Lecture 3 (14-03-2022)

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Example: Evaluate

ⓐ $\displaystyle\int_1^3\int_2^4 (40-2xy)\,dy\,dx$ ⓑ $\displaystyle\int_2^4\int_1^3 (40-2xy)\,dx\,dy$

Soln: $\displaystyle\int_1^3\int_2^4 (40-2xy)\,dy\,dx = \int_1^3 \left[40y - xy^2\right]_2^4 dx$

$$= \int_1^3 \left[(160-16x)-(80-4x)\right] dx$$

$$= \int_1^3 (80-12x)\,dx$$

$$= \left[80x - 6x^2\right]_1^3$$

$$= (240-54)-(80-6) = 112$$

ⓑ $\displaystyle\int_{y=2}^4\int_{x=1}^3 (40-2xy)\,dx\,dy = \int_2^4 \left[40x - yx^2\right]_1^3 dy$

$$= \int_2^4 \left[(120-9y)-(40-y)\right] dy$$

$$= \int_2^4 (80-8y)\,dy$$

$$= \left[80y - 4y^2\right]_2^4$$

$$= (320-64)-(160-16) = 112$$

$$= (320 - 64) - (160 - 16) = 112$$

We notice that in above example we obtained the same answers wheather we integrate w.r. to y or x as first.

Fubini's First theorem: (For constant limits) (for rectangular region)

If $f(x,y)$ is continuous on the rectangular region $R = \{(x,y) : a \le x \le b;\ c \le y \le d\}$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

<u>Example:</u> Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x,y) : 0 \le x \le 2,\ 1 \le y \le 2\}$

<u>soln:</u>

$$\iint_R (x-3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx$$

$$= \int_0^2 \left[xy - y^3\right]_1^2 dx$$

$$= \int_0^2 \left[(2x - 8) - (x - 1)\right] dx$$

$$= \int_0^2 (x - 7)\, dx$$

$$= \left[\frac{x^2}{2} - 7x\right]_0^2$$

$$= (2-14) = -12$$

$$\iint_R (x-3y^2)\,dA = \int_1^2\int_0^2 (x-3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_0^2 dy$$

$$= \int_1^2 (2-6y^2)\,dy$$

$$= \left[2y - 2y^3\right]_1^2$$

$$= (4-16) - (2-2)$$

$$= -12$$

$\boxed{\text{HW}}$ ① verify that $\displaystyle\int_1^2\int_0^1 (x^2+y^2)\,dx\,dy = \int_0^1\int_1^2 (x^2+y^2)\,dy\,dx$

<u>Answer</u>: $8/3$

② calculate $\displaystyle\iint_R (1-6x^2y)\,dA$, where

$R = \{(x,y) : 0 \le x \le 2, -1 \le y \le 1\}$

Answer $= 4$

$\boxed{\text{Ex:}}$ show that $\displaystyle\int_0^1\int_0^1 \frac{(x-y)}{(x+y)^3}\,dx\,dy \neq \int_0^1\int_0^1 \frac{(x-y)}{(x+y)^3}\,dy\,dx$

$\displaystyle\int\int \frac{(x-y)}{} \; dx \; dy$

Sol^n: Let $I_1 = \int_0^1 \int_0^1 \frac{(x-y)}{(x+y)^3}\, dx \, dy$

$\int \frac{dx}{x^2} = -\frac{1}{x}$

$\int \frac{dx}{x^3} = -\frac{1}{2x^2}$

$$= \int_0^1 \int_0^1 \frac{(x+y)-2y}{(x+y)^3}\, dx \, dy$$

$$= \int_0^1 \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right] dx \, dy$$

$$= \int_0^1 \left[-\frac{1}{(x+y)} + 2y \frac{}{2(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{(x+y)} + \frac{y}{(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[\left\{ -\frac{1}{y+1} + \frac{y}{(y+1)^2} \right\} - \left\{ -\cancel{\frac{1}{y}} + \cancel{\frac{1}{y}} \right\} \right) dy$$

$$= \int_0^1 \left[-\frac{1}{y+1} + \frac{(1+y)-1}{(y+1)^2} \right] dy$$

$$= \int_0^1 \left[-\cancel{\frac{1}{y+1}} + \cancel{\frac{1}{y+1}} - \frac{1}{(y+1)^2} \right) dy$$

$$= -\int_0^1 \frac{1}{(y+1)^2}\, dy$$

$$= \left[\frac{1}{y+1} \right]^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$= \left[\frac{1}{y+1}\right]_0 = \frac{1}{2} - 1 = -\,2$$

