Unit - 2 (Probability Distribution)

Discrete Distn → Binomial
Poisson
Geometric

Continuous distn → Uniform
exponential
normal

★ Binomial Distribution

It has parameter n and p and its probability mass f^n is

$$P(x) = P(X=x) = {}^nC_x p^x q^{n-x} \quad (x = 0,1,2,\dots n)$$

p → prob. of success   where $q = 1-p$

n → no. of trials

★ Moment Generating f^n

$$M_x(t) = \sum e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, {}^nC_x p^x q^{n-x} = \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x} = (q+pe^t)^n$$

$$\left[(x+a)^n = \sum_{r} {}^nC_r x^{n-r} a^r \right]$$

To find Mean of Variance

$$E(X) = \left[M_x'(t) \right]_{t=0}$$

$$M_x'(t) = n(q+pe^t)^{n-1}(pe^t)$$

Put $t=0$,

$$= n(q+p)^{n-1} p \quad (\because p+q=1)$$

$$= np$$

$$E(X^2) = \left[M_x''(t) \right]_{t=0}$$

$$M_x'(t) = np(q+pe^t)^{n-1}e^t$$

$$M_x''(t) = np\left[(q+pe^t)^{n-1}e^t + e^t(n-1)(q+pe^t)^{n-2}pe^t\right]$$

Put $t=0$

$$= np\left[1+(n-1)p\right]$$

$$var(X) = E(X^2) - (E(X))^2$$

$$= np[1+np-p] - (np)^2$$

$$= np[q+np] - n^2p^2 \quad (\because 1-p = q)$$

$$\Rightarrow \quad Var(X) = npq$$

<u>Que</u> – The mean of a binomial distribution is 20 and standard deviation is 4. Find the parameters.

<u>Soln.</u> –

$$np = 20$$

$$\sqrt{npq} = 4 \Rightarrow npq = 16$$

$$q = \frac{16}{20} = \frac{4}{5}$$

$$p = 1-q = \frac{1}{5}$$

$$np = 20 \Rightarrow n\left(\frac{1}{5}\right) = 20$$

$$\boxed{n = 100}$$

<u>Que</u> – Out of 800 families with 4 children each, how many family would be expected to have

(i) 2 boys and 2 girls

(ii) atleast one boy

(iii) atmost 2 girls

(iv) children of both boys and girls

<u>Soln.</u> $X \rightarrow$ no. of boys

$$p = \frac{1}{2}, \; q = \frac{1}{2}, \; n = 4$$

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

(i) $P(X=2) = {}^4C_2 \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{12}{2} \times \frac{1}{16} = \frac{3}{8}$

For 800 family $= \frac{3}{8} \times 800 = 300$

(ii) P(atleast 1 boy) $= P(X \geq 1)$

$$= P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= 1 - P(X=0)$$

$$= 1 - \left[{}^4C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4 \right] = 1 - \frac{1}{16} = \frac{15}{16}$$

For 800 family $= \frac{15}{16} \times 800 = 750$

(iii) P(atmost 2 girls) $= P(0G, 1G, 2G)$

$$= P(X=4) + P(X=3) + P(X=2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[{}^4C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4 + {}^4C_1 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^3 \right]$$

$$= 1 - \left[\frac{1}{16} + \frac{1}{4} \right] = 1 - \frac{5}{16} = \frac{11}{16}$$

For 800 family $= 800 \times \frac{11}{16} = 550$

(iv) P(both boys & girls) $= 1 - P(\text{all boys, all girls})$

$$= 1 - [P(X=4) + P(X=0)]$$

$$= 1 - \left[{}^4C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^0 + {}^4C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4 \right] = \frac{7}{8}$$

For 800 family $= 800 \times \frac{7}{8} = 700$

<u>Que</u> - The mean and variance of a binomial distn. are 4 and $\frac{4}{3}$. Find $P(X \geq 1)$

<u>Soln.</u>

$$np = 4, \quad npq = \frac{4}{3}$$

$$q = \frac{1}{3}, \quad p = \frac{2}{3}$$

$$n = 6$$

$$P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - \left[{}^6C_0 \left(\tfrac{2}{3}\right)^0 \left(\tfrac{1}{3}\right)^6 \right] = 1 - \left[\frac{1}{729} \right] = \frac{728}{729} = 0.9$$

Que — A binomial variable X satisfies

$$9P(X=4) = P(X=2) \quad \text{when } n=6$$

Find the parameter p.

