac{\pi}{3}) = -\cos(\frac{\pi}{3}) = -\frac{1}{2}$
$f'''(z) = \sin z$	$f'''(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

Taylor's series about $z = \frac{\pi}{3}$ is

$$f(z) = f(a) + f'(a)\frac{(z-a)}{1!} + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \cdots$$

$z = \frac{\pi}{3}$,

$$f(x) = f(\tfrac{\pi}{3}) + f'(\tfrac{\pi}{3})(z-\tfrac{\pi}{3}) + f''(\tfrac{\pi}{3})\frac{(z-\pi/3)^2}{2!} + \cdots$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}(z-\tfrac{\pi}{3}) - (\tfrac{1}{2})(\tfrac{1}{2!})(z-\tfrac{\pi}{3})^2 + \cdots$$

3) obtain Taylor's series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the region $|z| < 2$

Soln:

Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

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$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{(-5z-7)}{(z+2)(z+3)} \rightarrow ①$$

$$(z+2)(z+3) \quad z^2+5z+6 \overline{\big) \; z^2-1 \;} \; 1$$
$$\underline{\cancel{z^2} + 5z + 6}$$
$$-5z - 7$$

consider $\dfrac{-5z-7}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3} \rightarrow ②$.

$$A(z+3) + B(z+2) = -5z - 7$$

Put $z = -3$

$$A(0) + B(-3+2) = -5(-3) - 7 .$$

$$-B = 15 - 7 = 8$$

$$\boxed{B = -8}$$

Put $z = -2$

$$A(-2+3) + B(0) = -5(-2) - 7 .$$

$$A(1) + 0 = 10 - 7 .$$

$$\boxed{A = 3}$$

$\therefore$ ① $\Rightarrow$ $f(z) = 1 + \dfrac{(-5z-7)}{(z+2)(z+3)}$

from ② $\Rightarrow$ $f(z) = 1 + \dfrac{A}{z+2} + \dfrac{B}{z+3}$.

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} . \rightarrow ③$$

Given $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

Also, from ③ $\Rightarrow \left|\frac{z}{3}\right| < 1$.

$$\therefore f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$- \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left[\left(\frac{3}{2}\right)\left(\frac{z}{2}\right)^n - \left(\frac{8}{3}\right)\left(\frac{z}{3}\right)^n\right]$$

$$f(z) = 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n$$

---

4) Find the Taylor's expansion about $z=0$ of

$$f(z) = \frac{z}{(z+1)(z+3)}$$

soln:

Let $$f(z) = \frac{z}{(z+1)(z-3)}$$

consider, $$\frac{z}{(z+1)(z-3)} = \frac{A}{z+1} + \frac{B}{z-3}$$

$$A(z-3) + B(z+1) = z$$

Put $z = 3$.

$$A(0) + B(4) = 3.$$

$$\boxed{B = 3/4}$$

$z>1$

$z<1$

Put $z = -1$

$$A(-1-3) + B(0) = -1$$

$$\cancel{-}4A = \cancel{-}1$$

$$\boxed{A = 1/4}$$

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$$f(z) = \frac{1}{4(z+1)} + \frac{3}{4(z-3)}$$

Given $z=0$, $|z|<1. \Rightarrow |\frac{z}{3}|<1.$

$$f(z) = \frac{1}{4}(1+z)^{-1} + \frac{\cancel{3}}{-4(\cancel{3})(1-z/3)}$$

$$= \frac{1}{4}(1+z)^{-1} - \frac{1}{4}(1-z/3)^{-1}$$

$$= \frac{1}{4}\left[(1+z)^{-1} - (1-\frac{z}{3})^{-1}\right]$$

$$= \frac{1}{4}\left[(1-z+z^2-z^3+\cdots) - (1+\frac{z}{3}+(\frac{z}{3})^2+\cdots)\right]$$

$$= \frac{1}{4}[$$

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5) Find the Laurent's series expansion of $f(z) = \frac{1}{z^2+3z+2}$ in the region $1<|z|<2$.

soln

Given $f(z) = \frac{1}{z^2+3z+2} = \frac{1}{(z+1)(z+2)} \longrightarrow (1)$

$$\frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$A(z+2) + B(z+1) = 1. \longrightarrow (2)$$

Put $z=-2$ in (2)

$$A(0) + B(-2+1) = 1.$$

$$-B = 1.$$

$$\boxed{B=-1}$$

Put $z = -1$ in (2)

$$A(-1+2) + B(0) = 1$$

$$\boxed{A = 1}$$

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

Given $1 < |z| < 2$.