Again let $I_2 = \int_0^1\int_0^1 \frac{(x-y)}{(x+y)^3}\,dy\,dx$

$$= \int_0^1\int_0^1 \frac{x+y-2y-2x+2x}{(x+y)^3}\,dy\,dx$$

$$= \int_0^1\int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2}{(x+y)^2} + \frac{2x}{(x+y)^3}\right] dy\,dx$$

$$= \int_0^1\int_0^1 \left[-\frac{1}{(x+y)^2} + \frac{2x}{(x+y)^3}\right] dy\,dx$$

$$= \int_0^1 \left[\frac{1}{x+y} - \frac{x}{(x+y)^2}\right]_{y=0}^{1} dx$$

$$= \int_0^1 \left[\left\{\frac{1}{x+1} - \frac{x}{(x+1)^2}\right\} - \left\{\frac{1}{x} - \frac{1}{x}\right\}\right] dx$$

$$= \int_0^1 \left[\frac{1}{x+1} - \frac{x+1-1}{(x+1)^2}\right] dx$$

$$= \int_0^1 \left[\cancel{\frac{1}{x+1}} - \cancel{\frac{1}{(x+1)}} + \frac{1}{(x+1)^2}\right] dx$$

$$\int^1 \frac{dx}{\quad}$$

$$= \int_0^1 \frac{dx}{(1+x)^2}$$

$$= -\left[\frac{1}{(x+1)}\right]_0^1 = -\left(\frac{1}{2} - 1\right) = \frac{1}{2}$$

Hence $\displaystyle\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy \neq \int_0^1\int_0^1 \frac{(x-y)}{(x+y)^3}\,dy\,dx$

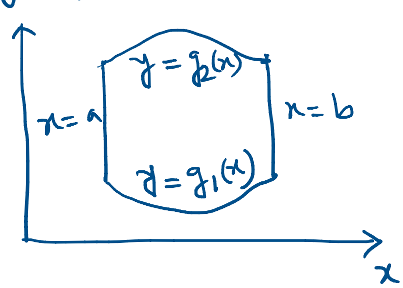
since the function or integrand has an infinite discontinuity at the origin $(0,0)$. Therefore, Fubini's theorem does not apply.

Evaluating double integral on non-rectangular domain:

(Stronger form of Fubini's theorem) (variable limits)

<u>Theorem</u>: Let $f(x,y)$ be a continuous function on the region R.

1. If R is defined by $a \le x \le b$; $g_1(x) \le y \le g_2(x)$ with $g_1(x)$ and $g_2(x)$ are continuous on $[a,b]$, then

