SRM Institute of Science and Technology Department of Mathematics 18MAB102T-Advanced Calculus and Complex Analysis 2020-2021 Even Unit – I: Multiple Integrals Tutorial Sheet - I		
S.No.	**Questions**	**Answers**
	Part – A [3 Marks]	
1	Evaluate $\int_0^3\int_0^2 xy(x+y)\,dxdy$	30
2	Evaluate $\int_2^4\int_1^2 \frac{\mathrm{dxdy}}{\mathrm{xy}}$	$(\log 2)^2$
3	Evaluate $\int_0^{\pi/2}\int_0^{2a\cos\theta} \mathrm{rdrd}\theta$	$\frac{\pi}{8}$
4	Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$	$\frac{3\pi a^2}{4}$
5	Change the order of integration $\int_0^3\int_1^{\sqrt{4-y}}(x+y)\,dxdy$	$\int_1^2\int_0^{4-x^2}(x+y)\,dydx$
	Part – B [6 Marks]	
6	Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2}$	$\frac{\pi}{4}\log(1+\sqrt{2})$
7	Evaluate $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$	a^3
8	Change the order of integration $\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dxdy$ and hence evaluate it	$\frac{\pi}{4}a$
9	Change the order of integration $\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dxdy$ and hence evaluate it	$\frac{a^2b^2}{8}$
10	Change the order of integration and hence find the value of $\int_0^1\int_x^1 \frac{x}{x^2+y^2}\,dxdy$	$\frac{1}{2}\log 2$

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT II - VECTOR CALCULUS

Part – A

1.	If $\varphi(x,y,z) = xyz$, then $\nabla\varphi =$ (A) $yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ (B) $xy\,\vec{i} + yz\,\vec{j} + xz\,\vec{k}$ (C) $xz\,\vec{i} + zy\,\vec{j} + xy\,\vec{k}$ (D) $x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$	ANS **A**	(CLO-2, Apply)
2.	Curl ($grad\ \varphi$) = (A) $\vec{0}$ (B) 1 (C) 2 (D) – 1	ANS **A**	(CLO-2, Remember)
3.	The maximum directional derivative of $\varphi(x,y,z) = x^2 + y^2 + z^2$ at $(1,1,1)$ is (A) 0 (B) 3 (C) 2 (D) $2\sqrt{3}$	ANS **D**	(CLO-2, Apply)
4.	If $\vec{r}$ is the position vector of the point (x,y,z) with respect to the origin, then $\nabla \bullet \vec{r} =$ (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Apply)
5.	If $\vec{u}$ and $\vec{v}$ are irrotational, then $\vec{u} \times \vec{v}$ is (A) irrotational (B) solenoidal (C) zero vector (D) constant	ANS **B**	(CLO-2, Remember)
6.	The condition for $\vec{F}$ to be conservative is (A) $\nabla \bullet \vec{F} = 0$ (B) 0 (C) $\nabla \times \vec{F} = \vec{0}$ (D) 1	ANS **C**	(CLO-2, Remember)

7.	The relation between the surface integral and the volume integral is given by (A) Green's theorem (B) Stoke's theorem (C) Gauss Divergence theorem (D) Cauchy's theorem	ANS **C**	(CLO-2, Remember)
8.	By Stoke's theorem, $\int_C \vec{F} \bullet d\vec{r} =$ (A) $\iint_S \nabla \times \vec{F}\, dS$ (B) $\iint_S \nabla \bullet \vec{F}\, dS$ (C) $\iint_S \left(\nabla \bullet \vec{F}\right) \hat{n}\ dS$ (D) $\iint_S \left(\nabla \times \vec{F}\right) \bullet \hat{n}\ dS$	ANS **D**	(CLO-2, Remember)
9.	The maximum value of the directional derivative is (A) $\lvert\nabla \varphi\rvert$ (B) curl φ (C) grad φ (D) $\lvert\nabla \times \varphi\rvert$	ANS **A**	(CLO-2, Remember)
10.	If $\vec{F}$ is irrotational, then $Curl\, \vec{F} =$ (A) 1 (B) 2 (C) 3 (D) $\vec{0}$	ANS **D**	(CLO-2, Apply)
11.	If the divergence of the vector is zero, then the vector is said to be (A) irrotational vector (B) constant vector (C) zero vector (D) solenoidal vector	ANS **D**	(CLO-2, Remember)
12.	The unit normal vector to the surface $x^2\, y + 2\, x\, z = 4$ at the point $(2, -2, 3)$ is (A) $-\frac{1}{3}\,\vec{\imath} + \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (B) $\frac{1}{3}\,\vec{\imath} + \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (C) $-\frac{1}{3}\,\vec{\imath} - \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (D) $\frac{1}{3}\,\vec{\imath} - \frac{2}{3}\,\vec{\jmath} - \frac{2}{3}\,\vec{k}$	ANS **A**	(CLO-2, Apply)
13.	If the vector $\vec{F} = (x + 3y)\,\vec{\imath} + (y - 2z)\,\vec{\jmath} + (x + a\,z)\,\vec{k}$ is solenoidal, then $a =$ (A) 2 (B) 0 (C) -2 (D) -1	ANS **C**	(CLO-2, Apply)
14.	The work done by the conservative force when it moves a particle around a closed curve is (A) $\nabla \bullet \vec{F} = 0$ (B) 0 (C) $\nabla \times \vec{F} = 0$ (D) $\nabla \bullet \left(\nabla \times \vec{F}\right) = 0$	ANS **C**	(CLO-2, Remember)

15.	The value of $\int_C x\,dy - y\,dx$ around the circle $x^2 + y^2 = 1$ is (A) π (B) 2π (C) 3π (D) 0	ANS **B**	(CLO-2, Apply)
16.	By Green's theorem, the area bounded by a simple closed curve is (A) $\int_C x\,dy - y\,dx$ (B) $\int_C x\,dy + y\,dx$ (C) $\int_C y\,dx - x\,dy$ (D) $\frac{1}{2}\left(\int_C x\,dy - y\,dx\right)$	ANS **D**	(CLO-2, Apply)
17.	To be conservative, $\vec{F}$ should be (A) solenoidal (B) irrotational (C) rotational (D) constant vector	ANS **B**	(CLO-2, Remember)
18.	The unit normal vector to the surface $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$ is (A) $\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$ (B) $\frac{\vec{i}+\vec{j}-\vec{k}}{\sqrt{3}}$ (C) $\frac{\vec{i}-\vec{j}-\vec{k}}{\sqrt{3}}$ (D) $\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{2}}$	ANS **B**	(CLO-2, Apply)
19.	If $\vec{r}$ is the position vector of the point (x, y, z) with respect to the origin, then $div\,\vec{r} =$ (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Remember)
20.	If φ is a scalar function, then $\nabla \times \nabla\varphi =$ (A) $\vec{0}$ (B) solenoidal (C) irrotational (D) constant	ANS **A**	(CLO-2, Remember)
21.	The value of line integral $\int_C \vec{F} \bullet d\vec{r}$ where C is the line $y = x$ in XY plane from (1, 1) to (2, 2) is (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Apply)
22.	Angle between two level surfaces $\varphi_1 = C$ and $\varphi_2 = C$ is given by (A) $\sin\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (B) $\cos\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (C) $\tan\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (D) $\tan\theta = \frac{\nabla\varphi_1 \times \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$	ANS **B**	(CLO-2, Apply)

23.	The condition for a vector $\vec{r}$ to be solenoidal is (A) $div\ \vec{r} = 0$ (B) $curl\ \vec{r} = 0$ (C) $div\ \vec{r} \neq 0$ (D) $curl\ \vec{r} \neq 0$	ANS **A**	(CLO-2, Remember)
24.	The unit normal vector to the surface $x^2 + 2\,y^2 + z^2 = 7$ at the point $(1, -1, 2)$ is (A) $\frac{\vec{\imath}-2\,\vec{\jmath}-2\,\vec{k}}{3}$ (B) $\frac{\vec{\imath}-2\,\vec{\jmath}+2\,\vec{k}}{3}$ (C) $\frac{\vec{\imath}+2\,\vec{\jmath}+2\,\vec{k}}{3}$ (D) $\frac{\vec{\imath}-2\,\vec{\jmath}+2\,\vec{k}}{3}$	ANS **D**	(CLO-2, Apply)
25.	If the integral $\int_A^B \vec{F} \bullet d\vec{r}$ depends only on the end points but not on the path C, then $\vec{F}$ is (A) neither solenoidal nor irrotational (B) solenoidal (C) irrotational (D) conservative	ANS **D**	(CLO-2, Remember)
26.	According to Gauss divergence theorem, $\int_C (P\,dx + Q\,dy) =$ (A) $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ (B) $\iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dx\,dy$ (C) $\iint_R \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy$ (D) $\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$	ANS **A**	(CLO-2, Apply)
27.	By Green's theorem, $\frac{1}{2}\left(\int_C x\,dy - y\,dx\right) =$ (A) Area of a closed curve (B) 2 × Area of a closed curve (C) Volume of a closed curve (D) 3 × Volume of a closed curve	ANS **A**	(CLO-2, Apply)
28.	The value of $\iint_S \vec{r} \bullet \vec{n}\,dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ is (A) $2\,\pi\,a^3$ (B) $3\,\pi\,a^3$ (C) $4\,\pi\,a^3$ (D) $5\,\pi\,a^3$	ANS **C**	(CLO-2, Apply)
29.	The maximum directional derivative of $\varphi\,(x, y, z) = xyz^2$ at (1, 0, 3) is (A) 9 (B) 1 (C) – 9 (D) 0	ANS **A**	(CLO-2, Apply)
30.	The relation between line integral and double integral is given by (A) Gauss divergence theorem (B) Cauchy's theorem (C) Green's theorem (D) Convolution theorem	ANS **C**	(CLO-2, Remember)

31.	If $\varphi\,(x, y, z) = x^2 + y^2 + z^2$, then $\nabla\varphi$ at $(1, 1, 1)$ = (A) $2\,\vec{i} + 2\,\vec{j} + 2\,\vec{k}$ (B) $2\,\vec{i} - 2\,\vec{j} + \vec{k}$ (C) $\vec{i} + \vec{j} + \vec{k}$ (D) $2\,\vec{i} - 2\,\vec{j} - 2\,\vec{k}$	ANS **A**	(CLO-2, Apply)
32.	If $\varphi\,(x, y, z) = xyz$, then $\nabla\varphi$ at $(1, 1, 1)$ is (A) $\vec{i} + \vec{j} + \vec{k}$ (B) $2\,\vec{i} + 2\,\vec{j} + 2\,\vec{k}$ (C) $2\,\vec{i} - 2\,\vec{j} + \vec{k}$ (D) $2\,\vec{i} - 2\,\vec{j} - 2\,\vec{k}$	ANS **A**	(CLO-2, Apply)
33.	The unit normal vector to the surface $\varphi = xy - yz - zx$ at the point (– 1, 1, 1) is (A) $-2\,\vec{j}$ (B) $-\vec{j}$ (C) $3\,\vec{i}$ (D) $4\,\vec{i}$	ANS **B**	(CLO-2, Apply)
34.	. $\nabla\, r^n =$ (A) $n\,\vec{r}$ (B) $n\,(n-1)\,\vec{r}$ (C) $n\,r^{n-2}\,\vec{r}$ (D) $n\,r^{n+2}\,\vec{r}$	ANS **C**	(CLO-2, Apply)
35.	The directional derivative of $\varphi = 2xy + z^2$ at (1, –1, 3) in the direction of $\vec{i} + 2\,\vec{j} + 2\,\vec{k}$ is (A) $\frac{14}{3}$ (B) $-\frac{14}{3}$ (C) $\frac{4}{3}$ (D) $\frac{3}{14}$	ANS **A**	(CLO-2, Apply)
36.	If $\vec{F} = (3x - 2y + z)\,\vec{i} + (4x + ay - 2)\,\vec{j} + (x - y + 2)\,\vec{k}$ is solenoidal, then $a =$ (A) 3 (B) 0 (C) -3 (D) -1	ANS **C**	(CLO-2, Apply)

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 2 – Vector Calculus

Part – B (Each question carries 3 Marks)

1. Find $\nabla\phi$ if $\phi = \log\left(x^2+y^2+z^2\right)$.

Solution

$$\nabla\phi = \vec{i}\ \frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right) + \vec{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$$

$$= \vec{i}\frac{2x}{(x^2+y^2+z^2)} + \vec{j}\frac{2y}{(x^2+y^2+z^2)} + \vec{k}\frac{2z}{(x^2+y^2+z^2)}$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{i}+y\vec{j}+z\vec{k}\right) = \frac{2\vec{r}}{r^2} \qquad \because (\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}\ \&\ r^2 = x^2+y^2+z^2)$$

2. Find the unit normal vector to the surface $x^2+y^2=z$ at the point $(1,-2,5)$.

Solution

Given

$$\phi = x^2+y^2-z$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi \text{ at } (1,-2,5) = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = 3$$

Unit Normal vector is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{3}$$

3. Prove that $curl(grad\phi) = 0$.

Solution

$$grad\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$Curl\,(grad\,\varphi) = \nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right) - \vec{j}\left(\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial x}\right) + \vec{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right)$$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$ **(Since mixed partial derivatives are equal.)**

4. Find $curl\vec{F}$ if $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$.

Solution

Given $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$

$$curl\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{i}\,(0-y) - \vec{j}(z-0) + \vec{k}\,(0-x)$$

$$= -y\,\vec{i} - z\,\vec{j} - x\,\vec{k}$$

5. In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.

Solution

Given $\phi = x^2y^2z^4$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xy^2z^4\vec{i} + 2x^2yz^4\vec{j} + 4x^2y^2z^3\vec{k}$$

$$\nabla\phi \; at \; (3,1,-2) = 92\vec{i} + 144\vec{j} - 92\vec{k}$$

$$|\nabla\phi| = \sqrt{92^2 + 144^2 + 92^2} = \sqrt{37664}$$

The directional derivative is maximum in the direction $\nabla\phi$ and the magnitude of this maximum is $|\nabla\phi| = \sqrt{37664}$.

6. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, – 2 , – 1) in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution

Given $\phi = x^2yz + 4xz^2$,

$$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}, \quad |\vec{a}| = \sqrt{4+1+4} = 3$$

$$\nabla\phi = (2xyz + 4z^2)\vec{i} + x^2z\vec{j} + (x^2y + 8xz)\vec{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\text{D.D.} = \nabla\phi.\frac{\vec{a}}{|\vec{a}|} = (8\vec{i} - \vec{j} - 10\vec{k}).\frac{2\vec{i} - \vec{j} - 2\vec{k}}{3} = \frac{37}{3}$$

7. Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at P (1, 2 , 3) in the direction of line PQ where Q is (5, 0, 4).

Solution

$$\nabla\varphi = grad\,\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = grad\,\varphi = \vec{i}\,2x + \vec{j}\,(-2y) + \vec{k}\,4z$$

$$\nabla\varphi \text{ at } (1,2,3) = 2\vec{i} - 4\vec{j} + 12\vec{k}$$

$$\vec{a} = OQ - OP = (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$$

Directional derivative = $\nabla\varphi \bullet \dfrac{\vec{a}}{|\vec{a}|}$

$$= (2\vec{i} - 4\vec{j} + 12\vec{k}) \bullet \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

8. Find the angle between the normals to the surfaces $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).

Solution

Given $\varphi = x^2 - yz$

$$\nabla\varphi = 2x\vec{i} - z\vec{j} - y\vec{k}$$

$$\nabla\varphi_1\,/(1,1,1) = 2\vec{i} - \vec{j} - \vec{k} \qquad \nabla\varphi_2\,/(2,4,1) = 4\vec{i} - \vec{j} - 4\vec{k}$$

$$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6} \qquad |\nabla\varphi_2| == \sqrt{16+1+16} = \sqrt{33}$$

$$\cos\theta = \frac{\nabla\varphi_1 \circ \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|} = \frac{(2\vec{i} - \vec{j} - \vec{k}) \circ (4\vec{i} - \vec{j} - 4\vec{k})}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}.$$

9. Find a **such that** $\vec{F} = (3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$ **is solenoidal.**

Solution

Given $\nabla \circ \vec{F} = 0 \Rightarrow \frac{\partial}{\partial x}(3x-2y+z) + \frac{\partial}{\partial y}(4x+ay-z) + \frac{\partial}{\partial z}(x-y+2z) = 0$

$$3 + a + 2 = 0 \Rightarrow a + 5 = 0 \ \Rightarrow a = -5$$

10. Find the constant a, b, c so that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ **is irrotational.**

Solution

Given $\vec{F}$ is irrotational **i.e.,** $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\vec{i}\left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right)$$
$$+\vec{k}\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right) = \vec{0}$$

$$= i.e., \quad \vec{i}\,(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$$
$$= \therefore c+1 = 0, 4-a = 0, and\; b-2 = 0$$
$$\Rightarrow a = 4, b = 2, c = -1$$

11. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, then find div curl $\vec{F}$.

Solution $div\ curl\,\vec{F} = \nabla.(\nabla\times\vec{F})$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$$

$$\nabla\times\vec{F} = \vec{0}$$

$$\therefore \nabla.(\nabla\times\vec{F}) = 0$$

12. Prove that $div\,\vec{r} = 3$**.**

Solution

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

$$div\,\vec{r} = \nabla\bullet\vec{r} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)\bullet\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

13. Show that the vector $\vec{F} = \left(6xy+z^3\right)\vec{i} + \left(3x^2-z\right)\vec{j} + \left(3xz^2-y\right)\vec{k}$ is irrotational.

Solution

Given $\vec{F} = \left(6xy+z^3\right)\vec{i} + \left(3x^2-z\right)\vec{j} + \left(3xz^2-y\right)\vec{k}$

$$curl\vec{F} = \nabla\times\vec{F} = \vec{0}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2-3z^2) + \vec{k}(6x-6x) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

14. If $F=(3x^2+6y)\vec{i}-14yz\vec{j}+20xz^2\vec{k}$. **Evaluate** $\int_C \vec{F}\bullet d\vec{r}$ **from (0,0,0) to (1,1,1) along the curve**

x = t, y = t², z = t³.

Solution

The end points are (0,0,0) and (1,1,1).

These points correspond to t = 0 and t = 1.

$\therefore$ dx = dt, dy = 2t dt, dz = $3t^2$dt

$$\int_C \vec{F}\bullet d\vec{r}=\int_C (3x^2+6y)dx-14yzdy+20xz^2dz$$

$$=\int_0^1 (3t^2+6t^2)dt-14t^5(2tdt)+20t^7(3t^2)dt=\int_0^1 (9t^2-28t^6+60t^9)dt=5$$

15. If $F=ax\vec{i}+by\vec{j}+cz\vec{k}$, **a, b, c are constants, show that** $\iint_S \vec{F}\bullet\hat{n}\,ds=\frac{4\pi}{3}(a+b+c)$ **where S is the surface of a unit sphere.**

Solution

W.K.T. Gauss's divergence theorem

$$\iint_S \vec{F}\bullet\hat{n}ds=\iiint_V \nabla\bullet\vec{F}dV=\iiint_V\left(\frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz)\right)dV$$

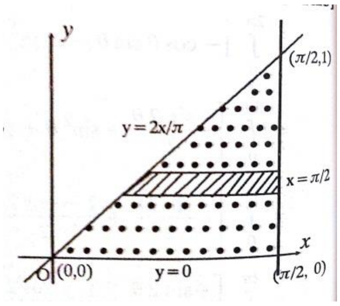
$$=\iiint_V (a+b+c)dV=(a+b+c)V=(a+b+c)\frac{4}{3}\pi(1)^3$$

$$\iint_S \vec{F}\bullet\hat{n}ds=\frac{4}{3}\pi(a+b+c)$$

16. Using Green's theorem, evaluate $\int_c (y-\sin x)\,dx+\cos x\,dy$ **where** c **is the triangle formed by** $y=0, x=\frac{\pi}{2}, y=\frac{2x}{\pi}$.

Solution

Using Green's theorem, we convert the line integral to double integral over the given region. $ie., \int_C u\,dx+v\,dy=\iint_R\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy$



$u=y-\sin x$ $\quad\quad v=\cos x$

$\frac{\partial u}{\partial y}=1$ $\quad\quad \frac{\partial v}{\partial x}=-\sin x$

Hence, $\int_C\{(y-\sin x)dx+\cos x dy\}=\iint_R(-\sin x-1)dxdy$

$$=\int_0^1\int_{\pi y/2}^{\pi/2}(-\sin x-1)dxdy=\int_0^1[\cos x-x]_{\pi y/2}^{\pi/2}$$

$$=\int_0^1\left(0-\frac{\pi}{2}-\cos\frac{\pi y}{2}+\frac{\pi y}{2}\right)dy$$

$$=\left[\frac{-\pi y}{2}-\frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}}+\frac{\pi}{2}.\frac{y^2}{2}\right]_0^1=\frac{-\pi}{2}-\frac{2}{\pi}+\frac{\pi}{4}$$

$$=\frac{-\pi^2-8}{4\pi}=-\left[\frac{\pi}{4}+\frac{2}{\pi}\right].$$

17. Using Green's theorem, evaluate $\int_c(3x^2-8y^2)\,dx+(4y-6xy)\,dy$ **where** c **is the boundary of the triangle formed by the lines** $x=0, y=0, x+y=1$ **in the** xy **plane.**

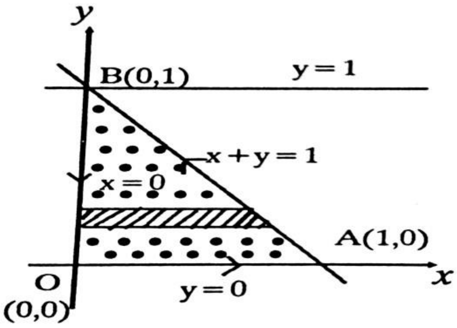
Solution

Using Green's theorem, we convert the line integral to double integral over the given region. $ie., \int_C u\,dx+v\,dy=\iint_R\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy$

$u = 3x - 8y^2$ $\qquad v = 4y - 6xy$

$\dfrac{\partial u}{\partial y} = -16y$ $\qquad \dfrac{\partial v}{\partial x} = -6y$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -6y + 16y = 10y$$



Hence, $\displaystyle\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = \iint_R (10y)\, dxdy$

$$= 10\int_0^1 \int_0^{1-y} (y)\, dxdy = \int_0^1 y[x]_0^{1-y}\, dy$$

$$= 10\int_0^1 y(1-y)dy = 10\int_0^1 (y - y^2)dy$$

$$= 10\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1$$

$$= 10\left(\frac{1}{2} - \frac{1}{3}\right).$$

$$= 10\,\frac{3-2}{6} = \frac{10}{6} = \frac{5}{3}$$

18. Using Gauss divergence theorem evaluate $\displaystyle\iiint_V \nabla \circ \vec{F} dv$ **where** $\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ **taken over the cube bounded by the planes** $x = 0,\ x = 1, y = 0,\ y = 1, z = 0, z = 1.$

Solution

$\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 4z - 2y + y = 4z - y$$

$$\iiint_V \nabla \circ \vec{F} dv = \int_0^1\int_0^1\int_0^1 (4z-y)dxdydz = \int_0^1\int_0^1 [4zx-yx]_0^1 dydz = \int_0^1\int_0^1 [4z-y]dydz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left[4z - \frac{1}{2}\right] dz = \left[4\frac{z^2}{2} - \frac{z}{2}\right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

19. Using Gauss divergence theorem theorem evaluate $\iiint_V \nabla \circ \vec{F} dv$ where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cube bounded by the planes $x=0,\ x=1, y=0,\ y=1, z=0, z=1.$

Solution

$$\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 2x + 2y + 2z = 2(x+y+z)$$

$$\iiint_V \nabla \circ \vec{F} dv = 2\int_0^1\int_0^1\int_0^1 (x+y+z)dxdydz = 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + xy + xz\right]_0^1 dydz = 2\int_0^1\int_0^1 \left[\frac{1}{2} + y + z\right]dydz$$

$$= 2\int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + yz\right]_0^1 dz = 2\int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z\right]_0^1 dz = 2\int_0^1 [1+z]_0^1 dz = 2\left[z + \frac{z^2}{2}\right]_0^1$$

$$= 2\left(1 + \frac{1}{2}\right) = 2\left(\frac{3}{2}\right) = 3$$

20. Using Stokes theorem find $\iint_S curl\,\vec{F}\,ds$ where $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\,\vec{j}$ in the rectangular region of $x=0,\ y=0,\ x=a$ and $y=a.$

Solution Stokes theorem $\int_c \vec{F}.d\vec{r} = \iint_s curl\,\vec{F}.d\vec{s}$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\,\vec{k}$$

Here $\hat{n} = \vec{k}$

$$\iint_S curl\,\vec{F}.\hat{n}ds = \iint_S 4ydxdy = \int_0^b\int_0^a 4ydxdy = 2ab^2$$

21. Prove that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\int_C (xdy - ydx).$$

Solution

W.K.T. Green's theorem

$$\int_C (udx + vdy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy \qquad \ldots 1$$

Here $v = \frac{x}{2}$ $u = -\frac{y}{2}$

$\frac{\partial v}{\partial x} = \frac{1}{2}$ $\frac{\partial u}{\partial y} = -\frac{1}{2}$

$$(1) \Rightarrow \int_C \left(\frac{x}{2}dy - \frac{y}{2}dx\right) = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right)dxdy$$

$$\frac{1}{2}\int_C (xdy - ydx) = \iint_R dxdy$$

22. Find the area of the ellipse x = a cos θ, y = b sin θ using Green's theorem.

Solution

Given x = a cos θ, y = b sin θ

dx = -a sin θ dθ, dy = b cos θ dθ

θ varies from 0 to 2π.

Area of the ellipse $= \frac{1}{2}\oint_C xdy - ydx$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(-b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} [ab\cos\theta\cos\theta + ab\sin\theta\sin\theta]d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)d\theta = \frac{ab}{2}\int_0^{2\pi} d\theta = \frac{ab}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

Area of the ellipse $= \frac{ab}{2}[2\pi] = \pi ab$

* * * * *

18MAB102T
Advanced Calculus and Complex Analysis
Unit II -Vector Calculus

Dr. P. GODHANDARAMAN & Dr. S. SABARINATHAN

Assistant Professor

Department of Mathematics

Faculty of Engineering and Technology

SRM Institute of Science and Technology, Kattankulathur- 603 203.

Scalar and Vector Fields:

- A physical quantity expressible as a continuous function and which can assume one or more definite values at each point of a region of space, is called point function in the region and the region concerned is called a field.
- Point functions are classified as scalar point function and vector point function according as the nature of the quantity concerned is a scalar or a vector.
- At each point P of the field if the function denoted by f(P) is a scalar, it is known as scalar point function while if $\vec{f}(P)$ is a vector, then the function $\vec{f}(P)$ is called a vector point function. The concerned field is called a scalar field or a vector field respectively.

Example of Scalar Fields:

- The temperature distribution in a medium, the gravitational potential of a system of masses and the electrostatic potential of a system of charges.

Example of Vector Fields:

- The velocity of a moving particle, the electrostatic, the magneto static and gravitational fields.

Vector Differential Operator DEL(∇):

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Gradient:

Let $\phi(x, y, z)$ defines a differentiable scalar field. (i.e) ϕ is differentiable at each point (x, y, z) is a certain region of space. Then the gradient of ϕ denoted by $\nabla\phi$ (or) grad ϕ is defined by

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \sum \vec{i}\frac{\partial\phi}{\partial x}$$

Divergence :

If $\vec{F}(x, y, z)$ is defined and differentiable vector point function at each point (x, y, z) is a certain region of space, then the divergence of $\vec{F}$ denoted by $\nabla.\vec{F}$ (or) div$\vec{F}$ is defined by

$$\text{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{F} = \sum \vec{i}.\frac{\partial \vec{F}}{\partial x}$$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then div$\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Solenoidal :

If $\vec{F}$ is a vector such that $\nabla.\vec{F} = 0$ for all points is a given region, then it is said to be a solenoidal vector in that region.

Curl :

If $\vec{F}(x, y, z)$ is a differentiable vector point function in a certain region of space, then the curl or rotation of $\vec{F}$ denoted by $\nabla \times \vec{F}$ (or) curl $\vec{F}$ (or) rot $\vec{F}$ is defined by

$$\nabla \times \vec{F} = curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Irrotational :

If $\vec{F}$ is vector such that $\nabla \times \vec{F} = 0$ for all points in the region, then it is called an irrotational vector (or) Lamellar vector in that region.

Directional derivation : $\dfrac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Unit normal vector : $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Angle between the surfaces :

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

Problem: 1

If $\phi = xyz$, find $\nabla\phi$ at $(1, 2, 3)$

Solution:

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xyz)$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}\,yz + \vec{j}\,xz + \vec{k}\,xy$$

$$\nabla\phi = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}\text{v}$$

$$\nabla\phi_{(1,2,3)} = 6\vec{i} + 3\vec{j} + 2\vec{k}.$$

Problem: 2

Prove that $\nabla(r^n) = nr^{n-2}\vec{r}$

Solution:

$$\nabla(r^n) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(r^n)$$

$$= \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$\nabla(r^n) = nr^{n-1}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right) \qquad (1)$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r}.\vec{r} = r^2 = x^2 + y^2 + z^2$$

$2r\dfrac{\partial r}{\partial x} = 2x$	$2r\dfrac{\partial r}{\partial y} = 2y$	$2r\dfrac{\partial r}{\partial z} = 2z$
$\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$	$\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$	$\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

(2)

Sub (2) in (1),

$$\nabla(r^n) = nr^{n-1}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$\nabla(r^n) = nr^{n-2}\vec{r}.$$

Problem: 3

Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1,2,3)$ in the direction of $2\vec{i} + \vec{j} - \vec{k}$.

Solution:

Directional derivation $= \dfrac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Given $\phi = x^2yz + 4xz^2 + xyz$

$$\nabla\phi = (2xyz + 4z^2 + yz)\vec{i} + (x^2z + xz)\vec{j} + (x^2 + 8xz + xy)\vec{k}$$

$$\nabla\phi_{(1,2,3)} = 54\vec{i} + 6\vec{j} + 28\vec{k}$$

Let $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$

$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-1)^2}$$

$$|\vec{a}| = \sqrt{6}$$

$$\text{Directional derivation} = \frac{\nabla\phi.\vec{a}}{|\vec{a}|}$$

$$\text{Directional derivation} = \frac{\left(54\vec{i}+6\vec{j}+28\vec{k}\right).\left(2\vec{i}+\vec{j}-\vec{k}\right)}{\sqrt{6}}$$

$$\text{Directional derivation} = \frac{86}{\sqrt{6}}$$

Problem: 4

Find a unit normal to the surface $x^2 + 2xz^2 = 8$ at the point $(1, 0, 2)$.

Solution:

Let $\phi = x^2 + 2xz^2 - 8$

$$\nabla\phi = (2xy + 2x^2)\vec{i} + x^2\vec{j} + 4xz\vec{k}$$

$$\nabla\phi_{(1,0,2)} = 8\vec{i} + \vec{j} + 8\vec{k}$$

$$|\nabla\phi| = \sqrt{8^2 + 1^2 + 8^2} = \sqrt{129}$$

Unit normal$= \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\vec{i}+\vec{j}+8\vec{k}}{\sqrt{129}}$.

Problem: 5

Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at (2, -1, 2).

Solution:

Given $\phi_1 = x^2 + y^2 - 2 - 3$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi_1\,(2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi_1| = \sqrt{21}$$

$$\phi_2 = x^2 + y^2 + z^2 - 9$$

$$\nabla\phi_2 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_2\,(2,-1,2) = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi_2| = 6$$

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{(\nabla\phi_1)(\nabla\phi_2)}$$

$$= \frac{(4\vec{i} - 2\vec{j} - \vec{k}).(4\vec{i} - 2\vec{j} + 4\vec{k})}{(\sqrt{21})\,(6)}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

Problem: 6

If $\nabla\phi = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$, find ϕ.

Solution:

$$\nabla\phi = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$$

$$\frac{\partial\phi}{\partial x} = yz$$

function not involving x.

$$\frac{\partial \phi}{\partial y} = zx$$

$\phi = xyz + a$, function not involving y.

$$\frac{\partial \phi}{\partial z} = xy$$

$\phi = xyz + a$, function not involving z.

From the last three statements,

we conclude

$\phi = xyz + a$ is a constant.

Problem: 7

If $\vec{F} = x^2\vec{\imath} + y^2\vec{\jmath} + z^2\vec{k}$, then find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.

Solution:

$$\nabla \cdot \vec{F} = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2\vec{\imath} + y^2\vec{\jmath} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\nabla \cdot \vec{F} = 2\mathrm{x} + 2\mathrm{y} + 2\mathrm{z}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \vec{i}[0] - \vec{j}[0] + \vec{k}[0]$$

$$\nabla \times \vec{F} = 0.$$

Problem: 8

Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

Solution:

$$\nabla . \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) . \left(z\vec{i} + x\vec{j} + y\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$\nabla . \vec{F} = 0$$

$\therefore \vec{F}$ is solenoidal.

Problem: 9

If $\vec{F} = (x+3y)\vec{\imath} + (y-2z)\vec{\jmath} + (x+\lambda z)\vec{k}$ is solenoidal, find the value of λ.

Solution:

$\nabla . \vec{F} = 0$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+\lambda z) = 0$$

$1 + 1 + \lambda = 0$

$\lambda = -2.$

Problem: 10

Show that $\vec{F} = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$ is irrotational.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$\nabla \times \vec{F} = 0$$

$\therefore \vec{F}$ is irrotational.

Laplace operator :

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Problem: 11

Prove that $\nabla^2 \mathrm{r}^n = n(n+1)\mathrm{r}^{n-2}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ and deduce $\nabla^2\left(\frac{1}{r}\right)$.

Solution:

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2 r^n = \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left[n r^{n-1}\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1}\frac{x}{r}\right] = \sum \frac{\partial}{\partial x}\left[n r^{n-2} x\right]$$

$$= \sum n\left[\left((n-2) r^{n-3}\frac{\partial r}{\partial x}\right)x + r^{n-2}\right]$$

$$= \sum n\left[\left((n-2) r^{n-3}\frac{x}{r}\right)x + r^{n-2}\right]$$

$$= \sum n[(\mathrm{x}^2(\mathrm{n}-2)\mathrm{r}^{n-4}) + \mathrm{r}^{n-2}]$$

$$= \sum[(n(\mathrm{n}-2)\mathrm{r}^{n-4}\mathrm{x}^2) + n\mathrm{r}^{n-2}]$$

$$= n(n-2)\mathrm{r}^{n-4}(x^2 + y^2 + z^2) + 3\mathrm{nr}^{n-2}$$

$$= n(n-2)\mathrm{r}^{n-4}r^2 + 3\mathrm{nr}^{n-2}$$

$$= n(n-2)\mathrm{r}^{n-2} + 3\mathrm{nr}^{n-2}$$

$$= n\mathrm{r}^{n-2}[n-2+3]$$

$$\nabla^2(\mathrm{r}^n) = n(n+1)\mathrm{r}^{n-2}.$$

Line Integral

Problem: 12

Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{\imath} + (2xz - y)\vec{\jmath} - z\vec{k}$ from $t = 0$ to $t = 1$ along the cone $x = 2t^2, y = t, z = 4t^3$.

Solution:

Work done $= \int_c \vec{F}.\overrightarrow{dr}$

$$\vec{F} = 3x^2\vec{\imath} + (2xz - y)\vec{\jmath} - z\vec{k}$$

$$\overrightarrow{dx} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$$

$$\vec{F}.\overrightarrow{dr} = 3x^2dx + (2xz - y)dy - zdz$$

$x = 2t^2$ | $y = t$ | $z = 4t^3$

$dx = 4t\ dt$ | $dy = dt$ | $dz = 12t^2\ dt$

$$\vec{F}.\overrightarrow{dr} = 48\ t^5\ dt + (16\ t^5 - t)dt - 48\ t^5\ dt$$

$$\int_c \vec{F}.\overrightarrow{dr} = \int_0^1 (16\ t^5 - t)dt$$

$$= \left[16\ \frac{t^6}{6} - \frac{t^2}{2}\right]_0^1$$

$$= \frac{16}{6} - \frac{1}{2}$$

$$= \frac{13}{6}$$

Surface Integrals

$$\iint_S \vec{F}.\hat{n}\ ds = \iint_R \frac{\vec{F}.\hat{n}}{|\vec{n}.\hat{k}|}\ ds\ dy$$

Problem: 13

Evaluate $\iint_S \vec{F}.\hat{n}\ ds$ where $\vec{F} = z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}$ and S is the surface of the cylinder $x^2 + y^2 = 1$ included in the first octant between the planes $z = 0$ and $z = 2$.

Solution :

$$\vec{F} = z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}$$

$$\varphi = x^2 + y^2 - 1$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$$

$$= \frac{2x\,\vec{i}+2y\,\vec{j}}{2}$$

$$\hat{n} = x\,\vec{i} + y\,\vec{j}$$

$$\vec{F}.\hat{n} = \left(z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}\right).(x\,\vec{i} + y\,\vec{j}) = xz + xy$$

$$\iint_S \vec{F}.\hat{n}\ ds = \iint_R \frac{\vec{F}.\hat{n}}{|\vec{n}.\hat{i}|}\ dy\,dz$$

Where R is the projection of S on yz plane.

$$= \iint_R (xz + xy) \ \frac{dy\,dz}{x}$$

$$= \iint_R (z + y) \ dy\,dz$$

$$= \int_0^2 \int_0^1 (z + y) \ dy\,dz$$

$$= \int_0^2 \left[zy + \frac{y^2}{2} \right]_0^1 \ dz$$

$$= \int_0^2 (z + \frac{1}{2}) \ dz$$

$$= \left[\frac{z^2}{2} + \frac{z}{2} \right]_0^2 = 3.$$

Volume Integrals

Problem: 14

If $\vec{F} = (2x^2 - 3x)\vec{i} - 2xy\vec{j} - 4x\vec{k}$. Evaluate $\iiint_v \nabla \times \vec{F}\, dv$ where v is the region bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$\nabla \times \vec{F} = \vec{j} - 2y\vec{k}$$

$$\iiint_v \nabla \times \vec{F}\, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} \left(\vec{j} - 2y\vec{k}\right) dzdydx$$

$$= \int_0^2 \int_0^{2-x} \left[z\vec{j} - 2yz\vec{k}\right]_0^{4-2x-2y} dydx$$

$$= \int_0^2 \int_0^{2-x} \left[(4 - 2x - 2y)\vec{j} - 2y(4 - 2x - 2y)\vec{k}\right] dydx$$

$$= \int_0^2 \left[\left(4y - 2xy - \frac{2y^2}{2}\right)\vec{j} - \left(4y^2 - 2xy^2 - \frac{4y^3}{3}\right)\vec{k}\right]_0^{2-x} dx$$

$$= \int_0^2 \left\{ [4(2-x) - 2x(2-x) - (2-x)^2]\vec{j} - \left[4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3\right]\vec{k} \right\} dx$$

$$= \int_0^2 \left[(4 - 4x + x^2)\vec{i} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3) \right] dx$$

$$\iiint_v \nabla \times \vec{F}\, dv = \left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2 \vec{i} - \frac{\vec{k}}{3}\left[16x - 12x^2 + 4x^3 - \frac{x^4}{2}\right]_0^2$$

$$= \left(8 - 8 + \frac{8}{3}\right)\vec{i} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8)$$

$$= \frac{8}{3}\left(\vec{j} - \vec{k}\right).$$

Thank You

Dr. P. GODHANDARAMAN & Dr. S. SABARINATHAN

Assistant Professor

Department of Mathematics

Faculty of Engineering and Technology

SRM Institute of Science and Technology, Kattankulathur- 603 203.

18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS; Unit II (Part-3) - Green's, Stoke's and Gauss Divergence theorem

Dr. Sahadeb Kuila

Assistant Professor
Department of Mathematics, SRMIST, Kattankulathur

Outline

Statement (Green's theorem):

Let C be a positively oriented, piecewise smooth, simple, closed curve and let R be the region enclosed by the curve C in the xy-plane. If $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives on R, then

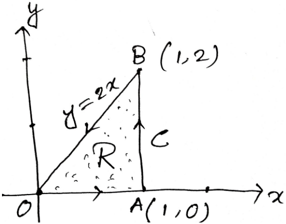
$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Applications of Green's theorem

Example 1:

Use Green's theorem to evaluate $\oint_C xydx + x^2y^3dy$, where C is the triangle with vertices $(0,0), (1,0), (1,2)$ with positive orientation.

Solution: Let $P = xy, Q = x^2y^3$ and the positive orientation curve C is as shown in the figure.



Applications of Green's theorem

Using Green's theorem,

$$\oint_C xydx + x^2y^3dy = \oint_C Pdx + Qdy$$

$$= \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint\limits_R (2xy^3 - x)dxdy$$

$$= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \int_0^1 \left[\frac{xy^4}{2} - xy\right]_0^{2x} dx$$

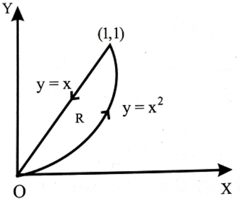
$$= \int_0^1 (8x^5 - 2x^2)dx = \left[\frac{4x^6}{3} - \frac{2x^3}{3}\right]_0^1 = \frac{2}{3}.$$

Applications of Green's theorem

Example 2:

Verify Green's theorem in the plane for $\oint_C [(xy + y^2)dx + x^2 dy]$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution: Let $P = xy + y^2$, $Q = x^2$ and the positive orientation curve C is as shown in the figure. The curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$.



Applications of Green's theorem

Using Green's theorem,

$$\oint_C [(xy+y^2)dx + x^2 dy] = \oint_C Pdx + Qdy$$

$$= \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint\limits_R (2x - x - 2y)dxdy$$

$$= \iint\limits_R (x-2y)dxdy = \int_0^1 \int_{y=x^2}^{x} (x-2y)dydx$$

$$= \int_0^1 \left[xy - y^2\right]_{y=x^2}^{x} dx = \int_0^1 (x^4 - x^3)dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = -\frac{1}{20}.$$

Applications of Green's theorem

Now let us evaluate the line integral along C. Along $y = x^2, dy = 2xdx$ and the line integral equals

$$\begin{aligned}\int_0^1 [(x(x^2) + x^4)dx + x^2(2x)dx] &= \int_0^1 (3x^3 + x^4)dx \\ &= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{19}{20}.\end{aligned}$$

Along $y = x, dy = dx$ and the line integral equals

$$\int_1^0 [(x(x) + x^2)dx + x^2dx] = \int_1^0 (3x^2)dx = \left[\frac{3x^3}{3}\right]_1^0 = -1.$$

Therefore, the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$. Hence the theorem is verified.