Take, $1 < |z|$ and $|z| < 2$

$\frac{1}{|z|} < 1$ and $\left|\frac{z}{2}\right| < 1$.

(ie) $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

$$= \frac{1}{z(1+\frac{1}{z})} - \frac{1}{2(1+\frac{z}{2})}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$- \frac{1}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \cdots\right]$$

---

b) Expand $f(z) = \frac{1}{z(z-1)}$ as Laurent's series in powers of z and state the respective region of validity.

soln:

$$f(z) = \frac{1}{z(z-1)}$$

$f(z)$ is not analytic at $z=0$ and $z=1$.

The function $f(z)$ is not analytic at $z=0$ and $z=1$. but it is analytic in the region - $0 < |z| < 1$.

- $|z| > 1$.

<u>case (i)</u> : $0 < |z| < 1$.

Now $\frac{1}{z(z-1)} = \frac{-1}{z(1-z)} = \frac{-1}{z}(1-z)^{-1}$.

$$= \frac{-1}{z}\left[1 + z + z^2 + \cdots\right]$$

$$= -\left[\frac{1}{z} + 1 + z + z^2 + \cdots\right]$$

the region of validity of this expansion is $0 < |z| < 1$

<u>case -(ii)</u> : $|z| > 1$

$\Rightarrow \quad 1 < |z|. \quad \Rightarrow \quad \left|\frac{1}{z}\right| < 1.$

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2(1-1/z)}$$

$$= \frac{1}{z^2}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

The region of validity of this expansion is $|z| > 1$.

---

7) obtain Laurent's expansion for $f(z) = \frac{1}{(z-1)(z-2)}$ valid in the region (i) $|z-1| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 2$

<u>soln</u> :

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$A(z-2) + B(z-1) = 1$$

Put $z = 2$

$$A(0) + B(2-1) = 1.$$

$$\boxed{B = 1}$$

Put $z = 1$.

$$A(1-2) + B(0) = 1.$$

$$-A = 1$$

$$\boxed{A = -1}$$

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

(i) $|z-1| < 1$.

Let $u = z-1$.

(ie) $|u| < 1$.

$$f(z) = \frac{-1}{u} + \frac{1}{u-1}$$

$z-2 = (z-1)-1$
$\Rightarrow (u-1)$

$$= \frac{-1}{u} - \frac{1}{1-u}.$$

$$= \frac{-1}{u} - (1-u)^{-1}$$

$$= \frac{-1}{u} - (1 + u + u^2 + u^3 + \cdots)$$

$$= \frac{-1}{z-1} - (1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots)$$

$$= -\left[\frac{1}{z-1} + 1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots\right]$$

is valid in the region $|z-1| < 1$.

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(ii) $1 < |z| < 2$.

(ie) Take $1 < |z|$ and $|z| < 2$.

(ie) $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$.

$$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(\frac{z}{2}-1\right)}$$

$$= \frac{-1}{z(1-1/z)} + \frac{1}{-2(1-z/2)}$$

$$= -\frac{1}{z}(1-1/z)^{-1} - \frac{1}{2}(1-z/2)^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right] - \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$

Valid in the region $1 < |z| < 2$.

(iii) $|z| > 2$.

$\Rightarrow$ $2 < |z|$

$\Rightarrow$ $\left|\frac{2}{z}\right| < 1$.

Also $\left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= -\frac{1}{z}(1-1/z)^{-1} + \frac{1}{z}(1-2/z)^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right] + \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right]$$

is valid in the region $|z| > 2$.