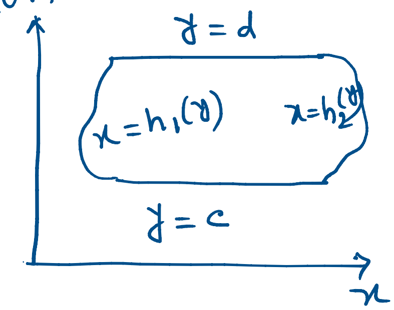


$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

2. If R is defined by $c \le y \le d$; $h_1(y) \le x \le h_2(y)$ with ... on $[c,d]$, then

$y = d$

2. If R is defined by $c \le y \le d$, $h_1(y)$ and $h_2(y)$ are continuous on $[c,d]$, then



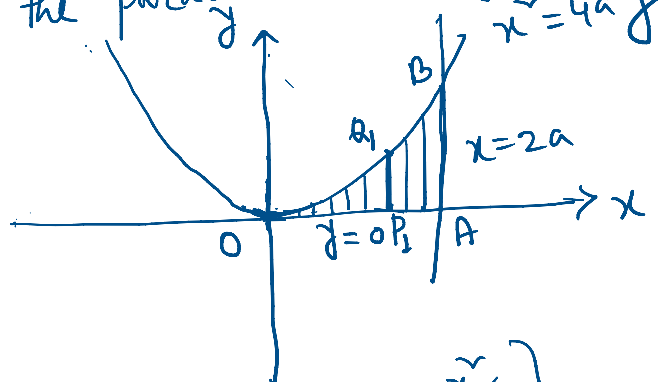
$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

$\boxed{\text{Note}}$ Integral with variable limits should be the innermost integral and it should be integrated first and then the constant limits.

<u>Example:</u> Evaluate $\iint_R xy\, dA$ where R is the domain bounded by x-axis, $\underline{x = 2a}$ and the parabola $x^2 = 4ay$

<u>Soln:</u>

$P_1(x, 0)$
$Q_1(x, \frac{x^2}{4a})$



$$\iint_R xy\,dA = \iint xy\, dy$$

$R =$ shaded region $OAB = \left\{(x,y) : 0 \le x \le 2a,\ 0 \le y \le \frac{x^2}{4a}\right\}$

$$\iint_R xy\, dA = \int_0^{2a} \int_0^{x^2/4a} xy\, dy\, dx$$

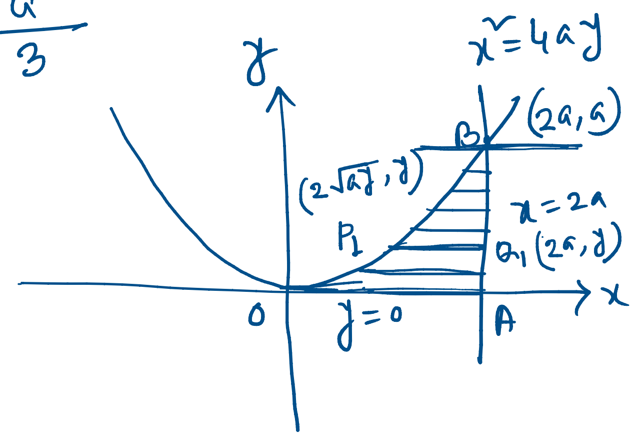
$$= \int_0^{2a} x \left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \int_0^{2a} x \cdot \frac{x^4}{\;}\, dx$$

$$= \int_0^{2a} x \cdot \frac{x^4}{32a^2}\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{1}{\cancel{32}a^2}\,\frac{\cancel{64}^{\,2}a^6}{\cancel{6}_{\,3}} = \frac{a^4}{3}$$



<u>Alternate:</u>

$$\iint_R xy\,dA = \iint_R xy\,dx\,dy$$

R = shaded region OAB = $\left\{(x,y):\ 0 \le y \le a;\ 2\sqrt{ay} \le x \le 2a\right\}$

$$\iint_R xy\,dA = \int_0^a \int_{2\sqrt{ay}}^{2a} xy\,dx\,dy$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{ay}}^{2a} dy$$

$$= \int_0^a \frac{y}{2}\left(4a^2-4ay\right) dy$$

$$= 2\int_0^a \left(a^2y - ay^2\right) dy$$

$$= 2\left[\frac{a^2y^2}{2} - \frac{ay^3}{3}\right]_0^a$$

a^4

$$= 2\left[\frac{a\,y}{2} - \frac{\quad}{3}\right]_0$$

$$= 2\left(\frac{a^4}{2} - \frac{a^4}{3}\right) = 2a^4\left(\frac{3-2}{6}\right) = \frac{a^4}{3}$$