Statement (Stoke's theorem):

Let S be a smooth surface that is bounded by a simple closed, smooth boundary curve C with positive orientation and $\overrightarrow{F} = F_1\overrightarrow{i} + F_2\overrightarrow{j} + F_3\overrightarrow{k}$ be any vector function having continuous first order partial derivatives, then

$$\oint_C \overrightarrow{F}.\overrightarrow{dr} = \iint_S curl\overrightarrow{F}.\hat{n}ds,$$

where $\hat{n}$ is the outward normal unit vector at any point of S.

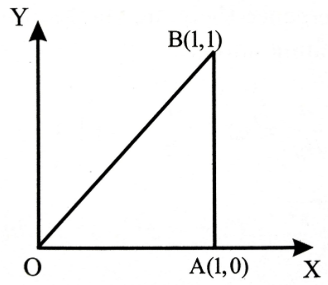
Applications of Stoke's theorem

Example 1:

Use Stoke's theorem to evaluate $\oint_C \overrightarrow{F}.\overrightarrow{dr}$, where $\overrightarrow{F} = y^2\overrightarrow{i} + x^2\overrightarrow{j} - (x+z)\overrightarrow{k}$ and C is the boundary of the triangle with vertices $(0,0,0), (1,0,0), (1,1,0)$ with positive orientation.

Solution: We note that z-coordinate of each vertex of the triangle is 0. Therefore, the triangle lies in the xy-plane. So $\widehat{n} = \overrightarrow{k}$ and the positive orientation curve C is as shown in the figure.

Applications of Stoke's theorem



Let $F_1 = y^2, F_2 = x^2, F_3 = -(x+z)$ and we have

$$curl\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0\,\overrightarrow{i} + \overrightarrow{j} + 2(x-y)\,\overrightarrow{k}$$

Applications of Stoke's theorem

and $curl\,\overrightarrow{F}.\hat{n} = [\overrightarrow{j} + 2(x-y)\overrightarrow{k}].\overrightarrow{k} = 2(x-y)$.
The equation of the line OB is $y = x$. Using Stoke's theorem,

$$\oint_C \overrightarrow{F}.\overrightarrow{dr} = \iint_S curl\,\overrightarrow{F}.\hat{n}ds = \int_0^1 \int_{y=0}^{x} 2(x-y)dxdy$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1 \frac{x^2}{2}dx = \frac{1}{3}.$$

Applications of Stoke's theorem

Example 2:

Verify Stoke's theorem for $\overrightarrow{F} = (2x - y)\overrightarrow{i} - yz^2\overrightarrow{j} - y^2 z\overrightarrow{k}$ over the upper half surface S of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane and C is its boundary.

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre at origin. Let $x = \cos t, y = \sin t$, $z = 0, 0 \leq t \leq 2\pi$ are parametric equations of C.

Applications of Stoke's theorem

Now

$$\oint_C \overrightarrow{F}.\overrightarrow{dr}$$

$$= \oint_C [(2x-y)\overrightarrow{i} - yz^2\overrightarrow{j} - y^2z\overrightarrow{k}].[dx\overrightarrow{i} + dy\overrightarrow{j} + dz\overrightarrow{k}]$$

$$= \oint_C (2x-y)dx - yz^2dy - y^2zdz = \oint_C (2x-y)dx$$

$$= -\int_0^{2\pi} (2\cos t - \sin t)\sin t dt = \pi. \quad (1)$$

Applications of Stoke's theorem

Also $\widehat{n} = \overrightarrow{k}$, $ds = dxdy$,

$$curl\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \overrightarrow{k}$$

and $curl\,\overrightarrow{F}.\widehat{n} = \overrightarrow{k}.\overrightarrow{k} = 1$.
Using Stoke's theorem,

$$\iint_S curl\,\overrightarrow{F}.\widehat{n}ds = \iint_S dxdy = \pi, \tag{2}$$

where $\pi(1)^2$ is the area of the circle C.
Hence from (1) and (2), the theorem is verified.

Statement (Gauss divergence theorem):

If V is the volume bounded by a closed surface S and $\overrightarrow{F}$ is a vector point function with continuous derivatives in V, then

$$\iint\limits_S curl\,\overrightarrow{F}.\hat{n}ds = \iiint\limits_V div\,\overrightarrow{F}\,dV,$$

where $\hat{n}$ is the outward normal unit vector at any point of S.

Applications of Gauss divergence theorem

Example 1:

Use Gauss divergence theorem to evaluate $\iint\limits_S [(x^3 - yz)dydz - 2x^2ydzdx + zdxdy]$ over the surface S of a cube bounded by the coordinate planes and the plane $x = y = z = a$.

Solution: Let $F_1 = x^3 - yz, F_2 = -2x^2y, F_3 = z$. Using Gauss divergence theorem,

$$\iint\limits_S curl\,\overrightarrow{F}.\widehat{n}ds = \iiint\limits_V div\,\overrightarrow{F}\,dV$$

$$= \iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

Applications of Gauss divergence theorem

$$= \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a}(x^2+1)dxdydz = \int_{z=0}^{a}\int_{y=0}^{a}\left[\frac{x^3}{3}+x\right]_{x=0}^{a} dydz$$

$$= \left[\frac{a^3}{3}+a\right]\int_{z=0}^{a}\int_{y=0}^{a} dydz = a\left[\frac{a^3}{3}+a\right]\int_{z=0}^{a} dz = a^2\left[\frac{a^3}{3}+a\right].$$

Applications of Gauss divergence theorem

Example 2:

Use Gauss divergence theorem to evaluate $\iint_S [(x+z)dydz + (y+z)dzdx + (x+y)dxdy]$ over the surface S of the sphere $x^2+y^2+z^2=4$.

Solution: Let $F_1 = x+z, F_2 = y+z, F_3 = x+y$. Using Gauss divergence theorem,

$$\iint_S curl\,\overrightarrow{F}.\hat{n}ds = \iiint_V div\,\overrightarrow{F}\,dV$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV$$

Applications of Gauss divergence theorem

$$= \iiint_V 2dV = 2 \iiint_V dV = 2V,$$

where V is the volume of the sphere $x^2 + y^2 + z^2 = 2^2$ ($\because$ the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$).

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64}{3}\pi.$$

$= yz\vec{i} + xz\vec{j} + xy\vec{k}$

2) If $\phi = \log(x^2+y^2+z^2)$, find $\nabla\phi$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\left(\frac{1}{x^2+y^2+z^2}(2x)\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}$$

$$= 2\left[\frac{x\vec{i}+y\vec{j}+z\vec{k}}{x^2+y^2+z^2}\right]$$

$$= 2\left(\frac{\vec{r}}{r^2}\right)$$

3) If $\phi = x^2y + y^2x + z^2$, find $\nabla\phi$ at $(1,1,1)$.

$$\nabla\phi = \vec{i}(2xy+y^2) + \vec{j}(x^2+2yx) + \vec{k}(2z)$$

$$= \vec{i}(3) + \vec{j}(3) + (2)\vec{k}$$

$$= 3\vec{i} + 3\vec{j} + 2\vec{k}$$

4) Find ∇r.

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r^2 = |\vec{r}|^2 = x^2+y^2+z^2$ $\qquad 2r\frac{\partial r}{\partial x} = 2x$

$$\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{\vec{r}}{r}$$

5) Find the unit normal vector to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$.

Unit normal vector $(\hat{n}) = \dfrac{\nabla\phi}{|\nabla\phi|}$

$\phi = xy^3z^2 - 4$

$$\nabla\phi = \vec{i}(y^3z^2) + \vec{j}(3xy^2z^2) + \vec{k}(2xy^3z)$$

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT III - LAPLACE TRANSFORMS

Part – A

1.	$L[t] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) s (D) s^2	ANS **B**	(CLO-3, Apply)
2.	$L[\cos t] =$ (A) $\frac{1}{s^2-1}$ (B) $\frac{1}{s^2+1}$ (C) $\frac{s}{s^2-1}$ (D) $\frac{s}{s^2+1}$	ANS **D**	(CLO-3, Apply)
3.	$L[e^{3t}] =$ (A) $\frac{1}{s-3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s-\log 9}$ (D) $\frac{9}{s}$	ANS **A**	(CLO-3, Apply)
4.	If $L[f(t)] = F(s)$, then $L[e^{at} f(t)] =$ (A) $F(s+a)$ (B) $F(s-a)$ (C) $e^{as}F(s)$ (D) $e^{-as}F(s)$	ANS **B**	(CLO-3, Remember)
5.	$L[f(t) * g(t)] =$ (A) $F(s) - G(s)$ (B) $F(s) + G(s)$ (C) $F(s)\ G(s)$ (D) $F(s) \div G(s)$	ANS **C**	(CLO-3, Remember)
6.	$L[\sin t] =$ (A) $\frac{1}{s^2-1}$ (B) $\frac{1}{s^2+1}$ (C) $\frac{s}{s^2-1}$ (D) $\frac{s}{s^2+1}$	ANS **B**	(CLO-3, Apply)

7.	$L[e^{-3\,t}] =$ (A) $\frac{1}{s+3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s-\log 3}$ (D) $\frac{3}{s}$	ANS **A**	(CLO-3, Apply)
8.	$L^{-1}\left[\frac{1}{s}\right] =$ (A) t (B) s (C) 1 (D) $\delta(t)$	ANS **C**	(CLO-3, Apply)
9.	$L^{-1}\left[\frac{1}{s^2+9}\right] =$ (A) $\frac{\cos 3t}{3}$ (B) $\frac{\sin 3t}{3}$ (C) $\sin 3t$ (D) $\cos 3t$	ANS **B**	(CLO-3, Apply)
10.	$L^{-1}\left[\frac{s}{s^2+9}\right] =$ (A) $\frac{\cos 3t}{3}$ (B) $\frac{\sin 3t}{3}$ (C) $\sin 3t$ (D) $\cos 3t$	ANS **D**	(CLO-3, Apply)
11.	If $L[f(t)] = F(s)$, then $L[e^{-at}\, f(t)] =$ (A) $F(s+a)$ (B) $F(s-a)$ (C) $e^{as}F(s)$ (D) $e^{-as}F(s)$	ANS **A**	(CLO-3, Remember)
12.	$L[t^2] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) $\frac{2}{s^3}$ (D) $\frac{1}{s^3}$	ANS **C**	(CLO-3, Apply)
13.	$L[1] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) $\frac{2}{s^3}$ (D) $\frac{1}{s^3}$	ANS **A**	(CLO-3, Apply)
14.	$L[e^{-2\,t}] =$ (A) $\frac{1}{s+2}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 4}$ (D) $\frac{4}{s}$	ANS **A**	(CLO-3, Apply)

15.	$L[\sin 3\,t] =$ (A) $\frac{1}{s^2-9}$ (B) $\frac{3}{s^2+9}$ (C) $\frac{s}{s^2-9}$ (D) $\frac{s}{s^2+9}$	ANS **B**	(CLO-3, Apply)
16.	$L[\sinh 2\,t] =$ (A) $\frac{2}{s^2-4}$ (B) $\frac{2}{s^2+4}$ (C) $\frac{1}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **A**	(CLO-3, Apply)
17.	$L[2^{\,t}] =$ (A) $\frac{1}{s-2}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 2}$ (D) $\frac{2}{s}$	ANS **C**	(CLO-3, Apply)
18.	$L[t\,e^{2t}] =$ (A) $\frac{1}{s-2}$ (B) $\frac{1}{(s-2)^2}$ (C) $\frac{2}{(s-2)^3}$ (D) $\frac{1}{s^3}$	ANS **B**	(CLO-3, Apply)
19.	If $L[f(t)] = F(s)$, then $L[\,f(at)] =$ (A) $\frac{1}{a}\,F\left(\frac{s}{a}\right)$ (B) $F\left(\frac{s}{a}\right)$ (C) $F(s+a)$ (D) $F(s-a)$	ANS **A**	(CLO-3, Remember)
20.	$L^{-1}\left[\frac{s-2}{s^2-4s+13}\right] =$ (A) $e^{-2t}\sin 3t$ (B) $e^{-2t}\cos 3t$ (C) $e^{2t}\sin 3t$ (D) $e^{2t}\cos 3t$	ANS **D**	(CLO-3, Apply)
21.	If $L[f(t)] = F(s)$, then $L\left[\int_0^t f(u)du\right] =$ (A) $\frac{F(s)}{s}$ (B) $F\left(\frac{s}{a}\right)$ (C) $\frac{f(t))}{t}$ (D) $F\,(u)$	ANS **A**	(CLO-3, Remember)
22.	$L^{-1}\,[1\,] =$ (A) $\frac{1}{s}$ (B) s (C) 1 (D) $\delta(t)$	ANS **D**	(CLO-3, Apply)

23.	$L^{-1}\left[\frac{s-3}{s^2-6s+13}\right] =$ (A) $e^{-3t}\cos\ 3t$ (B) $e^{2t}\cos 3t$ (C) $e^{3t}\cos 2t$ (D) $e^{-2t}\cos 2t$	ANS **C**	(CLO-3, Apply)
24.	$L[4^{t}] =$ (A) $\frac{1}{s-4}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 4}$ (D) $\frac{4}{s}$	ANS **C**	(CLO-3, Apply)
25.	$L[\cosh 3t] =$ (A) $\frac{s}{s^2+9}$ (B) $\frac{1}{s^2-9}$ (C) $\frac{s}{s^2-9}$ (D) $\frac{s}{s^2+9}$	ANS **C**	(CLO-3, Apply)
26.	$L[\mathrm{t}\cos at] =$ (A) $\frac{s^2+a^2}{(s^2-a^2)^2}$ (B) $\frac{s^2-a^2}{(s^2-a^2)^2}$ (C) $\frac{s^2-a^2}{(s^2+a^2)^2}$ (D) $\frac{s}{s^2+9}$	ANS **C**	(CLO-3, Apply)
27.	$L[t\sin 2t] =$ (A) $\frac{4s}{(s^2+4)^2}$ (B) $\frac{4s}{(s^2-4)^2}$ (C) $\frac{s}{(s^2+4)^2}$ (D) $\frac{4s}{(s^2-4)^2}$	ANS **A**	(CLO-3, Apply)
28.	$L[t\,e^{t}] =$ (A) $\frac{1}{s-1}$ (B) $\frac{1}{(s-2)^2}$ (C) $\frac{1}{(s-1)^2}$ (D) $\frac{1}{(s-1)^3}$	ANS **C**	(CLO-3, Apply)
29.	$L[2\,e^{-3t}] =$ (A) $\frac{2}{s+3}$ (B) $\frac{2}{(s-3)^2}$ (C) $\frac{1}{(s-3)^2}$ (D) $\frac{2}{(s-1)^3}$	ANS **A**	(CLO-3, Apply)
30.	$L[3] =$ (A) $\frac{1}{s-3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s+3}$ (D) $\frac{3}{s}$	ANS **D**	(CLO-3, Apply)

31.	$L[\sin 5t] =$ (A) $\frac{5}{s^2+29}$ (B) $\frac{5}{s^2+25}$ (C) $\frac{1}{s^2+29}$ (D) $\frac{s}{s^2+29}$	ANS **B**	(CLO-3, Apply)
32.	$L[\cos 2t] =$ (A) $\frac{1}{s^2-4}$ (B) $\frac{1}{s^2+4}$ (C) $\frac{s}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **D**	(CLO-3, Apply)
33.	$L[\cosh 2t] =$ (A) $\frac{s}{s^2+4}$ (B) $\frac{1}{s^2-4}$ (C) $\frac{s}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **C**	(CLO-3, Apply)
34.	$L^{-1}\left[\frac{1}{s-3}\right] =$ (A) e^{3t} (B) e^{-3t} (C) $\cos 3t$ (D) $\sin 3t$	ANS **A**	(CLO-3, Apply)
35.	$L^{-1}\left[\frac{s}{s^2-9}\right] =$ (A) $\cos 3t$ (B) $\sin 3t$ (C) $\cosh 3t$ (D) $\sinh 3t$	ANS **C**	(CLO-3, Apply)
36.	$L^{-1}\left[\frac{1}{(s-1)^2}\right] =$ (A) $t\,e^{t}$ (B) e^{t} (C) e^{-t} (D) $t\,e^{-t}$	ANS **A**	(CLO-3, Apply)

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 3 – Laplace Transforms

Part – B (Each question carries 3 Marks)

1. Find $L[2e^{-3t}]$.

Solution

$$L[e^{-at}] = \frac{1}{s+a}$$

$$L[2e^{-3t}] = 2L[e^{-3t}] = 2\left(\frac{1}{s+3}\right)$$

2. Find $L[e^{3t+5}]$.

Solution

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[e^{3t}.e^{5}] = e^{5}L[e^{3t}] = e^{5}\left(\frac{1}{s-3}\right)$$

3. Find the Laplace transform of $f(t) = cos^2(3t)$.

Solution

$$L[\cos^2 3t] = L\left[\frac{1+\cos 6t}{2}\right] = \frac{L(1)+L(\cos 6t)}{2} \qquad \because \cos^2 t = \frac{1+\cos 2t}{2}$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+36)} \qquad \because L(1) = \frac{1}{s}, L(\cos at) = \frac{s}{s^2+a^2}$$

$\therefore$ $\quad$ **L[cos² 3t]** $= \dfrac{s^2+18}{s(s^2+36)}$

4. Find $L\left(t^2-4\sin 2t+2\cos 3t\right)$**.**

Solution

$$L\left(t^2-4\sin 2t+2\cos 3t\right)=\frac{2}{s^3}-4\left(\frac{2}{s^2+4}\right)+2\left(\frac{s}{s^2+9}\right)$$

5. Find the Laplace transform of e^{-t} sin 2t.

Solution

$$L[e^{-t}\sin 2t] = L[e^{-at} f(t)] = F(s+a) = F(s+1)$$

$$F(s) = L[f(t)] = L(\sin 2t) = \frac{2}{s^2+4}$$

$$F(s+1) = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$$

6. Obtain the Laplace transform of $\boldsymbol{\sin 2t - 2t\cos 2t}$ **.**

Solution

$$L[\sin 2t-2t\cos 2t] = L[\sin 2t]-2L[t\cos 2t]=L[\sin 2t]-2\left(-\frac{d}{ds}L[\cos 2t]\right)$$

$$=\frac{2}{s^2+4}+2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4}+2\left(\frac{\left(s^2+4\right)(1)-s(2s)}{\left(s^2+4\right)^2}\right)$$

$$=\frac{2\left(s^2+4\right)+2\left(4-s^2\right)}{\left(s^2+4\right)^2}$$

$$\therefore\ L[\sin 2t-2t\cos 2t] = \frac{16}{\left(s^2+4\right)^2}$$

7. Find $L\left(t\,e^t\right)$**.**

Solution

$$L\left(t\ f(t)\right)=-\frac{d}{ds}L\left(f(t)\right)$$

$$L\left(t\,e^{t}\right)=-\frac{d}{ds}L(e^{t})$$

$$=-\frac{d}{ds}L\left(\frac{1}{s-1}\right)=\frac{1}{(s-1)^{2}}$$

8. Find $L(t\sin 2t)$**.**

Solution

$$L\left(t\,f(t)\right)=-\frac{d}{ds}L(f(t))$$

$$L\left(t\sin 2t\right)=-\frac{d}{ds}L(\sin 2t)$$

$$=-\frac{d}{ds}\left(\frac{2}{s^{2}+4}\right)=\frac{4s}{\left(s^{2}+4\right)^{2}}$$

9. Find the Laplace transform of $f(t)=\boldsymbol{t^{2}\cos t}$.

Solution

$$L\left[t^{2}\cos t\right]=\left[\frac{d^{2}}{ds^{2}}L\left[\cos t\right]\right]=\frac{d^{2}}{ds^{2}}\left(\frac{s}{s^{2}+1}\right)$$

$$=\frac{d}{ds}\left(\frac{\left(s^{2}+1\right).1-1.2s.s}{\left(s^{2}+1\right)^{2}}\right)=\frac{d}{ds}\left(\frac{1-s^{2}}{\left(s^{2}+1\right)^{2}}\right)$$

$$=\frac{\left(s^{2}+1\right)^{2}(-2s)-\left(1-s^{2}\right)2\left(s^{2}+1\right)2s}{\left(s^{2}+1\right)^{3}}=\frac{-2s\left(3-s^{2}\right)}{\left(s^{2}+1\right)^{3}}$$

10. Find the Laplace transform of $\boldsymbol{f(t)=te^{-3t}cos2t}$

Solution

$$L[f(t)]=L[te^{-3t}cos2t]=-\frac{d}{ds}L[\cos 2t]_{s\to s+3}=-\frac{d}{ds}\left[\frac{s}{s^{2}+4}\right]_{s\to s+3}$$

$$=-\left[\frac{(s^{2}+4)(1)-s(2s)}{(s^{2}+4)^{2}}\right]_{s\to s+3}=\left[\frac{s^{2}-4}{(s^{2}+4)^{2}}\right]_{s\to s+3}$$

$$= \frac{(s+3)^2-4}{((s+3)^2+4)^2}$$
$$= \frac{s^2+6s+5}{(s^2+6s+13)^2}$$

11. Find the Laplace Transform of $f(t) = e^{-t}t\, cost.$

Solution

$$L[e^{-t}tcost] = -\frac{d}{ds}L[\cos t]_{s\to s+1} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= -\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]_{s\to s+1}$$
$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to s+1}$$
$$= \frac{(s+1)^2-1}{((s+1)^2+1)^2} = \frac{s^2+2s}{(s^2+2s+2)^2}$$

$$= \frac{s(s+2)}{(s^2+2s+2)^2}$$

12. Find $L\left[\frac{sint}{t}\right]$.

Solution

$$L\left[\frac{sint}{t}\right]=L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(s)ds$$

$$\text{F(s)} = \text{L}\ [sint]=\frac{1}{s^2+1^2}$$

$$\int_s^\infty F(s)ds=\int_s^\infty \frac{1}{s^2+1}ds=[tan^{-1}(s)]_s^\infty$$

$$=[tan^{-1}\infty - tan^{-1}s]=\left[\frac{\pi}{2} - tan^{-1}s\right] = cot^{-1}s$$

13. Find the Laplace transform of $f(t) = \dfrac{e^{-t}\sin t}{t}$.

Solution

$$L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L\left(e^{-t}\sin t\right)ds$$

$$= \int_s^\infty L\left(\sin t\right)_{s+1} ds = \int_s^\infty \left(\frac{1}{s^2+1}\right)_{s+1} ds = \int_s^\infty \frac{1}{\left(s+1\right)^2+1} ds$$

$$= \left[\tan^{-1}\left(s+1\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(s+1\right) = \cot^{-1}\left(s+1\right)$$

14. Find the Laplace Transform of $f(t) = \frac{1-cost}{t}$.

Solution

$$L[1-cost] = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[1-cost]ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}[\log(s^2+1) - logs^2]_s^\infty$$

$$= -\frac{1}{2}\left[log\frac{s^2+1}{s^2}\right]_s^\infty = -\frac{1}{2}\left[log\left(1+\frac{1}{s^2}\right)\right]_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}log\left[1+\frac{1}{s^2}\right] = \frac{1}{2}log\left(\frac{s^2+1}{s^2}\right)$$

15. Find $L\left[\frac{cos\ at - cos\ bt}{t}\right]$.

Solution

$$L\left[\frac{cosat - cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}log(s^2+a^2) - \frac{1}{2}log(s^2+b^2)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty=\frac{1}{2}\left[\log\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$=\frac{1}{2}\left[\log 1-\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]=\frac{1}{2}\ \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

16. Evaluate $\int_0^\infty t\, e^{-2t} \sin t\, dt$ using Laplace transform.

Solution

$\int_0^\infty te^{-2t}\sin t\,dt=\int_0^\infty e^{-st}f(t)dt=F(s)$ Here s = 2.

$F(s)\ =\ L\,[f\ (t)], F(s)\ =\ L\,[t\sin t]$

$$=-\frac{d}{ds}\left[\frac{1}{s^2+1}\right]=\frac{2s}{(s^2+1)^2}$$

$$\int_0^\infty \boldsymbol{t\, e^{-2t} \sin t\, dt}\ =[\boldsymbol{F(s)}]_{s\,=\,2}=\frac{\mathbf{4}}{(\mathbf{4+1})^2}=\frac{\mathbf{4}}{\mathbf{25}}$$

17. Verify initial value theorem for the function *f* (*t*) = 2 – cos t.

Solution

Initial value theorem states that $\lim\limits_{t\to 0} f(t)=\lim\limits_{s\to\infty} sF(s)$

L. H. S. = $\lim\limits_{t\to 0} f(t)=2-\cos 0=1$

R. H. S. = $\lim\limits_{s\to\infty} s\,L(f(t))=\lim\limits_{s\to\infty} s\,L(2-\cos t)$

$$=\lim_{s\to\infty} s\left(2-\frac{s^2}{s^2+1}\right)==\lim_{s\to\infty} s\left(2-\frac{1}{1+\frac{1}{s^2}}\right)=2-1=1$$

L.H.S=R.H.S

Initial value theorem verified.

18. Verify final value theorem for the function f(t) = 1 + e^{-t} (sin t + cos t).

Solution

$$L\left[f(t)\right] = F(s)$$

$$= \frac{1}{s} + L\left[\sin t + \cos t\right]_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

Final value theorem states that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\text{L.H.S.} = \lim_{t\to\infty}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + 0 = 1$$

$$\text{R. H. S.} = \lim_{s\to 0} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right] = \lim_{s\to 0}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1$$

L.H.S.=R.H.S

Hence final value theorem verified

19. Find $L^{-1}\left(\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-9}\right)$.

Solution

$$L^{-1}\left(\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-9}\right) = e^{3t} + 1 + \cosh 3t$$

20. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$.

Solution

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right) = L^{-1}\left(\frac{1}{(s+2)}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2}\right) = e^{-2t} - 2te^{-2t}$$

21. Find $L^{-1}\left(\dfrac{1}{s^2+2s+5}\right)$**.**

Solution

$$L^{-1}\left(\frac{1}{s^2+2s+5}\right)=L^{-1}\left(\frac{1}{(s+1)^2+4}\right)=\frac{e^{-t}\,\sin 2t}{2}$$

22. Find $\boldsymbol{L^{-1}\left(\dfrac{s}{s^2+4s+5}\right)}$**.**

Solution

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right)=L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right)=e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$

$$=e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right)-2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$=e^{-2t}\left[\cos t-2\sin t\right]$$

23. Find $L^{-1}\left(\dfrac{s-5}{s^2-3s+2}\right)$**.**

Solution:

$$L^{-1}\left(\frac{s-5}{s^2-3s+2}\right)=L^{-1}\left(\frac{A}{s-1}+\frac{B}{s-2}\right)=L^{-1}\left(\frac{4}{s-1}\right)+L^{-1}\left(\frac{-3}{s-2}\right)=4e^{t}-3e^{2t}$$

24. Find $\boldsymbol{L^{-1}\left[\dfrac{s+2}{s^2+2s+2}\right]}$**.**

Solution:

$$L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right] \because\ L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]$$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right]+L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{1}{s^2+1}\right]\right)=\mathrm{e}^{-\mathrm{t}}\ (\cos \mathrm{t}+\sin \mathrm{t})$$

25. Find $L^{-1}\left[\frac{1}{s^2+6s+13}\right]$.

Solution

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right]=L^{-1}\left[\frac{1}{(s+3)^2+4}\right]=L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$
$$=\frac{1}{2}L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]=\frac{1}{2}\mathrm{e}^{-3t}\sin 2t.$$

26. Find $L^{-1}\left[\cot^{-1}(s+1)\right]$.

Solution:

Let $L^{-1}\left[\cot^{-1}(s+1)\right]=f(t)$

$$\therefore\ L\left[f(t)\right]=\cot^{-1}(s+1)$$

$$L\left[tf(t)\right]=-\frac{d}{ds}\left[\cot^{-1}(s+1)\right]=\frac{1}{(s+1)^2+1}$$

$$tf(t)=L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]=e^{-t}\sin t$$

$$\therefore\ f(t)=\frac{e^{-t}\sin t}{t}$$

27. Find the inverse Laplace transform of $\frac{s}{(s+2)^2}$.

Solution

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$
$$=\frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right)=\frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$
$$=\frac{d}{dt}\left(e^{-2t}\,t\right)=e^{-2t}+t\left(-2e^{-2t}\right)=e^{-2t}(1-2t)$$

* * * * *

Module – 3 Laplace Transforms

Laplace Transforms of standard functions – Transforms properties – Transforms of Derivatives and Integrals – Initial value theorems (without proof) and verification for some problems – Final value theorems (without proof) and verification for some problems – Inverse Laplace transforms using partial fractions – Inverse Laplace transforms using second shifting theorem – LT using Convolution theorem – problems only – ILT using Convolution theorem – problems only – LT of periodic functions – problems only – Solve linear second order ordinary differential equations with constant coefficients only – Solution of Integral equation and integral equation involving convolution type – Application of Laplace Transform in Engineering.

Periodic function:

A function $f(t)$ is said to be periodic function if $f(t+p) = f(t)$ for all t. The least value of $p > 0$ is called the period of $f(t)$. For example, $\sin t$ and $\cos t$ are periodic functions with period 2π .

Laplace Transform:

Let f(t) be a given function which is defined for all positive values of t, if

$$L\{f(t)\}= F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists, then F(s) is called *Laplace transform* of f(t).

Sufficient condition for the existence of Laplace transform:

The Laplace transform of $f(t)$ exists if

- **i.** $f(t)$ is piecewise continuous in $[a, b]$ where $a > 0$.
- **ii.** $f(t)$ is of exponential order.

Laplace transform for some basic functions

S.No	f(t)	L{f(t)}
1	e^{at}	$\frac{1}{s-a},\ s-a>0$
2	e^{-at}	$\frac{1}{s+a},\ s+a>0$
3	$\sin at$	$\frac{a}{s^2+a^2},\ s>0$
4	$\cos at$	$\frac{s}{s^2+a^2},\ s>0$

5	$\sinh at$	$\dfrac{a}{s^2-a^2}$, $s>\lvert a\rvert$
6	$\cosh at$	$\dfrac{s}{s^2-a^2}$, $s>\lvert a\rvert$
7	1	$\dfrac{1}{s}$
8	t	$\dfrac{1}{s^2}$
9	t^n	$\dfrac{n!}{s^{n+1}}$
10	Periodic function with period 'p'	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-at} f(t)\,dt$

Properties of Laplace transform:

Sl. No.	Property	Laplace Transform
1	Linear Property	$L\left(af(t)\pm bg(t)\right)=aL\left(f(t)\right)\pm bL\left(g(t)\right)$
2	First shifting theorem	$L\left(e^{-at}f(t)\right)=F(s+a)$ $L\left(e^{at}f(t)\right)=F(s-a)$
3	Change of scale property	$L\left(f(at)\right)=\dfrac{1}{a}F\left(\dfrac{s}{a}\right),\ a>0$
4	Multiplication by t	$L\left(t^n f(t)\right)=(-1)^n\dfrac{d^n}{ds^n}F(s)$
5	Division by t	$L\left(\dfrac{f(t)}{t}\right)=\int_s^\infty F(s)\,ds$, provided $\lim\limits_{t\to 0}\dfrac{f(t)}{t}$ exists
6	Transforms of integrals	$L\left(\int_0^t f(t)dt\right)=\dfrac{L\left[f(t)\right]}{s}$

Inverse Laplace transform for some basic functions:

S.No	F(s)	$f(t) = L^{-1}(F(s))$
1	$\frac{1}{s-a},\ s-a>0$	e^{at}
2	$\frac{1}{s+a},\ s+a>0$	e^{-at}
3	$\frac{a}{s^2+a^2},\ s>0$	$\sin at$
4	$\frac{s}{s^2+a^2},\ s>0$	$\cos at$
5	$\frac{a}{s^2-a^2},\ s>\|a\|$	$\sinh at$
6	$\frac{s}{s^2-a^2},\ s>\|a\|$	$\cosh at$
7	$\frac{1}{s}$	1
8	$\frac{1}{s^2}$	t
9	$\frac{n!}{s^{n+1}}$	t^n

Initial Value theorem:

If $L(f(t)) = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Final value theorem:

If $L(f(t)) = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Convolution:

The convolution of two functions f(t) and g(t) is defined as $\int_0^t f(u)g(t-u)\,du = f(t)*g(t)$

Convolution theorem:

The Laplace transform of convolution of two functions is equal to the product of their Laplace transforms.

(i.e) $L\left[f(t)*g(t)\right]=L\left[f(t)\right]L\left[g(t)\right]$.

1. **Obtain the Laplace transform of $\sin 2t-2t\cos 2t$.**

Solution: $L[\sin 2t-2t\cos 2t] = L[\sin 2t]-2L[t\cos 2t]=L[\sin 2t]-2\left(-\frac{d}{ds}L[\cos 2t]\right)$

$$=\frac{2}{s^2+4}+2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4}+2\left(\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right)$$

$$=\frac{2(s^2+4)+2(4-s^2)}{(s^2+4)^2}$$

$\therefore L[\sin 2t-2t\cos 2t] = \frac{16}{(s^2+4)^2}$

2. **Find the Laplace transform $\sin^3(2t)$**

Solution: $L\left[\sin^3(2t)\right]=\frac{1}{4}\text{L}[3\sin 2t-\sin 6t] = \frac{3}{4}\text{L}[\sin 2t]-\frac{1}{4}\text{L}[\sin 6t]$

$\left(\because \sin^3 t=\frac{1}{4}[3\sin t-\sin 3t]\right)$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right)-\frac{1}{4}\left(\frac{6}{s^2+36}\right) = \frac{6}{4}\left(\frac{1}{s^2+4}-\frac{1}{s^2+36}\right).$$

3. **Find the Laplace transform of $f(t)=\cos^2(3t)$.**

Solution: $L[\cos^2 3t] = L\left[\frac{1+\cos 6t}{2}\right]=\frac{L(1)+L(\cos 6t)}{2}$ $\because \cos^2 t=\frac{1+\cos 2t}{2}$

$$= \frac{1}{2s}+\frac{s}{2(s^2+36)} \quad \because L(1)=\frac{1}{s}, L(\cos at)=\frac{s}{s^2+a^2}$$

$\therefore \quad L[\cos^2 3t] = \frac{s^2+18}{s(s^2+36)}$

4. **Find the Laplace transform of unit step function**

Solution: The Unit step function is $u_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a, \ a \ge 0 \end{cases}$

The Laplace transform $L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\,dt = \int_a^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-as}\right] = \frac{e^{-as}}{s}.$

5. **Find the Laplace transform of the following functions (i) $\frac{e^{-t}\sin t}{t}$ (ii) $t^2 \cos t$**

Solution:

(i) To find $\frac{e^{-t}\sin t}{t}$

$$L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L\left(e^{-t}\sin t\right) ds$$

$$= \int_s^\infty L(\sin t)_{s+1}\, ds = \int_s^\infty \left(\frac{1}{s^2+1}\right)_{s+1} ds = \int_s^\infty \frac{1}{(s+1)^2+1}\, ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

(ii) $t^2 \cos t$

$$L\left[t^2\cos t\right] = \left[\frac{d^2}{ds^2} L[\cos t]\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2+1).1 - 1.2s.s}{(s^2+1)^2}\right) = \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right)$$

$$= \frac{(s^2+1)^2(-2s) - (1-s^2)2(s^2+1)2s}{(s^2+1)^3} = \frac{-2s(3-s^2)}{(s^2+1)^3}$$

6. **Find the Laplace transform of $e^{-2t} t^{1/2}$.**

Solution: $L\left[e^{-2t} t^{1/2}\right] = L\left[t^{1/2}\right]_{s \to s+2}$

$$\because \textit{If } L[f(t)] = F(s), \text{then } L\left[e^{-at} f(t)\right] = F(s)\big|_{s \to s+a}$$

$$= \left[\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}\right]_{s\to s+2} = \left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}\right]_{s\to s+2}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{(s+2)^{3/2}} \left(\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \Gamma n+1 = n\Gamma n\right)$$

7. **Find $L\left[t^2 e^{-t}\cos t\right]$**

Solution:

$$L\left[t^2 e^{-t}\cos t\right] = L\left[t^2\cos t\right]_{s\to s+1}$$

$$= \left[(-1)^2\frac{d^2}{ds^2}L[\cos t]\right]_{s\to s+1} = \left[\frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{\left(s^2+1\right)1 - s.2s}{\left(s^2+1\right)^2}\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{1-s^2}{\left(s^2+1\right)^2}\right]_{s\to s+1}$$

$$= \left[\frac{2s^3-6s}{\left(s^2+1\right)^3}\right]_{s\to s+1}$$

$$= \frac{2(s+1)^3 - 6(s+1)}{\left((s+1)^2+1\right)^3}$$

8. **Find $L\left[t^2 e^{t}\sin t\right]$**

Solution:

$$L\left[t^2 e^t \sin t\right] = (-1)^2\frac{d^2}{ds^2}L\left[e^t\sin t\right] \quad \ldots (1)$$

Now $L\left[e^t \sin t\right] = \left[L[\sin t]\right]_{s \to (s-1)} = \dfrac{1}{(s-1)^2+1}$ ... (2)

Substituting (2) in (1) we get

$$L\left[t^2 e^t \sin t\right] = \frac{d}{ds}\left[\frac{0-2(s-1)}{\left((s-1)^2+1\right)^2}\right] = \frac{d}{ds}\left[\frac{-2(s-1)}{\left(s^2-2s+2\right)^2}\right]$$

$$= \frac{\left(s^2-2s+2\right)^2(-2)+2(s-1)2\left(s^2-2s+2\right)(2s-2)}{\left(s^2-2s+2\right)^4}$$

$$= \frac{2\left(s^2-2s+2\right)\left[-\left(s^2-2s+2\right)+4(s-1)^2\right]}{\left(s^2-2s+2\right)^4}$$

$$= \frac{2\left(s^2-2s+2\right)\left[-s^2+2s-2+4s^2+4-8s\right]}{\left(s^2-2s+2\right)^4}$$

$$\therefore F(s) = \frac{2\left(s^2-2s+2\right)\left[3s^2-6s+2\right]}{\left(s^2-2s+2\right)^4} = \frac{2\left(3s^2-6s+2\right)}{\left(s^2-2s+2\right)^3}$$

9. **Find** $\boldsymbol{L\left[\dfrac{\sin^2 t}{t}\right]}$

Solution:

$$L\left[\frac{\sin^2 t}{t}\right] = L\left[\frac{1-\cos 2t}{2t}\right] = \frac{1}{2}L\left[\frac{1-\cos 2t}{t}\right] = \frac{1}{2}\int_s^\infty L[1-\cos 2t]\ ds$$

$$= \frac{1}{2}\int_s^\infty \left\{L[1]-L[\cos 2t]\right\} ds = \frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+4}\right]ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log\left(s^2+4\right)\right]_s^\infty = \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{1}{\sqrt{1+\frac{4}{s^2}}}\right]_s^\infty = \frac{1}{2}\left[\log 1 - \log \frac{1}{\sqrt{1+\frac{4}{s^2}}}\right] = \frac{1}{2}\left[0 - \log \frac{s}{\sqrt{s^2+4}}\right]$$

$$F(s) = \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)^{-1} = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

10. **Using Laplace transform, Evaluate $\int_0^\infty t\, e^{-2t} \sin t\, dt$**

Solution: $\int_0^\infty e^{-2t} f(t) dt = \left[\int_0^\infty e^{-st} f(t) dt\right]_{s=2} = [L[t \sin t]]_{s=2} = \left[-\frac{d}{ds} L[\sin t]\right]_{s=2}$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{-2s}{(s^2+1)^2}\right) = \frac{4}{25}$$

11. **Evaluate $\int_0^t \sin u \cos(t-u) du$ using Laplace Transform.**

Solution: Let $L\left[\int_0^t \sin u \cos(t-u) du\right] = L[\sin t * \cos t]$

$= L[\sin t]\ \ L[\cos t]$ (by Convolution theorem)

$$= \frac{1}{(s^2+1)} \frac{s}{(s^2+1)} = \frac{s}{(s^2+1)^2}$$

$$\int_0^t \sin u \cos(t-u) du = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2} L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = \frac{t}{2}\sin t \quad \left(\because L^{-1}\left[\frac{2s}{(s^2+a^2)^2}\right] = t \sin at\right)$$

12. **Find the Laplace transform of $\int_0^t t\, e^{-t} \sin t\, dt$**

Solution:

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[t\sin t] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{(s^2+1)0-2s}{(s^2+1)^2}\right) = \frac{2s}{(s^2+1)^2}$$

$$\therefore\ L\left[te^{-t}\ \sin t\right] = \left.\frac{2s}{(s^2+1)^2}\right|_{s\to s+1} = \frac{2(s+1)}{\left((s+1)^2+1\right)^2} = \frac{2(s+1)}{(s^2+2s+2)^2}$$

$$L\left[\int_0^t t\,e^{-t}\sin t\,dt\right] = \frac{1}{s}L\left[te^{-t}\ \sin t\right]$$

$$\therefore\quad = \frac{1}{s}\frac{2(s+1)}{(s^2+2s+2)^2}$$

13. **Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$**

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1} = \left[\frac{1}{s}L(t\cos t)\right]_{s\to(s+1)}$$

$$= \left[\frac{1}{s}\left(-\frac{d}{ds}L(\cos t)\right)\right]_{s\to(s+1)} = \left[-\frac{1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to(s+1)}$$

$$= \left[-\frac{1}{s}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)\right]_{s\to(s+1)} = \left[-\frac{1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to(s+1)}$$

$$\therefore F(s) = \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to(s+1)} = \left[\frac{(s+1)^2-1}{(s+1)\left[(s+1)^2+1\right]^2}\right] = \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

14. **Find the Laplace transform of** $e^{-4t}\int_0^t t\ \sin 3t dt$

Solution:

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L[t\sin 3t] = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -\left(\frac{(s^2+9)0 - 3(2s)}{(s^2+9)^2}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left(\int_0^t t\sin 3t dt\right) = \frac{L(t\sin 3t)}{s} = \frac{6}{(s^2+9)^2}$$

$$L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = L\left(\int_0^t t\sin 3t dt\right)\Bigg|_{s\to s+4} = \frac{6}{\left((s+4)^2+9\right)^2} = \frac{6}{\left(s^2+8s+16+9\right)^2}$$

$$\therefore L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = \frac{6}{\left(s^2+8s+25\right)^2}$$

15. **Verify initial and final value theorems for the function f(t) = 1 + e^{-t} (sin t + cos t)**

Solution:

Initial value theorem states that $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$L[f(t)] = F(s)$$

$$= \frac{1}{s} + L[\sin t + \cos t]_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

L.H.S. $= \lim_{t\to 0} f(t) = 1+1 = 2$

R.H.S $= \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right] = \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$

$$= \lim_{s\to\infty}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right] = \lim_{s\to\infty}\left[1 + \frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right] = 1+1 = 2$$

L.H.S=R.H.S

Initial value theorem verified.