Soln. $P(X=x) = {}^nC_x \, p^x q^{n-x}$

$$9\left[{}^6C_4 \, p^4 q^2\right] = {}^6C_2 \, p^2 q^4$$

$$9p^2 = q^2$$

$$9p^2 = (1-p)^2$$

$$9p^2 = 1+p^2-2p$$

$$8p^2+2p-1 = 0$$

$$8p^2+4p-2p-1 = 0$$

$$4p(2p+1)-(2p+1) = 0$$

$$(4p-1)(2p+1) = 0$$

$$\boxed{p = \frac{1}{4}}$$

Que — The probability of a bomb hitting a target is $\frac{1}{5}$. Two bombs are enough to destroy. If the six bombs are aimed at the bridge, find the probability that the bridge is destroyed.

$$p = \frac{1}{5} \quad q = \frac{4}{5}, \quad n = 6$$

$$P(X \geq 2) = 1 - \left[P(X=0) + P(X=1)\right]$$

$$= 1 - \left[{}^6C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^6 + {}^6C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^5\right]$$

$$= 1 - \left[\frac{4096}{15625} + \frac{6}{5} \times \frac{1024}{3125}\right]$$

$$= 1 - \left[\frac{4096+6144}{15625}\right]$$

$$= \frac{15625-10240}{15625} = \frac{5385}{15625} = \frac{1077}{3125}$$

$$\therefore P(X \geq 2) = \frac{1077}{3125}$$

Que — An irregular 6 faced die is such that the pbty that it gives 3 even no. in 5 throws is twice the pbty that it gives 2 even no. in 5 throws. How many sets of exactly 5 trials can be expected to give no even no. of 2500 sets.

Soln. — Let X denote the even no.

$$n = 5$$

$$P(X=3) = 2P(X=2)$$

$$^5C_3\, p^3 q^2 = 2\left(^5C_2\, p^2 q^3\right)$$

$$10p^3q^2 = 20p^2q^3$$

$$p = 2q = 2(1-p)$$

$$p = \frac{2}{3}, \quad q = \frac{1}{3}$$

$$P(X=0) = {^5C_0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

For 2500 sets,

$$= 2500 \times \frac{1}{243} \simeq 10$$

Que — Fit the binomial distn for the following data

x	0	1	2	3	4	5	6
f	5	18	28	12	7	6	4

Soln. Theoretical freq. $= NP(x)$

$N \rightarrow$ Total freq/– $(N = 80)$

x	0	1	2	3	4	5	6	
f	5	18	28	12	7	6	4	= 80
fx	0	18	56	36	28	30	24	= 192

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{192}{80} = 2.4$$

$$np = 2.4 \quad (n = 6)$$

$$p = \frac{2.4}{6} = 0.4 \qquad q = 0.6$$

x	0	1	2	3	4	5	6

Convert into whole no.'s

$P(x) = {}^nC_x p^x q^{n-x}$	0·0466	0·1866	0·3170	0·276	0·138	0·036	0·004
$NP(x) = 80P$	3·732	14·92	25·36	22·11	11·05	2·88	0·328

x	0	1	2	3	4	5	6
f	4	15	25	22	11	3	0

$$P(0) = {}^6C_0 (0\cdot 4)^0 (0\cdot 6)^6$$
$$= 0\cdot 0466$$
$$P(1) = {}^6C_1 (0\cdot 4)^1 (0\cdot 6)^5 = 0\cdot 187$$
$$P(2) = {}^6C_2 (0\cdot 4)^2 (0\cdot 6)^4 = 0\cdot 3170$$
$$P(3) = {}^6C_3 (0\cdot 4)^3 (0\cdot 6)^3 = 0\cdot 276$$
$$P(4) = {}^6C_4 (0\cdot 4)^4 (0\cdot 6)^2 = 0\cdot 138$$
$$P(5) = {}^6C_5 (0\cdot 4)^5 (0\cdot 6)^1 = 0\cdot 036$$
$$P(6) = {}^6C_6 (0\cdot 4)^6 (0\cdot 6)^0 = 0\cdot 004$$

★ Poisson Distribution

Its pmf is $P(x) = P(X = x) = \dfrac{e^{-\lambda} \lambda^x}{x!}$ $(x = 0, 1, 2, ---)$ $\lambda > 0$

$\lambda \rightarrow$ parameter

Note

1) Poisson distn is a limiting case of binomial distn.