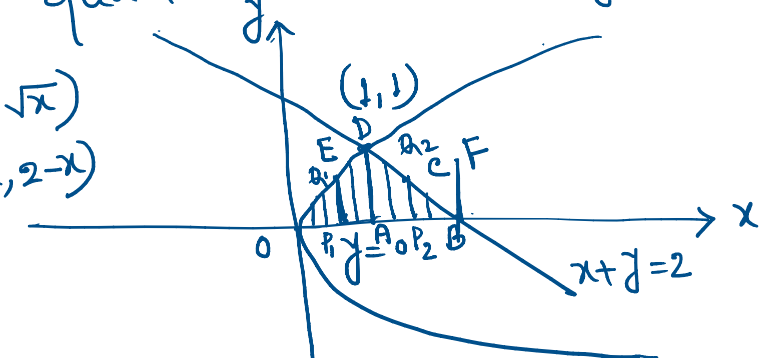
<u>Example:</u> Evaluate $\iint_R xy\, dA$, where R is the region bounded by the parabola $y^2 = x$ and the lines $y = 0$ and $x + y = 2$, lying in the first quadrant.

<u>Sol^n:</u> $x + y = 2$

$\Rightarrow \frac{x}{2} + \frac{y}{2} = 1$

$P_1(x, 0)$, $Q_1 = (x, \sqrt{x})$

$P_2(x, 0)$ $Q_2 = (x, 2-x)$



If the line is drawn the region OADE, the upper end of the line will lie on the parabola $y^2 = x$, on the other hand, if the line is drawn the region ABCD, the upper end of the line will lie on the line $x + y = 2$.

Hence in order to cover the entire region R, it should be divided into two, namely OADE and ABCD and the line P_1Q_1 move from y axis to AD and the line P_2Q_2 should move from AD to BF.

The given integral is given by

$$\iint_R xy\, dy\, dx = \int_0^1 \int_{y=0}^{\sqrt{x}} xy\, dy\, dx + \int_1^2 \int_0^{2-x} xy\, dy\, dx$$

$$\iint_R xy\,dy\,dx = \int_0 \int_{y=0} \quad + \int_1 \int_0$$

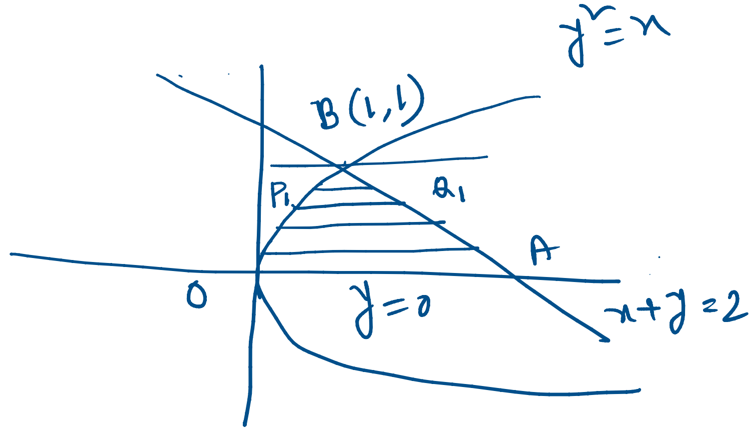
$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx + \int_1^2 x\left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^1 \frac{x}{2}(x)\,dx + \int_1^2 \frac{x}{2}(2-x)^2\,dx \qquad \Big|\, 4-4x+x^2$$

$$= \left[\frac{x^3}{6}\right]_0^1 + \frac{1}{2}\int_1^2 \left(4x - 4x^2 + x^3\right) dx$$

$$= \frac{1}{6} + \frac{1}{2}\left[2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4}\right]_1^2$$

$$= 3/8$$



Alternate procedure

$P_1(y^2, y)$

$Q_1(2-y, y)$

$$\iint_R xy\,dA = \int_{y=0}^{1}\int_{x=y^2}^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_{y^2}^{2-y} dy$$

$$= 3/8$$

[HW] Evaluate $\iint y\,dA$ over the region R bounded

$\boxed{\text{HW}}$ Evaluate $\iint_R y\, dA$ over the region bounded by $x=0$, $y=x^2$, $x+y=2$ in the first quadrant.

Answer: $\frac{16}{15}$