Final value theorem states that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

L.H.S. $= \lim_{t\to\infty}\left[1+e^{-t}(\sin t+\cos t)\right] = 1+0 = 1$

R.H.S $= \lim_{s\to 0}\left[1+\dfrac{s(s+2)}{(s+1)^2+1}\right] = 1+0 = 1$

L.H.S.=R.H.S

Hence final value theorem verified

16. **Find the Laplace transform of the square wave function defined by** $\boldsymbol{f(t)=\begin{cases} E, & 0<t<\dfrac{a}{2} \\ -E, & \dfrac{a}{2}<t<a \end{cases} \;\&\; f(t+a)=f(t)}$

Solution:

$$L\left[f(t)\right] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} Ee^{-st}\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right] = \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-\left(e^{-\frac{as}{2}}-1\right)+\left(e^{-as}-e^{-\frac{as}{2}}\right)\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-e^{-\frac{as}{2}}+1+e^{-as}-e^{-\frac{as}{2}}\right]$$

$$= \frac{E}{s\left(1-e^{-\frac{as}{2}}\right)\left(1+e^{-\frac{as}{2}}\right)}\left(1-e^{-\frac{as}{2}}\right)^2 = \frac{E}{s}\left(\frac{1-e^{-\frac{as}{2}}}{1+e^{-\frac{as}{2}}}\right)$$

$$\therefore F(s) = \frac{E}{s}\left[\frac{e^{sa/4} - e^{-sa/4}}{e^{sa/4} + e^{-sa/4}}\right] = \frac{E}{s}\tanh\left(\frac{sa}{4}\right)$$

17. **Find the Laplace transform of the rectangular wave given by** $\boldsymbol{f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}}$

Given $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

This function is periodic in the interval $(0, 2b)$ with period $2b$.

$$L[f(t)] = \frac{1}{1 - e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1 - e^{-2bs}}\left[\int_0^{b} e^{-st} f(t)\,dt + \int_b^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1 - e^{-2bs}}\left[\int_0^{b} e^{-st}\,dt + \int_b^{2b} e^{-st}(-1)\,dt\right] = \frac{1}{1 - e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{b} - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{s\left(1 - e^{-2bs}\right)}\left[-\left(e^{-bs} - 1\right) + \left(e^{-2bs} - e^{-bs}\right)\right]$$

$$= \frac{1}{s\left(1 - e^{-2bs}\right)}\left[-e^{-bs} + 1 + \left(e^{-bs}\right)^2 - e^{-bs}\right]$$

$$= \frac{1}{s\left(1 - e^{-bs}\right)\left(1 + e^{-bs}\right)}\left(1 - e^{-bs}\right)^2 = \frac{1}{s}\left(\frac{1 - e^{-bs}}{1 + e^{-bs}}\right)$$

$$\therefore F(s) = \frac{1}{s}\left[\frac{e^{sb/2} - e^{-sb/2}}{e^{sb/2} + e^{-sb/2}}\right] = \frac{1}{s}\tanh\left(\frac{sb}{2}\right)$$

18. **Find the Laplace transform of f(t) =** $\boldsymbol{\begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a \end{cases}}$ **and *f*(*t*+2*a*) = *f*(*t*) for all *t***

Solution:

$$L[f(t)] = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\, t\, dt + \int_a^{2a} e^{-st}(2a-t)dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[-t\left(\frac{e^{-st}}{s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[-(2a-t)\left(\frac{e^{-st}}{s}\right)+\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[\left(-a\frac{e^{-as}}{s}-\frac{e^{-as}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]+\left[\frac{e^{-2as}}{s^2}-\left(-\frac{ae^{-as}}{s}+\frac{e^{-as}}{s^2}\right)\right]\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right] = \frac{\left(1-e^{-sa}\right)^2}{s^2\left(1-e^{-as}\right)\left(1+e^{-as}\right)}$$

$$\therefore\ F(s) = \frac{1-e^{-sa}}{s^2\left(1+e^{-as}\right)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

19. **Find the Laplace transform of the rectangular wave given by** $f(t)=\begin{cases}\sin\omega t, & 0<t<\dfrac{\pi}{\omega}\\[4pt] 0, & \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega}\end{cases}$

Solution:

This function is periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0,\frac{2\pi}{\omega}\right)$

$$L[f(t)] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}\left[-s\sin\omega t-\omega\cos\omega t\right]\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega+\omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)} = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

20. **Find $L^{-1}\left(\frac{s}{s^2+4s+5}\right)$**

Solution:

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right) = L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right) = e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$

$$= e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$= e^{-2t}\left[\cos t - 2\sin t\right]$$

21. **Find $L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]$**

Solution: $L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right] \because L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$\therefore\ L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]=e^{-t}\,(\cos t+\sin t)$$

22. **Find** $L^{-1}\left(\dfrac{s}{(s+2)^3}\right)$

Solution: $L^{-1}\left(\dfrac{s}{(s+2)^3}\right)=L^{-1}\left(\dfrac{s+2-2}{(s+2)^3}\right)$

$$=L^{-1}\left(\frac{1}{(s+2)^2}\right)-2\,L^{-1}\left(\frac{1}{(s+2)^3}\right)$$

$$=e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)-e^{-2t}L^{-1}\left(\frac{2}{s^3}\right)$$

$$=\ e^{-2t}\left(t-t^2\right).$$

23. **Find** $L^{-1}\left[\tan^{-1}\left(\dfrac{1}{s}\right)\right]$

Solution: Let F(s) = $\tan^{-1}\left(\dfrac{1}{s}\right)$

$$F'(s)=\frac{1}{1+(1/s)^2}\left(\frac{-1}{s^2}\right)=\frac{-1}{s^2+1}$$

By property $L^{-1}\left[F'(s)\right]=-L^{-1}\left[\dfrac{1}{s^2+1}\right]=-\sin t$

$$\therefore\ L^{-1}(F'(s))=-\sin t\,;\ L^{-1}(F(s))=\frac{-1}{t}L^{-1}\left[F'(s)\right]$$

$$\therefore L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]=\frac{\sin t}{t}$$

24. **Find the inverse Laplace transform of** $\dfrac{s}{(s+2)^2}$

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$

$$=\frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right)=\frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$=\frac{d}{dt}\left(e^{-2t}\,t\right)=e^{-2t}+t\left(-2e^{-2t}\right)=e^{-2t}\left(1-2t\right)$$

25. **Find $L^{-1}\left[\cot^{-1}(s+1)\right]$**

Let $L^{-1}\left[\cot^{-1}(s+1)\right]=f(t)$

$\therefore\; L\left[f(t)\right]=\cot^{-1}(s+1)$

$$L\left[tf(t)\right]=-\frac{d}{ds}\left[\cot^{-1}(s+1)\right]=\frac{1}{(s+1)^2+1}$$

$$tf(t)=L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]=e^{-t}\sin t$$

$$\therefore\; f(t)=\frac{e^{-t}\sin t}{t}$$

26. **Find the inverse Laplace transform of $\log\left(\frac{1+s}{s^2}\right)$**

Solution:

Let $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]=f(t)$

$\therefore\; L\left[f(t)\right]=\log\left(\frac{1+s}{s^2}\right)$

$$L\left[t\,f(t)\right]=\frac{-d}{ds}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{-d}{ds}\left[\log(1+s)-\log\left(s^2\right)\right]=-\frac{1}{1+s}+\frac{1}{s^2}2s$$

$$L\left[t\,f(t)\right]=\frac{2}{s}-\frac{1}{s+1}$$

$$t\,f(t) = L^{-1}\left[\frac{2}{s} - \frac{1}{s+1}\right] = 2L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] = 2(1) - e^{-t}$$

$$\therefore\ f(t) = \frac{2-e^{-t}}{t}$$

$$\therefore\ L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right] = \frac{2-e^{-t}}{t}$$

27. **Find** $\boldsymbol{L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]}$

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

Put $s=-1 \Rightarrow \boxed{A = -\frac{1}{3}}$

Equating the coefficients of $s^3 \Rightarrow \boxed{B = \frac{1}{3}}$

Put $s=2 \Rightarrow \boxed{D=-7}$

Put $s=0 \Rightarrow \boxed{C=4}$

$$\therefore\ \frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{-1/3}{s+1} + \frac{1/3}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = -\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right] - 7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right] - 7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}\,t - \frac{7}{2}e^{2t}L^{-1}\left[\frac{2}{s^3}\right]$$

$$\therefore\ f(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}\,t - \frac{7}{2}e^{2t}t^2$$

28. **Using Convolution theorem find** $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos at*\frac{1}{a}\sin at=\frac{1}{a}\left[\cos at*\sin at\right]$$

$$=\frac{1}{a}\int_0^t \cos au\ \sin a(t-u)\,du=\frac{1}{a}\int_0^t \sin(at-au)\cos au\ du$$

$$=\frac{1}{a}\int_0^t \frac{\sin(at-au+au)+\sin(at-au-au)}{2}\,du$$

$$=\frac{1}{2a}\int_0^t\left[\sin at+\sin a(t-2u)\right]du$$

$$=\frac{1}{2a}\left[\sin at\ u+\left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[u\ \sin at+\left(\frac{\cos a(t-2u)}{2a}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[t\ \sin at+\left(\frac{\cos at}{2a}\right)-\left(0+\frac{\cos at}{2a}\right)\right]$$

$$f(t)=\frac{1}{2a}\left[t\sin at+\frac{\cos at}{2a}-\frac{\cos at}{2a}\right]=\frac{1}{2a}t\sin at$$

29. **Find the inverse Laplace transform of** $\frac{s}{(s^2+a^2)(s^2+b^2)}$ **using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$\therefore\ L^{-1}\left[\frac{s}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$=\frac{1}{b}\cos at*\sin bt$$

$$=\frac{1}{b}\int_0^t\cos au\ \ \sin b(t-u)\ du$$

$$=\frac{1}{2b}\int_0^t\left[\sin(au+bt-bu)-\sin(au-bt+bu)\right]\ du$$

$$=\frac{1}{2b}\int_0^t\left[\sin((a-b)u+bt)-\sin((a+b)u-bt)\right]\ du$$

$$=\frac{1}{2b}\left[\frac{-\cos(bt+(a-b)u)}{a-b}+\frac{\cos((a+b)u-bt)}{a+b}\right]_0^t$$

$$=\frac{1}{2b}\left[\left(\frac{-\cos(bt+at-bt)}{a-b}+\frac{\cos(at+bt-bt)}{a+b}\right)-\left(\frac{-\cos bt}{a-b}+\frac{\cos bt}{a+b}\right)\right]$$

$$=\frac{1}{2b}\left[\left(\frac{-\cos(at)}{a-b}+\frac{\cos(at)}{a+b}\right)-\left(\frac{-\cos bt}{a-b}+\frac{\cos bt}{a+b}\right)\right]$$

$$=\frac{1}{2b}\left(\frac{-2b\cos at}{a^2-b^2}+\frac{2b\cos bt}{a^2-\mathrm{b}^2}\right)$$

$$f(t)=\frac{\cos bt-\cos at}{a^2-b^2}$$

30. **Find the inverse Laplace transform of** $\boldsymbol{\frac{s^2}{(s^2+a^2)(s^2+b^2)}}$ **using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{s}{s^2+b^2}\right]=\cos at*\cos bt$$

$$=\int_0^t \cos au\ \cos b(t-u)\ du$$

$$=\frac{1}{2}\int_0^t\left[\cos(au+bt-bu)+\cos(au-bt+bu)\right]\ du$$

$$=\frac{1}{2}\int_0^t\left[\cos((a-b)u+bt)+\cos((a+b)u-bt)\right]\ du$$

$$=\frac{1}{2}\left[\frac{\sin(bt+(a-b)u)}{a-b}+\frac{\sin((a+b)u-bt)}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\left(\frac{\sin(bt+at-bt)}{a-b}+\frac{\sin(at+bt-bt)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left[\left(\frac{\sin(at)}{a-b}+\frac{\sin(at)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left(\frac{2a\sin(at)}{a^2-b^2}-\frac{2b\sin(bt)}{a^2-\mathrm{b}^2}\right)$$

$$f(t)=\frac{a\sin(at)-b\sin(bt)}{a^2-b^2}$$

31. **Find the inverse Laplace transform of** $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)}$

Solution:

$$L^{-1}\left[\frac{s}{\left(s^2+1\right)\left(s^2+4\right)}\right]=L^{-1}\left[\frac{s}{s^2+1}\ \frac{1}{s^2+4}\right]=L^{-1}\left[\frac{s}{s^2+1}\right]*\frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=\frac{1}{2}\cos t*\sin 2t$$

$$= \frac{1}{2}\int_0^t \cos u \, \sin 2(t-u)\, du$$

$$= \frac{1}{4}\int_0^t \left[\sin(u+2t-2u) - \sin(u-2t+2u)\right] du \qquad \left(\because 2\cos A \sin B = \sin(A+B) - \sin(A-B)\right)$$

$$= \frac{1}{4}\int_0^t \left[\sin(2t-u) - \sin(u-2t)\right] du$$

$$= \frac{1}{4}\left[\frac{-\cos(2t-u)}{-1} + \frac{\cos(u-2t)}{1}\right]_0^t$$

$$= \frac{1}{4}\left[\cos t - \cos 2t + \cos t - \cos 2t\right]$$

$$= \frac{1}{4}\left[2\cos t - 2\cos 2t\right]$$

$$\therefore\ f(t) = \frac{1}{2}\left[\cos t - \cos 2t\right]$$

32. **Using Convolution theorem find the inverse Laplace transform of** $\boldsymbol{\frac{2}{(s+1)(s^2+4)}}$

Solution:

$$L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{1}{s+1}\ \frac{2}{s^2+4}\right] = L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= e^{-t} * \sin 2t$$

$$= \int_0^t e^{-u} \ \sin 2(t-u)\, du$$

$$= \int_0^t e^{-u} \ \sin(2t-2u)\, du$$

$$= \int_0^t e^{-u} \ \left[\sin 2t \ \cos 2u - \cos 2t \ \sin 2u\right] du$$

$$= \int_0^t e^{-u} \ \sin 2t \ \cos 2u \ du - \int_0^t e^{-u} \ \cos 2t \ \sin 2u \ du$$

$$= \sin 2t \int_0^t e^{-u} \cos 2u \, du - \cos 2t \int_0^t e^{-u} \sin 2u \, du$$

$$= \sin 2t \left[\frac{e^{-u}}{1+4} \left(-\cos 2u + 2\sin 2u \right) \right]_0^t - \cos 2t \left[\frac{e^{-u}}{1+4} \left(-\sin 2u - 2\cos 2u \right) \right]_0^t$$

$$= \sin 2t \left[\left(\frac{e^{-t}}{5} \left(-\cos 2t + 2\sin 2t \right) \right) - \left(\frac{1}{5}(-1) \right) \right] - \cos 2t \left[\frac{e^{-t}}{5} \left(-\sin 2t - 2\cos 2t \right) - \left(\frac{1}{5}(-2) \right) \right]$$

$$= \sin 2t \left[\frac{e^{-t}}{5} \left(-\cos 2t + 2\sin 2t \right) + \frac{1}{5} \right] - \cos 2t \left[\frac{e^{-t}}{5} \left(-\sin 2t - 2\cos 2t + \frac{2}{5} \right) \right]$$

$$= \frac{e^{-t}}{5} \left[-\sin 2t \cos 2t + 2\sin^2 2t + \sin 2t \cos 2t + 2\cos^2 2t \right] + \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

$$= \frac{e^{-t}}{5} \left[2(1) \right] + \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

$$f(t) = \frac{1}{5} \left[2e^{-t} + \sin 2t - 2\cos 2t \right]$$

33. **Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+1)(s^2+4)}$**

Solution:

$$L^{-1}\left[F(s) G(s) \right] = L^{-1}\left[F(s) \right] * L^{-1}\left[G(s) \right]$$

$$\therefore \; L^{-1}\left[\frac{s^2}{\left(s^2+1^2\right)\left(s^2+2^2\right)} \right] = L^{-1}\left[\frac{s}{s^2+1^2} \right] * L^{-1}\left[\frac{s}{s^2+2^2} \right]$$

$$= \cos t * \cos 2t$$

$$= \int_0^t \cos u \, \cos 2(t-u) \, du$$

$$= \frac{1}{2} \int_0^t \left[\cos(u + 2t - 2u) + \cos(u - 2t + 2u) \right] du$$

$$= \frac{1}{2} \int_0^t \left[\cos(-u + 2t) + \cos(3u - 2t) \right] du$$

$$= \frac{1}{2}\left[\frac{\sin(2t-u)}{-1} + \frac{\sin(3u-2t)}{3}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin t}{-1} + \frac{\sin t}{3}\right) - \left(\frac{\sin 2t}{-1} - \frac{\sin 2t}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2\sin t}{-3} - \frac{4\sin 2t}{-3}\right)$$

$$f(t) = \frac{\sin t - 2\sin 2t}{-3}$$

34. Find $L^{-1}\left(\dfrac{e^{-2s}}{\left(s^2+s+1\right)^2}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{\left(s^2+s+1\right)^2}\right) = L^{-1}\left(\frac{e^{-s}}{s^2+s+1}\,\frac{e^{-s}}{s^2+s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1} * L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1}$$

$$= L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1} * L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1}$$

$$= e^{-t/2}L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1} * e^{-t/2}L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1}$$

$$= \left[e^{-t/2}\,\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}} * e^{-t/2}\,\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}\right]_{t\to t-1}$$

$$= \frac{2}{\sqrt{3}} e^{-(t-1)/2} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right) * \frac{2}{\sqrt{3}} e^{-(t-1)/2} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right)$$

$$= \frac{4}{3}\left[e^{-(t-1)/2} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right) * e^{-(t-1)/2} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\right]$$

$$= \frac{4}{3}\int_0^t e^{-\frac{u-1}{2}} e^{-\frac{t-u-1}{2}} \sin\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}\right) \ \sin\left(\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}\right) du$$

$$= \frac{4}{3}\int_0^t e^{-\left(\frac{t}{2}-1\right)} \frac{1}{2} \ \cos\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}t\right) - \cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right) du$$

$$= \frac{2}{3} e^{-\left(\frac{t-2}{2}\right)} \left[\frac{\sin\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}} - \cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right) u \right]_0^t$$

$$= e^{-\left(\frac{t-2}{2}\right)} \left[\frac{4}{3\sqrt{3}} \sin\frac{\sqrt{3}}{2}t - \frac{2}{3} t \cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right)\right]$$

35. **Solve using Laplace transform $\frac{dy}{dt} + y = e^{-t}$ given that y(0) = 0.**

Solution: Taking L.T. on both sides, we get $L\left[y'(t)\right] + L\left[y(t)\right] = L\left[e^{-t}\right]$

$$sL\left[y(t)\right] - y(0) + L\left[y(t)\right] = L\left[e^{-t}\right]$$

$$sL\left[y(t)\right] - 0 + L\left[y(t)\right] = \frac{1}{s+1}$$

$$(s+1)\ L[y(t)] = \frac{1}{s+1}$$

$$L[y(t)] = \frac{1}{(s+1)^2}$$

$\therefore y(t) = L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t} L\left(\frac{1}{s^2}\right) =$ e^{-t} t $\qquad \left(\because \ L\left[e^{-at} f(t)\right] = F(s+a)\right)$

36. **Using Laplace transform to solve the differential equation** $\boldsymbol{y'' + y' = t^2 + 2t}$**, given** $\boldsymbol{y = 4,\ y' = -2}$ **when** $\boldsymbol{t = 0}$

Solution:

Given $y'' + y' = t^2 + 2t$

$$L\left[y'' + y'\right] = L\left[t^2 + 2t\right]$$

$$\left[s^2 L\left[y(t)\right] - sy(0) - y'(0)\right] + \left[sL\left[y(t)\right] - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L\left[y(t)\right]\left(s^2 + s\right) = \frac{2}{s^3} + \frac{2}{s^2} + 4s - 2 + 4$$

$$L\left[y(t)\right]s(s+1) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2$$

$$L\left[y(t)\right] = \frac{2 + 2s + 4s^4 + 2s^3}{s^4(s+1)}$$

$$L\left[y(t)\right] = \frac{2}{s} + \frac{2}{s^4} + \frac{2}{s+1}$$

$$y(t) = L^{-1}\left[\frac{2}{s} + \frac{2}{s^4} + \frac{2}{s+1}\right]$$

$$= 2 + 2\frac{t^3}{6} + 2e^{-t}$$

$$y(t) = 2 + \frac{t^3}{3} + 2e^{-t}$$

37. **Solve** $\boldsymbol{\left(D^2 + 3D + 2\right)y = e^{-3t}}$**, given** $\boldsymbol{y(0) = 1,}$ **and** $\boldsymbol{y'(0) = -1}$ **using Laplace Transforms**

Solution:

Given $y'' + 3y' + 2y = e^{-3t}$

Taking Laplace transforms on both side

$$L\left(y'' + 3y' + 2y\right) = L\left(e^{-3t}\right)$$

$$L\left[y''(t)\right] + 3L\left[y'(t)\right] + 2L\left[y(t)\right] = \frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y^{'}(0)\right]+3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-s(1)-(-1)\right]+3\left[sL\left[y(t)\right]-1\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$L\left[y(t)\right]\left[s^2+3s+2\right]=\frac{1}{s+3}+s+2$$

$$L\left[y(t)\right]=\frac{s^2+5s+7}{(s+3)\left(s^2+3s+2\right)},\ y(t)=L^{-1}\left[\frac{s^2+5s+7}{(s+1)(s+2)(s+3)}\right]$$

$$y(t)=L^{-1}\left[\frac{3/2}{s+1}-\frac{1}{s+2}+\frac{1/2}{s+3}\right]$$

$$y(t)=\frac{3}{2}L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{s+2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$y(t)=\frac{3}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t}$$

38. **Solve $y^{''}+2y^{'}-3y=\sin t$, given $y(0)=0,\ y^{'}(0)=0$**

Solution:

Given $y^{''}+2y^{'}-3y=\sin t$

$$L\left[y^{''}(t)+2y^{'}(t)-3y(t)\right]=L[\sin t]$$

$$L\left[y^{''}(t)\right]+2L\left[y^{'}(t)\right]-3L\left[y(t)\right]=L[\sin t]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y^{'}(0)\right]+2\left[sL\left[y(t)\right]-y(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$\left[s^2L\left[y(t)\right]-s(0)-0\right]+2\left[sL\left[y(t)\right]-(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$s^2L\left[y(t)\right]+2sL\left[y(t)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]\left(s^2+2s-3\right)=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]=\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}$$

$$y(t) = L^{-1}\left[\frac{1}{(s^2+1)(s^2+2s-3)}\right] == L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]$$

Now

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{(s^2+1)}$$

$$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)$$

Put $s = 1 \Rightarrow \boxed{A = \frac{1}{8}}$

Put $s = -3 \Rightarrow \boxed{B = \frac{-1}{40}}$

Equating coeff. of $s^3 \Rightarrow \boxed{C = \frac{-1}{10}}$

Equating the constant terms $\Rightarrow \boxed{D = \frac{-1}{5}}$

$$\therefore \frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1/8}{s-1} + \frac{-1/40}{s+3} + \frac{(-1/10)s - 1/5}{(s^2+1)}$$

$$L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right] = L^{-1}\left[\frac{1/8}{s-1} + \frac{-1/40}{s+3} + \frac{(-1/10)s-1/5}{(s^2+1)}\right]$$

$$= \frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s+2}{s^2+1}\right]$$

$$= \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\left[L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{2}{s^2+1}\right]\right]$$

$$= \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}[\cos t + 2\sin t]$$

39. **Solve the equation $y'' + 9y = \cos 2t$ with $y(0) = 1$, $y\left(\frac{\pi}{2}\right) = -1$**

Solution:

Given $\left(D^2+9\right)y=\cos 2t$

Taking Laplace transforms on both sides

$$L\left[y''(t)\right]+9L\left[y(t)\right]=L[\cos 2t]$$

$$s^2L\left[y(t)\right]-sy(0)-y'(0)+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

Using the initial conditions

$y(0)=1,$ and taking $y'(0)=k$

We have

$$s^2L\left[y(t)\right]-(s)(1)-k+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

$$\Rightarrow L\left[y(t)\right]=\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}+\frac{s+k}{s^2+9}$$

$$=\frac{s}{5\left(s^2+4\right)}-\frac{s}{5\left(s^2+9\right)}+\frac{s}{s^2+9}+\frac{k}{s^2+9}$$

$$\therefore\ y(t)=\frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right]-\frac{1}{5}L^{-1}\left[\frac{s}{s^2+9}\right]+L^{-1}\left[\frac{s}{s^2+9}\right]+k\,L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{k}{3}\sin 3t$$

Put $t=\frac{\pi}{2}$ we get $y\left(\frac{\pi}{2}\right)=\frac{1}{5}(-1)-\frac{1}{5}(0)+0+\frac{k}{3}(-1)=-\frac{1}{5}-\frac{k}{3}$

But given $y\left(\frac{\pi}{2}\right)=-1$

$$\therefore\ -1=-\frac{1}{5}-\frac{k}{3}$$

$$\Rightarrow k=\frac{12}{5}$$

$$\therefore\ y(t)=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{4}{5}\sin 3t$$

$$y(t)=\frac{4}{5}\left[\cos 3t+\sin 3t\right]+\frac{1}{5}\cos 2t$$

40. **Solve $x'' + 2x' + 5x = e^{-t}\sin t$, where $x(0) = 0,\ x'(0) = 1$ using Laplace Transforms**

Solution:

Given $x'' + 2x' + 5x = e^{-t}\sin t$

Taking Laplace transforms on both side

$$L\left[x'' + 2x' + 5x\right] = L\left[e^{-t}\sin t\right]$$

$$L\left[x''(t)\right] + 2L\left[x'(t)\right] + 5L\left[x(t)\right] = \frac{1}{s^2+2s+2}$$

$$\left[s^2L\left[x(t)\right] - sx(0) - x'(0)\right] + 2\left[sL\left[x(t)\right] - x(0)\right] + 5L\left[x(t)\right] = \frac{1}{s^2+2s+2}$$

$$\left[s^2L\left[x(t)\right] - s(0) - 1\right] + 2\left[sL\left[x(t)\right] - (0)\right] + 5L\left[x(t)\right] = \frac{1}{s^2+2s+2}$$

$$L\left[x(t)\right]\left[s^2+2s+5\right] = \frac{1}{s^2+2s+2} + 1$$

$$L\left[x(t)\right]\left[s^2+2s+5\right] = \frac{s^2+2s+3}{s^2+2s+2}$$

$$L\left[x(t)\right] = \frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)} = \frac{(s+1)^2+2}{\left((s+1)^2+1\right)\left((s+1)^2+4\right)}$$

$$x(t) = L^{-1}\left[\frac{(s+1)^2+2}{\left((s+1)^2+1\right)\left((s+1)^2+4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{s^2+2}{\left(s^2+1\right)\left(s^2+4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{1/3}{s^2+1} + \frac{2/3}{s^2+4}\right]$$

$$= e^{-t}\left[\frac{1}{3}\sin t + \frac{1}{3}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}\left[\sin t + \sin 2t\right]$$

41. **Using Laplace transform to solve the differential equation**

$\boldsymbol{y'' - 3y' + 2y = 4t + e^{3t}}$**, where** $\boldsymbol{y(0) = 1,\ y'(0) = -1}$

Solution:

Given $y'' - 3y' + 2y = 4t + 3e^{t}$

$$L\left[y'' - 3y' + 2y\right] = L\left[4t + 3e^{t}\right]$$

$$L\left[y''(t)\right] - 3L\left[y'(t)\right] + 2L\left[y(t)\right] = 4L[t] + 3L\left[e^{3t}\right]$$

$$\left[s^2 L\left[y(t)\right] - sy(0) - y'(0)\right] - 3\left[sL\left[y(t)\right] - y(0)\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2 L\left[y(t)\right] - s(1) - (-1)\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2 L\left[y(t)\right] - s + 1\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2 - 3s + 2\right) = s - 4 + \frac{4}{s^2} + \frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2 - 3s + 2\right) = \frac{(s-4)s^2(s-3) + 4(s-4) + 3s^2}{s^2(s-3)}$$

$$L\left[y(t)\right] = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{\left(s^2 - 3s + 2\right)s^2(s-3)}$$

$$L\left[y(t)\right] = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{(s-2)(s-1)s^2(s-3)}$$

$$y(t) = L^{-1}\left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{(s-2)(s-1)s^2(s-3)}\right]$$

$$= L^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{s-2}+\frac{E}{s-3}\right]$$

$$= L^{-1}\left[\frac{3}{s}+\frac{2}{s^2}+\frac{-1/2}{s-1}+\frac{-2}{s-2}+\frac{1/2}{s-3}\right]$$

$$y(t) = 3+2t-\frac{1}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}$$

42. **Solve** $\boldsymbol{y'' - 3y' + 2y = e^{2t},\ y(0) = -3,\ y'(0) = 5}$

Solution:

Given $y'' - 3y' + 2y = e^{2t}$

$$L\left[y'' - 3y' + 2y\right] = L\left[e^{2t}\right]$$

$$L\left[y''\right] - 3L\left[y'\right] + 2L[y] = L\left[e^{2t}\right]$$

$$\left[s^2L\left[y(t)\right] - sy(0) - y'(0)\right] - 3\left[sL\left[y(t)\right] - y(0)\right] + 2L\left[y(t)\right] = \frac{1}{s-2}$$

$$\left[s^2L\left[y(t)\right] - s(-3) - 5\right] - 3\left[sL\left[y(t)\right] - (-3)\right] + 2L\left[y(t)\right] = \frac{1}{s-2}$$

$$s^2L\left[y(t)\right] + 3s - 5 - 3sL\left[y(t)\right] - 9 + 2L\left[y(t)\right] = \frac{1}{s-2}$$

$$L\left[y(t)\right]\left[s^2 - 3s + 2\right] + 3s - 14 = \frac{1}{s-2}$$

$$\therefore \quad L\left[y(t)\right]\left[s^2 - 3s + 2\right] = \frac{1}{s-2} - 3s + 14$$

$$\therefore \quad L\left[y(t)\right] = \frac{-3s^2 + 20s - 27}{(s-2)\left(s^2 - 3s + 2\right)}$$

$$y(t) = L^{-1}\left[\frac{-3s^2 + 20s - 27}{(s-2)\left(s^2 - 3s + 2\right)}\right]$$

$$y(t) = L^{-1}\left[\frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2}\right]$$

$$\frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$-3s^2+20s-27=A(s-2)^2+B(s-1)(s-2)+C(s-1)$$

Put $s=1 \Rightarrow \boxed{A=-10}$

Put $s=2 \Rightarrow \boxed{C=1}$

Equating the coeff. of $s^2 \Rightarrow \boxed{B=7}$

$$\therefore \frac{-3s^2+20s-27}{(s-1)(s-2)^2}=\frac{-10}{s-1}+\frac{7}{s-2}+\frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]=L^{-1}\left[\frac{-10}{s-1}\right]+L^{-1}\left[\frac{7}{s-2}\right]+L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$=-10e^t+7e^{2t}+e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=-10e^t+7e^{2t}+te^{2t}$$

43. **Solve $\frac{dx}{dt}-2x+3y=0;\ \frac{dy}{dt}-y+2x=0$ with $x(0)=8,\ y(0)=3$**

The given differential equation canbe written as

$$x'(t)-2x+3y=0 \quad y'(t)-y+2x=0$$

Taking Laplace transforms weget,

$$L\left[x'(t)-2x+3y\right]=L[0]$$

$$sL[x(t)]-x(0)-2L[x(t)]+3L[y(t)]=0$$

$$sL[x(t)]-8-2L[x(t)]+3L[y(t)]=0$$

$$L[x(t)](s-2)+3L[y(t)]=8 \qquad (1)$$

And $\quad L\left[y'(t)-y+2x\right]=L[0]$

$$sL[y(t)]-y(0)-L[y(t)]+2L[x(t)]=0$$

$$sL[y(t)]-3-L[y(t)]+2L[x(t)]=0$$

$$2L[x(t)]+(s-1)L[y(t)]=3 \qquad (2)$$

Solving (1) and (2) we get,

$$L\left[x(t)\right]=\frac{8s-17}{(s+1)(s-4)}=\frac{5}{s+1}+\frac{3}{s-4},$$

$$\therefore\ x(t)=L^{-1}\left[\frac{5}{s+1}+\frac{3}{s-4}\right],$$

$$\boxed{x(t)=5e^{-t}+3e^{4t}}$$

and $$L\left[y(t)\right]=\frac{3s-22}{(s+1)(s-4)}=\frac{5}{s+1}-\frac{2}{s-4}$$

$$y(t)=L^{-1}\left[\frac{5}{s+1}-\frac{2}{s-4}\right]=5e^{-t}-2e^{4t}$$

44. **Determine y which satisfies the equation** $\boldsymbol{\frac{dy}{dt}+2y+\int_0^t y\,dt=2\cos t,\ \ y(0)=1}$

Solution:

Given $y'(t)+2y(t)+\int_0^t y(t)\,dt=2\cos t,\ \ y(0)=1$

$$L\left[y'(t)\right]+2L\left[y(t)\right]+L\left[\int_0^t y(t)\,dt\right]=L\left[2\cos t\right]$$

$$sL\left[y(t)\right]-y(0)+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$sL\left[y(t)\right]-1+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$L\left[y(t)\right]=\frac{s}{s^2+1}$$

$$y(t)=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$

	SRM Institute of Science and Technology Kattankulathur	
	DEPARTMENT OF MEATHEMATICS	
	18MAB102T ADVANCED CALCULUS & COMPLEX ANALYSIS	
	UNIT –IV ANALYTIC FUNCTIONS	
	Tutorial Sheet -2	**Answers**
	Part – A	
1	Find the image of the circle |z|=3 under the transformation w=2z	6
2	Find a function w such that w=u+iv is analytic, if $u = e^x \sin y$	$f(z) = -ie^z + c$
3	Determine the analytic function u+iv whose real part $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$	$f(z) = z^3 + 3z^2 + c$
	Part – B	
4	Find the analytic function $f(z) = u + iv$ if $u - v = e^x(cosy - siny)$	$f(z) = e^z + c$
5	Find the analytic function $f(z) = u + iv$ if $u - v = \frac{sin2x}{cosh2y - cos2x}$	$f(z) = \frac{\cot z}{1+i} + c$
6	Determine the region D' of the w-plane into which the triangular region D enclosed by the lines x=0, y=0, x+y=1 is transformed under the transformation w=2z	
7	Find an analytic function $f(z) = u + iv$, given that $2u + 3v = \frac{sin\ 2x}{cos\ h2y - \cos x}$	$f(z) = \frac{(2+3i)\cot z}{13} + c$

Module – 5 Complex Integration

Cauchy's integral formulae – Problems – Taylor's expansions with simple problems – Laurent's expansions with simple problems – Singularities – Types of Poles and Residues – Cauchy's residue theorem (without proof) – Contour integration: Unit circle, semicircular contour – Application of Contour integration in Engineering.

Cauchy's Integral Theorem

If $f(z)$ is analytic at every point of the region R bounded by a simple closed curve C and if $f'(z)$ is continuous at all points inside and on C, then $\int_C f(z)\,dz = 0$

Cauchy's integral formula for n^{th} derivative

If $f(z)$ is analytic inside and on a simple closed curve C and $z = a$ is any interior point of the region R enclosed by C, then $f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

$$(i.e.)\quad \boxed{\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a)}$$

Taylor's series

If $f(z)$ is analytic inside a circle C with centre at a then Taylor's series about $z = a$ is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots$$

Laurent's series

If C_1, C_2 are two concentric circles with centre at z = a and radii r_1 and r_2 ($r_1 < r_2$) and if f(z) is analytic inside and on the circles and within the annular region between C_1 and C_2, then for any z in the annular region, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n},$$

where $a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz$ and $b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-a)^{-n+1}}dz$

Cauchy's Residue theorem

If f(z) is analytic inside a closed curve C except at a finite number of isolated singular points $a_1, a_2, \ldots a_n$ inside C, then

$\int\limits_C f(z)dz = 2\pi i \times$(sum of the residues of f(z) at these singular points).

Contour Integration

Type I:

$$\boxed{\int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) d\theta}$$

$$\text{Let } z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta = izd\theta \Rightarrow d\theta = \frac{dz}{iz}$$

Then we have

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right); \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\cos 2\theta = \text{Real part of } z^2 \; ; \qquad \cos n\theta = \text{Real part of } z^n$$

$$\sin 2\theta = \text{Im part of } z^2; \qquad \sin n\theta = \text{Im part of } z^n$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} = \text{Real part of } \left[\frac{1+z^2}{2}\right];$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} = \text{Real part of } \left[\frac{1-z^2}{2}\right]$$

$\therefore$

$\int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) d\theta = \int\limits_C f\left(z\right) dz$, where C *is* $|z| = 1$ and solve by known method.

Type II:

$$\boxed{\int_{-\infty}^{\infty} f\left(x\right) dx}$$

Using Cauchy's integral formula, find $\int\limits_C \frac{z+4}{z^2+2z+5} dz$**, where C is** $|z+1-i| = 2$

Solution:

$|\ z + 1 - i\ | = 2$

$|x + iy + 1 - i| = 2$

$|(x+1) + i(y-1)| = 2, \qquad \sqrt{(x+1)^2 + (y-1)^2} = 2$

Squaring on both sides,

$$(x+1)^2+(y-1)^2=4$$

This is equation of circle with centre $(-1,1)$ and radius 2.

$$z^2+2z+5=0$$

$$z=\frac{-2\pm\sqrt{4-4(1)(5)}}{2(1)}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\int_C \frac{z+4}{\left[z-(-1+2i)\right]\left[z-(-1-2i)\right]}\,dz$$

Here –1+2i lies inside the circle c and –1–2i lies outside the circle c.

Let $a=-1+2i$

By Cauchy's integral formula, $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

Substituting for a, $f(-1+2i)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-(-1+2i)}\,dz$ ......(1)

Comparing equation (1) with given problem,

$$f(z)=\frac{z+4}{z-(-1-2i)}$$

$$f(-1+2i)=\frac{-1+2i+4}{-1+2i-(-1-2i)}=\frac{2i+3}{-1+2i+1+2i}=\frac{2i+3}{4i}$$

Substituting for $f(-1+2i)$ in (1)

$$\frac{2i+3}{4i}=\frac{1}{2\pi i}\int_C \frac{z+4}{z^2+2z+5}\,dz$$

Cross multiplying

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\frac{(2i+3)(2\pi i)}{4i}=\frac{\pi}{2}(3+2i)$$

Using Cauchy's integral formula, evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)\ (z-1)}dz$, where C is $|z|=3$

Solution:

We know that, Cauchy's integral formula is $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

$(i.e)\ 2\pi i\ f(a)=\int_C \frac{f(z)}{z-a}\,dz$

Given: $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz$ Here, $f(z)=\sin \pi z^2+\cos \pi z^2$

The points $a_1=1, a_2=2$ lies inside $|z|=3$

Now, $\frac{1}{(z-1)(z-2)}=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$ (by Partial fraction method)

$$\therefore \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz=-\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)}dz+\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)}dz$$
$$=-2\pi i\ f(1)+2\pi i\ f(2)$$

$$\mathrm{f}(z)=\sin \pi z^2+\cos \pi z^2$$
$$\mathrm{f}(1)=\sin \pi+\cos \pi=-1 \text{ and } \mathrm{f}(2)=\sin 4\pi+\cos 4\pi=1$$

$$\therefore \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz=-2\pi i(-1)+2\pi i(1)=4\pi i$$

Using Cauchy's integral formula, evaluate $\int_C \frac{1}{(z-2)\ (z+1)^2}dz$, where C is $|z|=\frac{3}{2}$

Solution:

Here $z=-1$ is a pole lies inside the circle
$z=2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{(z+1)^2(z-2)}=\int \frac{\frac{1}{z-2}}{(z+1)^2}\,dz$$

Here $\mathrm{f}(z)=\frac{1}{z-2}$, $f'(z)=-\frac{1}{(z-2)^2}$

Hence by Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz=\frac{2\pi i}{n!}f^n(a)$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2}\,dz = \frac{2\pi i}{1!} f'(-1)$$

$$= 2\pi i\left[\frac{-1}{(-1-2)^2}\right] \left(\because f'(z) = \frac{-1}{(z-2)^2}\right) = 2\pi i\left[\frac{-1}{9}\right]$$

$$\int_C \frac{1}{(z-2)\ (z+1)^2}dz = \frac{-2}{9}\pi i.$$

Using Cauchy's integral formula, evaluate $\int_c \frac{z}{z^2+1}dz$ where C is $|z+i|=1$.

Solution:

Consider the curve

$|z+i| = 1 \Rightarrow |x+iy+i| = 1$

$|x+i(y+1)| = 1 \Rightarrow x^2+(y+1)^2 = 1$

Which is a circle with centre (0,–1) and radius 1

The poles are obtained by $z^2+1=0$

$\Rightarrow z=i$ is a simple pole which lies outside C.