(i) n is large

(ii) p is small

(iii) np is finite (i.e., $np = \lambda$)

★ Mgf of poisson distn

$$M_X(t) = \sum e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$$

$$= e^{-\lambda} \left[1 + \frac{e^t \lambda}{1!} + \frac{(e^t \lambda)^2}{2!} + --- \right]$$

$$= e^{-\lambda} e^{e^t \lambda}$$
$$= e^{\lambda(e^t - 1)}$$

To find mean and variance

$$M_x(t) = e^{-\lambda} e^{e^t \lambda}$$
$$M_x'(t) = e^{-\lambda} e^{e^t \lambda} (\lambda e^t)$$

Put $t = 0$

$$= e^{-\lambda} e^{\lambda} (\lambda)$$
$$= \lambda$$

Mean $E(X) = [M_x'(t)]_{t=0}$
$$= \lambda$$

$$M_x'(t) = \lambda e^{-\lambda} (e^{\lambda e^t} e^t)$$
$$M_x''(t) = \lambda e^{-\lambda} [e^{\lambda e^t} e^t + e^t e^{\lambda e^t} \lambda e^t]$$
$$E(X^2) = [M_x''(t)]_{t=0} = \lambda e^{-\lambda} [e^{\lambda} + e^{\lambda}(\lambda)]$$
$$= \lambda e^{-\lambda} e^{\lambda} (1+\lambda) = \lambda(1+\lambda)$$
$$\text{var}(X) = E(X^2) - (E(X))^2$$
$$= \lambda(1+\lambda) - (\lambda)^2$$
$$= \lambda$$
$$S.D = \sqrt{\text{Var}(X)} = \sqrt{\lambda}$$

Que – The no. of accidents in a year attributed to taxi drivers in a city follows a poisson distn. with mean 3, out of 1000 taxi drivers, find the no. of drivers with (i) no accident in a year
(ii) more than 3 accident in a year

soln. – $n = 1000$, Mean $\lambda = 3$

$$P(X = x) = p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

Let $X \to$ no. of accidents in a year

(i) $P(X=0) = \dfrac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} = e^{-3} = 0.0498$

For 1000 taxi drivers $= 0.0498 \times 1000 = 49.8$

(ii) $P(X > 3) = 1 - P(X \leq 3)$

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{e^{-3}(3)^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!}$$

$$= 0.0498\left[1+3+\frac{9}{2}+\frac{27}{3!}\right] \qquad = 0.6474$$

$$P(X>3) = 1 - 0.6474 \quad = 0.3526$$

For 1000 drivers,

$$= 1000 \times 0.3526$$

$$= 35.26$$

Que – An insurance company found that 0.01% of the pbty is involved in a road accident in a city. If 1000 policy holders are randomly selected, what is the pbty that not more than 2 of their clients will be involved in such a accident in a year.

Soln. $p = 0.01\% \ = 0.0001$, $n = 1000$

$\lambda = np \ = 0.1$, $P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$X \rightarrow$ no. of clients will be involved in a accident

$$P(\text{not more than } 2) = P(X \le 2)$$

$$= P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!} + \frac{e^{-0.1}(0.1)^2}{2!}$$

$$= 0.985$$

Que – If X is a poisson random variable such that

$$P(X=2) = 9P(X=4) + 90P(X=6)$$

Find the variance.

Soln. $P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$$\frac{e^{-\lambda}\lambda^2}{2!} = \frac{9e^{-\lambda}\lambda^4}{4!} + \frac{90e^{-\lambda}\lambda^6}{6!}$$

$$\frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$\lambda^2 = 1, -4$$

$$\lambda = \pm 1 \,, \pm 2i$$

$$\lambda = 1$$

Que — Fit a poisson distⁿ for the following data

x	0	1	2	3	4	5
f	142	156	69	27	5	1

soln.

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400
fx	0	156	138	81	20	5	400

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{400}{400} = 1$$

Theoretical freq. $= NP(x)$

$$P(x) = \frac{e^{-\lambda}\lambda^2}{x!}$$

$$N = \Sigma f = 400$$

x	0	1	2	3	4	5
$P(x) = \frac{e^{-\lambda}\lambda^2}{x!}$	e^{-1} 0.3678	e^{-1} 0.3678	$e^{-1}/2$ 0.1839	$e^{-1}/6$ 0.0613	$e^{-1}/4!$ 0.0153	$e^{-1}/5!$ 0.0030
$NP(x) = 400\,P(x)$	147.15	147.15	73.57	24.54	6.13	1.23

convert into whole numbers with condⁿ. that $N = 400$

x	0	1	2	3	4	5
f	147	147	74	25	6	1

☆ Geometric distⁿ

Its pmf is

$$P(x) = P(X = x) = q^{x-1}p \quad (x = 1, 2, ---)$$

1^{st} $(x-1)$ trial — failure

x^{th} trial — success

Mgf

$$M_X(t) = E(e^{tX}) = \sum e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \sum_{x=1}^{\infty} e^{t(x-1)} q^{x-1} p \cdot e^{t}$$