8) Find the Laurent's series expansion of the function $\frac{z-1}{(z+2)(z+3)}$ valid in the region $2<|z|<3$

Soln:

$$f(z) = \frac{z-1}{(z+2)(z+3)}$$

$$\frac{z-1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$z-1 = A(z+3) + B(z+2)$$

Put $z=-3$

$$-4 = A(0) + B(-1)$$

$$-B = -4$$

$$\boxed{B=4}$$

Put $z=-2$.

$$-3 = A(1) + B(0).$$

$$\boxed{A=-3}$$

$$\therefore f(z) = \frac{z-1}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{4}{z+3}$$

Given $2<|z|<3$

$2<|z|$ and $|z|<3$.

$\left|\frac{2}{z}\right|<1$ $\quad \left|\frac{z}{3}\right|<1$

$$f(z) = \frac{-3}{z\left(1+\frac{2}{z}\right)} + \frac{4}{3\left(\frac{z}{3}+1\right)}$$

$$= \frac{-3}{z}\left(1+2/z\right)^{-1} + \frac{4}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{-3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right] + \frac{4}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

9) Find the Taylor's series and Laurent's series which represents the function $\frac{z}{(z+1)(z+2)}$ in (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $|z+1|<1$

Soln:

$$f(z) = \frac{z}{(z+1)(z+2)}$$

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}.$$

$$A(z+2) + B(z+1) = z.$$

Put $z=-2$.

$$A(0) + B(-1) = -2.$$

$$-B = -2$$

$$\boxed{B = 2}$$

Put $z=-1$.

$$A(1) + B(0) = -1.$$

$$\boxed{A = -1}$$

(i) $|z|<1$. $\Rightarrow$ $\left|\frac{z}{2}\right|<1$

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{-1}{(1+z)} + \frac{\cancel{2}}{\cancel{2}(1+\frac{z}{2})}$$

$$= -1(1+z)^{-1} + (1+\frac{z}{2})^{-1}$$

$$= -1[1-z+z^2-z^3+\cdots] + \left[1-\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\cdots\right]$$

(ii) $1<|z|<2$.

Take $1<|z|$ and $|z|<2$.

$\Rightarrow$ $\left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{2}\right|<1$

$$f(z) = \frac{-1}{1+z} + \frac{2}{z+2}.$$

$$= \frac{-1}{z\left(\frac{1}{z}+1\right)} + \frac{\cancel{2}}{\cancel{2}\left(\frac{z}{2}+1\right)}.$$

$$= \frac{-1}{z}\left(1+ 1/z\right)^{-1} + \left(1+ z/2\right)^{-1}$$

$$= \frac{-1}{z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^{2}+\cdots\right] + \frac{\cancel{2}}{\cancel{4}}\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^{2}-\left(\frac{z}{2}\right)^{3} + \cdots\right]$$

(iii) $|z| > 2.$

$\Rightarrow \quad 2 < |z|.$

$$\frac{2}{|z|} < 1 \quad \Rightarrow \quad \left|\frac{2}{z}\right| < 1. \quad \Rightarrow \quad \left|\frac{1}{z}\right| < 1.$$

$$f(z) = \frac{-1}{1+z} + \frac{2}{z+2}.$$

$$= \frac{-1}{z(1+1/z)} + \frac{2}{z\left(1+\frac{2}{z}\right)}.$$

$$= -\frac{1}{z}\left(1+1/z\right)^{-1} + \frac{2}{z}\left(1+2/z\right)^{-1}.$$

$$= -\frac{1}{z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^{2}+\cdots\right] + \frac{2}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^{2}-\cdots\right]$$

(iv) $|z+1| < 1$

Let $u = z+1. \quad \Rightarrow \quad |u| < 1.$

$u+1 = z+2.$

$$f(z) = \frac{-1}{z+1} + \frac{2}{z+2} = \frac{-1}{u} + \frac{2}{u+1}$$

$$= \frac{-1}{u} + 2(1+u)^{-1}. = -\frac{1}{u} + 2\left[1-u+u^{2}-u^{3}+\cdots\right]$$

$$= \frac{-1}{z+1} + 2\left[1-(z+1)+(z+1)^{2}-(z+1)^{3}+\cdots\right]$$