$z=-i$ is a simple pole which lies inside C.

$$\int_c \frac{z}{z^2+1}dz = \int_c \frac{z}{(z+i)(z-i)}dz = \int_c \frac{\frac{z}{(z-i)}}{(z+i)} = 2\pi i f(-i) \ldots (1)$$

$$f(z) = \frac{z}{(z-i)},\ f(-i) = \frac{-i}{(-i-i)} = \frac{-i}{-2i} = \frac{1}{2}$$

$$(1) \Rightarrow \int_c \frac{z}{z^2+1}dz = 2\pi i f(-i) = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

Expand $f(z) = \log(1+z)$ in Taylor's series about $z=0$

Solution: Let $f(z) = \log(1+z)$ $f(0) = \log 1 = 0$

$$f'(z) = \frac{1}{1+z} \quad f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \qquad f''(0) = -1$$

$$f'''(z)=\frac{2}{(1+z)^3} \quad f'''(0)=2$$

$$f^{iv}(z)=\frac{-6}{(1+z)^4} \qquad f^{iv}(0)=-6$$

$$\log(1+z)= f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+...=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+....$$

Find the Taylor's series expansion of f(z) = $\frac{z}{(z+1)(z-3)}$, in the region $|z|<1$

Solution:

Splitting f(z) into partial fractions, we have

$$f(z)=\frac{z}{(z+1)(z-3)}=\frac{A}{(z+1)}+\frac{B}{(z-3)}$$

$$\Rightarrow z=A(z-3)+B(\ z+1)$$

$$\text{put } z=-1, \text{we get } \ A=\frac{1}{4}$$

$$\text{put } z=3, \text{we get } \ B=\frac{3}{4}$$

$$\therefore \quad f(z)=\frac{1}{4}\left(\frac{1}{z+1}\right)+\frac{3}{4}\left(\frac{1}{z-3}\right)=\frac{1}{4}\left(\frac{1}{1+z}\right)+\frac{3}{4}\left(\frac{1}{-3}\right)\left(\frac{1}{1-\frac{z}{3}}\right)$$

$$=\frac{1}{4}\left[(1+z)^{-1}-\left(1-\frac{z}{3}\right)^{-1}\right]$$

$$=\frac{1}{4}\left[\left(1-z+z^2-....\right)-\left(1+\frac{z}{3}+\frac{z^2}{9}+...\right)\right]$$

$$=\frac{1}{4}\left[\left((-1)-\frac{1}{3}\right)z+\left((-1)^2-\left(\frac{1}{3}\right)^2\right)z^2+...\right]$$

$$\therefore f(z)=\frac{1}{4}\sum_{n=1}^{\infty}\left((-1)^n-\left(\frac{1}{3}\right)^n\right)z^n$$

Obtain Taylor's Series to represent the function $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ in the region $|z|<2$

Solution:

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$$

Since the degree of the numerator and denominator are same we have to divide and apply partial fractions.

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

$|z|<2 \Rightarrow \frac{|z|}{2}<1$ and $\therefore \frac{|z|}{3}<1$

Consider

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{3}{z+2}-\frac{8}{z+3}=\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}=\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

$$\therefore \frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Find the Laurent's series expansion of $\frac{1}{(z-2)(z-1)}$ valid in the regions $|z|>2$ and $0<|z-1|<1$

Solution:

$$f(z)=\frac{1}{(z-2)(z-1)}=\frac{A}{(z-1)}+\frac{B}{(z-2)}=\frac{A(z-2)+B(z-1)}{(z-2)(z-1)}$$

$$\Rightarrow 1=A(z-2)+B(z-1)$$

Put $z=1,\ A=-1$

$z=2,\ B=1$

$$\therefore f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$$

Region 1:

$$|z|>2 \Rightarrow 2<|z|$$

$$\Rightarrow \left|\frac{2}{z}\right|<1$$

$$f(z)=\frac{-1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$=-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=-\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots\right)+\frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\ldots\right)$$

$$=-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n+\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

Region 2:

Put $z-1=t \Rightarrow z=1+t$

$$0<|z-1|<1 \Rightarrow 0<|t|<1$$

$$\Rightarrow |t|<1$$

$$f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$$

$$=\frac{-1}{t}+\frac{1}{t-1}$$

$$=\frac{-1}{t}+\frac{1}{-(1-t)}$$

$$=\frac{-1}{t}-(1-t)^{-1}$$

$$=\frac{-1}{t}-\left(1+t+t^2+\ldots\right)$$

$$= \frac{-1}{(z-1)} - \left(1 + (z-1) + (z-1)^2 + \ldots\right)$$

$$= \frac{-1}{(z-1)} - \sum_{n=0}^{\infty} (z-1)^n$$

Expand f(z) = $\frac{z^2-1}{z^2+5z+6}$ in a Laurent's series expansion for $|z| > 3$ and $2 < |z| < 3$

Solution:

$$\frac{z^2-1}{z^2+5z+6} = 1 + \frac{-5z-7}{z^2+5z+6} = 1 + \frac{-5z-7}{(z+3)(z+2)}$$

Consider $\frac{-5z-7}{(z+3)(z+2)}$

$$\frac{-5z-7}{(z+3)(z+2)} = \frac{A}{z+2} + \frac{B}{z+3} = \frac{A(z+3)+B(z+2)}{(z+3)(z+2)}$$

$$-5z-7 = A(z+3)+B(z+2)$$

Put $z=-2$ then $A=3$

Put $z=-3$ then $B=-8$

Substituting we get, $\frac{-5z-7}{(z+3)(z+2)} = \frac{3}{z+2} - \frac{8}{z+3}$

$$\frac{z^2-1}{z^2+5z+6} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) **Given** $|z| > 3 \quad \Rightarrow \quad \frac{3}{|z|} < 1$

$$\frac{z^2-1}{z^2+5z+6} = 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \ldots\right) - \frac{8}{z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \ldots\right)$$

(ii) **Given** $2 < |z| < 3 \Rightarrow \quad \frac{2}{|z|} < 1$ and $\frac{|z|}{3} < 1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\ldots\right)$$

Obtain the Laurent's series expansion for the function f(z) = $\frac{4z}{\left(z^2-1\right)(z-4)}$ in

$|z-1|>4$ **and** $2<|z-1|<3$

Solution:

Put $z-1=u \Rightarrow z=u+1$

Now, $f(z)=\frac{4z}{\left(z^2-1\right)(z-4)}=\frac{4z}{(z-1)(z+1)(z-4)}$

Hence $f(u)=\frac{4(u+1)}{u(u+2)(u-3)}$

$$\frac{4(u+1)}{u(u+2)(u-3)}=\frac{A}{u}+\frac{B}{u+2}+\frac{C}{u-3}=\frac{A(u+2)(u-3)+Bu(u-3)+Cu(u+2)}{u(u+2)(u-3)}$$

$$4(u+1)=A(u+2)(u-3)+Bu(u-3)+Cu(u+2)$$

Put $u=0$ then $A=\frac{-2}{3}$

Put $u=-2$ then $B=\frac{-2}{5}$

Put $u=3$ then $C=\frac{16}{15}$

$$f(u)=\frac{4(u+1)}{u(u+2)(u-3)}=\frac{-2/3}{u}+\frac{-2/5}{u+2}+\frac{16/15}{u-3}$$

(i) $|u|>4 \quad \Rightarrow \quad \frac{4}{|u|}<1$

$$f(u)=\frac{-2/3}{u}-\frac{2/5}{u+2}+\frac{16/15}{u-3}$$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{u\left(1-\frac{3}{u}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} + \frac{16}{15}\left(\frac{1}{u}\right)\left(1-\frac{3}{u}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{u}+\frac{4}{u^2}-....\right) + \frac{16}{15}\left(1+\frac{3}{u}+\frac{9}{u^2}+....\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-....\right) + \frac{16}{15}\left(1+\frac{3}{(z-1)}+\frac{9}{(z-1)^2}+....\right)\right]$$

(ii) $2 < |u| < 3 \Rightarrow \quad \frac{2}{|u|} < 1 \text{ and } \frac{|u|}{3} < 1$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{-3\left(1-\frac{u}{3}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} - \frac{16}{45}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{u}+\frac{4}{u^2}-....\right)\right] - \frac{16}{45}\left(1+\frac{u}{3}+\frac{u^2}{9}+....\right)$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-....\right)\right] - \frac{16}{45}\left(1+\frac{(z-1)}{3}+\frac{(z-1)^2}{9}+....\right)$$

Find the Laurent's series expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ **in 1 < | z + 1| < 3**

Solution:

The singular points are z = 0, z = 2, z = –1

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$\Rightarrow$ 7z – 2 = A(z – 2) (z + 1) + B z (z + 1) + C z (z – 2)

Put z = 0, –2 = A(–2) $\Rightarrow$ A = 1

z = 2, 14 – 2 = B 2(2 + 1) $\Rightarrow$ B = 2

$z = -1, \quad -7 - 2 = C(-1)(-1-2) \Rightarrow C = -3$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Put t = z + 1 $\Rightarrow$ z = t – 1

$\therefore 1 < |t| < 3$

$1 < |t| \Rightarrow \left|\frac{1}{t}\right| < 1$ and $\left|\frac{t}{3}\right| < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t}$$

$$= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{2}{(-3)\left(1-\frac{t}{3}\right)} - \frac{3}{t}$$

$$= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t}$$

$$= \frac{1}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots\right] - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots\right] - \frac{3}{t}$$

$$= -\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots-\frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots\right]$$

$$= -2(z+1)^{-1}+(z+1)^{-2}+(z+1)^{-3}+\ldots-\frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\ldots\right]$$

Evaluate $\int_C \frac{z\ \ dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2| = \frac{1}{2}$ by Cauchy Residue theorem.

Solution:

The poles are obtained by $(z-1)(z-2)^2 = 0$

$\Rightarrow$ z = 1 is a simple pole and z = 2 is a pole of order 2.

C is the circle $|z-2| = \frac{1}{2}$

Here z = 1 lies outside C and z = 2 lies inside C.

Residue at z=2: (Pole of order 2)

$$\text{Res } f(z) = \underset{z\to 2}{lt} \frac{d}{dz}(z-2)^2 \frac{z}{(z-1)(z-2)^2} = \underset{z\to 2}{lt} \frac{z-1-z}{(z-1)^2} = -1$$

By Cauchy Residue theorem,

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i(-1) = -2\pi i$$

Using Cauchy's residue theorem evaluate $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$, where C is $|z|=2$

Solution:

$|z|=2$ is the equation of the circle with centre at origin and radius 2.

$$(z^2-1)(z-3)=0$$

$$(z^2-1)=0, \qquad (z-3)=0$$

$$z^2=1, \qquad z=3$$

$$z=\pm 1, \qquad z=3$$

$z=1,-1$ lies inside the circle and $z=3$ lies outside the circle

Residue at $z=1$ is

$$= Lt_{z\to 1}\left((z-1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to 1}\left(\frac{3z^2+z-1}{(z+1)(z-3)}\right) = -\frac{3}{4}$$

Residue at $z=-1$ is

$$= Lt_{z\to -1}\left((z+1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to -1}\left(\frac{3z^2+z-1}{(z-1)(z-3)}\right) = \frac{1}{8}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{3z^2+z-1}{\left(z^2-1\right)(z-3)}dz \quad = 2\pi i\left(\frac{1}{8}-\frac{3}{4}\right) = -\frac{5\pi i}{4}$$

Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is $|z-i|=2$ using Cauchy's residue theorem

Solution:

Let $f(z)=\dfrac{z-1}{(z+1)^2(z-2)}$

poles of $f(z)$ are z = - 1 (pole of order 2) and z = 2 (simple pole)

Given: $|z-i|=2$

$$|x+iy-i|=2 \Rightarrow |x+i(y-1)|=2$$

Squaring on both sides $\sqrt{x^2+(y-1)^2}=2 \Rightarrow x^2+(y-1)^2=4$

This is equation of circle with centre $(0,1)$ and radius 2

Hence, The pole $z=2$ lies outside C and $z=-1$ lies inside C

Residue of f(z) at $z=-1$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left((z+1)^2\frac{(z-1)}{(z+1)^2(z-2)}\right)$$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left(\frac{(z-1)}{(z-2)}\right) = Lt_{z\to -1}\left(\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right)$$

$$= Lt_{z\to -1}\left(\frac{-1}{(z-2)^2}\right) = -\frac{1}{9}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{(z-1)}{(z+1)^2(z-2)}\,dz = 2\pi i\left(0-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Using Cauchy's residue theorem, find $\displaystyle\int_C \frac{z+1}{(z-3)(z-1)}\,dz$ **, where C is** $|z| = 2$

Solution:

The singular points are given by (z – 1) (z – 3) = 0 $\Rightarrow$ z = 1 , 3

Given C is | z | = 2

If z = 1 then | z | = | 1 | = 1 < 2

If z = 3 then | z | = | 3 | = 3 > 2

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

Residue at z=1:

$$\text{Res}\Big|_{z=1} = \lim_{z\to 1}(z-1)\,f(z) = \lim_{z\to 1}(z-1)\frac{z+1}{(z-3)(z-1)} = -1$$

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)}\,dz = 2\pi i(-1) = -2\pi i$$

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **by using Contour integration.**

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\dfrac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz},\ \sin\theta = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+5\frac{(z^2-1)}{2iz}} = \int_C \frac{\frac{dz}{iz}}{\frac{26iz+5z^2-5}{2iz}} = 2\int_C \frac{dz}{5z^2+26iz-5}$$

Let $f(z) = \dfrac{1}{5z^2+26iz-5}$ $\qquad \therefore I = 2\int_C f(z)dz$

The poles of f(z) are given by $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i \pm \sqrt{(26i)^2 - 4\ .\ 5(-5)}}{10} = \frac{-26i \pm \sqrt{-676+100}}{10} = \frac{-26i \pm \sqrt{-576}}{10} = \frac{-26i \pm 24i}{10}$$

$$z = -\frac{i}{5}\ ,\ -5i$$

which are simple poles.

Now $5z^2 + 26iz - 5 = 5\left(z+\frac{i}{5}\right)(z+5i)$

Since $\left|\frac{-i}{5}\right| = \frac{1}{5} < 1,$ the pole $z = \frac{-i}{5}$ lies inside C

and $|-5i| = 5 > 1,$ $\therefore$ the pole $z = -5i$ lies outside C.

Now $$R\left(-\frac{i}{5}\right) = \lim_{z\to\frac{-i}{5}}\left(z+\frac{i}{5}\right)f(z) = \lim_{z\to\frac{-i}{5}}\left(z+\frac{i}{5}\right)\frac{1}{5\left(z+\frac{i}{5}\right)(z+5i)} = \lim_{z\to\frac{-i}{5}}\frac{1}{5(z+5i)}$$

$$= \lim_{z\to\frac{-i}{5}}\frac{1}{5\left(-\frac{i}{5}+5i\right)} = \frac{1}{24i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\therefore I = 2.\frac{\pi}{12} = \frac{\pi}{6}$$

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$ by using Contour integration.

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\dfrac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+12\frac{(z^2+1)}{2z}} = \int_C \frac{dz}{\frac{iz(13z+6z^2+6)}{z}} = \int_C \frac{dz}{i(6z^2+13z+6)} = \frac{1}{i6}\int_C \frac{dz}{(z^2+\frac{13}{6}z+1)}$$

Let $f(z) = \int_C \frac{dz}{(z^2+\frac{13}{6}z+1)}$ $\qquad \therefore I = \frac{1}{6i}\int_C f(z)dz$

The poles of f(z) are given by $z^2 + \frac{13}{6}z + 1 = 0$

By solving we get $z = -\frac{2}{3}, \ -\frac{3}{2}$

which are simple poles.

Now $z^2 + \frac{13}{6}z + 1 = \left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)$

Since $\left|\frac{-2}{3}\right| = \frac{2}{3} < 1,$ the pole $z = \frac{-2}{3}$ lies inside C

and $\left|\frac{-3}{2}\right| = 1.5 > 1,$ $\therefore$ the pole $z = \frac{-3}{2}$ lies outside C.

Now $R\left(-\frac{2}{3}\right) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)f(z) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)\frac{1}{\left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)} = \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)}$

$$= \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)} = \frac{6}{5}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{6}{5}\right) = \frac{12\pi i}{5}, \qquad \therefore I = \frac{1}{6i}\times\left(\frac{12\pi i}{5}\right) = \frac{2\pi}{5}.$$

Evaluate $\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$ by using Contour integration

Solution:

Consider the unit circle | z | = 1 as contour C.

$Put\ z = e^{i\theta},\ then\ \frac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

cos3θ=R.P. of $e^{i3\theta}$=R.P. of $(e^{i\theta})^3$=R.P. of z^3

$$\therefore \qquad I = \int_C \frac{R.P.of\ z^3 \frac{dz}{iz}}{5-4\frac{(z^2+1)}{2z}} = R.P.\ of\ \int_C \frac{z^3 dz}{\frac{iz(5z-2z^2-2)}{z}}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{i(-2z^2+5z-2)}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{-i(2z^2-5z+2)}$$

$$= R.P.of\ \frac{-1}{2i}\int_C \frac{z^3 dz}{(2z-1)(z-2)}$$

Let $\int_C f(z)dz = \int_C \frac{z^3 dz}{(2z-1)(z-2)}$ $\quad \therefore I = R.P.$ of $\frac{-1}{2i}\int_C f(z)dz$

The poles of f(z) are given by

$(2z-1)(z-2) = 0$

$z = \frac{1}{2},\ z = 2$

$z = \frac{1}{2},\ z = 2$ (simple poles)

$z = \frac{1}{2}$ is a pole lies inside c.

$z = 2$ is a pole lies outside c.

Now $\quad \operatorname{Res}\left(z=\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right) f(z) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{\left(z-\frac{1}{2}\right)\ (z-2)} = \frac{-1}{12}$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\ \left(\frac{-1}{12}\right) = \frac{-\pi i}{6}$$

$$\therefore I = R.P.of\ \frac{-1}{2i}\ .\frac{-\pi i}{6} = R.P.of\ \frac{\pi}{12} = \frac{\pi}{12}$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2}$ **, | p | < 1**

Solution: Let $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{z^2-1}{2iz}$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2} = \int_C \frac{\left(dz/iz\right)}{1-2p\left(\frac{z^2-1}{2iz}\right)+p^2} \quad ,C\text{ is } |z|=1$$

$$= \int_C \frac{dz}{iz - p(z^2-1) + izp^2} = -\int_C \frac{dz}{pz^2 - iz(p^2+1) - p} = -\frac{1}{p}\int_C \frac{dz}{z^2 - iz\left(p+\frac{1}{p}\right) - 1}$$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2} = -\frac{1}{p}\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} \quad \text{........(1)}$$

The poles are given by $z = ip \;\&\; z = \frac{i}{p}$

| z | = | i p | = p < 1. $\therefore$ z= ip lies inside C and $z = \frac{i}{p}$ lies outside C.

$$\therefore \left[\text{Res of } f(z)\right]_{z=ip} = \underset{z\to ip}{\text{Lt}}\,(z-ip)\left[\frac{1}{(z-ip)\left(z-\frac{i}{p}\right)}\right] = \underset{z\to ip}{\text{Lt}}\left(\frac{1}{z-\frac{i}{p}}\right) = \frac{1}{i\left(p-\frac{1}{p}\right)} = \frac{ip}{1-p^2}$$

By Cauchy Residue Theorem $\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} = 2\pi i\left(\frac{ip}{1-p^2}\right) = \frac{-2\pi p}{1-p^2}$

From (1) $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2} = -\frac{1}{p}\left(-\frac{2\pi p}{1-p^2}\right) = \frac{2\pi}{1-p^2}$

Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}, (a>0)$ **using contour integration**

Solution:

Let $f(z)=\dfrac{1}{(z^2+a^2)^2}$. Consider $\int\limits_c f(z)dz$

where C is the contour consists of the upper half circle c_1 of $|z|=R$ & the real axix from –R to R.

$$\therefore \int\limits_c f(z)dz=\int\limits_{c_1} f(z)dz+\int\limits_{-R}^{R} f(z)dz \text{.......... ....} \quad (1)$$

The poles of $f(z)$ are given by $(z^2+a^2)^2=0 \Rightarrow z=\pm ai$ (twice)

z = ai is a pole of order 2 & lies inside C

z = -ai is a pole of order 2 & lies outside C

$$\text{Res}\left[f(z),ai\right]=\lim_{z\to ai}\frac{d}{dz}\left[(z-ai)^2\frac{1}{(z+ai)^2(z-ai)^2}\right]=\lim_{z\to ai}\frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]=\frac{-2}{(2ai)^3}=\frac{1}{4a^3i}$$

By Cauchy's Residue Theorem $\int f(z)dz=2\pi i(\frac{1}{4a^3i})=\frac{\pi}{2a^3}$

In (1) $R\to\infty$, then $\int\limits_{c_1} f(z)dz=0$

$$\therefore (1) \Rightarrow \int\limits_c f(z)dz=\int\limits_{-\infty}^{\infty} f(x)dx$$

$$=\int\limits_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{2a^3}$$

$$=2\int\limits_{0}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{2a^3}$$

$$\int\limits_{0}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{4a^3}$$

Evaluate $\int\limits_0^{\infty}\dfrac{\cos ax\,dx}{x^2+1}$, $a>0$, using contour integration.

Solution:

$$\int\limits_0^{\infty}\frac{\cos ax\;dx}{1+x^2}=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\cos ax\;dx}{1+x^2}$$

Now $\int\limits_{-\infty}^{\infty}\dfrac{\cos ax\;dx}{1+x^2}=\int\limits_{-\infty}^{\infty}\dfrac{\text{RP of } e^{iax}}{1+x^2}dx \qquad \left\{\because e^{i\theta}=\cos\theta+i\sin\theta\right\}$

Consider $\int_c f(z)\,dz = \text{R.P}\int_c \frac{e^{iaz}}{1+z^2}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [–R, R]. By Cauchy's residue theorem, we have

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

The poles of f(z) are at $1+z^2=0$

$z^2=-1 \Rightarrow z=\pm i$

The point $z=i$ lies inside the semi-circle and the point $z=-i$ lies outside the semi-circle

Residue at $z=i$ is given by

$$Lt_{z\to i}(z-i)\,f(z) = Lt_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)}$$

$$=Lt_{z\to i}\frac{e^{iaz}}{(z+i)} = \frac{e^{ia(i)}}{i+i} = \frac{e^{ai^2}}{2i} = \frac{e^{-a}}{2i}$$

By Cauchy Residue theorem,

$$R.P\int_c \frac{e^{iaz}\,dz}{1+z^2} = \text{R.P of } 2\pi i\left(\frac{e^{-a}}{2i}\right) = \text{R.P of } \pi e^{-a} = \pi e^{-a}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \pi e^{-a}$$

If $R\to\infty$, then $\int_{\Gamma} f(z)\,dz \to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-a}$

$$\int_0^{\infty}\frac{\cos ax\;dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax\;dx}{1+x^2} = \frac{\pi e^{-a}}{2}$$

Evaluate $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}\,dx$**, using contour integration.**

Solution:

Let $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$

Consider $\int_c f(z)\,dz = \int_c \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [-R, R]. By Cauchy's residue theorem, we have

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

The poles f(z) are at $z^4 + 10z^2 + 9 = 0$

$$\left(z^2+1\right)\left(z^2+9\right) = 0$$

$$z^2 = -1; \qquad z^2 = -9$$

$$z = \pm i; \qquad z = \pm 3i$$

The poles are at $3i,\ -3i,\ i,\ -i$

Here the poles $3i$ and i lie inside the semi-circle.

Residue at $z = 3i$ is given by

$$= Lt_{z\to 3i}\left(z-3i\right) f\left(z\right)$$

$$= Lt_{z\to 3i}\left(z-3i\right)\frac{z^2 - z + 2}{\left(z^2+9\right)\left(z^2+1\right)}$$

$$= Lt_{z\to 3i}\left(z-3i\right)\frac{z^2 - z + 2}{\left(z-3i\right)\left(z+3i\right)\left(z^2+1\right)}$$

$$= Lt_{z\to 3i}\frac{z^2 - z + 2}{\left(z+3i\right)\left(z^2+1\right)} = \frac{7+3i}{48i}$$

Residue at $z = i$ is given by

$$= Lt_{z\to i}\left(z-i\right) f\left(z\right)$$

$$= Lt_{z\to i}\left(z-i\right)\frac{z^2 - z + 2}{\left(z^2+9\right)\left(z^2+1\right)}$$

$$= Lt_{z\to i}\left(z-i\right)\frac{z^2 - z + 2}{\left(z-i\right)\left(z+i\right)\left(z^2+9\right)}$$

$$= Lt_{z\to i} \frac{z^2 - z + 2}{(z+i)(z^2+9)} = \frac{1-i}{16i}$$

By Cauchy Residue theorem,

$$\int_c \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = 2\pi i\left[\frac{7+3i}{48i} + \frac{1-i}{16i}\right] = 2\pi i\left[\frac{7+3i+3-3i}{48i}\right] = 2\pi i\left[\frac{10}{48i}\right] = \frac{5\pi}{12}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{5\pi}{12}$$

If $R \to \infty$, then $\int_{\Gamma} f(z)\,dz \to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \frac{5\pi}{12}$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$$

Evaluate $\int_0^{\infty} \frac{x \sin mx \quad dx}{(x^2 + a^2)}$, where a > 0 , m > 0

Solution:

Let $f(z) = \int_0^{\infty} \frac{x \sin mx dx}{(x^2+a^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin mx dx}{(x^2+a^2)}$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin mx dx}{(x^2+a^2)} = \frac{1}{2}\mathrm{IP}\int_{-\infty}^{\infty} \frac{x e^{imx} dx}{(x^2+a^2)} = \frac{1}{2}\mathrm{IP}(\mathrm{I}_1)$$

$$\mathrm{I}_1 = \int_{-\infty}^{\infty} \frac{x e^{imx}}{x^2+a^2} dx = \int_{-\infty}^{\infty} F(x) dx$$

Here $F(x) = \frac{x e^{imx}}{x^2+a^2}$ let $F(z) = \frac{z e^{imx}}{z^2+a^2}$

The poles of F(z) are given by

$\Rightarrow$ z = ±ia are poles of order 1

$\Rightarrow$ z = ia lies inside C

Consider $\int_C f(z)\, dz$ where C is the contour consists of the upper half circle C, of | z| = R. and the real axis from –R to R.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad ----(1)$$

$$\therefore [\text{Res of } f(z)]_{z=ai} = \underset{z\to ia}{\text{Lt}} (z-ia)\,\frac{ze^{imz}}{(z+ib)(z-ib)}$$

$$= \frac{e^{-ma}(ia)}{2ia} = \frac{e^{-ma}}{2}$$

$$I_1 = 2\pi i\left(\frac{e^{-ma}}{2}\right) + \pi i(0) = i\pi e^{-ma}$$

$$I = \frac{1}{2}IP(I_1) = \frac{1}{2}IP(i\pi e^{-ma}) = \frac{\pi e^{-ma}}{2}$$

By Cauchy's Residue Theorem

$$\therefore (1) \Rightarrow \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \qquad \because \int_C f(z)\,dz \to 0 \text{ as } R \to \infty$$

$$\Rightarrow \int_0^{\infty} f(x)\,dx = \frac{\pi e^{-ma}}{2}$$

Evaluate $\int_0^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$**, a > 0 , b > 0**

Solution:

Let $f(z) = \text{Real Part of } \frac{e^{iz}}{(z^2+a^2)\;(z^2+b^2)}$

Consider $\int_C f(z)\,dz$ where C is the contour consists of the upper half circle C, of | z| = R. and the real axis from –R to R.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad ----(1)$$

The poles of $f(z)$ are given by $(z^2+a^2)(z^2+b^2) = 0$

$\Rightarrow z = \pm ia, \pm ib$

$\Rightarrow z = ia, ib$ lies inside C and $z = -ia, -ib$ lies in lower half plane

$$\therefore \left[\text{Res of } f(z)\right]_{z=ai} = \underset{z\to ia}{\text{Lt}} (z-ia)\,\frac{e^{iz}}{(z+ia)(z-ia)(z^2+b^2)}$$

$$= \frac{e^{-a}}{2ia(b^2-a^2)}$$

$$\left[\text{Res of } f(z)\right]_{z=bi} = \underset{z\to ia}{\text{Lt}} (z-ib)\,\frac{e^{iz}}{(z+ib)(z-ib)(z^2+a^2)}$$

$$= \frac{e^{-a}}{2ib(a^2-b^2)}$$

By Cauchy's Residue Theorem

$$\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz = 2\pi i\left[\frac{e^{-a}}{2ia(b^2-a^2)} + \frac{e^{-b}}{2ib(a^2-b^2)}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

In (1) if $R\to\infty$, $\int_{C_1} f(z)\,dz \to 0$

$$\therefore (1) \Rightarrow \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \text{Real Part of } \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

SRM OF INSTITUTE OF SCIENCE AND TECHNOLOGY
FACULTY OF ENGINEERING AND TECHNOLOGY
18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS
PART - A : MULTIPLE CHOICE QUESTIONS

UNIT – I: MULTIPLE INTEGRALS

1. Evaluation of $\int_0^1\int_0^1 dxdy$ is

   (a) 1 (b) 2 (c) 0 (d) 4

2. The curve $y^2 = 4x$ is a

   (a) parabola (b) hyperbola (c) straight line (d) ellipse

3. Evaluation of $\int_0^\pi\int_0^\pi d\theta d\phi$ is

   $a)\ 1$ $b)\ 0$ $c)\ \pi/2$ $d)\ \pi^2$

4. The area of an ellipse is

   $a)\ \pi r^2$ $b)\ \pi a^2 b$ $c)\ \pi a b^2$ $d)\ \pi ab$

5. $\int_1^b\int_1^a \frac{dxdy}{xy}$ is equal to

   a) $\log a + \log b$ b) $\log a$ c) $\log b$ d) $\log a \log b$

6. $\int_0^1\int_0^x dxdy$ is equal to

   $a)\ 1$ $b)\ 1/2$ $c)\ 2$ $d)\ 3$

7. $\int_0^1\int_0^2 dxdy$ is equal to

   $a)\ \int_0^2\int_0^1 dydx$ $b)\ -\int_0^1\int_0^2 dxdy$ $c)\ \int_2^0\int_0^1 dydx$ $d)\ \int_1^0\int_0^2 dydx$

8. If R is the region bounded $x = 0,\ y = 0,\ x + y = 1$ then $\iint_R dxdy$ is equal to

   $a)\ 1$ $b)\ 1/2$ $c)\ 1/3$ $d)\ 2/3$

9. Area of the double integral in cartesian co-ordinate is equal to

   $a)\ \iint_R dydx$ $b)\ \iint_R rdrd\theta$ $c)\ \iint_R xdxdy$ $d)\ \iint_R x^2dxdy$

10. Change the order of integration in $\int_0^a \int_0^x dxdy$ is

$a)\ \int_0^a \int_0^x dxdy \qquad b)\ \int_0^a \int_0^x xdydx \qquad c)\ \int_0^a \int_y^a dxdy \qquad d)\ \int_0^a \int_0^y dxdy$

11. Area of the double integral in polar co-ordinate is equal to

$a)\ \iint_R drd\theta \qquad b)\ \iint_R r^2 drd\theta \qquad c)\ \iint_R (r+1)drd\theta \qquad d)\ \iint_R rdrd\theta$

12. $\int_0^1 \int_0^2 \int_0^3 dxdydz$ is equal to

$a)\ 3 \qquad b)\ 4 \qquad c)\ 2 \qquad d)\ 6$

13. The name of the curve $r = a(1+\cos\theta)$ is

a) lemniscate b) cycloid c) cardioid d) hemicircle

14. The volume integral in cartesian coordinates is equal to

$a)\ \iiint_V dxdydz \qquad b)\ \iiint_V drd\theta d\phi \qquad c)\ \iint_R drd\theta \qquad d)\ \iint_R rdrd\theta$

15. $\int_0^1 \int_0^2 x^2 ydxdy$ is equal to

$a)\ \frac{2}{3} \qquad b)\ \frac{1}{3} \qquad c)\ \frac{4}{3} \qquad d)\ \frac{8}{3}$

16. $\int_0^1 \int_0^1 (x+y)dxdy$ is equal to

$a)\ 1 \qquad b)\ 2 \qquad c)\ 3 \qquad d)\ 4$

17. After changing the double integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$ into polar coordinates, we have

$a)\ \int_0^{\pi/2} \int_0^\infty e^{-r^2} drd\theta \qquad b)\ \int_0^{\pi/4} \int_0^\infty e^{-r} drd\theta \qquad c)\ \int_0^{\pi/2} \int_0^\infty e^{-r^2} rdrd\theta \qquad d)\ \int_0^{\pi/2} \int_0^\infty e^{-r} drd\theta$

18. $\int_0^\infty \int_0^y \frac{e^{-y}}{y} dxdy$ is equal to

$a)\ 1 \qquad b)\ 0 \qquad c)\ -1 \qquad d)\ 2$

19. The value of the integral $\int_0^2 \int_0^1 xydxdy$ is

(a) 1 (b) 2 (c) 3 (d) 4

20. The value of the integral $\int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta+\phi)d\theta d\phi$

(a) 1 (b) 2 (c) 3 (d) 4

21. The region of integration of the integral $\int_{-b}^{b}\int_{-a}^{a} f(x,y)dxdy$ is

(a) square (b) circle (c) rectangle (d) triangle

22. The region of integration of the integral $\int_0^1\int_0^x f(x,y)dxdy$ is

(a) square (b) rectangle (c) triangle (d) circle

23. The limits of integration is the double integral $\iint_R f(x,y)dxdy,$ where R is in the first quadrant and bounded by $x=0,\ \ y=0,\ \ x+y=1$ are

$(a)\ \int_{x=0}^{1}\int_{y=0}^{1-x} f(x,y)dydx$ $(b)\ \int_{y=1}^{2}\int_{x=0}^{1-y} f(x,y)dxdy$

$(c)\ \int_{y=0}^{1}\int_{x=1}^{y} f(x,y)dxdy$ $(d)\ \int_{y=0}^{2}\int_{x=0}^{1-y} f(x,y)dxdy$

ANSWERS:

1	a	6	b	11	d	16	a	21	c
2	a	7	a	12	d	17	c	22	c
3	d	8	b	13	c	18	a	23	a
4	d	9	a	14	a	19	a		
5	d	10	c	15	c	20	b		

UNIT – II: VECTOR CALCULUS

1. The directional derivative of $\phi = xy + yz + zx$ at the point (1,2,3) along x - axis is
   (a) 4 (b) 5 (c) 6 (d) 0

2. In what direction from (3, 1, -2) is the directional derivative of $\phi = x^2 y^2 z^4$ maximum?
   a) $\frac{1}{\sqrt{19}}(\vec{i} + 3\vec{j} - \vec{k})$ (b) $19(\vec{i} + 3\vec{j} - 3\vec{k})$
   (c) $96(\vec{i} + 3\vec{j} - 3\vec{k})$ d) $\frac{1}{\sqrt{19}}(3\vec{i} + 3\vec{j} - \vec{k})$

3. If $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin, then $\nabla \cdot \vec{r}$ is
   (a) 2 (b) 3 (c) 0 (d) 1

4. If $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin, then $\nabla \times \vec{r}$ is
   a) $\nabla \times \vec{r} = 0$ b) $x\vec{i} + y\vec{j} + z\vec{k} = 0$ c) $\nabla \times \vec{r} \neq 0$ d) $\vec{i} + \vec{j} + \vec{k} = 0$

5. The unit vector normal to the surface $x^2 + y^2 - z^2 = 1$ at (1, 1, 1) is
   a) $\frac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}}$ b) $\frac{2\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{2}}$ c) $\frac{3\vec{i} + 3\vec{j} - 3\vec{k}}{2\sqrt{3}}$ d) $\frac{\vec{i} + \vec{j} - \vec{k}}{3\sqrt{2}}$

6. If $\phi = xyz,$ then $\nabla\phi$ is
   a) $yz\vec{i} + zx\vec{j} + xy\vec{k}$ b) $xy\vec{i} + yz\vec{j} + zx\vec{k}$ c) $zx\vec{i} + xy\vec{j} + yz\vec{k}$ d) 0

7. If $\vec{F} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$ then $\vec{F}$ is
   a) solenoidal b) irrotational c) constant vector
   d) both solenoidal and irrotational

8. If $\vec{F} = \left(axy - z^3\right)\vec{i} + (a-2)x^2\vec{j} + (1-a)xz^2\vec{k}$ is irrotational then the value of a is
   a) 0 b) 4 c) -1 d) 2

9. If $\vec{u}$ and $\vec{v}$ are irrotational then $\vec{u} \times \vec{v}$ is
   a) solenoidal b) irrotational c) constant vector d) zero vector

10. If ϕ and ψ are scalar functions then $\nabla\phi\times\nabla\psi$ is

a) solenoidal b) irrotational c) constant vector

d) both solenoidal and irrotational

11. If $\vec{F} = \left(y^2 - z^2 + 3yz - 2x\right)\vec{i} + \left(3xz + 2xy\right)\vec{j} + \left(3xy - 2xz + 2z\right)\vec{k}$ then $\vec{F}$ is

a) solenoidal b) irrotational c) both solenoidal and irrotational

d) neither solenoidal nor irrotational

12. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $grad(\vec{a}\cdot\vec{r})$ is

a) 0 b) 1 c) $\vec{a}$ d) $\vec{r}$

13. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $div(\vec{a}\times\vec{r})$ is

a) 0 b) 1 c) $\vec{a}$ d) $\vec{r}$

14. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $curl(\vec{a}\times\vec{r})$ is

a) 0 b) 1 c) $2\vec{a}$ d) $2\vec{r}$

15. If ϕ scalar functions then $curl(grad\phi)$ is

a) solenoidal b) irrotational c) constant vector d) 0

16. If the value of $\int_A^B \vec{F}\cdot d\vec{r}$ does not depend on the curve C, but only on the terminal points A and B then $\vec{F}$ is called

a) solenoidal vector b) irrotational vector c) conservative vector

d) neither conservative nor irrotational

17. The condition for $\vec{F}$ to be Conservative is, $\vec{F}$ should be

a) solenoidal vector b) irrotational vector c) rotational

d) neither solenoidal nor irrotational

18. The value of $\int_C \vec{r} \cdot d\vec{r}$ where C is the line $y = x$ in the xy-plane from (1,1) to (2,2) is

a) 0 b) 1 c) 2 d) 3

19. The work done by the conservative force when it moves a particle around a closed curve is

a) $\nabla \cdot \vec{F} = 0$ b) $\nabla \times \vec{F} = 0$ c) 0 d) $\nabla \cdot (\nabla \times \vec{F}) = 0$

20. The connection between a line integral and a double integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) convolution theorem

21. The connection between a line integral and a surface integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) Residue theorem

22. The connection between a surface integral and a volume integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) Cauchy's theorem

23. Using Gauss divergence theorem, find the value of $\iint_S \vec{r}\,\vec{ds}$ where $\vec{r}$ is the position vector and V is the volume

a) $4V$ b) 0 c) $3V$ d) volume of the given surface

24. If S is any closed surface enclosing the volume V and if $\vec{F} = ax\,\vec{i} + by\,\vec{j} + cz\,\vec{k}$ then the value of $\iint_S \vec{F} \cdot \vec{n}\, dS$ is

a) $abcV$ b) $(a+b+c)V$ c) 0 d) $abc(a+b+c)V$

ANSWERS:

1	b	6	a	11	c	16	c	21	b
2	c	7	a	12	c	17	b	22	c
3	b	8	b	13	a	18	d	23	c
4	a	9	a	14	a	19	c	24	b
5	a	10	a	15	d	20	a		

UNIT-III LAPLACE TRANSFORMS

1. L(1) =

   $(a)\ \frac{1}{s}\quad (b)\ \frac{1}{s^2}\quad (c)\ 1\quad (d)\ s$

2. $L(e^{3t}) =$

   $(a)\ \frac{1}{s+3}\qquad (b)\ \frac{1}{s-3}\qquad (c)\ \frac{3}{s+3}\qquad (d)\ \frac{s}{s-3}$

3. $L(e^{-at}) =$

   $(a)\ \frac{1}{s+1}\qquad (b)\ \frac{1}{s-1}\qquad (c)\ \frac{1}{s+a}\qquad (d)\ \frac{1}{s-a}$

4. $L(\cos 2t) =$

   $(a)\ \frac{s}{s^2+4}\qquad (b)\ \frac{s}{s^2+2}\qquad (c)\ \frac{2}{s^2+2}\qquad (d)\ \frac{4}{s^2+4}$

5. $L(t^4) =$

   $(a)\ \frac{4!}{s^5}\quad (b)\ \frac{3!}{s^4}\quad (c)\ \frac{4!}{s^4}\quad (d)\ \frac{5!}{s^4}$

6. $L(a^t) =$

   $(a)\ \frac{1}{s-\log a}\qquad (b)\ \frac{1}{s+\log a}\qquad (c)\ \frac{1}{s-a}\qquad (d)\ \frac{1}{s+a}$

7. $L(\sinh \omega t) =$

   $(a)\ \frac{s}{s^2+\omega^2}\qquad (b)\ \frac{\omega}{s^2+\omega^2}\qquad (c)\ \frac{s}{s^2-\omega^2}\qquad (d)\ \frac{\omega}{s^2-\omega^2}$

8. An example of a function for which the Laplace transforms does not exists is

   $(a)\ f(t)=t^2\quad (b)\ f(t)=\tan t\quad (c)\ f(t)=\sin t\qquad (d)\ f(t)=e^{-at}$

9. If $L(f(t)) = F(s)$, then $L(e^{-at} f(t)) =$

   $(a)\ F(s+a)\qquad (b)\ F(s-a)\qquad (c)\ F(s)\quad (d)\ \frac{1}{a}F\left(\frac{s}{a}\right)$

10. $L(e^{-at}\cos bt) =$

    $(a)\ \frac{s+b}{(s+b)^2+a^2}\qquad (b)\ \frac{s+a}{(s+a)^2+b^2}\qquad (c)\ \frac{a}{s^2+a^2}\qquad (d)\ \frac{s}{s^2+b^2}$