$$= pe^{t} \sum_{x=1}^{\infty} (e^{t} q)^{x-1}$$

$$= pe^{t} \left[1 + e^{t}q + (e^{t}q)^2 + - - - \right]$$

$$= pe^{t} (1 - qe^{t})^{-1}$$

$$= \frac{pe^{t}}{1 - qe^{t}}$$

$$M_X(t) = \frac{pe^{t}}{1 - qe^{t}}$$

$$M_X'(t) = \frac{(1-qe^{t})pe^{t} - pe^{t}(-qe^{t})}{(1-qe^{t})^2}$$

Put $t = 0$

$$= \frac{(1-q)p + pq}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E(X) = \frac{1}{p}$$

$$M_X'(t) = \frac{pe^{t}}{(1-qe^{t})^2}$$

$$M_X''(t) = \frac{(1-qe^{t})^2 pe^{t} - pe^{t}(2)(1-qe^{t})(-qe^{t})}{(1-qe^{t})^4}$$

Put $t = 0$

$$= \frac{(1-q)^2 p + 2pq(1-q)}{(1-q)^4}$$

$$= \frac{p^2 + 2pq}{p^3} = \frac{p + 2q}{p^2} = \frac{1 - q + 2q}{p^2}$$

$$E(X^2) = \frac{1+q}{p^2}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

$p^2 \quad p \quad p$

★ Memoryless Property

If X has a geometric distⁿ, then for any +ve integers m and n

$$P\{X > m+n \mid X > m\} = P\{X > n\}$$

Que – If the probability that the applicant will pass the road test is 0.8. what is the probability that he will finally pass the test.

(i) on 4th trial

(ii) less than 4th trial

Soln. $P(x) = q^{x-1}p \quad (x = 1, 2, ---)$

$X \rightarrow$ Pass the test

$p = 0.8, \quad q = 0.2$

(i) $P(X=4) = q^3 p = (0.2)^3(0.8)$

$= 0.0064$

(ii) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= q^0 p + qp + q^2 p$

$= 0.8 + 0.2 \times 0.8 + (0.2)^2 0.8$

$= 0.8 + 0.16 + 0.032$

$= 0.992$

Que – Find the mean and variance of geometric distⁿ $P(X=x) = \frac{2}{3^x}$; $x = 1, 2, \ldots$

Soln. – $P(X=x) = \frac{2}{3^{x-1}} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^{x-1} \frac{2}{3}$ $\quad \left(P(X=x) = q^{x-1}p\right)$

$q = \frac{1}{3}$; $p = 1-q = \frac{2}{3}$

$\text{Mean} = \frac{1}{p} = \frac{3}{2}$

$\text{variance} = \frac{q}{p^2} = \frac{9}{3 \times 4} = \frac{3}{4}$

Que – The pbty to destroy the bridge is 0.6. What is the pbty that it should be destroyed by

(i) 5th attempt

(ii) more than 2 attempt

Soln. $X \rightarrow$ to destroy the bridge

$p = 0.6 \quad q = 0.4$

$P(X=x) = q^{x-1}p$

(i) $P(X=5) =$

(ii) $P(X>2) = 1 - P(X=1)$

<u>Que</u> – A and B shoot independently until each hits his own target. The pbty of hitting the target at each shot was 3/5 and 5/7 respectively. Find the probability that B will require more shots than A.

<u>Soln</u>. Let $X \to$ no. of trials required for A

$Y \to$ no. of trials required for B.

$$P(X=x) = q^{x-1}p$$

<u>For A</u> $p_1 = 3/5$, $q_1 = \frac{2}{5}$

<u>For B</u> $p_2 = 5/7$, $q_2 = 2/7$

$$P(Y>X) = \sum_{x=1}^{\infty} P(X=x \text{ and } Y = x+1, x+2, \ldots)$$

$$= \sum_{x=1}^{\infty} P(X=x)\,P(Y = x+1, x+2, \ldots)$$

$$= \sum_{x=1}^{\infty} P(X=x) \sum_{i=1}^{\infty} P(Y=x+i)$$

$$= \sum_{x=1}^{\infty} \left(\frac{2}{5}\right)^{x-1}\left(\frac{3}{5}\right) \sum_{i=1}^{\infty}\left(\frac{2}{7}\right)^{x+i-1}\left(\frac{5}{7}\right)$$

$$= \sum_{x=1}^{\infty}\left(\frac{2}{5}\right)^{x-1}\left(\frac{3}{5}\right)\left(\frac{2}{7}\right)^{x-1}\left(\frac{5}{7}\right)\sum_{i=1}^{\infty}\left(\frac{2}{7}\right)^{i}$$