11. $L(te^{t}) =$

$(a)\ \dfrac{1}{(s+1)^2}$ $(b)\ \dfrac{1}{s+1}$ $(c)\ \dfrac{1}{s-1}$ $(d)\ \dfrac{1}{(s-1)^2}$

12. $L(t\sin at) =$

$(a)\ \dfrac{2as}{(s^2+a^2)^2}$ $(b)\ \dfrac{2s}{(s^2+a^2)^2}$ $(c)\ \dfrac{s^2-a^2}{(s^2+a^2)^2}$ $(d)\ \dfrac{1}{s^2+a^2}$

13. $L(\sin 3t) =$

$(a)\ \dfrac{3}{s^2+3}$ $(b)\ \dfrac{3}{s^2+9}$ $(c)\ \dfrac{s}{s^2+3}$ $(d)\ \dfrac{s}{s^2+9}$

14. $L(\cosh t) =$

$(a)\ \dfrac{s}{s^2+1}$ $(b)\ \dfrac{s}{s^2-1}$ $(c)\ \dfrac{1}{s^2+1}$ $(d)\ \dfrac{1}{s^2-1}$

15. $L(t^{1/2}) =$

$(a)\ \dfrac{\Gamma(3/2)}{s^{1/2}}$ $(b)\ \dfrac{\Gamma(1/2)}{s^{3/2}}$ $(c)\ \dfrac{\Gamma(1/2)}{s^{1/2}}$ $(d)\ \dfrac{\Gamma(3/2)}{s^{3/2}}$

16. $L(t^{-1/2}) =$

$(a)\ \sqrt{\dfrac{\pi}{s}}$ $(b)\ \sqrt{\dfrac{\pi}{2s}}$ $(c)\ \sqrt{\dfrac{1}{s}}$ $(d)\ \dfrac{1}{s}$

17. $L[te^{2t}] =$

$(a)\ \dfrac{1}{(s-2)^2}$ $(b)\ -\dfrac{1}{(s-2)^2}$ $(c)\ \dfrac{1}{(s-1)^2}$ $(d)\ \dfrac{1}{(s+1)^2}$

18. If $L[f(t)] = F(s)$ then $L\left\{f\left(\dfrac{t}{a}\right)\right\}$ is

(a) $aF(as)$ (b) $\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$ (c) $F(s+a)$ (d) $\dfrac{1}{a}F(as)$

19. $L\left(\int_0^t \sin t\,dt\right)$ is

(a) $\dfrac{1}{s^2+1}$ (b) $\dfrac{s}{s^2+1}$ (c) $\dfrac{1}{(s^2+1)^2}$ (d) $\dfrac{1}{s(s^2+1)}$

20. $L(\sin t \cos t)$ is

(a) $L(\sin t)..L(\cos t)$ (b) $L(\sin t)+L(\cos t)$ (c) $L(\sin t)-L(\cos t)$ (d) $\dfrac{L(\sin 2t)}{2}$

21. If $L[f(t)]=F[s]$ then $L[tf(t)]=$

$(a)\ \frac{d}{ds}F(s)$ $(b)\ -\frac{d}{ds}F(s)$ $(c)\ (-1)^n\frac{d}{ds}F(s)$ $(d)\ -\frac{d^2}{ds^2}F(s)$

22. If $L[f(t)]=F[s]$ then $L\left[\frac{f(t)}{t}\right]=$

$(a)\ \int_0^\infty F(s)\,ds$ $(b)\ \int_s^\infty F(s)\,ds$ $(c)\ \int_{-\infty}^\infty F(s)\,ds$ $(d)\ \int_a^\infty F(s)\,ds$

23. $L\left[\frac{\cos t}{t}\right]=$

$(a)\ \frac{s}{s^2+a^2}$ $(b)\ \frac{1}{s^2+a^2}$ (c) *does not exist* $(d)\ \frac{s^2-a^2}{(s^2+a^2)^2}$

24. If $L[f(t)]=F[s]$ then $L[t^n f(t)]=$

$(a)\ (-1)^n\frac{d^n}{ds^n}F(s)$ $(b)\ \frac{d^n}{ds^n}F(s)$ $(c)\ -\frac{d^n}{ds^n}F(s)$ $(d)\ (-1)^{n-1}\frac{d^n}{ds^n}F(s)$

25. $L\left[\frac{1-e^{-t}}{t}\right]=$

$(a)\ \log\left(\frac{s}{s-1}\right)$ $(b)\ \log\left(\frac{s}{s+1}\right)$ $(c)\ \log\left(\frac{s+1}{s}\right)$ $(d)\ \log\left(\frac{s-1}{s}\right)$

26. $L(u_a(t))$ is

(a) $\frac{e^{as}}{s}$ (b) $\frac{e^{-as}}{s}$ (c) $-\frac{e^{-as}}{s}$ (d) $-\frac{e^{as}}{s}$

27. If $L[f(t)]=F[s]$ then $L[f^{'}(t)]=$

(a) $sL[f(t)]-f(0)$ (b) $sL[f(t)]-sf(0)$ (c) $L[f(t)]-f(0)$ (d) $sL[f(t)]-f^{'}(0)$

28. Using the initial value theorem, find the value of the function $f(t)=ae^{-bt}$

$(a)\ a$ $(b)\ a^2$ $(c)\ ab$ $(d)\ 0$

29. Using the initial value theorem, find the value of $f(t)=e^{-2t}\sin t$

$(a)\ 0$ $(b)\ \infty$ $(c)\ 1$ $(d)\ 2$

30. Using the initial value theorem, find the value of the function $f(t) = \sin^2 t$

(a) 0 (b) ∞ (c) 1 (d) 2

31. Using the initial value theorem, find the value of the function $f(t) = 1 + e^{-t} + t^2$

(a) 2 (b) 1 (c) 0 (d) ∞

32. Using the initial value theorem, find the value of the function $f(t) = 3 - 2\cos t$

(a) 3 (b) 2 (c) 1 (d) 0

33. Using the final value theorem, find the value of the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

(a) 1 (b) 0 (c) ∞ (d) -2

34. Using the final value theorem, find the value of the function $f(t) = t^2 e^{-3t}$

(a) 0 (b) ∞ (c) 1 (d) -1

35. Using the final value theorem, find the value of the function $f(t) = 1 - e^{-at}$

(a) 0 (b) 1 (c) 2 (d) ∞

36. The period of $\tan t$ is

(a) π (b) $\frac{\pi}{2}$ (c) 2π (d) $\frac{\pi}{4}$

37. The period of $|\sin \omega t|$ is

(a) $\frac{\pi}{\omega}$ (b) $\frac{2\pi}{\omega}$ (c) 2π (d) $2\pi\omega$

38. Inverse Laplace transform of $\frac{1}{(s-1)^2}$ is

(a) te^{-t} (b) te^{t} (c) t^2e^{t} (d) t

39. Inverse Laplace transform of $\frac{2}{s-b}$ is

(a) $2e^{-bt}$ (b) $2e^{bt}$ (c) $2te^{bt}$ (d) $2bt$

40. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{F(s)}{s}\right)$ is

(a) $\int_0^\infty f(t)dt$ (b) $\int_0^t f(t)dt$ (c) $\int_{-\infty}^\infty f(t)dt$ (d) $\int_{-a}^a f(t)dt$

41. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{1}{s^2+4}\right)$ is

(a) $\frac{\sin 2t}{2}$ (b) $\frac{\sin\sqrt{2}t}{\sqrt{2}}$ (c) $\sin 2t$ (d) $\sin\sqrt{2}t$

42. Inverse Laplace transform of $\frac{1}{s^2-a^2}$ is

(a) $\frac{\sin at}{a}$ (b) $\frac{\sinh at}{a}$ (c) $\sin at$ (d) $\sinh at$

43. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{1}{s^2}\right)$ is

(a) t (b) $2t$ (c) $3t$ (d) t^2

44. Inverse Laplace transform of $\frac{s}{s^2-9}$ is

(a) $\cos 9t$ (b) $\cos 3t$ (c) $\cosh 9t$ (d) $\cosh 3t$

45. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}(F(as))$ is

(a) $\frac{f(t)}{a}$ (b) $\frac{1}{a}f\left(\frac{t}{a}\right)$ (c) $f\left(\frac{t}{a}\right)$ (d) $f(at)$

46. Inverse Laplace transform of $\frac{1}{s^3}$ is

(a) $\frac{t}{2}$ (b) t (c) $\frac{t^2}{2}$ (d) t^2

47. Inverse Laplace transform of $\frac{s+3}{(s+3)^2+9}$ is

(a) $e^{3t}\cos 3t$ (b) $e^{-3t}\cos 3t$ (c) $e^{-3t}\cosh 3t$ (d) $e^{-3t}\cos 9t$

48. Inverse Laplace transform of $\frac{b}{s+a}$ is

(a) ae^{-bt} (b) be^{-bt} (c) ae^{bt} (d) be^{at}

49. The value of $e^{-t} * \sin t =$

$(a)\left(\frac{\sin t-\cos t}{2}\right)$ $(b)\left(\frac{\cos t-\sin t}{2}\right)$ $(c)\left(\frac{e^{-t}}{2}\right)+\left(\frac{\sin t-\cos t}{2}\right)$ $(d)\left(\frac{e^{-t}}{2}\right)$

50. The value of $1 * e^t$ is

(a) $\mathrm{e}^{\mathrm{t}}-1$ (b) $\mathrm{e}^{\mathrm{t}}+1$ (c) e^{t} (d) e

ANSWERS:

1	a	11	d	21	b	31	a	41	a
2	b	12	a	22	b	32	c	42	b
3	c	13	b	23	c	33	a	43	a
4	a	14	b	24	a	34	a	44	d
5	a	15	d	25	c	35	b	45	b
6	a	16	a	26	b	36	a	46	c
7	d	17	a	27	a	37	a	47	b
8	b	18	a	28	a	38	b	48	b
9	a	19	d	29	a	39	b	49	c
10	b	20	d	30	a	40	b	50	a

UNIT– IV: ANALYTIC FUNCTIONS

1. Cauchy – Riemann equation in polar co-ordinates are
   $(a)\ ru_r = v_\theta, u_\theta = -rv_r$ $(b)\ -ru_r = v_\theta, u_\theta = rv_r$
   $(c)\ -ru_r = v_\theta, u_\theta = rv_r$ $(d)\ u_r = rv_\theta, ru_\theta = v_r$

2. If $w = f(z)$ is analytic function of z, then
   $(a)\ \frac{\partial w}{\partial z} = i\frac{\partial w}{\partial x}$ $(b)\ \frac{\partial w}{\partial z} = i\frac{\partial w}{\partial y}$ $(c)\ \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$ $(d)\ \frac{\partial w}{\partial \bar{z}} = 0$

3. The function $f(z) = u + iv$ is analytic if
   $(a)\ u_x = v_y, u_y = -v_x$ $(b)\ u_x = -v_y, u_y = v_x$
   $(c)\ u_x + v_y = 0, u_y - v_x = 0$ $(d)\ u_y = v_y, u_x = v_x$

4. The function $w = \sin x \cosh y + i \cos x \sinh y$ is
   (a) need not be analytic (b) analytic (c) discontinuous
   (d) differentiable only at origin

5. If u and v are harmonic, then $u + iv$ is
   (a) harmonic (b) need not be analytic (c) analytic (d) continuous

6. If a function $u(x, y)$ satisfies $u_{xx} + u_{yy} = 0$, then u is
   (a) analytic (b) harmonic (c) differentiable (d) continuous

7. If $u + iv$ is analytic, then the curves $u = c_1$ and $v = c_2$ are
   (a) cut orthogonally (b) intersect each other (c) are parallel
   (d) coincides

8. The invariant point of the transformation $w = \frac{1}{z - 2i}$ is
   (a) $z = i$ $(b)\ z = -i$ $(c)\ z = 1$ $(d)\ z = -1$

9. The transformation $w = cz$ where c is real constant represents
   (a) rotation (b) reflection (c) magnification (d) magnification and rotation

10. The complex function $w = az$ where a is complex constant represents
    (a) rotation (b) magnification and rotation (c) translation (d) reflection
11. The values of $C_1 \& C_2$ such that the function $f(z) = C_1 xy + i[C_2 x^2 + y^2]$ is analytic are
    $(a)\ C_1 = 0, C_2 = 1$ $(b)\ C_1 = 2, C_2 = -1$
    $(c)\ C_1 = -2, C_2 = 1$ $(d)\ C_1 = -2, C_2 = 0$

12. The real part of $f(z)=e^{2z}$ is

$(a)\ e^{x}\cos y$ $(b)\ e^{x}\sin y$ $(c)\ e^{2x}\cos 2y$ $(d)\ e^{2x}\sin 2y$

13. If *f(z)* is analytic where $f(z)=r^{2}\cos 2\theta+ir^{2}\sin p\theta$, the value of p is

$(a)\ p=1$ $(b)\ p=-2$ $(c)\ p=-1$ $(d)\ p=2$

14. The points at which the function $f(z)=\dfrac{1}{z^{2}+1}$ fails to be analytic an

$(a)\ z=\pm 1$ $(b)\ z=\pm i$ $(c)\ z=0$ $(d)\ z=\pm 2$

15. The critical point of transformation $w=z^{2}$ is

$(a)\ z=2$ $(b)\ z=0$ $(c)\ z=1$ $(d)\ z=-2$

16. An analytic function with constant modulus is

(a) zero (b) analytic (c) constant (d) harmonic

17. The image of the rectangular region in the z-plane bounded by the lines $x=0$, $y=0$, $x=2$ and $y=1$ under the transformation $w=2z$.

(a) parabola (b) circle (c) straight line (d) rectangle is magnified twice

18. If $f(z)$ and $\overline{f(z)}$ are analytic function of z, then $f(z)$ is

(a) analytic (b) zero (c) constant (d) discontinuous

19. The invariant points of the transformation $w=-\left(\dfrac{2z+4i}{iz+1}\right)$ are

$(a)\ z=4i,-i$ $(b)\ z=-4i,i$ $(c)\ z=2i,i$ $(d)\ z=-2i,i$

20. The function $|z|^{2}$ is

(a) differentiable at the origin (b) analytic (c) constant (d) differentiable everywhere

21. If f(z) is regular function of z then,

$(a)\left(\dfrac{\partial^{2}}{\partial x^{2}}+\dfrac{\partial^{2}}{\partial y^{2}}\right)|f(z)|^{2}=|f'(z)|^{2}$ $(b)\left(\dfrac{\partial^{2}}{\partial x^{2}}+\dfrac{\partial^{2}}{\partial y^{2}}\right)|f(z)|^{2}=4|f'(z)|^{2}$

$(c)\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}\right)|f(z)|^{2}=4|f'(z)|^{2}$ $(d)\left(\dfrac{\partial^{2}}{\partial x^{2}}+\dfrac{\partial^{2}}{\partial y^{2}}\right)|f(z)|^{2}=4|f'(z)|$

22. The transformation $w=z+c$ where c is a complex constant represents

(a) rotation (b) magnification (c) translation (d) magnification & rotation

23. The mapping $w=\dfrac{1}{z}$ is

(a) conformal (b) not conformal at $z=0$ (c) conformal every where
(d) orthogonal

24. The function $u+iv=\dfrac{x-iy}{x-iy+a}$ $(a\neq 0)$ is not analytic function of z where as $u-iv$ is

(a) need not be analytic (b) analytic at all points (c) analytic except at $z=a$
(d) continuous everywhere

25. If z_1, z_2, z_3, z_4 are four points in the z-plane then the cross-ratio of these point is

(a) $\dfrac{(z_1-z_2)(z_4-z_3)}{(z_1-z_4)(z_2-z_3)}$ (b) $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$

(c) $\dfrac{(z_1-z_2)(z_4-z_3)}{(z_1-z_4)(z-z_3)}$ (d) $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_4-z_1)(z_3-z_2)}$

26. The invariant points of the transformation $w=\dfrac{1-iz}{z-i}$

(a) 0 (b) $\pm i$ (c) ± 2 (d) ± 1

ANSWERS:

1	a	6	b	11	b	16	c	21	b	26	d
2	d	7	a	12	c	17	d	22	c		
3	a	8	a	13	d	18	c	23	b		
4	b	9	c	14	b	19	a	24	c		
5	b	10	b	15	b	20	a	25	b		

UNIT – V: COMPLEX INTEGRATION

1. A curve which does not cross itself is called a
   (a) curve (b) closed curve (c) simple closed curve (d) multiple curve

2. The value of $\int_c \frac{zdz}{z-2}$ where c is the circle $|z|=1$ is
   (a) 0 (b) $\frac{\pi}{2}i$ (c) $\frac{\pi}{2}$ (d) 2

3. The value of $\int_c \frac{z}{(z-1)^2}dz$ where c is the circle $|z|=2$ is
   (a) πi (b) $2\pi i$ (c) $4\pi i$ (d) 0

4. The value of $\int_c (z-2)^n dz;\ (n \neq 1)$ where c is the circle $|z-2|=4$ is
   a. 2^n (b) n^2 (c) 0 (c) n

5. The value of $\int_c \frac{1}{2z+1}dz$ where c is the circle $|z|=1$ is
   (a) 0 (b) πi (c) $\frac{\pi}{2}i$ (d) 2

6. The value of $\int_c \frac{1}{3z+1}dz$ where c is the circle $|z|=1$ is
   (a) 0 (b) πi (c) $\frac{2\pi}{3}i$ (d) 2

7. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{z-a}dz$, where c is the simple closed curve and a is any point within c, is
   (a) $f(a)$ (b) $2\pi i f(a)$ (c) $\pi i f(a)$ (d) 0

8. If $f(z)$ is analytic inside and on c, the value of $\int_c f(z)dz$, where c is the simple closed curve, is
   (a) $f(a)$ (b) $2\pi i f(a)$ (c) $\pi i f(a)$ (d) 0

9. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{(z-a)^2}dz$, where c is the simple closed curve and a is any point within c, is
   (a) $f'(a)$ (b) $2\pi i f'(a)$ (c) $\pi i f'(a)$ (d) 0

10. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{(z-a)^3}dz$, where c is the simple closed curve and a is any point within c, is

(a) $f''(a)$ (b) $2\pi i f''(a)$ (c) $\pi i f''(a)$ (d) 0

11. Let $C: |z-a| = r$ be a circle, the f(z) can be expanded as a Taylor's series if

(a) $f(z)$ is a defined function within c

(b) $f(z)$ is a analytic function within c

(c) $f(z)$ is not a analytic function within c

(d) $f(z)$ is a analytic function outside c

12. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$ be two concentric circles ($R_2 < R_1$), the $f(z)$ can be expanded as a Laurent's series if

(a) $f(z)$ is analytic within C_2

(b) $f(z)$ is not analytic within C_2

(c) $f(z)$ is analytic in the annular region

(d) $f(z)$ is not analytic in the annular region

13. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$ be two concentric circles ($R_2 < R_1$), the annular region is defined as

(a) within C_1 (b) within C_2

(c) within C_2 and outside C_1 (d) within C_1 and outside C_2

14. The part $\sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral powers of $(z-a)$ is called as

(a) The analytic part of the Laurent's series

(b) The principal part of the Laurent's series

(c) The real part of the Laurent's series

(d) The imaginary part of the Laurent's series

15. The part $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral powers of $(z-a)$ is called as

(a) The analytic part of the Laurent's series

(b) The principal part of the Laurent's series

(c) The real part of the Laurent's series

(d) The imaginary part of the Laurent's series

16. The annular region for the function $f(z) = \frac{1}{z(z-1)}$ is

(a) $0 < |z| < 1$ (b) $1 < |z| < 2$ (c) $1 < |z| < 0$ (d) $|z| < 1$

17. The annular region for the function $f(z)=\dfrac{1}{(z-1)(z-2)}$ is

(a) $0<|z|<1$ (b) $1<|z|<2$ (c) $1<|z|<0$ (d) $|z|<1$

18. The annular region for the function $f(z)=\dfrac{1}{z^2-z-6}$ is

(a) $0<|z|<1$ (b) $1<|z|<2$ (c) $2<|z|<3$ (d) $|z|<3$

19. If $f(z)$ is not analytic at $z=z_0$ and there exists a neighborhood of $z=z_0$ containing no other singularity, then

(a) The point $z=z_0$ is isolated singularity of $f(z)$

(b) The point $z=z_0$ is a zero point of $f(z)$

(c) The point $z=z_0$ is nonzero of $f(z)$

(d) The point $z=z_0$ is non isolated singularity of $f(z)$

20. If $f(z)=\dfrac{\sin z}{z}$, then

(a) $z=0$ is a simple pole (b) $z=0$ is a pole of order 2

(c) $z=0$ is a removable singularity (d) $z=0$ is a zero of $f(z)$

21. If $f(z)=\dfrac{\sin z-z}{z^3}$, then

(a) $z=0$ is a simple pole (b) $z=0$ is a pole of order 2

(c) $z=0$ is a removable singularity (d) $z=0$ is a zero of $f(z)$

22. If $\lim_{z\to a}(z-a)^n f(z)\neq 0$ then

(a) $z=a$ is a simple pole (b) $z=a$ is a pole of order n

(c) $z=a$ is a removable singularity (d) $z=a$ is a zero of $f(z)$

23. If $f(z)=\dfrac{1}{(z-4)^2(z-3)^3(z-1)}$, then

(a) 4 is a simple pole, 3 is a pole of order 3 and 1 is a pole of order 2

(b) 3 is a simple pole, 1 is a pole of order 3 and 4 is a pole of order 2

(c) 1 is a simple pole, 3 is a pole of order 3 and 4 is a pole of order 2

(d) 3 is a simple pole, 4 is a pole of order 1 and 4 is a pole of order 2

24. If $f(z)=e^{\frac{1}{z-4}}$ then

(a) $z=4$ is removable singularity (b) $z=4$ is pole of order 2

(c) $z=4$ is an essential singularity (d) $z=4$ is zero of $f(z)$

25. Let $z = a$ is a simple pole for $f(z)$ and $b = \lim_{z \to a}(z-a)f(z)$, then

(a) b is a simple pole (b) b is a residue at a

(c) b is removable singularity (d) b is a residue at a of order n

26. The residue of $f(z) = \dfrac{1-e^{2z}}{z^3}$ is

(a) 0 (b) 2 (c) -2 (d) 1

27. The residue of $f(z) = \dfrac{e^{2z}}{(z+1)^2}$ is

(a) e^{-2} (b) -2 e^{-2} (c) -1 (d) 2 e^{-2}

28. The residue of $f(z) = \cot z$ is

(a) π (b) 1 (c) -1 (d) 0

ANSWERS:

1	c	6	c	11	b	16	a	21	c	26	c
2	a	7	b	12	c	17	b	22	b	27	d
3	b	8	d	13	d	18	c	23	c	28	b
4	c	9	b	14	a	19	a	24	c		
5	b	10	b	15	b	20	c	25	b		

SRM University
Department of Mathematics
Complex Integration- Multiple Choice questions
UNIT V

Slot-B

1. A continuous curve which does not have a point of self-intersection is called a
   a. Curve
   b. Closed curve
   c. Simple closed curve
   d. Multiple curve

   Answer: c. Simple closed curve

2. The zero's of $f(z) = \frac{z^2+1}{1-z^2}$ are
   a. 0
   b. $\pm i$
   c. ± 1
   d. 1

   Answer: b. $\pm i$

3. If $f(z)$ is analytic inside and on C, then the value of $\oint_C \frac{f(z)}{z-a} dz$, where C is the simple closed curve and a is any point within C is

   a. $f(a)$
   b. $2\pi i f(a)$
   c. $\pi i f(a)$
   d. 0

   Answer: b. $2\pi i$f(a)

4. If $f(z)$ is analytic inside and on C, then the value of $\oint_C \frac{f(z)}{(z-a)^5} dz$, where C is the simple closed curve and a is any point within C is

   a. $2\pi i\ \frac{f^v(a)}{5!}$
   b. $2\pi i\ f(a)$
   c. $2\pi i\ \frac{f^{iv}(a)}{4!}$
   d. 0

   Answer:c. $2\pi i\ \frac{f^{iv}(a)}{4!}$

5. The value of $\oint_C \frac{e^{-z}}{z+1} dz$ where C is the circle $|z| = \frac{1}{3}\ is$

   a. 0
   b. $2\pi ie$
   c. $\frac{\pi}{2} ie$
   d. πie

   Answer: a. 0

6. The value of $\oint_C \frac{e^{2z}}{(z+1)^3} dz$ where C is the circle $|z| = 2\ is$

   a. 0
   b. $2\pi ie^{-2}$
   c. $8\pi ie^{-2}$
   d. $4\pi ie^{-2}$

   Answer: d. $4\pi ie^{-2}$

7. The value of $\oint_C \frac{1}{2z-3} dz$ where C is the circle $|z| = 1\ is$

   a. 0
   b. $2\pi i$
   c. $\frac{\pi}{2} i$
   d. πi

Answer: a. 0

8. The value of $\oint_C \frac{z^2}{(z-2)^2}dz$ where C is the circle $|z| = 3$ *is*
    a. 0
    b. $2\pi i$
    c. $4\pi i$
    d. $8\pi i$

Answer: d. $8\pi i$

9. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$ be two concentric circles $(R_2 < R_1)$, the annular region is defined as
    a. Within C_1
    b. Within C_2
    c. Within C_2 and outside C_1
    d. Within C_1 and outside C_2

Answer: d. Within C_1 and outside C_2

10. The part $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral powers of $(z-a)$ is called as
    a. The analytic part of the Laurent's series
    b. The principal part of the Laurent's series
    c. The real part of the Laurent's series
    d. The imaginary part of the Laurent's series

Answer: b. The principal part of the Laurent's series

11. Let $C: |z-a| = r$ be a circle, the $f(z)$ can be expanded as a Taylor's series if
    a. $f(z)$ is a function on C
    b. $f(z)$ is an analytic function within C
    c. $f(z)$ is not an analytic function within C
    d. $f(z)$ is an analytic function outside C

Answer: b. f(z) is an analytic function within c

12. Expansion of $\frac{\sin z}{(z-\pi)}$ in Taylor's series about $z = \pi$ is

a. $\frac{(z-\pi)}{1!} - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - \cdots$
b. $\frac{(z-\pi)^2}{2!} - \frac{(z-\pi)^4}{4!} + \frac{(z-\pi)^6}{6!} - \cdots$
c. $-1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \cdots$
d. $\frac{(z-\pi)}{2!} + \frac{(z-\pi)^3}{4!} - \frac{(z-\pi)^5}{6!} + \cdots$

Answer :c. $-1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \cdots$

13. The annular region for the function $f(z) = \frac{1}{z^2-z-6}$ is

a. $0 < |z| < 1$
b. $1 < |z| < 2$
c. $2 < |z| < 3$
d. $|z| < 3$

Answer :c. $2 < |z| < 3$

14. The Laurent's series expansion $-\frac{1}{2}\sum\frac{(z+2)^n}{4^n} - \sum\frac{3^n}{(z+2)^n}$ for the function $f(z) = \frac{z}{(z-1)(z-2)}$ is valid in the region

a. $|z+2| < 3$
b. $1 < |z+2| < 2$
c. $3 < |z+2| < 4$
d. $|z+2| > 4$

Answer :c. $3 < |z+2| < 4$

15. If $f(z)$ is not analytic at $z = z_0$ and there exists $\lim_{z\to z_0} f(z)$ and is finite then,

a. The point $z = z_0$ is isolated singularity of $f(z)$
b. The point $z = z_0$ is a removable singularity of $f(z)$
c. The point $z = z_0$ is essential singularity of $f(z)$
d. The point $z = z_0$ is non isolated singularity of $f(z)$

Answer : b. The point $z = z_0$ is a removable singularity of $f(z)$

16. Let $z = a$ is a simple pole for $f(z)$ and $b = \lim_{z \to a} (z - a)f(z)$, then

a. b is a simple pole
b. b is removable singularity
c. b is a residue at a of order n
d. b is a residue at $z = a$

Answer : d. b is a residue at $z = a$

17. Let z = a is a pole of order m for $f(z)$, then the residue is

a. $\lim_{z \to a} [(z - a)f(z)]$
b. $\lim_{z \to a} [(z - a)f''(z)]$
c. $\lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)]$
d. $\lim_{z \to a} \frac{1}{m!} \frac{d^m}{dz^m} [(z - a)^m f(z)]$

e. Answer: c. $\lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)]$

18. The residue of $f(z) = \frac{z}{(z-1)^2}$ at $z = 1$ is

a. π
b. 1
c. -1
d. 0

Answer: b. 1

19. The residue of $f(z) = \frac{z}{z^2+1}$ at $z = i$ is

a. 1
b. -1
c. 0
d. 1/2

Answer : d. 1/2

20. If $f(z) = \frac{sinz}{z}$, then

a. $z = 0$ is a simple pole
b. $z = 0$ is a pole of order 2
c. $z = 0$ is a removable singularity
d. $z = 0$ is a zero of $f(z)$

Answer: c. $z = 0$ is a removable singularity

21. The value of the integral $\oint_C e^z dz$ where $|z| = 1$ is

a. $2\pi i$
b. $\frac{\pi}{2}i$
c. πi
d. 0

Answer: d. 0

22. If $f(z) = \frac{-1}{(z-1)} - 2[1 + (z-1) + (z-1)^2 + \cdots]$ then the residue of $f(z)$ at $\mathrm{z} = 1$is

a. 1
b. -1
c. 0
d. -2

Answer: b. -1

23. If the integral $\oint_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \oint_C f(z)dz$, C is $|z| = 1$, then
(A) $z = -\frac{1}{3}$ lies inside C and
(B) $z = 3$ lies outside C. Which of the following is true.

a. Both A and B
b. Only A
c. Only B
d. Neither A nor B

Answer: a. Both A and B

24. In Cauchy's Lemma for contour integration, if $f(z)$ be continuous function such that $|zf(z)| \to 0$ as $|z| \to \infty$, for C is the circle $|z| = R$, then

a. $\oint_C f(z)dz \to \infty \ as \ R \to \infty.$

b. $\oint_C f(z)dz \to 0 \ as \ R \to \infty.$

c. $\oint_C f(z)dz \to 0 \ as \ R \to 0.$

d. $\oint_C f(z)dz \to \infty \ as \ R \to 0.$

Answer : b. $\oint_C f(z)dz \to 0 \ as \ R \to \infty.$

25. If $\oint_C \frac{e^z}{z^2} dz = 0,$ then C is

a. $|z| = 1$
b. $|z - 1| = 2$
c. $|z - 2| = 1$
d. $|z| = 2$

Answer: c. $|z - 2| = 1$

SRM University
Department of Mathematics
Complex Integration- Multiple Choice questions
UNIT V

Slot-C

1. A contour integral is an integral along a ---------------- curve.
   a. Open Curve
   b. Closed curve
   c. Simple closed curve
   d. Multiple curve

   Answer: c. Simple closed curve

2. If f(z) is analytic inside and on C , the value of $\oint_C f(z)\, dz$,where C is the simple closed curve is
   a. f(a)
   b. $2\pi i$f(a)
   c. πif(a)
   d. 0

   Answer: d. 0

3. If f(z) is analytic inside and on C , the value of $\oint_C \frac{f(z)}{(z-a)^n}\, dz$,where C is the simple closed curve and a is any point within c is
   a. $2\pi i\ \frac{f^n(a)}{n!}$
   b. $2\pi i\ f(a)$
   c. $2\pi i\ \frac{f^{n-1}(a)}{(n-1)!}$
   d. 0

   Answer: c. $2\pi i\ \frac{f^{n-1}(a)}{(n-1)!}$

4. The value of $\oint_C \frac{\sin z}{z+1}\, dz$ where C is the circle $|z| = \frac{1}{3}\ is$
   a. 0

b. $2\pi i$
c. $\frac{\pi}{2}i$
d. πi

Answer: a. 0

5. The value of $\oint_C \frac{e^z}{(z-2)^2}\, dz$ where C is the circle $|z| = 3\ is$
   a. 0
   b. $2\pi i e^{-2}$
   c. $2\pi i e^{2}$
   d. $4\pi i e^{-2}$

   Answer: c. $2\pi i e^{2}$

6. The value of $\oint_C \frac{z}{2z-1}\, dz$ where C is the circle $|z| = 1\ is$
   a. 0
   b. $2\pi i$
   c. $\frac{\pi}{2}i$
   d. πi

   Answer: d. πi

7. The value of $\oint_C \frac{1}{(z-3)^2}\, dz$ where C is the circle $|z| = 1\ is$
   a. 0
   b. $2\pi i$
   c. $\frac{\pi}{2}i$
   d. πi

   Answer: a. 0

8. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$be two concentric circles($R_2 > R_1$), the annular region is defined as
   a. Within C_1
   b. Within C_2
   c. Within C_2 and outside C_1

d. Within C_1 and outside C_2

Answer: c. Within C_2 and outside C_1

9. $The\ part\ \sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral powers of (z - a) is called as
   a. The analytic part of the Laurent's series
   b. The principal part of the Laurent's series
   c. The real part of the Laurent's series
   d. The imaginary part of the Laurent's series

   Answer: a. The analytic part of the Laurent's series

10. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$ be two concentric circles ($R_2 < R_1$), the f(z) can be expanded as a Laurent's series if
    a. f(z) is analytic within C_2
    b. f(z) is not analytic within C_2
    c. f(z) is analytic in the annular region
    d. f(z) is not analytic in the annular region

    Answer: c. f(z) is analytic in the annular region

11. Expansion of $\frac{1-\cos z}{z}$ in Laurent's series about $z = 0$ is
    a. $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots$
    b. $\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots$
    c. $\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$
    d. $\frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots$

    Answer: a. $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots$

12. The annular region for the function f(z) = $\frac{1}{z^2-3z+2}$ is

a. $0 < |z| < 1$
b. $1 < |z| < 2$
c. $2 < |z| < 3$
d. $|z| < 3$

Answer :b. $1 < |z| < 2$

13. The Laurent's series expansion $1 + \frac{3}{z}\sum\frac{(-1)^n 2^n}{z^n} - \sum\frac{(-1)^n 3^n}{z^n}$ for the function $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ is valid in the region

a. $|z| < 3$
b. $|z| < 2$
c. $2 < |z| < 3$
d. $|z| > 3$

e. Answer :d. $|z| > 3$

14. If f(z) is not analytic at $z = z_o$ and there exists a neighborhood of $z = z_o$ containing no other singularity, then

a. The point $z = z_o$ is isolated singularity of f(z)
b. The point $z = z_o$ is a zero point of f(z)
c. The point $z = z_o$ is nonzero of f(z)
d. The point $z = z_o$ is non isolated singularity of f(z)

Answer : a. The point $z = z_o$ is isolated singularity of f(z)

15. If f(z) = $e^{\frac{1}{z+1}}$ then

a. z = -1 is removable singularity
b. z = -1 is pole of order 2
c. z = -1 is an essential singularity
d. z = -1 is zero of f(z)

Answer : c. z = -1 is an essential singularity

16. Let z = a is a simple pole for $f(z) = \frac{P(z)}{Q(z)}$, then the Residue of $f(z)$ is

a. $\frac{P'(a)}{Q(a)}$
b. $\frac{P(a)}{Q(a)}$
c. $\frac{P'(a)}{Q'(a)}$
d. $\frac{P(a)}{Q'(a)}$

Answer : d. $\frac{P(a)}{Q'(a)}$

17. Let z = a is a pole of order 3 for f(z), then the residue is

a. $\lim_{z \to a} [(z - a)f(z)]$
b. $\lim_{z \to a} [(z - a)f''(z)]$
c. $\lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} [(z - a)^3 f(z)]$
d. $\lim_{z \to a} \frac{1}{3!} \frac{d^3}{dz^3} [(z - a)^3 f(z)]$

Answer: c. $\lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} [(z - a)^3 f(z)]$

18. The residue of $f(z) = \frac{z}{(z-2)}$ is

a. $2\pi i$
b. 1
c. 2
d. 0

Answer: c. 2

19. The residue of $f(z) = \frac{1}{(z^2+1)^2}$ at $z = i$ is

a. 4i
b. 1/4i
c. 0
d. 1/2i

Answer :b. 1/4i

20. If f(z) = $\frac{sinz-z}{z^3}$, then

a. z= 0 is a simple pole
b. z= 0 is a pole of order 2
c. z= 0 is a removable singularity
d. z= 0 is a zero of f(z)

Answer: c. z= 0 is a removable singularity

21. The value of the integral $\oint_C \frac{1}{ze^z} dz$ where $|z| = 1$ is

a. $2\pi i$
b. $\frac{\pi}{2} i$
c. πi
d. 0

Answer: a. $2\pi i$

22. If $f(z) = \frac{1}{z} + [2 + 3z + 4z^2 + \cdots]$ then the residue of $f(z)$ at z=0 is

a. 1
b. -1
c. 0
d. -2

Answer: a. 1

23. If the integral $\oint_0^{2\pi} \frac{d\theta}{13+5\cos\theta} = \oint_C f(z)dz$, C is $|z| = 1$, then
(A) $z = -i/5$ lies inside C and
(B) $z = -5i$ lies outside C. Which of the following is true.

a. Both A and B
b. Only A
c. Only B
d. Neither A nor B

Answer: a. Both A and B

24. If the integral $\oint_{-\infty}^{\infty} \frac{\cos mx}{(x^2+1)^2} dx \;, m > 0$, then
(A) $z = i$ double pole lies in the upper half of the z-plane and
(B) $z = -i$ double pole does not lie in the upper half of the z-plane.
Which of the following is true.

a. Both A and B
b. Only A
c. Only B
d. Neither A nor B

Answer: a. Both A and B

25. If $f(z)$ be continuous function such that $|f(z)| \to 0$ as $|z| \to \infty$, for C is the semicircle $|z| = R$ above the real axis, then

a. $\oint_C e^{-imz} f(z)dz \;\to \infty \; as \; R \;\to \infty$.
b. $\oint_C e^{imz} f(z)dz \;\to 0 \; as \; R \;\to \infty$.
c. $\oint_C e^{imz} f(z)dz \;\to 0 \; as \; R \;\to 0$.
d. $\oint_C f(z)dz \;\to \infty \; as \; R \;\to 0$.

Answer : b. $\oint_C e^{imz} f(z)dz \;\to 0 \; as \; R \;\to \infty$.

Cycle Test-1

18MAB102T- Multiple Integral

1. The value of $\int_0^5 \int_0^{10} dydx$ is

   A. 50
   B. 15
   C. 0
   D. 25

   ANSWER: A

2. The value of $\int_0^1 \int_0^x dydx$ is

   A. x^2
   B. xy
   C. $\frac{x^2}{2}$
   D. $\frac{1}{2}$

   ANSWER: D

3. The value of $\int_1^2 \int_2^3 xy^2 dxdy$ is

   A. 6/35
   B. 35
   C. $35/6$
   D. 70

   ANSWER: C

4. In polar system, the name of the curve $r = a(1 - \cos\theta)$ is

   A. circle
   B. cardioid
   C. cycloid
   D. ellipsoid

   ANSWER: B

5. Area of the double integral in polar coordinate is equal to

   A. $\int\int(r+1)drd\theta$
   B. $\int\int r^3 drd\theta$
   C. $\int\int drd\theta$
   D. $\int\int rdrd\theta$

   ANSWER: D

6. The region of integration in $\int_{-2}^{2}\int_{-3}^{3} f(x,y)dydx$ is

A. rectangle

B. triangle

C. square

D. circle

ANSWER: A

7. The volume integration in cartesian coordinates is equal to

A. $\iiint rdrd\phi d\theta$

B. $\iiint dxdydz$

C. $\iint dydx$

D. $\iint drd\theta$

ANSWER: B

8. In cartesian co-ordinates $\iint_S ds$ is equal to

A. $\iint_S dxdy$

B. $\iint_S drd\theta$

C. $\iint_S dxd\theta$

D. $\iint_S dyd\theta$

ANSWER: A

9. The value of $\int_0^a\int_0^b\int_0^c dzdydx$ is

A. $abc \ \left(\frac{a+b+c}{2}\right)$

B. $\frac{a+b+c}{2}$

C. abc

D. $\frac{abc}{2}$

ANSWER: C

10. The value of $\int_0^1\int_1^2 x^2y^2dydx$

A. 7/3

B. 9/7

C. 17/9

D. 7/9

ANSWER: D

11. $\int_0^3\int_0^2 (x+y)dxdy$ is equal to

A. 15

B. 14

C. 13

D. 12

ANSWER: A

12. By changing into polar coordinates, $\int\limits_0^a \int\limits_y^a dxdy =$

A. $\int\limits_0^{\pi/4} \int\limits_0^{a/\cos\theta} rdrd\theta$

B. $\int\limits_0^{\pi/4} \int\limits_0^{\infty} drd\theta$

C. $\int\limits_0^{\infty} \int\limits_0^{a/2} rdrd\theta$

D. $\int\limits_0^{\pi/2} \int\limits_0^{\infty} drd\theta$

ANSWER: A

13. If R is the region bounded by $x = 0,\ y = 0, x + y = 1$, then $\iint\limits_R dydx$ is equal to

A. 1

B. 1/2

C. 1/3

D. 2/3

ANSWER: B

14. $\int\limits_0^{\pi} \int\limits_0^{a(1+\cos\theta)} rdrd\theta$ is equal to

A. $\pi a/2$

B. $\pi a/4$

C. $3\pi a^2/4$

D. $\pi/4$

ANSWER: C

15. $\int\limits_0^{\pi} \int\limits_0^{\cos\theta} rdrd\theta$ is equal to

A. 0

B. $\pi/2$

C. π

D. $\pi/4$

ANSWER: D

16. By changing into polar coordinates, $\int\limits_0^a \int\limits_0^{\sqrt{(a^2-x^2)}} e^{-(x^2+y^2)} dxdy =$

A. $\int\limits_0^{\pi/4} \int\limits_0^{\infty} e^{-r^2} drd\theta$

B. $\int\limits_0^{\infty} \int\limits_0^{a/2} e^{-r^2} drd\theta$

C. $\int\limits_0^{\pi/2} \int\limits_0^{a} e^{-r^2} rdrd\theta$

D. $\int\limits_0^{\pi/2} \int\limits_0^{\infty} e^{-r} drd\theta$

ANSWER: C

17. The value of $\int\limits_0^\infty \int\limits_0^y \frac{e^{-y}}{y} dxdy$ is

A. 1
B. 0
C. 2
D. 3

ANSWER: A

18. $\int\limits_0^4 \int\limits_0^2 \int\limits_0^1 (x+y+z) dzdydx =$

A. 22
B. 24
C. 26
D. 28

ANSWER: D

19. $\int\limits_0^1 \int\limits_0^2 \int\limits_0^3 e^{(x+y+z)} dzdydx =$

A. $(e-1)^3$
B. $(e-1)(e^2-1)(e^3-1)$
C. $3(e-1)$
D. e^3

ANSWER: B

20. The smaller region of integration of the area between $x^2 + y^2 = 9$ and $x + y = 3$ lies in the ___________ quadrant.

A. first
B. second
C. third
D. fourth

ANSWER: A

21. The area between two circles $r = a\cos\theta$ and $r = 2a\cos\theta$ lies between ___________ quadrant.

A. I and II
B. I and III
C. I and IV
D. III and IV

ANSWER: C

22. The upper limit of "y" for $\iint\limits_R dydx$ in the positive quadrant of the circle $x^2 + y^2 = a^2$ is

A. $-\sqrt{a^2 - y^2}$
B. $+\sqrt{a^2 - y^2}$
C. $-\sqrt{a^2 - x^2}$
D. $+\sqrt{a^2 - x^2}$

ANSWER: D

23. The area between the curve $y = x$ and $y = x^2$ is

A. $1/2$
B. $1/3$
C. $1/6$
D. 1
ANSWER: C

24. The value of $\int_{-a}^{a}\int_{0}^{x} dydx$.

A. 0
B. 1
C. 2
D. 3
ANSWER: A

25. On changing the order of integration, $\int_{0}^{1}\int_{0}^{x} f(x,y)dydx$ is equal to

A. $\int_{0}^{1}\int_{0}^{y} f(x,y)dxdy$
B. $\int_{0}^{1}\int_{y}^{1} f(x,y)dxdy$
C. $\int_{0}^{1}\int_{0}^{1} f(x,y)dxdy$
D. $\int_{0}^{x}\int_{0}^{1} f(x,y)dxdy$
ANSWER: B

Module – 1 Multiple Integrals

Evaluation of double integration Cartesian and plane polar coordinates – Evaluation of double integration by changing order of integration – Area as a double integral (Cartesian) – Area as a double integral (Polar) – Triple integration in Cartesian coordinates – Conversion from Cartesian to polar in double integrals – Volume using triple integral – Application of Multiple integral in Engineering.

Evaluation of double integration – Cartesian and Polar coordinates

Type – 1 Limits are constants

1. Evaluate $\int_0^1\int_1^2 \left(x^2+y^2\right)dx\,dy$ **.**

Solution:

$$\int_0^1\int_1^2 \left(x^2+y^2\right)dx\,dy = \int_0^1 \left(\frac{x^3}{3}+x\,y^2\right)_1^2 dy$$

$$= \int_0^1 \left[\left(\frac{8}{3}+2\,y^2\right)-\left(\frac{1}{3}+y^2\right)\right]dy$$

$$= \int_0^1 \left(\frac{7}{3}+y^2\right)dy$$

$$= \left(\frac{7}{3}y+\frac{y^3}{3}\right)_0^1 = \frac{8}{3}$$

Note: $\int_1^2\int_0^1 \left(x^2+y^2\right)dy\,dx = \frac{8}{3}$

If the limits of integration are constants, then the order of integration is insignificant.

2. Evaluate $\int_0^3\int_0^2 x\,y\left(x+y\right)dy\,dx$ **.**

Solution:

$$\int_0^3\int_0^2 x\,y\left(x+y\right)dy\,dx = \int_0^3\int_0^2 \left(x^2\,y+x\,y^2\right)dy\,dx$$

$$= \int_0^3 \left(\frac{x^2\,y^2}{2}+x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_0^3 \left(2\,x^2+\frac{8}{3}x\right)dx$$

$$= \left(2\frac{x^3}{3}+\frac{8}{3}\frac{x^2}{2}\right)_0^3 = 30$$

3. Evaluate $\int_2^a \int_2^b \frac{dx\,dy}{x\,y}$ **.**

Solution:

$$\int_2^a \int_2^b \frac{dx\,dy}{x\,y} = \int_2^a \left(\int_2^b \frac{dx}{x} \right) \frac{dy}{y}$$

$$= \int_2^a (\log x)_2^b \frac{dy}{y}$$

$$= (\log x)_2^b \; (\log x)_2^a$$

$$= \log\left(\frac{b}{2}\right) \log\left(\frac{a}{2}\right)$$

4. Evaluate $\int_0^3 \int_0^2 r\,dr\,d\theta$ **.**

Solution:

$$\int_0^3 \int_0^2 r\,dr\,d\theta = \int_0^3 \left(\frac{r^2}{2}\right)_0^2 d\theta = \int_0^3 2\,d\theta = 2(\theta)_0^3 = 6$$

Type – 2 Limits are variables

5. Evaluate $\int_0^1 \int_x^{\sqrt{x}} x\,y(x+y)\,dy\,dx$ **.**

Solution:

$$\int_0^1 \int_x^{\sqrt{x}} x\,y(x+y)\,dy\,dx = \int_0^1 \int_x^{\sqrt{x}} \left(x^2\,y + x\,y^2\right) dy\,dx$$

$$= \int_0^1 \left(\frac{x^2\,y^2}{2} + x\frac{y^3}{3} \right)_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^4}{2} - \frac{x^4}{3} \right) dx$$

$$= \left(\frac{x^4}{8} + \frac{x^{7/2}}{3\times\frac{7}{2}} - \frac{x^5}{10} - \frac{x^5}{15} \right)_0^1 = \frac{3}{56}$$

6. **Evaluate** $\int_0^a \int_0^{\sqrt{a^2-x^2}} y \; dy\, dx$ **.**

Solution:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y \; dy\, dx = \int_0^a \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(\frac{a^2-x^2}{2}\right) dx = \frac{a^3}{3}$$

7. **Evaluate** $\int_0^a \int_0^{\sqrt{ay}} x\,y \; dx\, dy$ **.**

Solution:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y \; dy\, dx = \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^a y\,a\,y \; dy = \frac{a^4}{6}$$

CHANGE THE ORDER OF INTEGRATION

For changing the order of integration in a given double integral

Step 1: Draw the region of integration by using the given limits.

Step 2: After changing the order, consider

- **dxdy as horizontal strip**
- **dydx as vertical strip**

Step 3: Find the new limits.

Step 4: Evaluate the double integral.

8. **Change the order of integration in** $\int_0^a \int_y^a \frac{\mathbf{x\,dydx}}{\mathbf{x^2+y^2}}$ **and hence evaluate it.**

Solution:

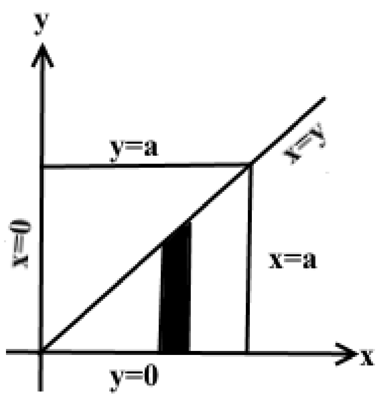
$\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$ **(Correct Form)**

Given limits $x: y \to a \;;\; y: 0 \to a$

After changing the order,

$dy\,dx \to vertical\ strip$

Now, limit $x: 0 \to a \;;\; y: 0 \to x$



$$\therefore \int_0^a \int_0^x \frac{\mathbf{x}}{\mathbf{x^2+y^2}} \;\mathbf{dydx} = \int_0^a \int_0^x x\left(\frac{1}{\mathbf{x^2+y^2}}\right) \mathbf{dy\,dx}$$

$$= \int_0^a x\left(\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right)_0^x \ \mathbf{dx}$$

$$= \int_0^a x\left(\tan^{-1}(1) - \tan^{-1}(0)\right) \ \mathbf{dx} \quad \because \tan^{-1}(1) = \frac{\pi}{4},\ \tan^{-1}(0) = 0$$

$$= \int_0^a \left(\frac{\pi}{4}\right) \ \mathbf{dx}$$

$$= \left(\frac{\pi}{4}\right)(x)_0^a$$

$$= \frac{\pi a}{4}$$

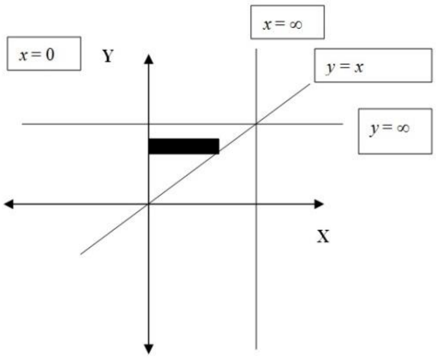
9. **Evaluate** $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ **by changing the order of integration.**

Solution:

Given limits:

$x : 0 \to \infty$

$y : x \to \infty$



After changing the order,

dxdy $\to$ *horizontal strip*

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \frac{e^{-y}}{y}(x)_0^y \ dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, y\, dy$$

$$= \int_0^\infty e^{-y} dy$$

$$= \left(\frac{e^{-y}}{-1}\right)_0^\infty$$

$$= \left(\frac{e^{-\infty}}{-1} - \left(\frac{e^{-0}}{-1}\right)\right)$$

$$= -e^{-\infty} + e^{-0}$$

$$= 1 \qquad \because e^{-\infty} = 0; \quad e^{-0} = e^{0} = 1$$

10. **Change the order of integration** $\int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} \, dx \, dy$ **and hence evaluate it.**

Solution

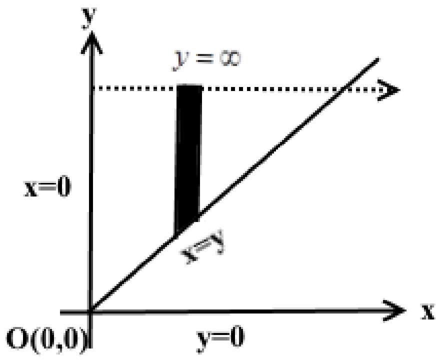
Given limits:

$x = 0, \quad x = y$

$y = 0, \quad y = \infty$

After changing the order,

$dy\,dx \rightarrow vertical\ strip$



$$\int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} \, dx \, dy = \frac{1}{2} \int_0^\infty \int_x^\infty 2 y e^{-\frac{y^2}{x}} \, dy \, dx$$

$$= \frac{1}{2} \int_0^\infty \left(\int_x^\infty 2 y e^{-\frac{y^2}{x}} \, dy \right) dx$$

$$= \frac{1}{2} \int_0^\infty \left(\int_x^\infty e^{-\frac{y^2}{x}} \, d(y^2) \right) dx$$

$$= \frac{1}{2} \int_0^\infty \left[\int_x^\infty -x e^{-\frac{y^2}{x}} \right]_x^\infty dx$$

(Or) Use Substitution $y^2 = t$, $2ydy = dt$, *Limits:* $t : x^2 \rightarrow \infty$

$$= \frac{1}{2} \int_0^\infty \left[0 - \left(-x e^{-\frac{x^2}{x}} \right) \right] dx$$

$$= \frac{1}{2} \int_0^\infty x e^{-x} \, dx$$

$$= \frac{1}{2} \left[\frac{x e^{-x}}{-1} - (1) \left(\frac{e^{-x}}{(-1)(-1)} \right) \right]_0^\infty$$

$$= \frac{1}{2} \left[-x e^{-x} - e^{-x} \right]_0^\infty$$

$$= \frac{1}{2} \left[(0+0) - (0+1) \right] \qquad \because e^{-\infty} = 0, \; e^{0} = 1$$

$$= \frac{1}{2}$$

11. **Evaluate** $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$ **by changing the order of integration.**

Solution:

Given $y=0,\ y=\sqrt{a^2-x^2}$

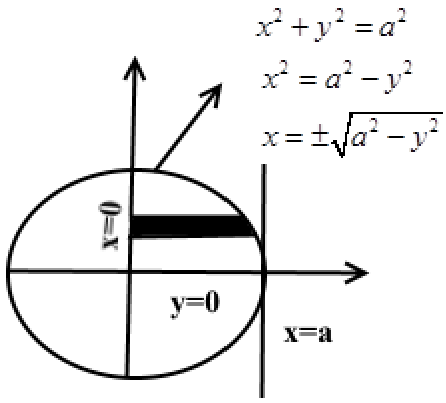
$y^2=a^2-x^2$

$x^2+y^2=a^2$

$x=0,\ x=a$

After changing the order,

dxdy $\rightarrow$ *horizontal strip*



$$\int_0^a \int_0^{\sqrt{a^2-y^2}} x\,y\ dx\,dy = \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$=\int_0^a \frac{y}{2}\left(a^2-y^2\right)dy$$

$$=\frac{a^2}{2}\int_0^a y\,dy-\frac{1}{2}\int_0^a y^3dy$$

$$=\frac{a^2}{2}\left(\frac{y^2}{2}\right)_0^a-\frac{1}{2}\left(\frac{y^4}{4}\right)_0^a$$

$$=\frac{a^4}{4}-\frac{a^4}{8}$$

$$=\frac{a^4}{4}-\frac{a^4}{8}$$

$$=\frac{a^4}{8}$$

12. **Changing the order of integration and hence evaluate** $\int_{0}^{1}\int_{x^2}^{2-x} xy\,dy\,dx$**.**

Solution:

Given limits:

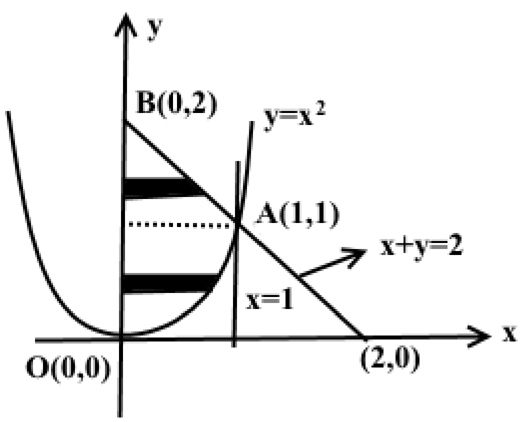
$y=x^2$

$y=2-x \Rightarrow x+y=2$

$x=0,\quad y=1$

After changing the order,

dxdy $\rightarrow$ *horizontal strip*



$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx=\int_0^1\int_0^{\sqrt{y}} xy\,dxdy+\int_1^2\int_0^{2-y} xy\,dxdy$$

$$=I_1+I_2\quad (say)------(1)$$

To find I_1:

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\, dxdy$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy$$

$$= \int_0^1 y\left(\frac{y}{2} - 0\right) dy$$

$$= \int_0^1 \frac{y^2}{2} dy$$

$$= \left(\frac{y^3}{6}\right)_0^1$$

$$= \left(\frac{1}{6} - 0\right)$$

$$I_1 = \frac{1}{6}$$

To find I_2:

$$I_2 = \int_1^2 \int_0^{2-y} xy\, dxdy$$

$$= \int_1^2 y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_1^2 y\left(\frac{(2-y)^2}{2}\right) dy$$

$$= \int_1^2 \frac{y}{2}\left(4 - 4y + y^2\right) dy$$

$$= \frac{1}{2}\int_1^2 \left(4y - 4y^2 + y^3\right) dy$$

$$= \frac{1}{2}\left(\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right)_1^2$$

$$= \frac{1}{2}\left(\frac{4(2)^2}{2} - \frac{4(2)^3}{3} + \frac{2^4}{4} - \left(\frac{4(1)^2}{2} - \frac{4(1)^3}{3} + \frac{1^4}{4}\right)\right)$$

$$= \frac{1}{2}\left(8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(10 - \frac{28}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{10(12) - 28(4) - 1(3)}{12}\right)$$

$$= \frac{1}{2}\left(\frac{5}{12}\right)$$

$$I_2 = \frac{5}{24}$$

$$(1) \Rightarrow I = \frac{1}{6} + \frac{5}{24}$$

$$= \frac{9}{24}$$

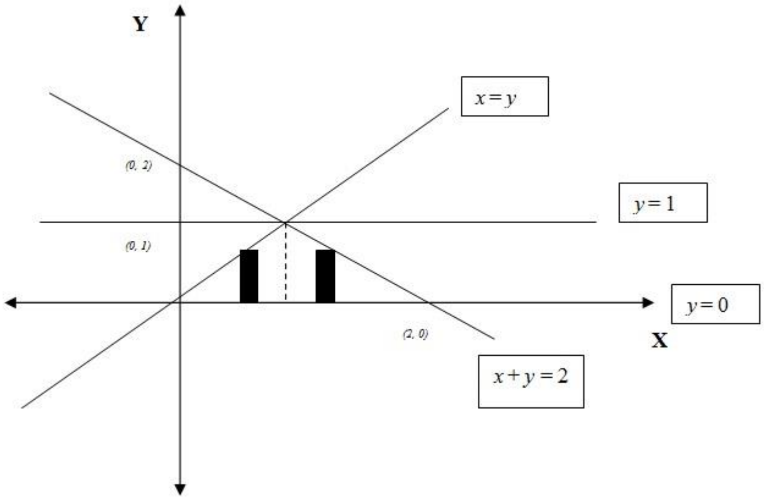
$$I = \frac{3}{8}$$

13. **Change the order of integration and hence evaluate** $\int_0^1 \int_y^{2-y} x\,y\,dx\,dy$ **.**

Solution:

Given limits: $\boldsymbol{x = y, x = 2 - y}$

$\boldsymbol{y = 0 , y = 1}$



After changing the order, *dy dx* $\rightarrow$ *vertical strip*

$$\int_0^1 \int_y^{2-y} x\,y\,dx\,dy = \int_0^1 \int_0^x x\,y\,dy\,dx + \int_1^2 \int_0^{2-x} x\,y\,dy\,dx$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^x dx + \int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \int_0^1 \frac{x^3}{2} dx + \frac{1}{2}\int_1^2 x\left(4 + x^2 - 4x\right)dx = \frac{1}{8} + \frac{5}{24} = \frac{1}{3}$$

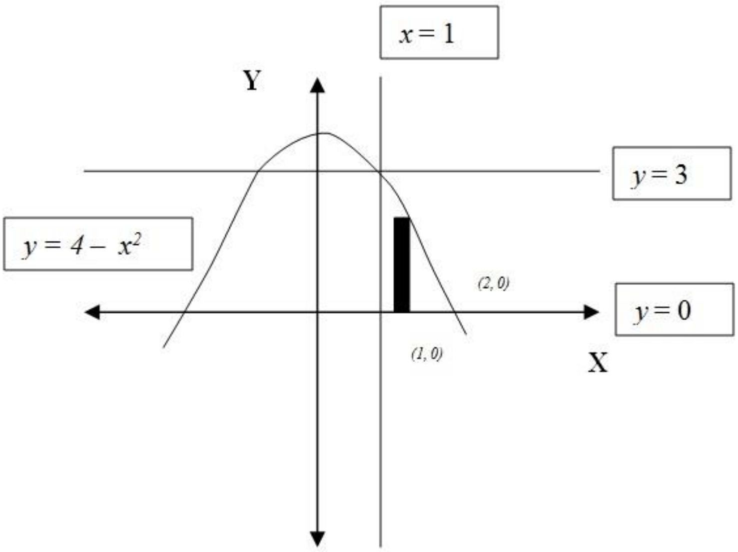
14. Change the order of integration and hence evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ **.**

Solution:

Given limits: $x = 1,\ x = \sqrt{4-y}$

$y = 0\ ,\ y = 3$

x	-2	-1	0	1	2
$y = 4 - x^2$	0	3	4	3	0



After changing the order, *dy dx* $\rightarrow$ *vertical strip*

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy = \int_1^2 \int_0^{4-x^2} (x+y)\,dy\,dx$$

$$= \int_1^2 \left(x\,y + \frac{y^2}{2} \right)_0^{4-x^2} dx$$

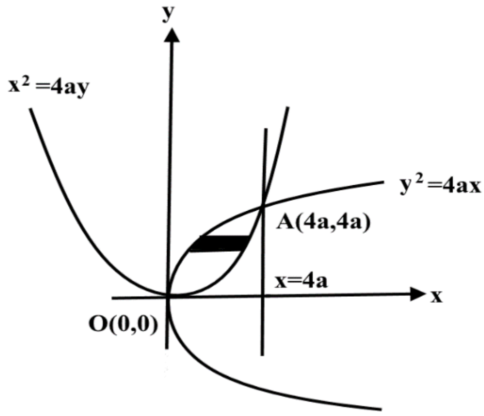
$$= \int_1^2 \left(4x - x^3 + 8 + \frac{x^4}{2} - 4x^2 \right) dx$$

$$= \frac{241}{60}$$

15. **Change the order of integration and hence evaluate** $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ **.**

Solution:

Given limits:



$$y=\frac{x^2}{4a}\Rightarrow x^2=4ay-------(1)$$

$$y=2\sqrt{ax}\Rightarrow y^2=4ax------(2)$$

$$x=0\quad,\qquad x=4a$$

Sub (1) in (2),

$$\left(\frac{y^2}{4a}\right)^2=4ay$$

$$\frac{y^4}{16a^2}=4ay$$

$$y^4=64a^3y$$

$$\left(y^4-64a^3y\right)=0$$

$$y\left(y^3-64a^3\right)=0$$

$$y=0\ \ and\ \ y^3-64a^3=0$$

$$y=0\quad and\quad y^3=64a^3$$

$$y=0\quad and\quad y\ =4a$$

$$when\ \ y=0\Rightarrow x=0$$

$$when\ \ y=4a\Rightarrow x=\frac{16a^2}{4a}=4a$$

After changing the order, *dx dy* $\rightarrow$ *horizontal strip*

dy dx $\rightarrow$ *vertical strip*

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx=\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dxdy$$

$$=\int_0^{4a}\left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$=\int_0^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dy$$

$$=\int_0^{4a}\left(2\sqrt{a}(y)^{1/2}-\frac{y^2}{4a}\right)dy$$

$$= \left(2\sqrt{a}\,\frac{(y)^{3/2}}{3/2} - y^3/12a \right)_0^{4a}$$

$$= \left(\frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{(4a)^3}{12a} \right)$$

$$= \left(\frac{32a^2}{3} - \frac{(4a)^3}{12a} \right) \quad \because (4)^{3/2} = 4\sqrt{4} = 8$$

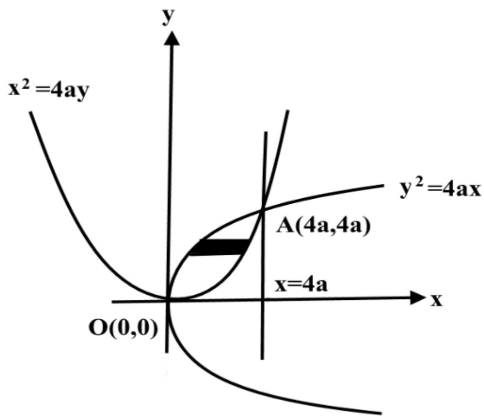
$$= \left(\frac{32a^2}{3} - \frac{64a^3}{12a} \right)$$

$$= \left(\frac{32a^2}{3} - \frac{16a^2}{3} \right) = \frac{16a^2}{3}$$

16. **Change the order of integration and hence evaluate** $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} x\,y\,dy\,dx$**.**

Solution:

Given limits:



$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay ------- (1)$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax ------ (2)$

$x = 0 \quad , \qquad x = 4a$

Sub (1) in (2),

$$\left(\frac{y^2}{4a} \right)^2 = 4ay$$

$$\frac{y^4}{16a^2} = 4ay$$

$$y^4 = 64a^3 y$$

$$\left(y^4 - 64a^3 y \right) = 0$$

$$y\left(y^3 - 64a^3 \right) = 0$$

$y = 0$ *and* $y^3 - 64a^3 = 0$

$y = 0$ *and* $y^3 = 64a^3$

$y = 0$ *and* $y = 4a$

when $y = 0 \Rightarrow x = 0$

when $y = 4a \Rightarrow x = \frac{16a^2}{4a} = 4a$

After changing the order,

dx dy $\rightarrow$ *horizontal strip*

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{a\,x}} x\,y\,dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{a\,y}} x\,y\,dx\,dy$$

$$= \int_0^{4a} y\left(\frac{x^2}{2}\right)_{y^2/4a}^{2\sqrt{a\,y}} dy$$

$$= \int_0^{4a}\left(2\,a\,y^2 - \frac{y^5}{32\,a^2}\right)dy = \frac{64}{3}a^4$$

Area as a double integral (Cartesian Coordinates) $\iint_R dx\,dy$ **or** $\iint_R dy\,dx$

17. Find the area of the circle $x^2 + y^2 = a^2$.

Solution:

Area of circle = 4 × Area in first quadrant

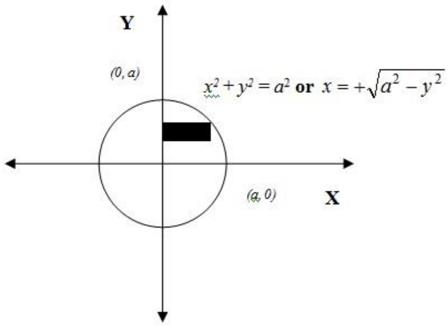
$$= 4\int_0^{a}\int_0^{\sqrt{a^2-y^2}} dx\,dy$$

$$= 4\int_0^{a} (x)_0^{\sqrt{a^2-y^2}}\,dy$$

$$= 4\int_0^{a} \sqrt{a^2-y^2}\,dy$$

$$= 4\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right]_0^{a}$$

$$= 4\left[\frac{a^2}{2}\,\frac{\pi}{2}\right] = \pi\,a^2$$



18. **Find the area bounded by the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **using double integration.**

Solution: By the symmetry of the curve the area of the ellipse is

Area = 4 Area in the first quardrant

$$= 4\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dydx$$

$$= 4\int_0^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\ dx$$

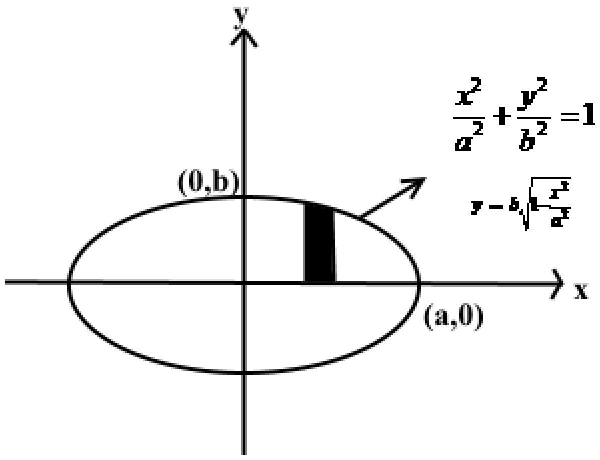
$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{\cancel{4}b}{a}\left[\frac{a^2}{\cancel{2}}\sin^{-1}\right]$$

$$= \cancel{2}ab\left(\frac{\pi}{\cancel{2}}\right)$$

$$= \pi ab.$$



19. **Find the double integration the area by the curves** $y^2 = 4ax$ **and** $x^2 = 4ay$ **.**

Solution:

The area is closed by the parabola

$y^2 = 4ax$ --------(1) and $x^2 = 4ay$ --------(2)

To find the limits solve (1) and (2)

$(2) \Rightarrow\ y = \frac{x^2}{4a}$

sub in (1)

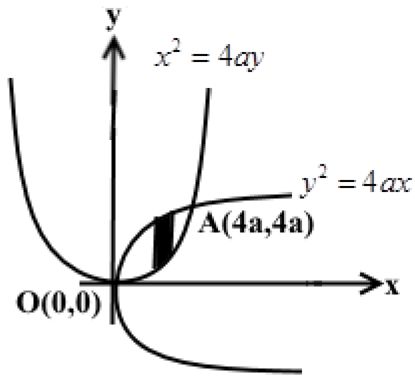
$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$x^4 = 64a^3$$

$$(x^4 - 64a^3) = 0$$

$x = 0$ **or** $(x^3 - 64a^3) = 0$

$x = 0$ **or** $x^3 = 64a^3 \Rightarrow x = 4a$



$$\therefore \textbf{Area} = \int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dydx = \int_0^{4a}[y]_{\frac{x^2}{4a}}^{\sqrt{4ax}}\,dx = \int_0^{4a}\left[\sqrt{4ax} - \frac{x^2}{4a}\right]dx$$

$$= \int_0^{4a}\left[2\sqrt{a}\,x^{1/2} - \frac{1}{4a}x^2\right]dx = \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

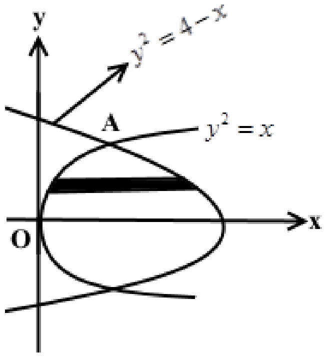
$$= \frac{4\sqrt{a}}{3}(4)^{3/2}(a)^{3/2} - \frac{1}{12a}64a^3 = \frac{4^{5/2}}{3}a^{4/2} - \frac{1}{12a}64a^3$$

$$= \frac{(2^2)^{5/2}}{3}a^2 - \frac{16}{3}a^2 = \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

20. **Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = x$ by double integration.**

Solution:



The area is bounded by

$y^2 = 4 - x$ ---------(1)

$y^2 = x$ ----------(2)

$y^2 = -(x - 4)$ is a parabola with vertex $(4,0)$ and in the direction of negative x-axis both the curves are symmetric about x-axis.

To find the limits solve (1) and (2)

$4 - x = x$

$2x = 4 \Rightarrow x = 2$

$y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

$$\textbf{Area} = 2\int_0^{\sqrt{2}}\int_{y^2}^{4-y^2} dxdy$$

$$= 2\int_0^{\sqrt{2}}[x]_{y^2}^{4-y^2}\,dy$$

$$= 2\int_0^{\sqrt{2}}(4 - y^2 - y^2)\,dy$$

$$= 2\int_0^{\sqrt{2}}(4 - 2y^2)\,dy$$

$$= 2\left[4y - \frac{2y^3}{3}\right]_0^{\sqrt{2}}$$

$$= 2\left[4\sqrt{2} - \frac{2\left(\sqrt{2}\right)^3}{3} - 0\right]$$

$$= 2\left[4\sqrt{2} - \frac{2(2)^{3/2}}{3}\right]$$

$$= 2\left[4\sqrt{2} - \frac{2(2)(2)^{1/2}}{3}\right]$$

$$= 2\left[4\sqrt{2} - \frac{4\sqrt{2}}{3}\right]$$

$$= 2\left(4\sqrt{2}\right)\left[1 - \frac{1}{3}\right]$$

$$= 8\sqrt{2}\left[\frac{2}{3}\right]$$

$$= \frac{16}{3}\sqrt{2}$$

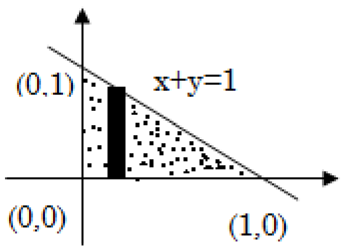
21. **Evaluate $\iint\limits_R (x^2 + y^2)dy\,dx$ over the region R for which** $x, y \geq 0,\ x + y \leq 1.$

Solution:

The region of integration is the triangle bounded by the lines

$x = 0,\ y = 0,\ x + y = 1$

Limits of y : 0 to $1 - x$; Limits of x : 0 to 1



$$\iint\limits_R (x^2 + y^2)dy\,dx = \int_0^1 \int_0^{1-x} \left(x^2 + y^2\right)dydx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{1}{6}$$

Area as a double integral (Polar Coordinates) $\iint\limits_R r\,dr\,d\theta$

22. **Find the area of the cardioid $r = a(1+\cos\theta)$ by using double integration.**

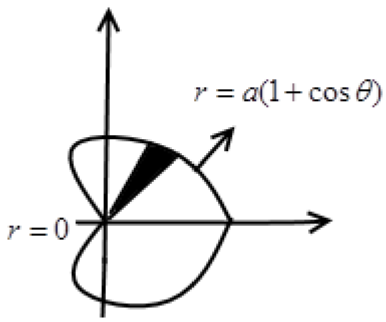
Solution:

Given the curve in polar co ordinates $r = a(1+\cos\theta)$

$\therefore$ **Area of the cardioid = 2(Area above the initial line)**

θ varies from 0 to π

r varies from 0 to $r = a(1+\cos\theta)$



$$\text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left[1+2\cos\theta+\left(\frac{1+\cos\theta}{2}\right)\right]d\theta$$

$$= a^2\int_0^{\pi}\left[\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right]d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\pi} \qquad \because \sin n\pi = 0,\ \forall n$$

$$= a^2\left[\frac{3}{2}\pi\right]$$

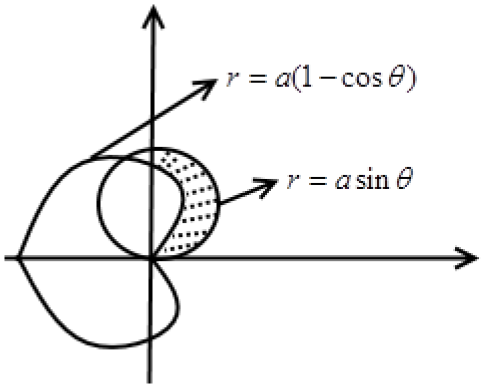
$$= \frac{3\pi a^2}{2}$$

23. **Find the area inside the circle $r = a\sin\theta$ but lying outside the cardioid $r = a(1-\cos\theta)$.**

Solution:

Given $r = a\sin\theta$ -------- (1)

and $r = a(1-\cos\theta)$ ---- (2)



Eliminating r from (1) and (2)

$$a\sin\theta = a(1-\cos\theta)$$

$$\sin\theta + \cos\theta = 1 \text{------}(3)$$

$$(3)^2 \Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$$

$$1 + 2\sin 2\theta = 1$$

$$\sin 2\theta = 0$$

$$2\theta = 0,\ \pi$$

$$\theta = 0,\ \frac{\pi}{2}$$

$$\text{Area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$\text{Area} = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta}$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[a^2\sin^2\theta - a^2(1-\cos\theta)^2\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[\sin^2\theta - \left(1 - 2\cos\theta + \cos^2\theta\right)\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[-1 + 2\cos\theta\right]d\theta \qquad \because \int_0^{\pi/2}\cos^2\theta\,d\theta = \int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\left[-\theta + 2\sin\theta\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\left(-\frac{\pi}{2} + 2\sin\frac{\pi}{2}\right) - 0\right]$$

$$= \frac{a^2}{2}\left(-\frac{\pi}{2} + 2\right) \qquad = \frac{a^2}{4}(4-\pi)$$

24. **Find the area bounded between $r = 2\cos\theta$ and $r = 4\cos\theta$.**

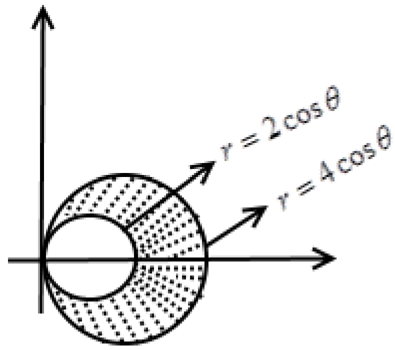
Solution:

$$\text{Area} = \iint_R r\,dr\,d\theta$$

Where the region R is the area between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$

$\therefore$ **r varies from $r = 2\cos\theta$ to $r = 4\cos\theta$**

θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$



$$\text{Area} = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[16\cos^2\theta - 4\cos^2\theta\right]d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 12\cos^2\theta\,d\theta$$

$$= 6\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 6(2)\int_{0}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 6(2)\frac{1}{2}\frac{\pi}{2} \qquad \because \int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}\frac{\pi}{2}$$

$$= 3\pi$$

Conversion from Cartesian to Polar in double integrals

Evaluation of double integrals by changing Cartesian coordinates to polar coordinates:

Changing from (x, y) *to* (r,θ), the variables are related by $x = r\cos\theta, y = r\sin\theta$

and $dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$

$$\therefore \iint f(x,y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Formula

$$\int_0^{\pi/2} \sin^n \theta \, d\theta = \int_0^{\pi/2} \cos^n \theta \, d\theta = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3} \times 1 & \text{if } n \text{ is odd} \\[2ex] \dfrac{(n-1)(n-3)(n-5)\cdots 1}{n(n-2)(n-4)\cdots 2} \times \dfrac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

25. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\, dy$ by changing to polar coordinates. And hence find $\int_0^\infty e^{-x^2} \, dx$

Solution:

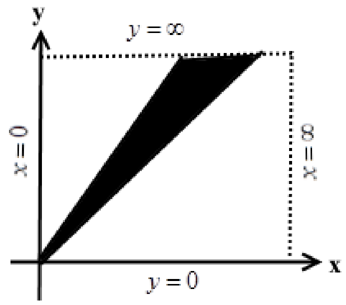
$x = r\cos\theta, y = r\sin\theta \quad and \quad dx\,dy = r\,dr\,d\theta$

r varies from 0 to ∞, θ varies from 0 to $\dfrac{\pi}{2}$

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\, dy$$

$$= \int_0^{\pi/2} \left(\int_0^\infty e^{-r^2} \, r\, dr \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\int_0^\infty e^{-t} \, dt \right) d\theta \quad \because let\, r^2 = t \Rightarrow 2r\,dr = dt \text{ and r: 0 to } \infty \Rightarrow \text{ t: 0 to } \infty$$



$$= \frac{1}{2} \int_0^{\pi/2} \left[-e^{-t} \right]_0^\infty d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[-e^{-\infty} + e^0 \right] d\theta \quad \because e^{-\infty} = 0, e^0 = 1$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{1}{2} [\theta]_0^{\pi/2} = \frac{1}{2} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{4}$$

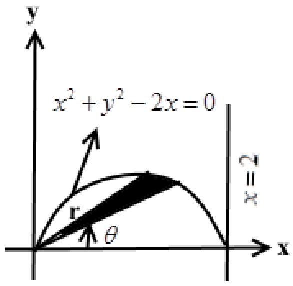
Since $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\, dy = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} \, dy = \dfrac{\pi}{4}$

$$\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-x^2} dx = \frac{\pi}{4}$$

$$\Rightarrow \left(\int_0^\infty e^{-x^2} dx \right)^2 = \frac{\pi}{4} \Rightarrow \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

26. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ by changing to polar coordinates.