$$= \frac{3}{7}\sum_{x=1}^{\infty}\left(\frac{4}{35}\right)^{x-1}\sum_{i=1}^{\infty}\left(\frac{2}{7}\right)^{i}$$

$$= \frac{3}{7}\left[1+\left(\frac{4}{35}\right)+\left(\frac{4}{35}\right)^2+---\right]\left[\frac{2}{7}+\left(\frac{2}{7}\right)^2+---\infty\right]$$

$$= \frac{3}{7}\left(1-\frac{4}{35}\right)^{-1}\left(\frac{2}{7}\right)\left(1+\frac{2}{7}+\left(\frac{2}{7}\right)^2+---\infty\right)$$

$$= \frac{3}{7}\left(1-\frac{4}{35}\right)^{-1}\left(\frac{2}{7}\right)\left(1-\frac{2}{7}\right)^{-1}$$

$$= \frac{6}{49}\left(\frac{35}{31}\right)\left(\frac{7}{5}\right) = \frac{6}{31}$$

☆ <u>Uniform Distribution</u> [Rectangular Distribution]

Its pdf is

$$f(x) = \frac{1}{\quad} \qquad (a \le x \le b)$$

$b-a$

☆ Moment generating function

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx = \frac{1}{b-a}\left(\frac{e^{tx}}{t}\right)_a^b = \frac{e^{bt}-e^{at}}{t(b-a)}$$

rth moment

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\, dx = \int_a^b x^r \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^{r+1}-a^{r+1}}{r+1}\right]$$

$r=1$

$$\text{(Mean)}\ \mu_1' = \frac{1}{b-a}\left(\frac{(b^2-a^2)}{2}\right) = \frac{b+a}{2}$$

$r=2$

$$\mu_2' = \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right]$$

$$= \frac{1}{b-a}\,\frac{(b-a)(b^2+ab+a^2)}{3} = \frac{b^2+ab+a^2}{3}$$

$$\text{variance} = \mu_2' - (\mu_1')^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{b^2+a^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4(b^2+a^2+ab) - 3(a^2+b^2+2ab)}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(b-a)^2}{12}$$

Que – If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$. Find $P(X<0)$.

Soln. Mean $= \frac{a+b}{2} = 1 \Rightarrow a+b = 2$ —— ①

variance $= \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a)^2 = 16$

$\Rightarrow b-a = \pm 4$ —— ②

$a+b = 2$ $\qquad$ $a+b=2$

$$b - a = 4$$

$$2b = 6$$

$$\boxed{b = 3 \Rightarrow a = -1}$$

$$b - a = -4$$

$$2b = -2$$

$b = -1 \Rightarrow a = 3$ (not possible since $a > b$)

$$P(X<0) = \int_{-\infty}^{0} f(x)\,dx$$

$$= \int_{-1}^{0} \frac{1}{4}\,dx \qquad \left(f(x) = \frac{1}{b-a} = \frac{1}{4} \right)$$

$$= \frac{1}{4}[x]_{-1}^{0} = \frac{1}{4}$$

Que – The no. of computer sold daily of a shop is uniformly distributed with minimum of 2000 pc and a max. of 4000 pc. Find the

(i) Pbty that sales fall between 2500 to 3000

(ii) Pbty that the shop will sale atleast 4000 pc

Soln. $a = 2000$, $b = 4000$ $\qquad X \rightarrow$ sales of computer

$$f(x) = \frac{1}{b-a} = \frac{1}{2000}$$

(i) $$P(2500 < X < 3000) = \int_{2500}^{3000} f(x)\,dx = \int_{2500}^{3000} \frac{1}{2000}\,dx = \frac{1}{4}$$

(ii) $$P(\text{atleast } 3000 \text{ Pc}) = P(X > 3000)$$

$$= \int_{3000}^{4000} f(x)\,dx = \int_{3000}^{4000} \frac{1}{2000}\,dx$$

$$= \frac{1000}{2000} = \frac{1}{2}$$

Que – Bus arrive at a stop at 15 min intervals starting at 7AM [i.e., 7, 7.15, 7.30, 7.45, -] If a passenger arrive at a stop is uniformly distributed between 7 and 7.30. Find the pbty that he will wait

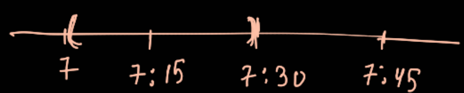
(i) less than 5 min

(ii) atleast 12 min for a bus

Soln. – $X \rightarrow$ arrival time of passenger at the stop after 7AM

It is uniformly distn over $(0, 30)$

$$f(x) = \frac{1}{30}$$



(i) P[wait less than 5 min]