Solution:



$x = r\cos\theta, y = r\sin\theta$ *and* $dx\,dy = r\,dr\,d\theta$

The limits of x are x=0 to x=2,

The limits of y are y=0 to y= $\sqrt{2x-x^2}$

$y = 0 \Rightarrow r\cos\theta = 0$

$\Rightarrow r = 0$ *and* $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2$ $\qquad$ $x = 0 \Rightarrow r\sin\theta = 0$

$x^2 + y^2 - 2x = 0$ $\qquad$ $\sin\theta = 0$

$\Rightarrow \theta = 0$

$\Rightarrow r^2 - 2r\cos\theta = 0$

$\Rightarrow r = 2\cos\theta$

r varies from 0 to $2\cos\theta$, θ varies from 0 to $\frac{\pi}{2}$

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4\cos^3\theta\, d\theta = 2\int_0^{\pi/2} \cos^3\theta\, d\theta = 2\left[\frac{2}{3}.1\right] = \frac{4}{3}$$

27. Evaluate $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$ by changing to polar coordinates.

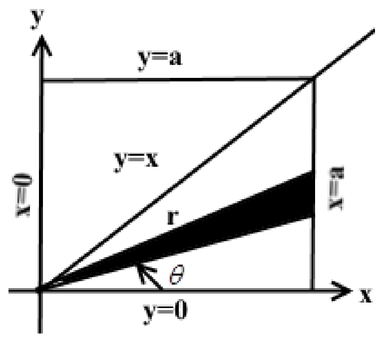
Solution:

$x = r\cos\theta, y = r\sin\theta$ *and* $dx\,dy = r\,dr\,d\theta$

The limits of x are x=y to x=a, The limits of y are y=0 to y=a

$x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$,

$x = a \Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta}$



$y = 0 \Rightarrow r\sin\theta = 0$

$\Rightarrow r = 0 \ and \sin\theta = 0 \Rightarrow \theta = 0$

r varies from 0 to $\frac{a}{\cos\theta}$, θ varies from 0 to $\frac{\pi}{4}$

$$I = \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \int_0^{\pi/4} \int_0^{a/\cos\theta} \frac{r^2\cos^2\theta}{r}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{a/\cos\theta} r^2\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \left[\frac{r^3}{3}\right]_0^{a/\cos\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \cos^2\theta \left[\frac{a^3}{\cos^3\theta} - 0\right] d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} \sec\theta\,d\theta$$

$$= \frac{a^3}{3}\left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4} = \frac{a^3}{3}\left[\log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \log(\sec 0 + \tan 0)\right] = \frac{a^3}{3}\left[\log\left(\sqrt{2}+1\right)\right]$$

28. Evaluate $\int_0^{2a}\int_0^{\sqrt{2x-x^2}} \left(x^2+y^2\right)dy\,dx$ by changing to polar coordinates

Solution:

$x = r\cos\theta, y = r\sin\theta \ \ and \ \ dx\,dy = r\,dr\,d\theta$

The limits of x are x=0 to x=2a, The limits of y are y=0 to y = $\sqrt{2ax-x^2}$

$y = 0 \Rightarrow r\sin\theta = 0$

$\Rightarrow r = 0 \ and \ \sin\theta = 0 \Rightarrow \theta = 0$

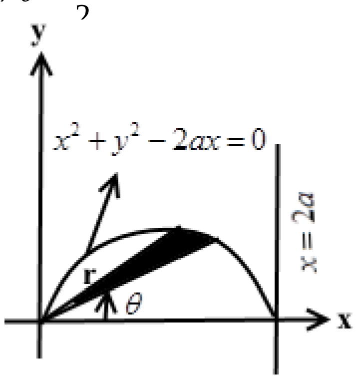
$y = \sqrt{2ax - x^2} \Rightarrow y^2 = 2ax - x^2$ $\qquad$ $x = 0 \Rightarrow r\cos\theta = 0$

$x^2 + y^2 - 2ax = 0$ $\qquad$ $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$\Rightarrow r^2 - 2ar\cos\theta = 0$

$\Rightarrow r = 2a\cos\theta$

r varies from 0 to $2a\cos\theta$, θ varies from 0 to $\frac{\pi}{2}$



$$I=\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\left(r^2\right)r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\left(r^3\right)\,dr\,d\theta$$

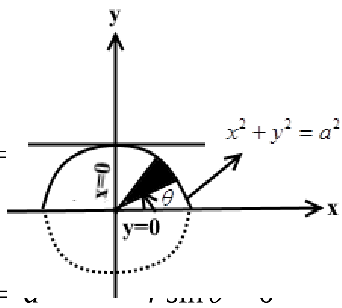
$$=\int_0^{\frac{\pi}{2}}\left(\frac{r^4}{4}\right)_0^{2a\cos\theta}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{\left(2a\cos\theta\right)^4}{4}-0\right)d\theta$$

$$=\frac{16a^4}{4}\int_0^{\frac{\pi}{2}}\left(\cos^4\theta\right)d\theta=4a^4.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}=\frac{3a^4\pi}{4}$$

29. Evaluate $\int_0^{a}\int_0^{\sqrt{a^2-y^2}}\left(x^2+y^2\right)dy\,dx$ by changing to polar coordinates.

Solution:



$x=r\cos\theta, y=r\sin\theta$ *and* $dx\,dy=r\,dr\,d\theta$

The limits of x are x=0 , $x=\sqrt{a^2-y^2}$, and limits of y are y=0, y=

$x=0\Rightarrow r\cos\theta=0$ $\qquad$ $x=\sqrt{a^2-y^2}\Rightarrow x^2+y^2$

$r=0\,\text{and}\cos\theta=0\Rightarrow\theta=\frac{\pi}{2}$ $\qquad$ $r^2=$

$\Rightarrow r=a$ $\qquad$ $\sin\theta=0\Rightarrow\theta=0$

r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$

$$I=\int_0^{a}\int_0^{\sqrt{a^2-y^2}}\left(x^2+y^2\right)dy\,dx$$

$$=\int_0^{\pi/2}\int_0^{a}\left(r^2\right)r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{a}\left(r^3\right)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^4}{4}\right)_0^a d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{a^4}{4}\right) d\theta = \frac{a^4}{4}(\theta)_0^{\pi/2} = \frac{a^4}{4}\frac{\pi}{2} = \frac{a^4\pi}{8}$$

Triple Integration (Cartesian Coordinates)

$$I = \int_{z=z_1}^{z_2} \int_{y=y_1}^{y_2} \int_{x=x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz$$

Also

$$I = \int_{x=x_1}^{x_2} \int_{y=y_1}^{y_2} \int_{z=z_1}^{z_2} f(x,y,z)\, dz\, dy\, dx$$

30. Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz\, dz\, dy\, dx$

Solution:

$$I = \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} xyz\, dz\, dy\, dx = \int_0^1 \int_0^2 \left(\frac{z^2}{2}\right)_0^3 dy\, dx$$

$$= \int_0^1 \int_0^2 \left(\frac{9}{2} - 0\right)_0^3 dy\, dx = \frac{9}{2}\int_0^1 x\left(\frac{y^2}{2}\right)_0^2 dx$$

$$= \frac{9}{2}\int_0^1 x\left(\frac{4}{2} - 0\right) dx = \frac{9}{2}\int_0^1 2x\, dx = 9\int_0^1 x\, dx = 9\left(\frac{x^2}{2}\right)_0^1 = 9\left(\frac{1}{2} - 0\right) = \frac{9}{2}$$

31. Evaluate $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dxdydz$

Solution:

$$I = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} e^{x+y+z}\, dx\, dy\, dz = \int_0^1\int_0^1 \left[e^{1+y+z} - e^{y+z}\right] dy\, dz$$

$$= \int_0^1 \left(e^{z+2} - 2e^{z+1} + e^z\right) dz$$

$$= e^3 - 3e^2 + 3e - 1$$

$$= (e-1)^3$$

32. **Evaluate** $\int_0^c \int_0^b \int_0^a (x+y+z)\,dx\,dy\,dz$ **.**

Solution:

$$\int_0^c \int_0^b \int_0^a (x+y+z)\,dx\,dy\,dz = \int_0^c \int_0^b \left(\frac{x^2}{2}+xy+xz\right)_0^a dy\,dz$$

$$= \int_0^c \int_0^b \left(\frac{a^2}{2}+ay+az\right) dy\,dz$$

$$= \int_0^c \left(\frac{a^2}{2}y+a\frac{y^2}{2}+azy\right)_0^b dz$$

$$= \int_0^c \left(\frac{a^2}{2}b+a\frac{b^2}{2}+azb\right) dz$$

$$= \left(\frac{a^2}{2}bz+a\frac{b^2}{2}z+ab\frac{z^2}{2}\right)_0^c$$

$$= \frac{abc(a+b+c)}{2}$$

33. **Evaluate** $\int_0^4 \int_0^x \int_0^{\sqrt{x+y}} z\;dx\,dy\,dz$ **.**

Solution:

$$I = \int_{x=0}^{4} \int_{y=0}^{x} \int_{z=0}^{\sqrt{x+y}} z\;dz\,dy\,dx$$

$$= \int_0^4 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dydx$$

$$= \frac{1}{2}\int_0^4 \int_0^x (x+y)dydx$$

$$= \frac{1}{2}\int_0^4 \left(xy+\frac{y^2}{2}\right)_0^x dx \quad = \frac{1}{2}\int_0^4 \left(x^2+\frac{x^2}{2}\right) dx \quad = \frac{3}{4}\int_0^4 x^2 dx = \frac{3}{4}\left(\frac{x^3}{3}\right)_0^4 = 16$$

34. **Evaluate** $\int_0^{\log 2} \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$ **.**

Solution:

$$\int_0^{\log 2} \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx = \int_0^{\log 2} \int_0^x \left(e^z\right)_0^{x+y} e^y\;e^x\;\;dy\,dx$$

$$= \int_0^{\log 2} \int_0^{x} \left(e^{2x}\, e^{2y} - e^{x}\, e^{y}\right) dy\, dx$$

$$= \int_0^{\log 2} \left(e^{2x}\, \frac{e^{2y}}{2} - e^{x}\, e^{y}\right)_0^{x} dx$$

$$= \int_0^{\log 2} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x}\right) dx = \frac{5}{8}$$

35. **Evaluate** $\displaystyle\int_0^{a} \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{a^2-x^2-y^2-z^2}}$

Solution:

Let $\displaystyle I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{a^2-x^2-y^2-z^2}}$

$$= \int_0^{a} \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \int_0^{a} \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\, dx$$

$$= \int_0^{a} \int_0^{\sqrt{a^2-x^2}} \left[\frac{\pi}{2} - 0\right] dy\, dx \;= \frac{\pi}{2} \int_0^{a} [y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \frac{\pi}{2} \int_0^{a} \sqrt{a^2-x^2}\, dx \;= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^{a}$$

$$= \frac{\pi}{2}\left[\left(0 + \frac{a^2}{2}\,\frac{\pi}{2}\right) - (0+0)\right] = \frac{\pi^2 a^2}{8}$$

36. **Evaluate** $\displaystyle\int\int\int \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$ **for all positive values of x,y,z for which the integral is real.**

Solution:

Let $\displaystyle I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$

$$= \int_0^{1} \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^{1} \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\, dx$$

$$=\int_0^1\int_0^{\sqrt{1-x^2}}\left[\frac{\pi}{2}-0\right]dy\,dx \quad =\frac{\pi}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}}\,dx$$

$$=\frac{\pi}{2}\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi^2}{8}$$

37. Evaluate $\iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region of integration bounded by the planes $\boldsymbol{x=0, y=0}$, $\boldsymbol{z=0, x+y+z=1}$

Solution:

Here z varies from $z=0$ to $z=1-x-y$

y varies from $y=0$ to $y=1-x$

x varies from $x=0$ to $x=1$

$$\therefore \iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}=\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}\frac{1}{(x+y+z+1)^3}dz\,dy\,dx$$

$$=\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z+1)^{-3}dz\,dy\,dx$$

$$=\int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y}dy\,dx$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[2^{-2}-(x+y+1)^{-2}\right]dy\,dx$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-(x+y+1)^{-2}\right]dy\,dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}y-\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x}dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}y+(x+y+1)^{-1}\right]_0^{1-x}dx$$

$$=-\frac{1}{2}\int_0^1\left[\left(\frac{1}{4}(1-x)+2^{-1}\right)-\left(0+(x+1)^{-1}\right)\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}-\frac{x}{4}+\frac{1}{2}-\frac{1}{1+x}\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{3}{4}-\frac{x}{4}-\frac{1}{1+x}\right]dx$$

$$=-\frac{1}{2}\left[\frac{3}{4}x-\frac{x^2}{8}-\log(1+x)\right]_0^1$$

$$=-\frac{1}{2}\left[\left(\frac{3}{4}-\frac{1}{8}-\log 2\right)-(0-0-0)\right]$$

$$=\frac{1}{2}\log 2-\frac{5}{16}$$

Volume using Triple Integral

38. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:

Since the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$is symmetric about the coordinate planes.

Volume of ellipsoid = $8\times$ volume in the first octant.

In the first octant,

z varies from 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

x varies from 0 to a

$$\text{volume } =8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ dy\ dx$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\ dy\ dx$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\ dx$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\frac{c}{b}\left(\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\right)dy\ dx$$

$$= \frac{8c}{b}\int_0^a \left[\left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right)\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} \right] dy\ dx$$

$$= \frac{4c}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)\left(\sin^{-1}1-\sin^{-1}0\right)dx$$

$$= \frac{4c}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}dx$$

$$= 2\pi bc\int_0^a \left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc\left[x-\frac{1}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= 2\pi bc\left[a-\frac{a^3}{3a^2}-0\right] = 2\pi bc\left(a-\frac{a}{3}\right) = 2\pi bc\left(\frac{2a}{3}\right) = \frac{4}{3}\pi abc$$

39. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate's planes.

Solution:

The region of integration is the region bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,\ x=0,\ y=0, z=0$

z varies from 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y varies from 0 to $b\left(1-\frac{x}{a}\right)$

x varies from 0 to a

$$\text{volume} = \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\ dy\ dx$$

$$= \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} (z)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\,dx$$

$$= \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right) b\left(1-\frac{x}{a}\right) - \frac{1}{2b} b^2 \left(1-\frac{x}{a}\right)^2 \right] dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right)^2 - \frac{b}{2}\left(1-\frac{x}{a}\right)^2 \right] dx$$

$$= c\int_0^a \left[\frac{b}{2}\left(1-\frac{x}{a}\right)^2 \right] dx$$

$$= \frac{bc}{2}\int_0^a \left[\left(1-\frac{x}{a}\right)^2 \right] dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{-3/a} \right]_0^a = \frac{-abc}{6}[0-1] = \frac{abc}{6}$$

40. Find the volume of sphere $x^2+y^2+z^2=a^2$ using triple integrals.

Solution:

Since the sphere $x^2+y^2+z^2=a^2$ is symmetric about the coordinate plane

Volume of sphere = $8 \times$ volume in the first octant.

Int the first octant,

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a

Volume of sphere = $= 8\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}}\, dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 8\int_0^a \left[\frac{y\sqrt{a^2-x^2-y^2}}{2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right) \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left[0 - \frac{a^2-x^2}{2}\sin^{-1}(1) \right] - \left[0 - \frac{a^2-x^2}{2}\sin^{-1}(0) \right] dx$$

$$= 8\int_0^a \left(\frac{a^2-x^2}{2} \right)\left[\sin^{-1}(1) - \sin^{-1}(0) \right] dx$$

$$= 4\int_0^a \left(a^2 - x^2\right)\left[\frac{\pi}{2} - 0\right] dx$$

$$= 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a = 2\pi\left(a^3 - \frac{a^3}{3}\right) = 2\pi\left(\frac{2a^3}{3}\right) = \frac{4\pi a^3}{3}$$

* * * * *

Module – 2 Vector Calculus

Review of vectors in 2, 3 dimensions – Gradient, divergence, curl – Solenoidal, Irrotational fields – Vector identities (without proof) – Directional derivatives – Line integrals, Surface integrals, Volume integrals – Green's theorem (without proof) – Gauss divergence theorem (without proof), Verification, Applications to Cubes, parallelopiped only – Stoke's theorem (without proof) – Verification, Applications to Cubes, parallelopiped only – Applications of Line and Volume integrals in Engineering.

Basic Formulae

1. $\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$

2. $\nabla\varphi = grad\ \varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$

3. Directional derivative = $\nabla\varphi \bullet \frac{\vec{a}}{|\vec{a}|}$

4. Normal derivative = $|\nabla\varphi|$

5. Unit normal vector $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$

6. Angle between the surfaces $\cos\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

7. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

Differentiate partially w.r.t. *x*

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiate partially w.r.t. *y* $\frac{\partial r}{\partial y} = \frac{y}{r}$

Differentiate partially w.r.t. z $\frac{\partial r}{\partial z} = \frac{z}{r}$

1. **Find $\nabla\phi$ if $\phi = \log\left(x^2+y^2+z^2\right)$.**

Solution:

$$\nabla\phi = \vec{i}\ \frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right) + \vec{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$$

$$= \vec{i}\frac{2x}{(x^2+y^2+z^2)} + \vec{j}\frac{2y}{(x^2+y^2+z^2)} + \vec{k}\frac{2z}{(x^2+y^2+z^2)}$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{i}+y\vec{j}+z\vec{k}\right) = \frac{2\vec{r}}{r^2} \qquad \because (\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}\ \&\ r^2 = x^2+y^2+z^2)$$

2. **Find $\nabla\phi$ if $\phi = x\,y\,z$ at the point (1, 2, 3).**

Solution:

$$\nabla\varphi = grad\ \varphi = \vec{i}\ \frac{\partial\varphi}{\partial x} + \vec{j}\ \frac{\partial\varphi}{\partial y} + \vec{k}\ \frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = \vec{i}\ y\,z + \vec{j}\ x\,z + \vec{k}\ x\,y$$

$$\nabla\varphi \text{ at } (1,2,3) = 6\vec{i} + 3\vec{j} + 2\vec{k}$$

3. **Find ∇r .**

Solution:

$$\nabla r = \vec{i}\ \frac{\partial r}{\partial x} + \vec{j}\ \frac{\partial r}{\partial y} + \vec{k}\ \frac{\partial r}{\partial z}$$

$$\nabla r = \vec{i}\ \frac{x}{r} + \vec{j}\ \frac{y}{r} + \vec{k}\ \frac{z}{r} = \frac{\vec{r}}{r}$$

4. **Find the unit normal vector to the surface $x^2 + xy + z^2 = 4$ at the point (1, –1, 2).**

Solution:

Let $\varphi = x^2 + xy + z^2 - 4$

$$\nabla\varphi = grad\ \varphi = \vec{i}\ \frac{\partial\varphi}{\partial x} + \vec{j}\ \frac{\partial\varphi}{\partial y} + \vec{k}\ \frac{\partial\varphi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x + y, \qquad \frac{\partial\phi}{\partial y} = x, \qquad \frac{\partial\phi}{\partial z} = 2z$$

$$[\nabla\phi]_{(1,-1,2)} = \left[(2x+y)\vec{i} + x\vec{j} + 2z\vec{k}\right]_{(1,-1,2)} = \vec{i} + \vec{j} + 4\vec{k}$$

The unit normal vector is

$$\therefore \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{1^2+1^2+4^2}} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{18}}.$$

5. **Find the unit normal vector to the surface $x^2 + y^2 + z^2 = 1$ at the point (1, 1, 1).**

Ans $\hat{n} = \dfrac{i+j+k}{\sqrt{3}}$

6. **Find the directional derivative of $\phi = 3x^2 + 2y - 3z$ at (1, 1, 1) in the direction $2\vec{i} + 2\vec{j} - \vec{k}$.**

Solution: The gradient of ϕ is $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = 6x, \qquad \frac{\partial\phi}{\partial y} = 2, \qquad \frac{\partial\phi}{\partial z} = -3$$

$$\nabla\phi = 6x\,i + 2\,j - 3k$$

Directional derivative of ϕ is

$$\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$|\vec{a}| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3$$

$$\nabla\phi.\frac{\vec{a}}{|\vec{a}|} = \left[(6x\vec{i} + 2\vec{j} - 3\vec{k})\cdot\left(\frac{2\vec{i}+2\vec{j}-\vec{k}}{3}\right)\right]_{(1,1,1)} = \frac{19}{3}$$

7. **Find the directional derivative of $\phi = 2xy + z^2$ at (1, – 1 , 3) in the direction $i + 2j + 2k$.**

Ans $\dfrac{14}{3}$

8. **Find the directional derivative of $\phi = x^2 + y^2 + 4\,x\,y\,z$ at (1, – 2 , 2) in the direction $2i - 2j + k$.**

Ans $-\dfrac{44}{3}$

9. **Find the directional derivative of $\phi = x^2 - y^2 + 2\,z^2$ at P (1, 2 , 3) in the direction of line PQ where Q is (5, 0, 4).**

Solution:

$$\nabla\varphi = grad\ \varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$\nabla\varphi = grad\ \varphi = \vec{i}\ 2x + \vec{j}\,(-2y) + \vec{k}\ 4z$

$\nabla\varphi$ at $(1, 2, 3) = 2\vec{i} - 4\vec{j} + 12\vec{k}$

$\vec{a} = OQ - OP = (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$

Directional derivative = $\nabla\varphi \bullet \dfrac{\vec{a}}{|\vec{a}|}$

$$= (2\vec{i} - 4\vec{j} + 12\vec{k}) \bullet \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

10. **In what direction from (3, 1, –2) is the directional derivative of $\phi = x^2y^2z^4$ a maximum? Find the magnitude of this maximum.**

Solution: Given $\phi = x^2y^2z^4$

$\dfrac{\partial\phi}{\partial x} = 2xy^2z^4, \quad \dfrac{\partial\phi}{\partial y} = 2x^2yz^4, \quad \dfrac{\partial\phi}{\partial z} = 4x^2y^2z^3$

$\nabla\phi = (2xy^2z^4)\vec{i} + (2x^2yz^4)\vec{j} + (4x^2y^2z^3)\vec{k}$

$[\nabla\phi]_{(3,\,1,\,-2)} = 96\vec{i} + 288\vec{j} - 288\vec{k} = 96(\vec{i} + 3\vec{j} - 3\vec{k})$

$\therefore$ The maximum directional derivative occurs in the direction of $\nabla\phi = 96(\vec{i} + 3\vec{j} - 3\vec{k})$

The magnitude of this maximum directional derivative is

$|\nabla\phi| = 96\sqrt{1^2 + 3^2 + (-3)^2} = 96\sqrt{1+9+9} = 96\sqrt{19}.$

11. **In what direction from (1, 1, –2) is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ a maximum? Find the magnitude of this maximum.**

Ans Directional derivative is maximum in the direction of $2\vec{i} - 4\vec{j} - 16\vec{k}$

Maximum directional derivative = $\sqrt{276}$

12. **Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1).**

Solution: Let $\phi_1 = y^2 - x\log z - 1$

$\dfrac{\partial\phi}{\partial x} = -\log z, \quad \dfrac{\partial\phi}{\partial y} = 2y, \quad \dfrac{\partial\phi}{\partial z} = -\dfrac{x}{z}$

$\nabla\phi_1 = -\log z\ \vec{i} + 2y\vec{j} - \dfrac{x}{z}\vec{k}$, $(\nabla\phi_1)_{(1,1,1)} = 2\vec{j} - \vec{k}$ and $|\nabla\phi_1| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

Let $\phi_2 = x^2y - 2 + z$

$$\frac{\partial\phi}{\partial x} = 2xy\ , \qquad \frac{\partial\phi}{\partial y} = x^2, \qquad \frac{\partial\phi}{\partial z} = 1$$

$\nabla\phi_2 = (2xy)\vec{i} + x^2\vec{j} + (1)\vec{k}$, $(\nabla\phi_2)_{(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}$ and $|\nabla\phi_2| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$

$$cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{\left(2\vec{j}-\vec{k}\right).\left(2\vec{i}+\vec{j}+\vec{k}\right)}{\left(\sqrt{5}\right)\left(\sqrt{6}\right)} = \frac{0+2-1}{\sqrt{30}} \Rightarrow \theta = cos^{-1}\left(\frac{1}{\sqrt{30}}\right).$$

13. **Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point (2, – 1 , 2).**

Ans $\cos\theta = \frac{8}{3\sqrt{21}}$

14. **Find the angle between the normals to the surface $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).**

Solution:

Given $\varphi = x^2 - yz$

$$\nabla\varphi = 2x\vec{i} - z\vec{j} - y\vec{k}$$

$\nabla\varphi_1 /(1,1,1) = 2\vec{i} - \vec{j} - \vec{k}$ $\qquad$ $\nabla\varphi_2 /(2,4,1) = 4\vec{i} - \vec{j} - 4\vec{k}$

$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6}$ $\qquad$ $|\nabla\varphi_2| == \sqrt{16+1+16} = \sqrt{33}$

$$\cos\theta = \frac{\nabla\varphi_1 \circ \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|} = \frac{(2\vec{i}-\vec{j}-\vec{k})\circ(4\vec{i}-\vec{j}-4\vec{k})}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}$$

15. **Find 'a' and 'b' so that the surfaces $\mathbf{ax^3 - by^2z = (a+3)x^2}$ and $\mathbf{4x^2y - z^3 = 11}$ cut orthogonally at (2, –1, –3).**

Solution:

$Let\ \phi_1 = ax^3 - by^2z - (a+3)x^2$

$$\frac{\partial\phi}{\partial x} = 3ax^2 - (a+3)2x, \qquad \frac{\partial\phi}{\partial y} = -2byz, \qquad \frac{\partial\phi}{\partial z} = -by^2$$

$$\therefore \nabla\phi_1 = [3ax^2 - (a+3)2x]\vec{i} - 2byz\vec{j} - by^2\vec{k}$$

$At\ (2,-1,-3)\ \nabla\phi_1 = (8a-12)\vec{i} - 6b\vec{j} - b\vec{k}$

$Let\ \phi_2 = 4x^2y - z^3 - 11$

$\frac{\partial\phi}{\partial x} = 8xy, \quad \frac{\partial\phi}{\partial y} = -4x^2, \quad \frac{\partial\phi}{\partial z} = -3z^2$

$\therefore \nabla\phi_2 = 8xy\vec{i} - 4x^2\vec{j} - 3z^2\vec{k}$

$At\ (2,-1,-3)\ \nabla\phi_2 = 16\vec{i} - 16\vec{j} - 27\vec{k}$

Since the surfaces cut orthogonally at $(2,-1,-3)$,

$\nabla\phi_1.\nabla\phi_2 = 0$

$\Rightarrow -16(8a-12) - 16(6b) + 27b = 0$

$\Rightarrow -128a + 192 - 69b = 0$

$\Rightarrow 128a + 69b = 192 \qquad \rightarrow (1)$

*Since the po*int *s* $(2,-1,-3)$ *lies on the surface* $\phi_1(x,y,z) = 0$, *we have*

$8a + 3b - 4a = 12$

$\Rightarrow 4a + 3b = 12 \qquad \rightarrow (2)$

Solving $(1)\,\&\,(2)$ *we get* $a = -2.333 \quad b = 7.111$

16. **Find a and b such that the surfaces ax^2 – byz = (a+2)x and 4x^{2}y+z^3=4 cut orthogonally at (1, –1, 2).**

Solution:

Let $\phi_1 = ax^2 - byz - (a+2)x$

$\nabla\varphi_1 = \vec{i}[2ax-(a-2)] + \vec{j}(-bz) + \vec{k}(-by)$

$\nabla\varphi_1$ **at (1, –1, 2)** $= \vec{i}[a-2] - 4b\vec{j} + b\vec{k}$

$|\nabla\varphi_1| = \sqrt{(a-2)^2 + 17b^2}$

$\phi_2 = 4x^2y + 2^3 - 4$

$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$

$\underset{(1,-1,2)}{\nabla\phi_2} = -8\vec{i} + 4\vec{j} + 12\vec{k}$

$|\nabla\varphi_2| = \sqrt{64+16+144} = \sqrt{224}$

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{-8(a-2)-16b+12b}{\sqrt{(a-2)^2+17b^2}\sqrt{224}}$$

Given $\theta = 90^0$, $\cos 90^0 = 0$

$$\therefore 0 = \frac{-8a+16-16b+12b}{\sqrt{(a-2)^2+17b^2}\sqrt{224}}$$

$$= -8a+16-16b+12b = 0$$

$$= 2a+b-4 = 0 \qquad \dots (1)$$

Since the point (1,-1,2) lies on the surface $\phi_1(x,y,z) = 0$,

$a - 2b - (a+2) = 0$

$b = -1$

$\therefore (1) \Rightarrow 2a + (-1)-4 = 0 \qquad a = \frac{5}{2}$

17. **Find** $\nabla(r^n)$

Solution:

$$\nabla\left(r^n\right) = \vec{i}\ \frac{\partial\left(r^n\right)}{\partial x} + \vec{j}\ \frac{\partial\left(r^n\right)}{\partial y} + \vec{k}\ \frac{\partial\left(r^n\right)}{\partial z}$$

$$= \vec{i}\ \ nr^{n-1}\frac{x}{r} + \vec{j}\ \ nr^{n-1}\frac{y}{r} + \vec{k}\ \ nr^{n-1}\frac{z}{r}$$

$$= \vec{i}\, nr^{n-2}x + \vec{j}\, nr^{n-2}y + \vec{k}\, nr^{n-2}z$$

$$= nr^{n-2}(x\vec{i} + y\vec{j} + z\vec{k}) \ (\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k})$$

$$\therefore \nabla\left(r^n\right) = nr^{n-2}\vec{r}.$$

DIVERGENCE, CURL, SOLENOIDAL, IRROTATIONAL

Let $\vec{F} = F_1\,\vec{i} + F_2\,\vec{j} + F_3\,\vec{k}$

1. $div\,\vec{F} = \nabla \bullet \vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

2. $Curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

3. Solenoidal $\nabla \bullet \vec{F} = 0$

4. Irrotation al $\nabla \times \vec{F} = \vec{0}$

18. Find $curl\vec{F}$ if $\vec{F} = xy\vec{\imath} + yz\vec{\jmath} + zx\vec{k}$.

Solution:

Given $\vec{F} = xy\vec{\imath} + yz\vec{\jmath} + zx\vec{k}$

$$curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{\imath}\,(0 - y) - \vec{\jmath}(z - 0) + \vec{k}\,(0 - x)$$

$$= -y\,\vec{\imath} - z\,\vec{\jmath} - x\,\vec{k}$$

19. **Find 'a', such that $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.**

Solution: We know that $\vec{F}$ is Solenoidal if $div\,\vec{F} = 0$ *or* $\nabla.\vec{F} = 0$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).[(3x-2y+z)\,\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}\,] = 0$$

$$\frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$\Rightarrow$ 3 + a + 2 =0

$\Rightarrow$ 5 + a = 0 $\therefore$ a = –5.

20. **Find the constant a, b, c so that** $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ **is irrotational.**

Solution:

Given $\vec{F}$ is irrotational **i.e.,** $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\vec{i}\left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right)$$
$$+\vec{k}\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right) = \vec{0}$$

$$= i.e., \quad \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$$
$$= \therefore c+1 = 0, 4-a = 0, and\ b-2 = 0$$
$$\Rightarrow a = 4, b = 2, c = -1$$

21. **Find the constant a, b, c so that** $\vec{F} = (a\,x\,y + b\,z^3)\vec{i} + (3x^2 - c\,z)\vec{j} + (3\,x\,z^2 - y)\vec{k}$ **is irrotational.**

Ans $\boldsymbol{a}$ **= 6,** $\boldsymbol{b}$ **= 1,** $\boldsymbol{c}$ **= 1**

22. **Prove that** $r^n\,\vec{r}$ **is an irrotational vector for any value of 'n' but is solenoidal only if n = – 3 .**

Solution:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$r = |r| = \sqrt{x^2+y^2+z^2}$$
$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similarly

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$
$$\frac{\partial r}{\partial z} = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

$$r^n = \left(x^2+y^2+z^2\right)^{n/2}$$
$$r^n r = r^n\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\nabla\times\left(r^n r\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ r^n x & r^n y & r^n z\end{vmatrix}$$

$$=\vec{i}\left(\frac{\partial}{\partial y}(r^n z)-\frac{\partial}{\partial z}\left(r^n y\right)\right)-\vec{j}\left(\frac{\partial}{\partial x}\left(r^n z\right)-\frac{\partial}{\partial z}\left(r^n x\right)\right)+\vec{k}\left(\frac{\partial}{\partial x}\left(r^n y\right)-\frac{\partial}{\partial y}\left(r^n x\right)\right)$$

$$=\vec{i}\left(znr^{n-1}\frac{\partial r}{\partial y}-ynr^{n-1}\frac{\partial r}{\partial z}\right)-\vec{j}\left(znr^{n-1}\frac{\partial r}{\partial x}-xnr^{n-1}\frac{\partial r}{\partial z}\right)+\vec{k}\left(ynr^{n-1}\frac{\partial r}{\partial x}-xnr^{n-1}\frac{\partial r}{\partial y}\right)$$

$$=\vec{i}\left(znr^{n-1}\frac{y}{r}-ynr^{n-1}\frac{z}{r}\right)-\vec{j}\left(znr^{n-1}\frac{x}{r}-xnr^{n-1}\frac{z}{r}\right)+\vec{k}\left(ynr^{n-1}\frac{x}{r}-xnr^{n-1}\frac{y}{r}\right)$$

$$=0\vec{i}+0\vec{j}+0\vec{k}=0$$

$\therefore$ ***r^n r*** is irrotational for all values of n.

$$div\left(r^n\,\vec{r}\right)=\nabla\bullet\left(r^n\,\vec{r}\right)=\left(\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}\right)\left(r^n\,(x\vec{i}+y\,\vec{j}+z\,\vec{k})\right)$$

$$=\frac{\partial}{\partial x}\left(r^n\,x\right)+\frac{\partial}{\partial y}\left(r^n\,y\right)+\frac{\partial}{\partial z}\left(r^n\,z\right)$$

$$=r^n+xnr^{n-1}\frac{\partial r}{\partial x}+r^n+ynr^{n-1}\frac{\partial r}{\partial y}+r^n+znr^{n-1}\frac{\partial r}{\partial z}$$

$$=3r^n+nr^{n-2}(x^2+y^2+z^2)=3r^n+nr^{n-2}(r^2)=3r^n+nr^n=(3+n)r^n$$

If n = -3 then $\nabla\bullet\left(r^n\,\vec{r}\right)=0$.

$\therefore$ ***r^n r*** is solenoidal only if n = – 3.

23. **If $\vec{F}=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$, then find div curl $\vec{F}$.**

Solution: $div\ curl\,\vec{F}=\nabla.\,(\nabla\times\vec{F})$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^3 & y^3 & z^3\end{vmatrix}$$

$$=\vec{i}(0-0)-\vec{j}(0-0)+\vec{k}(0-0)=\vec{0}$$

$$\nabla\times\vec{F}=\vec{0}$$

$$\therefore \nabla.(\nabla\times\vec{F})=0$$

24. **If $\nabla\phi=(y^2-2xyz^3)\vec{i}+(3+2xy-x^2z^3)\vec{j}+(6z^3-3x^2yz^2)\vec{k}$ find ϕ.**

Solution:

$$\nabla\varphi=\vec{i}\,\frac{\partial\varphi}{\partial x}+\vec{j}\,\frac{\partial\varphi}{\partial y}+\vec{k}\,\frac{\partial\varphi}{\partial z} \qquad \text{(1)}$$

$$\nabla\varphi=\left(y^2-2xyz^3\right)\vec{i}+\left(3+2xy-x^2z^3\right)\vec{j}+\left(6z^3-3x^2yz^2\right)\vec{k} \qquad \text{(2)}$$

Comparing (1) and (2)

$$\frac{\partial\varphi}{\partial x}=y^2-2xyz^3 \qquad \text{(3)}$$

$$\frac{\partial\varphi}{\partial y}=3+2xy-x^2z^3 \qquad \text{(4)}$$

$$\frac{\partial\varphi}{\partial z}=6z^3-3x^2yz^2 \qquad \text{(5)}$$

Integrating (3) w.r.t. *x* (keeping *y* and *z* as constant)

$$\varphi=y^2x-x^2yz^3+f_1(y,z)$$

Integrating (4) w.r.t. *y* (keeping *x* and *z* as constant)

$$\varphi=3y+xy^2-x^2yz^3+f_2(x,z)$$

Integrating (5) w.r.t. *z* (keeping *x* and *y* as constant)

$$\varphi=\frac{3}{2}z^4-x^2yz^3+f_3(x,y)$$

Hence $\varphi=y^2x-x^2yz^3+3y+\frac{3}{2}z^4+c$ where c is a constant, $c=f_1(y,z)+f_2(x,z)+f_3(x,y)$

25. **If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find $\phi(x,y,z)$ given that $\phi(1,-2,2) = 4$.**

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z} \qquad \rightarrow (1)$$

Given $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k} \qquad \rightarrow (2)$

$\therefore$*comparing* (1) & (2)

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \qquad \rightarrow (3)$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \qquad \rightarrow (4)$$

$$\frac{\partial\varphi}{\partial z} = 3x^2\,y\,z^2 \qquad \rightarrow \mathbf{(5)}$$

Integrating (3) w.r.t. *x* (keeping *y* and *z* as constant)

$\varphi = x^2\,y\,z^3 + f_1(y,z)$

Integrating (4) w.r.t. *y* (keeping *x* and *z* as constant)

$\varphi = x^2\,y\,z^3 + f_2(x,z)$

Integrating (5) w.r.t. *z* (keeping *x* and *y* as constant)

$\varphi = x^2\,y\,z^3 + f_3(x,y)$

Hence $\varphi = x^2\,y\,z^3 + c$ where c is a constant, $c = f_1(y,z) + f_2(x,z) + f_3(x,y)$

Given $\varphi(1,-2,2) = 4$

$\varphi(1,-2,2) = -16 + c = 4$

$c = 20$

Hence $\varphi = x^2\,y\,z^3 + 20$

26. **Show that the vector $\vec{F}=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$ is irrotational and find the scalar potential function.**

Solution:

$curl\vec{F}=\nabla\times\vec{F}=\vec{0}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2\end{vmatrix}$$

$$=\vec{i}\left(\frac{\partial}{\partial y}\left(3xz^2\right)-\frac{\partial}{\partial z}(2y\sin x-4)\right)-\vec{j}\left(\frac{\partial}{\partial x}\left(3xz^2\right)-\frac{\partial}{\partial z}\left(y^2\cos x+z^3\right)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial y}\left(y^2\cos x+z^3\right)-\frac{\partial}{\partial x}(2y\sin x-4)\right)$$

$$=\vec{i}\,(0-0)-\vec{j}\left(3z^2-3z^2\right)+\vec{k}\,(2y\cos x-2y\cos x)=\vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find Scalar potential ϕ we assume $\vec{F}=\nabla\phi$

$$\vec{F}=\nabla\phi=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$$

$$\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\ \frac{\partial\phi}{\partial y}+\vec{k}\ \frac{\partial\phi}{\partial z}\right)=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+3xz^2\vec{k}$$

comparing coefficient of $\vec{i},\vec{j}\ \&\ \vec{k}$

$$\frac{\partial\phi}{\partial x}=y^2\cos x+z^3 \qquad \rightarrow(1)$$

$$\frac{\partial\phi}{\partial y}=2y\sin x-4 \qquad \rightarrow(2)$$

$$\frac{\partial\phi}{\partial z}=3xz^2 \qquad \rightarrow(3)$$

Integrating (1) w.r.t. *x* (keeping *y* and *z* as constant)

$$\varphi=y^2(\sin x)+xz^3+f_1(y,z)$$

Integrating (2) w.r.t. *y* (keeping *x* and *z* as constant)

$$\varphi=y^2\sin x-4y+f_2(x,z)$$

Integrating (3) w.r.t. *z* (keeping *x* and *y* as constant)

$$\varphi=xz^3+f_3(x,y)$$

Hence $\varphi = y^2 \sin x + x z^3 - 4y + c$ where c is a constant, $c = f_1(y,z) + f_2(x,z) + f_3(x,y)$

27. **Show that the vector $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$ is irrotational and find the scalar potential function.**

Solution:

Given $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$

$curl\vec{F} = \nabla \times \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find scalar potential ϕ we assume $\vec{F} = \nabla\phi$

$$\vec{F} = \nabla\phi = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$$

$$\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$$

comparing coefficient of $\vec{i}, \vec{j}$ & $\vec{k}$

$$\frac{\partial\phi}{\partial x} = \left(6xy + z^3\right) \qquad \rightarrow (1)$$

$$\frac{\partial\phi}{\partial y} = \left(3x^2 - z\right) \qquad \rightarrow (2)$$

$$\frac{\partial\phi}{\partial z} = \left(3xz^2 - y\right) \qquad \rightarrow (3)$$

Integrating (1) w.r.t. *x* (keeping *y* and *z* as constant)

$\varphi = 3x^2 y + x z^3 + f_1(y,z)$

Integrating (2) w.r.t. *y* (keeping *x* and *z* as constant)

$\varphi = 3x^2 y - y z + f_2(x,z)$

Integrating (3) w.r.t. *z* (keeping *x* and *y* as constant)

$\varphi = x z^3 - y z + f_3(x,y)$

Hence $\varphi = 3x^2 y + x z^3 - y z + c$ where c is a constant, $c = f_1(y,z) + f_2(x,z) + f_3(x,y)$

28. If $\vec{A}$ and $\vec{B}$ are irrotational, then prove that $\vec{A}\times\vec{B}$ is solenoidal.

Solution:

$\vec{A}$ and $\vec{B}$ are irrotational.

$\therefore \nabla\times\vec{A}=\vec{0}$ and $\nabla\times\vec{B}=\vec{0}$

Now $\nabla\bullet\left(\vec{A}\times\vec{B}\right)=\left(\nabla\times\vec{A}\right)\bullet\vec{B}-\left(\nabla\times\vec{B}\right)\bullet\vec{A}=0-0=0$

$\therefore \vec{A}\times\vec{B}$ is solenoidal.

29. Prove that (i) $div\,\vec{r}=3$ (ii) $curl\,\vec{r}=0$.

Solution:

$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

$$div\,\vec{r}=\nabla\bullet\vec{r}=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\bullet\left(x\vec{i}+y\vec{j}+z\vec{k}\right)$$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3$$

$$Curl\,\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0\vec{i}+0\vec{j}+0\vec{k}$$

30. If $\phi=x^2-y^2$, then prove that $\nabla^2\varphi=0$.

Solution:

$\nabla^2\varphi=\nabla\bullet\nabla\varphi$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\bullet\left(\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}(2x)+\frac{\partial}{\partial y}(-2y)+\frac{\partial}{\partial z}(0)=2-2=0$$

31. Prove that $curl\,(grad\,\varphi) = 0$.

Solution:

$$Curl\,(grad\,\varphi) = \nabla \times \nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right) - \vec{j}\left(\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial x}\right) + \vec{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k}$$ (Since mixed partial derivatives are equal.)

32. State Green's Theorem.

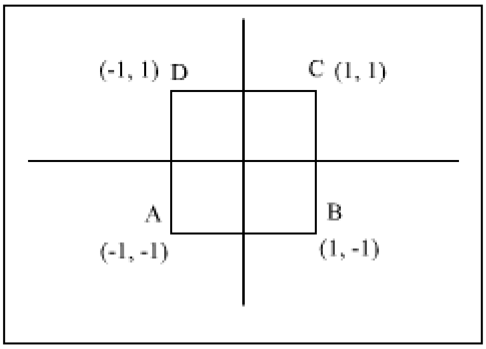
Statement: If $P(x, y)$ and $Q(x, y)$ are continuous functions of x, y with continuous partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ in a region R of the xy plane bounded by a simple closed curve C, then $\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ where C is the curve traversed in the counter clockwise direction.

33. Verify Green's theorem for $\int_c \left[x^2(1+y)dx + (x^3+y^3)dy\right]$ where C is the boundary of the region defined by the lines x = ± 1 and y = ± 1.

Solution:

By Green's theorem

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



Given $\int_c x^2(1+y)dx + (y^3 + x^3)dy$

$P = x^2(1+y)$ $\qquad$ $Q = y^3 + x^3$

$\frac{\partial P}{\partial y} = x^2$ $\qquad$ $\frac{\partial Q}{\partial x} = 3x^2$

Consider

$$\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$=\int_{-1}^{1}\int_{-1}^{1}(3x^2-x^2)dydx$$

$$=\int_{-1}^{1}\int_{-1}^{1}(2x^2)dydx$$

$$=\int_{-1}^{1}2\left[\frac{x^3}{3}\right]_{-1}^{1}dy$$

$$=\int_{-1}^{1}\frac{2}{3}\left[1^3-(-1)^3\right]dy=\int_{-1}^{1}\frac{2}{3}2\,dy=\int_{-1}^{1}\left[\frac{4}{3}\right]dy$$

$$=\left[\frac{4}{3}\right][y]_{-1}^{1}=\left[\frac{4}{3}\right]\left[1-(-1)\right]=\frac{8}{3} \qquad \to (1)$$

Consider

$$\int_c Pdx+Qdy=\int_{AB}+\int_{BC}+\int_{CD}+\int_{DA}$$

Along $AB,\ y=-1,\ dy=0$ *and* x *varies from* -1 *to* 1

$$\therefore \int_{AB} Pdx+Qdy=\int_{-1}^{1}x^2(1+y)dx=\int_{-1}^{1}x^2(1-1)dx=0$$

Along $BC,\ x=1,\ dx=0$ *and* y *varies from* -1 *to* 1

$$\therefore \int_{BC} Pdx+Qdy=\int_{-1}^{1}(x^3+y^3)dy=\int_{-1}^{1}(1+y^3)dy$$

$$=\left[y+\frac{y^4}{4}\right]_{-1}^{1}=\left[1+\frac{1}{4}\right]-\left[-1+\frac{1}{4}\right]$$

$$=1+\frac{1}{4}+1-\frac{1}{4}=2$$

Along $CD,\ y=1,\ dy=0$ *and* x *varies from* 1 *to* -1

$$\therefore \int_{CD} Pdx+Qdy=\int_{-1}^{1}x^2(1+y)dx=\int_{-1}^{1}2x^2dx=\left[\frac{2x^3}{3}\right]_{1}^{-1}=\frac{2}{3}\left[(-1)^3-(1)^3\right]=\frac{2}{3}\left[-1-1\right]=-\frac{4}{3}$$

Along $DA,\ x=-1,\ dx=0$ *and* y *varies from* 1 *to* -1

$$\therefore \int_{DA} Pdx+Qdy=\int_{1}^{-1}(x^3+y^3)dy=\int_{1}^{-1}(-1+y^3)dy$$

$$=\left[\frac{y^4}{4}-y\right]_{1}^{-1}=\frac{1}{4}+1-\frac{1}{4}+1=2$$

$$\int_C Pdx + Qdy = 0 + 2 - \frac{4}{3} + 2 = 4 - \frac{4}{3} = \frac{8}{3} \qquad \to (2)$$

$$\therefore (1) = (2)$$

Hence the theorem is verified.