(i) P[to wait less than 5 min]

$$= P\left[\begin{matrix}\text{arrival at the stop between} \\ 7{:}10 \text{ to } 7{:}15 \text{ or } 7{:}25 \text{ to } 7{:}30\end{matrix}\right] = P(10 < X < 15) + P(25 < X < 30)$$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx = \frac{1}{3}$$

(ii) P(atleast 12 min) = P(arrival of the stop between 7 to 7:03 or 7:15 to 7:18)

$$= P(0 < X < 3) + P(15 < X < 18)$$

$$= \int_{0}^{3} \frac{1}{30}\,dx + \int_{15}^{18} \frac{1}{30}\,dx = \frac{1}{5}$$

☆ Exponential Distribution

A continous RV X is said to follow an exponential distⁿ with parameter λ and its pdf is

$$f(x) = \lambda e^{-\lambda x} \quad (\lambda > 0)$$

Mgf

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{0}^{\infty} e^{tx}(\lambda e^{-\lambda x})\,dx = \lambda \int_{0}^{\infty} e^{-(\lambda - t)x}\,dx = \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty}$$

$$= \frac{\lambda}{\lambda - t}[0-1] = \frac{\lambda}{\lambda - t}$$

To find Mean and Variance

$$M_X(t) = \frac{\lambda}{\lambda - t} = \frac{\lambda}{\lambda\left(1 - \frac{t}{\lambda}\right)} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^3 + \cdots$$

(Mean) $\mu_1' = \text{coeff. of } \frac{t}{1!} \text{ in } M_X(t) = \frac{1}{\lambda}$

$\mu_2' = \text{coeff. of } \frac{t^2}{2!} \text{ in } M_X(t) = \frac{2}{\lambda^2}$

variance $\mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Memoryless Property of Exponential Distribution

If X is exponentially distributed then

$$P(X > s+t \mid X > s) = P(X > t)$$

Que – The mileage which car owners get with a certain kind of radial tyre is a r.v having an exponential distn with mean 40000 km. Find the pbty that one of the tyre will last

(i) atleast 20,000 km

(ii) atmost 30,000 km

Soln. $X \to$ kind of radial tyre

Mean $\frac{1}{\lambda} = 40,000 \Rightarrow \lambda = \frac{1}{40000}$

(i) $P(X > 20,000) = \int_{20000}^{\infty} f(x)\,dx$

$$= \int_{20000}^{\infty} \frac{1}{40000} e^{-\frac{1}{40000}x}\,dx = \frac{1}{40000}\left[\frac{e^{-x/40000}}{-1/40000}\right]_{20000}^{\infty}$$

$$= -\left[0 - e^{-1/2}\right] = e^{-0.5} = 0.6065$$

(ii) $P(X < 30000) = \int_{0}^{30000} \frac{1}{40000} e^{-x/40000}\,dx = \frac{1}{40000}\left[\frac{e^{-x/40000}}{-1/40000}\right]_{0}^{30000}$

$$= -\left[e^{-3/4} - 1\right] = 0.5270$$

Que – If X is r.v which follows an exponential distn with parameter λ and $P(X \le 1) = P(X > 1)$. Find Var(X)

Soln. – $P(X \le 1) = P(X > 1)$

$1 - P(X > 1) = P(X > 1)$

$P(X > 1) = \frac{1}{2}$

$\int_{1}^{\infty} f(x)\,dx = \frac{1}{2} \Rightarrow \int_{1}^{\infty} \lambda e^{-\lambda x}\,dx = \frac{1}{2}$

$\Rightarrow \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_{1}^{\infty} = \frac{1}{2} \Rightarrow -\left(0 - e^{-\lambda}\right) = \frac{1}{2}$

$\Rightarrow e^{-\lambda} = \frac{1}{2}$

$\Rightarrow -\lambda = -\log 2$

$\Rightarrow \lambda = \log 2$

$\text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{(\log 2)^2}$

Que – The time required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$.
(i) what is the pbty that the time exceeds 2 hr.
(ii) what is the pbty that a repair takes 10hr given that it extension exceeds 9 hr.

soln. – $\lambda = \frac{1}{2}$, $f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-x/2}$

$X \to$ repairing time

(i) $P(X > 2) = \int_2^\infty \frac{1}{2} e^{-x/2} = \frac{1}{2}\left(\frac{e^{-x/2}}{-1/2}\right)_2^\infty = -(0 - e^{-1}) = 0.36$

(ii) $P(X > s+t \mid X > s) = P(X > t)$

$P(X > 10 \mid X > 9) = P(X > 9+1 \mid X > 9) = P(X > 1)$

$= \int_1^\infty \frac{1}{2} e^{-x/2} dx = e^{-1/2} = 0.6$

Que – The length of the shower on a island has an exponential distn with mean 1/2. what is the pbty that the shower will last more than 3 min.
(ii) If the shower is already lasted for 2 min, what is the pbty that will last for one more minute.

soln – $\frac{1}{\lambda} = \frac{1}{2} \Rightarrow \lambda = 2$

$f(x) = \lambda e^{-\lambda x} = 2e^{-2x}$

$X \to$ length of shower

(i) $P(X > 3) = \int_3^\infty 2e^{-2x} dx = e^{-6} = 0.002$

(ii) $P(X > 2+1 \mid X > 2) = P(X > 1)$

$= \int_1^\infty 2e^{-2x} dx = e^{-2}$

Que – The daily consumption of milk in excess of 20,000 gallons is exp. dist will mean 3000. The city has a daily stock of 35000 gallons. what is the pbty that 2 days are selected at random and the stock is insufficient for both the days.

soln. – $X \to$ daily consumption

$Y = X - 20000$ (excess)

$\frac{1}{\lambda} = 3000$

$$P[X > 35000] = P[X + 20{,}000 > 35{,}000]$$

$$= P[Y > 15000] = \int_{15000}^{\infty} \lambda e^{-\lambda y}\, dy$$

$$= \int_{15000}^{\infty} \frac{1}{3000} e^{-y/3000}\, dy$$

★ Normal Distribution $[N(\mu, \sigma^2)]$

→ parameter μ (mean) and σ^2 (variance)

pdf $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\sigma > 0$

$-\infty < x < \infty$

$-\infty < \mu < \infty$

Properties

1) The curve is bell shaped and symmetrical about the line $x = \mu$
2) Mean = Median = Mode coincide at $x = \mu$
3) Linear combination of indep/- normal variate is also a normal variate.

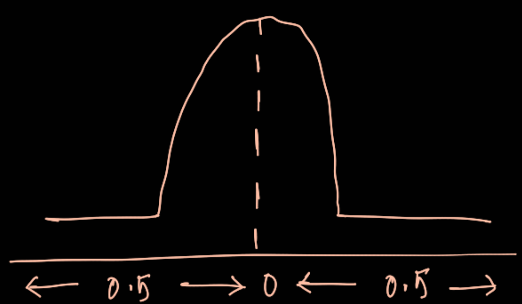
★ Standard Normal Distribution $(N(0,1))$

Mean $\mu = 0$, $\sigma = 1$

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

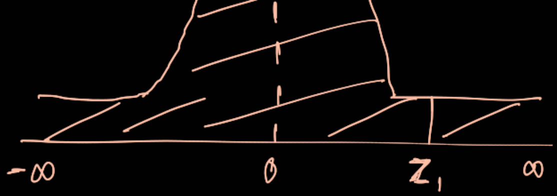
where $z = \frac{x-\mu}{\sigma}$

Area under normal curve



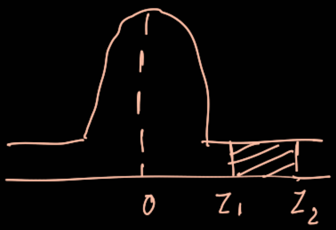
(i)

$P(z < z_1) = 0.5 + P(0 < z < z_1)$



(ii) $P(z > z_1) = 0.5 - P(0 < z < z_1)$

(iii) $P(z_1 < z < z_2) = P(0 < z < z_2) - P(0 < z < z_1)$



Que – If X is normally distributed with mean 8 and SD 4.

Find (i) $P(5 < X < 10)$

(ii) $P(X \geq 15)$

(iii) $P(X < 5)$

(iv) Find K if $P(X > K) = 0.24$

Soln. – $\mu = 8$, $\sigma = 4$

$$z = \frac{X - \mu}{\sigma} = \frac{X - 8}{4}$$

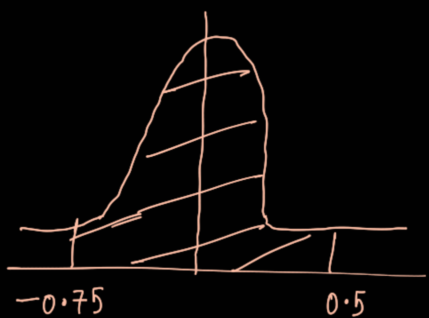
(i) $P(5 < X < 10) = P\left(\frac{5-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{10-\mu}{\sigma}\right)$

$= P(-0.75 < z < 0.5)$

$= P(0 < z < 0.75) + P(0 < z < 0.5)$

$= 0.2734 + 0.1915$

$= 0.4649$



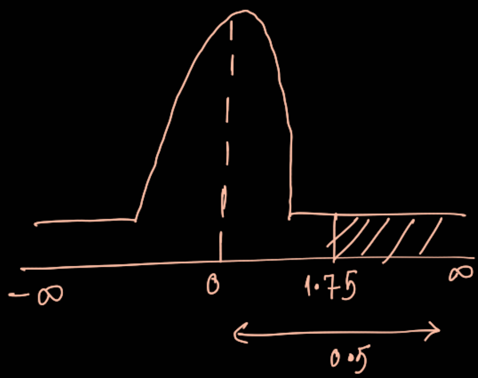
(ii) $P(X \geq 15) = P\left(\frac{X-\mu}{\sigma} \geq \frac{15-\mu}{\sigma}\right)$

$= P(z \geq 1.75)$

$= 0.5 - P(0 < z < 1.75)$

$= 0.5 - 0.4599$

$= 0.0401$



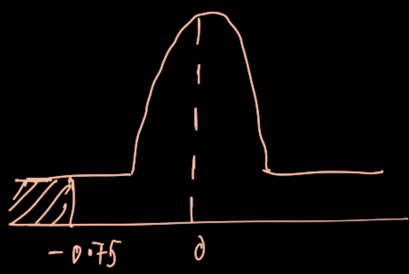
(iii) $P(X \le 5) = P\left(\frac{X-\mu}{\sigma} \le \frac{5-\mu}{\sigma}\right)$

$= P(z \le -0.75)$

$= 0.5 - P(0 < z < 0.75)$

$= 0.5 - 0.2734$

$= 0.2266$

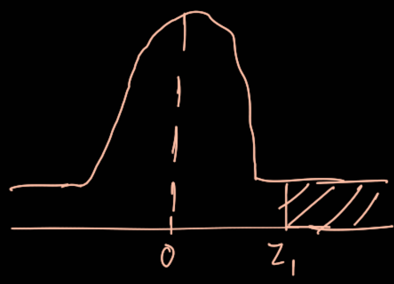


(iv) Given $P(X > K) = 0.24$

$P\left(\frac{X-\mu}{\sigma} > \frac{K-\mu}{\sigma}\right) = 0.24$

$P\left(z > \frac{K-8}{4}\right) = 0.24$

$P(z > z_1) = 0.24 \quad \left(z_1 = \frac{K-8}{4}\right)$



$\Rightarrow 0.5 - P(0 < z < z_1) = 0.24$

$P(0 < z < z_1) = 0.26$ (table value)

$z_1 = 0.7$

$\Rightarrow \frac{K-8}{4} = 0.7$

$K = 10.8$

Que — The mean of 1000 students are normally dist with mean 70 and S.D 5 estimate the no. of students whose mark will be

(i) between 60 and 75

(ii) more than 75

(iii) less than 68

Soln. — $\mu = 70$, $\sigma = 5$

$z = \frac{X-\mu}{\sigma} = \frac{X-70}{5}$

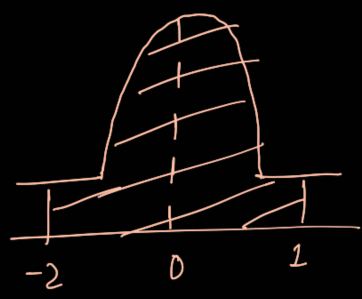
(i) $P(60 < X < 75) = P\left(\frac{60-\mu}{\sigma} < z < \frac{75-\mu}{\sigma}\right)$

$= P(-2 < z < 1)$

$= P(0 < z < 2) + P(0 < z < 1)$

$= 0.4772 + 0.3413$

$= 0.8185$



No. of students $= 0.8185 \times 1000 = 818.5 = 819$

(ii) $P(X > 75) = P\left(z > \frac{75-\mu}{\sigma}\right)$

(ii) $P(X > \ldots) = P\left(\ldots\right)$

$= P(Z > 1)$

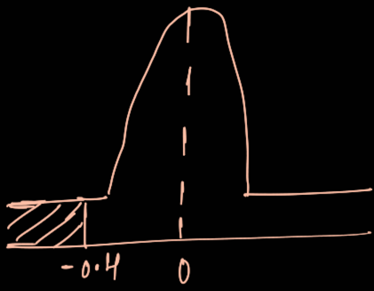
$= 0.5 - P(0 < Z < 1)$

$= 0.5 - 0.3413$

$= 0.1587$

No. of students $\simeq 159$

(iii) $P(X < 68) = P\left(Z < \frac{68-\mu}{\sigma}\right)$



$= P(Z < -0.4)$

$= 0.5 - P(0 < Z < 0.4)$

$= 0.5 - 0.1554$

$= 0.3446$