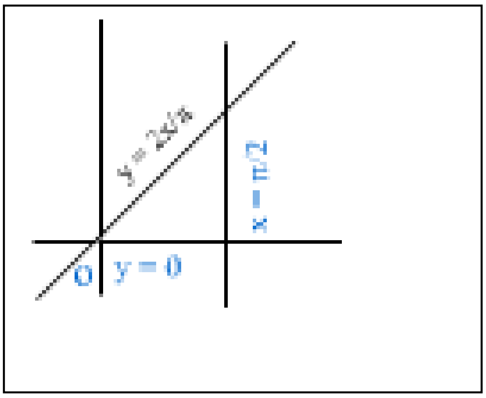
34. **Using Green's theorem , evaluate $\int_C (y - \sin x)dx + \cos x\, dy$ where C is the triangle bounded by the lines y = 0, $x = \frac{\pi}{2}$ and y = $\left(\frac{2}{\pi}\right)$x**

Solution:

Green's theorem states that

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Given $\int_C (y - \sin x)dx + \cos x\, dy$



$$P = y - \sin x \qquad\qquad Q = \cos x$$

$$\frac{\partial P}{\partial y} = 1 \qquad\qquad \frac{\partial Q}{\partial x} = -\sin x$$

$$\int_C (y - \sin x)dx + \cos x\, dy = \iint_R (-\sin x - 1)\, dx\, dy$$

$$\iint_R (-\sin x - 1)\, dx\, dy = \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (-\sin x - 1)\, dx\, dy$$

$$= \int_0^1 [\cos x - x]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$

$$= \int_0^1 \left[\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\frac{\pi y}{2} - \frac{\pi y}{2}\right)\right] dy$$

$$= \left[-\frac{\pi y}{2} - \frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}} + \frac{\pi y^2}{4}\right]_0^1 = \left[-\frac{\pi}{2} - \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} + \frac{\pi}{4}\right] - [0] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

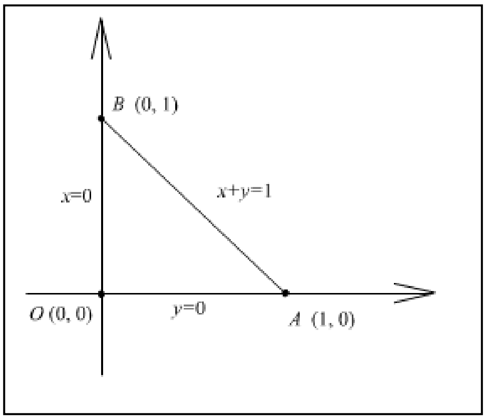
35. **Verify Green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by the lines x = 0, y = 0 and x + y =1.**

Solution:

Green's theorem states that

Given $\int_C \left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy$

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$



$P = 3x^2 - 8y^2$

$$\frac{\partial P}{\partial y} = -16y$$

$Q = 4y - 6xy$

$$\frac{\partial Q}{\partial x} = -6y$$

Evaluation of RHS:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_R \left(-6y + 16y\right)dxdy$$

$$= \int_0^1 \int_0^{1-y} 10y\,dx\,dy = \int_0^1 10y\left[x\right]_0^{1-y} dy$$

$$= \int_0^1 10y(1-y)dy$$

$$= 10\int_0^1 \left(y - y^2\right)dy$$

$$= 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= 10\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{10}{6}$$

$$= \frac{5}{3}$$

Evaluation of LHS:

$$\int_C (Pdx+Qdy) = \int_{OA} (Pdx+Qdy) + \int_{AB} (Pdx+Qdy) + \int_{BO} (Pdx+Qdy)$$

Along OA: $y=0 \Rightarrow dy=0$

$$\int_{OA} Pdx+Qdy = \int_{OA} \left(3x^2\right)dx$$

$$=\left[\frac{3x^3}{3}\right]_0^1 = 1-0=1$$

Along AB:

$$x+y=1 \Rightarrow y=1-x$$
$$\Rightarrow dy=-dx$$

$$\int_{AB} Pdx+Qdy = \int_{AB} \left(3x^2-8y^2\right)dx+\left(4y-6xy\right)dy$$

$$= \int_{AB} [3x^2-8(1-x)^2]dx+[4(1-x)-6x(1-x)](-dx)$$

$$= \int_1^0 \left(-11x^2+26x-12\right)dx$$

$$=\left[\frac{-11x^3}{3}+\frac{26x^2}{2}-12x\right]_1^0 = (0)-(\frac{-11}{3}+\frac{26}{2}-12) = \frac{11}{3}-1=\frac{8}{3}$$

Along BO: $x=0 \Rightarrow dx=0$

$$\int_{BO} Pdx+Qdy = \int_{BO} 4y\,dy$$

$$=\left[\frac{4y^2}{2}\right]_1^0 = 2\left[0-(1)\right]$$

$$=-2$$

$$\therefore \int_C Pdx+Qdy = 1+\frac{8}{3}-2=\frac{5}{3}$$

Hence Green’s theorem is verified.

36. Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_C (xdy - ydx)$. Hence find area of the ellipse x = a cos θ, y = b sin θ.

Solution: W.K.T. Green's theorem is

$$\int_C (udx + vdy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy \qquad \ldots 1$$

Here $v = \frac{x}{2}$ $\qquad u = -\frac{y}{2}$

$\frac{\partial v}{\partial x} = \frac{1}{2}$ $\qquad \frac{\partial u}{\partial y} = -\frac{1}{2}$

$$(1) \Rightarrow \int_C \left(\frac{x}{2}dy - \frac{y}{2}dx\right) = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dxdy$$

$$\frac{1}{2}\int_C (xdy - ydx) = \iint_R dxdy$$

$$\frac{1}{2}\int_C xdy - ydx = \text{Area of the ellipse} \qquad \ldots 2$$

Given x = a cos θ, $\qquad$ y = b sin θ

dx = -a sin θ dθ, $\qquad$ dy = b cos θ dθ

θ varies from 0 to 2π.

$$(2) \Rightarrow \text{Area of the ellipse} = \frac{1}{2}\int_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(-b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} [ab\cos\theta\cos\theta + ab\sin\theta\sin\theta] d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi}(\cos^2\theta+\sin^2\theta)d\theta = \frac{ab}{2}\int_0^{2\pi}d\theta = \frac{ab}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

Area of the ellipse $= \frac{ab}{2}[2\pi] = \pi ab$

37. State Stoke's theorem (Relation between Line and Surface Integrals).

Statement: If S is an open surface bounded by a simple closed curve C and if a vector function $\vec{F}$ is continuous and has continuous first order partial derivatives in S and on C, then

$\iint_S curl\vec{F}\cdot\hat{n}\,ds = \int_C \vec{F}\cdot d\vec{r}$ where $\hat{n}$ is the outward unit normal vector at any point of S.

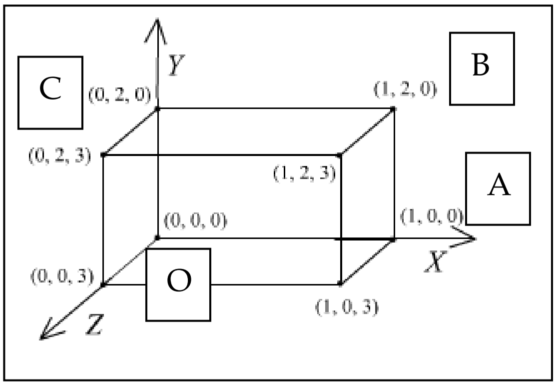
38. Verify Stoke's theorem for the vector $\vec{F} = xy\vec{i} - 2yz\,\vec{j} - xz\vec{k}$, where S is the open surface of the rectangular parallelopiped formed by the planes x = 0, y = 0, x = 1, y = 2 and z = 3 above the XOY plane.

Solution:

By Stoke's theorem

$$\oint_C \vec{F}\bullet d\vec{r} = \iint_S \nabla\times\vec{F}\bullet\hat{n}\,dS$$

$$\oint_C \vec{F}\bullet d\vec{r} = x\,y\,dx - 2\,y\,z\,dy - x\,z\,dz$$



Evaluation of $L.H.S$:

$$\int_C \vec{F}.\vec{dr} = \int_{OA}\vec{F}.\vec{dr} + \int_{AB}\vec{F}.\vec{dr} + \int_{BD}\vec{F}.\vec{dr} + \int_{DO}\vec{F}.\vec{dr}$$

Along OA : $y=0,\ z=0,\ dy=0,\ dz=0$

$$\int_{OA}\vec{F}.\vec{dr} = 0$$

Along AB : $x=1,\ z=0,\ dx=0,\ dz=0$

$$\int_{AB}\vec{F}.\vec{dr} = \int_{AB} 0 = 0$$

Along BC : $y=2,\ z=0,\ dy=0, dz=0$

$$\int_{BC}\vec{F}.\vec{dr} = \int_{BC}(2x)dx = \int_1^0 2x\,dx = \left[\frac{2x^2}{2}\right]_1^0 = 0-1 = -1$$

Along CO: $x=0,\ z=0, dx=0, dz=0$

$$\int_{CO} \vec{F}.\vec{dr} = \int_{CO} 0 = 0$$

$$\therefore \int_{C} \vec{F}.\vec{dr} = 0+0-1+0 = -1$$

Evaluation of RHS:

$$\iint_S \nabla\times\vec{F}.n\ ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

Given, $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix} = \vec{i}\left[0-(-2y)\right] - \vec{j}\left[-z-0\right] + \vec{k}\left[0-x\right]$$

$$= 2y\vec{i} + (z)\vec{j} - x\vec{k}$$

Over S_1: x = 0 , $n = -\vec{i}$

$$\iint_{S_1} \left(\nabla\times\vec{F}\right).n\ ds = \int_0^3\int_0^2 \left[2y\vec{i}\right].(-\vec{i})dydz$$

$$= \int_0^3\int_0^2 -2y dy\, dz$$

$$= \int_0^3\int_0^2 -2y\, dy\, dz = \int_0^3 \left[\frac{-2y^2}{2}\right]_0^2 dz$$

$$= -4(z)_0^3 = -12$$

Over S_2: x =1 , $n = \vec{i}$

$$\iint_{S_2} \left(\nabla\times\vec{F}\right).n\ ds = \int_0^3\int_0^2 \left[2y\vec{i}\right].\left(\vec{i}\right) dydz$$

$$= \int_0^3\int_0^2 2y\, dy\, dz = \int_0^3 \left[\frac{2y^2}{2}\right]_0^2 dz = 12$$

Over S_3: y =0 , $n = -\vec{j}$

$$\iint_{S_3} \left(\nabla\times\vec{F}\right).n\ ds = \int_0^3\int_0^1 \left[z\vec{j}\right].\left(-\vec{j}\right)dxdz = -\int_0^3\int_0^1 (z)\ dx\,dz$$

$$= -\int_0^3 (xz)_0^1 = -\int_0^3 (z)\,dz = -\left(\frac{z^2}{2}\right)_0^3 = -\frac{9}{2}$$

Over S_4: y = 1 , $n = \vec{j}$

$$\iint_{S_4} \left(\nabla\times\vec{F}\right).n\ ds = \int_0^3\int_0^1 z\,\vec{j}.\vec{j}\,dxdz$$

$$= \int_0^3\int_0^1 (z)\ dx\,dz = \int_0^3 (xz)_0^1\,dz$$

$$= \left(\frac{z^2}{2}\right)_0^3 = \frac{9}{2}$$

Over S_5 : z = 1 , $n = \vec{k}$

$$\iint_{S_5} \left(\nabla\times\vec{F}\right).n\ ds = \int_0^2\int_0^1 \left(-x\vec{k}\right).\vec{k}dxdy$$

$$= \int_0^2\int_0^1 (-x)dxdy = \int_0^2 \left(-\frac{x^2}{2}\right)_0^1 dy$$

$$= \int_0^2 \left(\frac{-1}{2}\right) dy = \left(\frac{-1}{2}\right)(y)_0^2 = -1$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} \quad = -12 + 12 - \frac{9}{2} + \frac{9}{2} - 1 = -1$$

$\therefore$ L.HS = R.HS.

Hence Stoke’s theorem is verified.

39. **Verify Stoke’s theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ in the rectangular region bounded by the lines x = 0, x = a, y = 0 and y = b.**

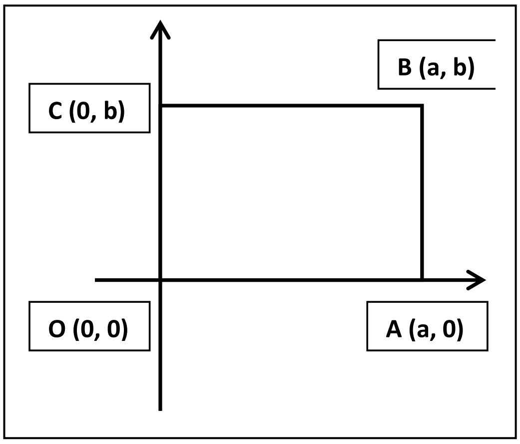
Solution:

Given $\vec{F} = \left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$

By Stoke's theorem $\int\limits_C \vec{F}.d\vec{r} = \iint\limits_S curl\,\vec{F}.\vec{n}\,ds$

Evaluation of LHS:

$$\int\limits_C \vec{F}.dr = \int\limits_{OA} + \int\limits_{AB} + \int\limits_{BC} + \int\limits_{CO}$$



Along OA : $y = 0 \Rightarrow dy = 0$, x varies from 0 to a

$$\therefore \int\limits_{OA} \vec{F}.dr = \int_0^a \left(x^2\right)dx$$

$$= \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along AB: $x = a \Rightarrow dx = 0$, y varies from 0 to b

$$\int\limits_{AB} \vec{F}.dr = \int_0^b -2ay\ dy$$

$$= -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2$$

Along BC: $y = b$, $dy = 0$, x varies from a to 0

$$\int\limits_{BC} \vec{F}.d\vec{r} = \int_a^0 \left(x^2 + b^2\right)dx$$

$$= \left(\frac{x^3}{3} + b^2 x\right)_a^0$$

$$= -\frac{a^3}{3} - ab^2$$

Along CO: $x = 0$, $dx = 0$, y varies from b to 0

$$\int\limits_{CO} \vec{F}.d\vec{r} = \int_b^0 \left(0 + y^2\right)0 + 0 = 0$$

$$\therefore \int\limits_C \vec{F}.d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0 = -2ab^2$$

Evaluation of RHS:

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y] = -4y\vec{k}$$

As the region is in the xy plane we can take $\vec{n} = \vec{k}$ and $ds = dxdy$

$$\iint_S curl\,\vec{F}.\vec{n}\,ds = \iint -4y\vec{k}.\vec{k}\,dx\,dy$$

$$= -4\int_0^b \int_0^a y\ dx\,dy$$

$$= -4\left(\frac{y^2}{2}\right)_0^b (x)_0^a$$

$$= -2ab^2$$

$$\therefore \int_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\vec{n}ds$$

Hence Stoke's theorem is verified.

40. State Gauss Divergence Theorem (Relation between Surface and Volume Integrals).

Statement: If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is having continuous first order partial derivatives on S , then $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\vec{F}\ dV$.

where $\hat{n}$ is the outward unit normal vector to the surface.

41. Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ taken over the cube bounded by the planes $x=0,\ x=1, y=0,\ y=1, z=0, z=1.$

Solution:

$$\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$$

$$\nabla\circ\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla\circ\vec{F} = 4z - 2y + y = 4z - y$$

$$RHS = \iiint_V \nabla \circ \vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz = \int_0^1\int_0^1 [4zx-yx]_0^1\,dy\,dz = \int_0^1\int_0^1 [4z-y]\,dy\,dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left[4z - \frac{1}{2}\right] dz = \left[4\frac{z^2}{2} - \frac{z}{2}\right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2} \text{ ....(1)}$$

Surface	$\hat{n}$	$\vec{F} \circ \hat{n}$	Equation	$\vec{F} \circ \hat{n}$ on S	dS	$\iint_s \vec{F} \circ \hat{n}\,dS$
S_1	$\vec{i}$	$4xz$	$x=1$	$4z$	$dydz$	$\int_0^1\int_0^1 4z\,dydz$
S_2	$-\vec{i}$	$-4xz$	$x=0$	0	$dydz$	0
S_3	$\vec{j}$	$-y^2$	$y=1$	-1	$dxdz$	$\int_0^1\int_0^1 (-1)\,dxdz$
S_4	$-\vec{j}$	y^2	$y=0$	0	$dxdz$	0
S_5	$\vec{k}$	yz	$z=1$	y	$dxdy$	$\int_0^1\int_0^1 y\,dxdy$
S_6	$-\vec{k}$	$-yz$	$z=0$	0	$dxdy$	0

$$LHS = \iint_S \vec{F} \circ \hat{n}\,dS = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6}\right) \vec{F} \circ \hat{n}\,dS$$

$$= \int_0^1\int_0^1 4z\,dydz + \int_0^1\int_0^1 (0)\,dydz + \int_0^1\int_0^1 (-1)\,dxdz + \int_0^1\int_0^1 (0)\,dxdz + \int_0^1\int_0^1 y\,dxdy + \int_0^1\int_0^1 (0)\,dxdy$$

$$= 4\int_0^1\int_0^1 z\,dydz + 0 - \int_0^1\int_0^1 dxdz + 0 + \int_0^1\int_0^1 y\,dxdy + 0$$

$$= 4\int_0^1 z(y)_0^1 dz - \int_0^1 (x)_0^1\,dz + \int_0^1 y(x)_0^1\,dy = 4\int_0^1 z\,dz - \int_0^1 dz + \int_0^1 y\,dy = \left(4\frac{z^2}{2}\right)_0^1 - (z)_0^1 + \left(\frac{y^2}{2}\right)_0^1$$

$$= \frac{4}{2} - 1 + \frac{1}{2} = 2 - 1 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} \text{..(2)}$$

From (1) and (2),

$$\iiint_V \nabla \circ \vec{F} dv = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n}\, dS$$

Hence Gauss Divergence theorem is verified.

42. Verify Gauss divergence theorem for $\vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$ taken over the cube bounded by the planes $x = 0,\ x = 1, y = 0,\ y = 1, z = 0, z = 1.$

Solution:

$$\vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 2y + 2z = 2(x + y + z)$$

$$RHS = \iiint_V \nabla \circ \vec{F} dv = 2\int_0^1\int_0^1\int_0^1 (x + y + z)dxdydz = 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + xy + xz \right]_0^1 dydz = 2\int_0^1\int_0^1 \left[\frac{1}{2} + y + z \right] dydz$$

$$= 2\int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + yz \right]_0^1 dz = 2\int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z \right]_0^1 dz = 2\int_0^1 [1 + z]_0^1 dz = 2\left[z + \frac{z^2}{2} \right]_0^1$$

$$= 2\left(1 + \frac{1}{2}\right) = 2\left(\frac{3}{2}\right) = 3 \text{.........................(1)}$$

Surface	$\hat{n}$	$\vec{F} \circ \hat{n}$	Equation	$\vec{F} \circ \hat{n}$ on S	dS	$\iint_s \vec{F} \circ \hat{n}\, dS$
S_1	$\vec{i}$	x^2	$x = 1$	1	$dydz$	$\int_0^1\int_0^1 dydz$
S_2	$-\vec{i}$	$-x^2$	$x = 0$	0	$dydz$	0
S_3	$\vec{j}$	y^2	$y = 1$	1	$dxdz$	$\int_0^1\int_0^1 dxdz$
S_4	$-\vec{j}$	$-y^2$	$y = 0$	0	$dxdz$	0
S_5	$\vec{k}$	z^2	$z = 1$	1	$dxdy$	$\int_0^1\int_0^1 dxdy$
S_6	$-\vec{k}$	$-z^2$	$z = 0$	0	$dxdy$	0

$$LHS = \iint_S \vec{F} \circ \hat{n}\, dS = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n}\, dS$$

$$= \int_0^1\int_0^1 dydz + \int_0^1\int_0^1 (0)dydz + \int_0^1\int_0^1 dxdz + \int_0^1\int_0^1 (0)dxdz + \int_0^1\int_0^1 dxdy + \int_0^1\int_0^1 (0)dxdy$$

$$= 1+1+1 = 3 \text{..}(2)$$

From (1) and (2),

$$\iiint_V \nabla \circ \vec{F} dv = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S6} \right) \vec{F} \circ \hat{n}\, dS$$

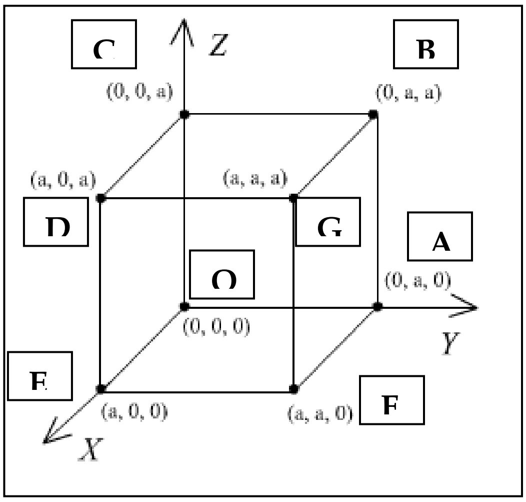
Hence Gauss Divergence theorem is verified.

43. **Verify Gauss divergence theorem for $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ taken over the cube bounded by x = 0, x = a, y = 0, y = a, z = 0 and z = a.**

Solution:

By Gauss Divergence theorem $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V div\vec{F}\ dV$

S_1	OABC
S_2	DEFG
S_3	OCDE
S_4	ABGF
S_5	OEFA
S_6	CDGB



Evaluation of LHS:

$$\iint_S \vec{F} \cdot nds = \iint_{S_1} \vec{F} \cdot \hat{n}\, ds + \iint_{S_2} \vec{F} \cdot nds + ... + \iint_{S_6} \vec{F} \cdot nds$$

Over S_1: x = 0, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(-\vec{i})\, dy\, dz = \int_0^a\int_0^a -x^3\, dy\, dz$$

$$= 0$$

Over S_2: x = a, $\hat{n} = \vec{i}$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\vec{i})\,dy\,dz = \int_0^a\int_0^a x^3\,dy\,dz$$

$$= \int_0^a\int_0^a a^3\,dy\,dz = a^3\int_0^a [y]_0^a dz = a^3\int_0^a a\,dz$$

$$= a^4\,[z]_0^a = a^4(a) = a^5$$

Over S_3: y = 0, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(-\vec{j})\,dx\,dz = \int_0^a\int_0^a -y^3\,dx\,dz = 0$$

Over S_4: y = a, $\hat{n} = \vec{j}$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\vec{j})\,dx\,dz = \int_0^a\int_0^a y^3\,dx\,dz$$

$$= \int_0^a\int_0^a a^3\,dx\,dz = a^3\int_0^a [x]_0^a\,dz = a^3\int_0^a [a-0]\,dz = a^4\,[z]_0^a = a^4\,(a) = a^5$$

Over S_5: z = 0, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\overrightarrow{-k})\,dx\,dy = \int_0^a\int_0^a -z^3\,dx\,dy = 0$$

Over S_6: z = a, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(\vec{k})\,dx\,dy = \int_0^a\int_0^a z^3\,dx\,dy$$

$$= a^3\int_0^a [x]_0^a dy = a^3\int_0^a a\;dy = a^4\,[y]_0^a = a^4(a) = a^5$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 0 + a^5 + 0 + a^5 + 0 + a^5 = 3a^5$$

Evaluation of RHS:

$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(x^3\vec{i} + y^3\vec{j} + z^3\vec{k}\right)$$

$$\nabla.\vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\iiint_V \nabla.\vec{F}\, dV = \int_0^a\int_0^a\int_0^a 3x^2 + 3y^2 + 3z^2\, dx\, dy\, dz$$

$$= 3\int_0^a\int_0^a\int_0^a x^2 + y^2 + z^2\, dx\, dy\, dz$$

$$= 3\int_0^a\int_0^a \left[\frac{x^3}{3} + (y^2 + z^2)x\right]_0^a dy\, dz$$

$$= 3\int_0^a\int_0^a \left[\frac{a^3}{3} + (y^2 + z^2)\mathrm{a}\right] dy\, dz$$

$$= 3\int_0^a \left[\frac{a^3}{3}y + a\frac{y^3}{3} + az^2y\right]_0^a dz$$

$$= 3\int_0^a \frac{a^4}{3} + \frac{a^4}{3} + a^2z^2\, dz$$

$$= 3\left[\frac{a^4}{3}z + \frac{a^4}{3}z + a^2\frac{z^3}{3}\right]_0^a$$

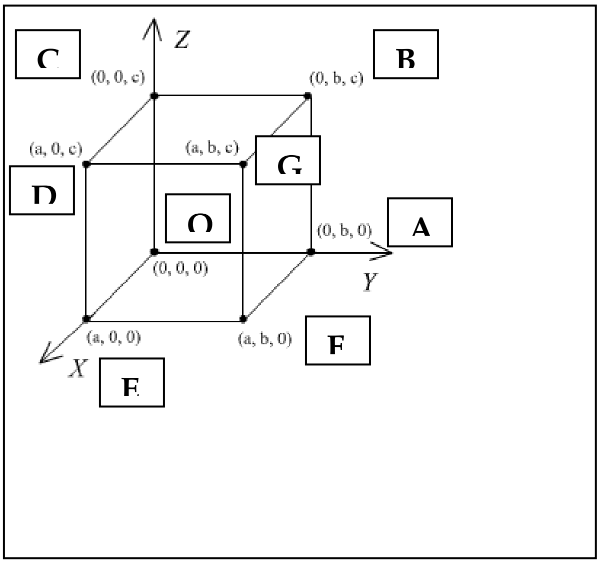
$$= 3\left[\frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3}\right]$$

$$= \frac{9a^5}{3} = 3a^5$$

Hence Gauss Divergence theorem is verified.

44. Verify Gauss divergence theorem for $\vec{F} = \left(x^2 - yz\right)\vec{i} + \left(y^2 - zx\right)\vec{j} + \left(z^2 - xy\right)\vec{k}$ taken over the rectangular parallelopiped bounded by the planes x = 0, x = a, y = 0, y = b, z = 0, and z = c.

Solution:



By Gauss Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\vec{F}\ dV$$

S1	OABC
S2	DEFG
S3	OCDE
S4	ABGF
S5	OEFA
S6	CDGB

Evaluation of LHS:

$$\iint_S \vec{F}\cdot nds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot nds + \ldots + \iint_{S_6} \vec{F}\cdot nds$$

Over S_1: x = 0, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (0-yz)(-1)dy\,dz = \int_0^c\int_0^b (yz)\,dy\,dz = \int_0^c \left[z\left(\frac{y^2}{2}\right)_0^b \right] dz = \frac{b^2}{2}\left(\frac{z^2}{2}\right)_0^c = \frac{b^2c^2}{4}$$

Over S_2: x = a, $\hat{n} = \vec{i}$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (-yz+a^2)\,dy\,dz = \int_0^b \left[-y\left(\frac{z^2}{2}\right)_0^c + a^2\,[z]_0^c \right] dy$$

$$= -\frac{c^2}{2}\left(\frac{y^2}{2}\right)_0^b + ca^2\,[y]_0^b = a^2bc - \frac{b^2c^2}{4}$$

Over S_3: y = 0, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (xz)\,dx\,dz = \int_0^c \left(\frac{x^2}{2}z\right)_0^a dz = \frac{a^2}{2}\left(\frac{c^2}{2}\right) = \frac{a^2c^2}{4}$$

Over S_4: y = b, $\hat{n} = \vec{j}$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (-xz+b^2)\,dx\,dz = \int_0^c \left[-z\left(\frac{a^2}{2}\right)+b^2a\right]dz = ab^2c - \frac{a^2c^2}{4}$$

Over S_5: z = 0, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (xy)\,dx\,dy = \int_0^b \left[y\left(\frac{x^2}{2}\right)_0^a\right]dy = \frac{a^2b^2}{4}$$

Over S_6: z = c, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (-xy+c^2)\,dx\,dy = \int_0^b \left[-y\left(\frac{a^2}{2}\right)+c^2a\right]dy = abc^2 - \frac{a^2b^2}{4}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \frac{b^2c^2}{4}+a^2bc-\frac{b^2c^2}{4}+\frac{a^2c^2}{4}+a\ b^2c-\frac{a^2c^2}{4}+\frac{a^2b^2}{4}+a\ bc^2-\frac{a^2b^2}{4}$$

$$= a^2bc+ab^2c+abc^2 = abc(a+b+c)$$

Evaluation of RHS:

$$\nabla.\vec{F} = 2(x+y+z)$$

$$\iiint_V \nabla.\vec{F}\,dV = \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2}+xy+xz\right]_0^a dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2}+ay+az\right]dy\,dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}y+a\frac{y^2}{2}+ayz\right]_0^b dz$$

$$= 2\left[\frac{a^2bz}{2}+\frac{ab^2z}{2}+\frac{abz^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]= a^2bc+ab^2c+abc^2 = abc(a+b+c)$$

Hence Gauss divergence theorem is verified.

45. **Evaluate** $\iint_S \vec{F}.\hat{n}ds$ **where** $\overline{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ **and S is the surface bounding the region** $x^2+y^2 = 4$**,** $z = 0$ **and** $z = 3$**.**

Solution:

Gauss divergence theorem is

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V \nabla.\vec{F}dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}\left(-2y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)\right]dv$$

$$= \iiint_V [4-4y+2z]dv \quad = \int_{-2}^{2}\int_{\sqrt{4-x_2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)dzdydx$$

$$= \int_{-2}^{2}\int_{\sqrt{4-x_2}}^{\sqrt{4-x^2}}\int_0^3 \left[4x-4yz+\frac{2z^2}{2}\right]_0^3 dydx \qquad = \int_{-2}^{2}\int_{\sqrt{4-x_2}}^{\sqrt{4-x^2}}[(12-12y+9)-0]dydx$$

$$= \int_{-2}^{2}\int_{\sqrt{4-x_2}}^{\sqrt{4-x^2}}[21-12y]dx = \int_{-2}^{2}\left[21y-12\frac{y^2}{2}\right]_{\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}dx = (42)(2)\int_0^2 \sqrt{4-x^2}dx = 84\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 84\pi$$

46. **If** $F = ax\vec{i} + by\vec{j} + cz\vec{k}$**, a, b, c are constants, show that** $\iint_S \vec{F}\bullet\hat{n}\,ds = \frac{4\pi}{3}(a+b+c)$ **where S is the surface of a unit sphere.**

Solution:

W.K.T. Gauss's divergence theorem

$$\iint_S \vec{F} \bullet \hat{n} ds = \iiint_V \nabla \bullet \vec{F} dV \;= \iiint_V \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \right) dV$$

$$= \iiint_V (a+b+c) dV \;= (a+b+c)V = (a+b+c)\frac{4}{3}\pi(1)^3$$

$$\iint_S \vec{F} \bullet \hat{n} ds = \frac{4}{3}\pi(a+b+c)$$

47. **Evaluate** $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ **and C is the straight line from A (0, 0, 0) to B (2, 1, 3).**

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\overrightarrow{F}.d\vec{r} = 3x^2 dx + (2xz - y)dy + zdz$

The equation of AB is $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say) $\qquad \left(\because \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \right)$

$\Rightarrow x = 2t \Rightarrow dx = 2dt$

$y = t \Rightarrow dy = dt$, $\qquad \int_C \overrightarrow{F}. \; \overrightarrow{dr} = \int_0^1 3x^2 dx + (2xz - y)dy + zdz$

$z = 3t \Rightarrow dz = 3dt$

$$= \int_0^1 \left(36t^2 + 8t\right) dt = \left[36\frac{t^3}{3} + 8\left(\frac{t^2}{2}\right) \right]_0^1 = 16$$

48. **Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle in the XY - plane from (0, 0) to (1,1) along the parabola $y^2 = x$.**

Solution:

Given $\overrightarrow{F} = \left(x^2 - y^2 + x\right)\vec{i} - \left(2xy + y\right)\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\overrightarrow{F}. \overrightarrow{dr} = \left(x^2 - y^2 + x\right)dx - \left(2xy + y\right)dy$.

Given $y^2 = x$

$2ydy = dx$

$$\therefore \vec{F}.d\vec{r} = \left(x^2 - x + x\right)dx - \left(2y^3 + y\right)dy$$
$$= x^2 dx - \left(2y^3 + y\right)dy$$

$$\int_C \overrightarrow{F}.\overrightarrow{dr} = \int_0^1 x^2 dx - \int_0^1 \left(2y^3 + y\right)dy$$
$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{2y^4}{4} + \frac{y^2}{2}\right]_0^1$$
$$= \left(\frac{1}{3} - 0\right) - \left[\left(\frac{2}{4} + \frac{1}{2}\right) - (0+0)\right] = \frac{-2}{3}$$

$\therefore$ Work done $= \dfrac{2}{3}$

49. **If** $F = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ **, evaluate** $\int_C \vec{F} \bullet d\vec{r}$ **from (0,0,0) to (1,1,1) along the curve x = t, y = t^2, z = t^3.**

Solution:

The end points are (0,0,0) and (1,1,1).

These points correspond to t = 0 and t = 1.

$\therefore$ dx = dt, dy = 2t dt, dz = 3t^{2}dt

$$\int_C \vec{F} \bullet d\vec{r} = \int_C \left(3x^2 + 6y\right)dx - 14yzdy + 20xz^2 dz$$

$$= \int_0^1 (3t^2 + 6t^2)dt - 14t^5(2tdt) + 20t^7(3t^2)dt = \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = 5$$

* * * * *

Module – 3 Laplace Transforms

Laplace Transforms of standard functions – Transforms properties – Transforms of Derivatives and Integrals – Initial value theorems (without proof) and verification for some problems – Final value theorems (without proof) and verification for some problems – Inverse Laplace transforms using partial fractions – Inverse Laplace transforms using second shifting theorem – LT using Convolution theorem – problems only – ILT using Convolution theorem – problems only – LT of periodic functions – problems only – Solve linear second order ordinary differential equations with constant coefficients only – Solution of Integral equation and integral equation involving convolution type – Application of Laplace Transform in Engineering.

Introduction

Laplace Transformation named after a Great French mathematician **PIERRE SIMON DE LAPLACE** (1749-1827) who used such transformations in his researches related to "Theory of Probability". The powerful practical Laplace transformation techniques were developed over a century later by the English electrical Engineer **OLIVER HEAVISIDE** (1850-1925) and were often called "Heaviside - Calculus".

Transformation

A "Transformation" is an operation which converts a mathematical expression to a different equivalent form.

Laplace Transform

Let $f(t)$ be a given function which is defined for all positive values of t.

If $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ exists, then F(s) is called ***Laplace transform*** of $f(t)$.

Exponential Order

A function $f(t)$ is said to be of **exponential order** if $\lim_{t\to\infty} e^{-st} f(t) = 0$.

Sufficient conditions for the existence of Laplace transforms

The Laplace transform of $f(t)$ exists if

- **i.** $f(t)$ is continuous or piecewise continuous in $[a, b]$ where $a > 0$.
- **ii.** $f(t)$ is of exponential order.

Example

L [tan t] does not exist since tan t is not piecewise continuous. i.e., tan t has infinite number of infinite discontinuities at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$

Note:

(i) Not all $f(t)$ are Laplace transformable.

(ii) The above two conditions are not *necessary*.

Laplace transform for some basic functions

S. No.	$f(t)$	L{ $f(t)$ }
1	e^{at}	$\frac{1}{s-a}$, $s-a>0$
2	e^{-at}	$\frac{1}{s+a}$, $s+a>0$
3	$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
4	$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
5	$\sinh at$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
6	$\cosh at$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$
7	1	$\frac{1}{s}$
8	K	$\frac{K}{s}$
9	t	$\frac{1}{s^2}$
10	t^n	$\frac{n!}{s^{n+1}}$, $n=0, 1, 2, \ldots.$
11	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, n is not an integer.
12	$t\,e^{at}$	$\frac{1}{(s-a)^2}$
13	Periodic function with period 'p'	$\frac{1}{1-e^{-sp}}\int_0^p e^{-st}\, f(t)\,dt$

Properties of Laplace transform:

S. No.	Property	Laplace Transform
1	Linear Property	$L\left(af(t)\pm bg(t)\right)=aL\left(f(t)\right)\pm bL\left(g(t)\right)$
2	First shifting theorem	$L\left(e^{-at}f(t)\right)=F(s+a)$ $L\left(e^{at}f(t)\right)=F(s-a)$
3	Second shifting theorem	If $L(f(t))=F(s)$ and $g(t)=\begin{cases} f(t-a), t>a \\ 0, t<a \end{cases}$, then $L(g(t))=e^{-as}F(s)$.
4	Change of scale property	$L\left(f(at)\right)=\frac{1}{a}F\left(\frac{s}{a}\right),\ a>0$
5	Multiplication by t	$L\left(t^n f(t)\right)=(-1)^n\frac{d^n}{ds^n}F(s)$
6	Division by t	$L\left(\frac{f(t)}{t}\right)=\int_s^\infty F(s)ds,$ provided $\lim\limits_{t\to 0}\frac{f(t)}{t}$ exists
7	Transforms of integrals	$L\left(\int_0^t f(t)dt\right)=\frac{L\left[f(t)\right]}{s}$
8	**Initial Value theorem:** If $L\left(f(t)\right)=F(s)$ then $\lim\limits_{t\to 0}f(t)=\lim\limits_{s\to\infty}sF(s)$	
9	**Final value theorem:** If $L\left(f(t)\right)=F(s)$ then $\lim\limits_{t\to\infty}f(t)=\lim\limits_{s\to 0}sF(s)$	
10	**Convolution of two functions:** The convolution of two functions *f*(*t*) and *g*(*t*) is defined as $\int_0^t f(u)g(t-u)du=f(t)*g(t)$	
11	**Convolution theorem:** The Laplace transform of convolution of two functions is equal to the product of their Laplace transforms. (i.e) $L\left[f(t)*g(t)\right]=L\left[f(t)\right]L\left[g(t)\right]$.	

Problems based on Laplace Transforms

1. **Find** $L\left(2e^{-3t}+3t^2-4\sin 2t+2\cos 3t\right)$.

Solution:

$$L\left(2e^{-3t}+3t^2-4\sin 2t+2\cos 3t\right)=\frac{2}{s+3}+3\left(\frac{2}{s^3}\right)-4\left(\frac{2}{s^2+4}\right)+2\left(\frac{s}{s^2+9}\right)$$

2. **Find** $\boldsymbol{L[e^{3t+5}]}$.

Solution:

$$\mathbf{L[e^{3t}.e^{5}]=e^{5}L[e^{3t}]=e^{5}\left(\frac{1}{s-3}\right)}$$

3. **Find the Laplace transform of** $\boldsymbol{f(t)=cos^2(3t)}$.

Solution: $L[\cos^2 3t] \quad = L\left[\frac{1+\cos 6t}{2}\right]=\frac{L(1)+L(\cos 6t)}{2} \because \cos^2 t=\frac{1+\cos 2t}{2}$

$$=\frac{1}{2s}+\frac{s}{2(s^2+36)} \because L(1)=\frac{1}{s}, L(\cos at)=\frac{s}{s^2+a^2}$$

$\therefore \qquad L[\cos^2 3t] = \dfrac{s^2+18}{s(s^2+36)}$

4. **Find the Laplace transform of** $\boldsymbol{sin^3(2t)}$.

Solution: $L\left[sin^3(2t)\right]=\frac{1}{4}L[3\sin 2t-\sin 6t]=\frac{3}{4}L[\sin 2t]-\frac{1}{4}L[\sin 6t]$

$$\left(\because \sin^3 t=\frac{1}{4}[3\sin t-\sin 3t]\right)$$

$$=\frac{3}{4}\left(\frac{2}{s^2+4}\right)-\frac{1}{4}\left(\frac{6}{s^2+36}\right)=\frac{6}{4}\left(\frac{1}{s^2+4}-\frac{1}{s^2+36}\right).$$

5. **Find** $L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]$.

Solution:

$$L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]=L\left[\sin 8t\cos 4t\right]+L\left[\cos^3 4t\right]+L[5]$$

$$L\left[\sin 8t+\cos 4t\right]=L\left[\frac{\sin 12t+\sin 4t}{2}\right]\ \left[\because \sin A\cos B=\frac{\sin(A+B)+\sin(A-B)}{2}\right]$$

$$=\frac{1}{2}\left\{L\left[\sin 12t\right]+L\left(\sin 4t\right)\right\}$$

$$=\frac{1}{2}\left\{\frac{12}{s^2+144}+\frac{4}{s^2+16}\right\}$$

$$L\left[\cos^3 4t\right]=L\left[\frac{\cos 12t+3\cos 4t}{4}\right]\left[\because \cos^3\theta=\frac{\cos 3\theta+3\cos\theta}{4}\right]$$

$$=\frac{1}{4}\left\{L\left(\cos 12t\right)+3L\left(\cos 4t\right)\right\}$$

$$=\frac{1}{4}\left[\frac{s}{s^2+144}+\frac{3s}{s^2+16}\right]$$

$$L[5]=5L[1]=5\left[\frac{1}{s}\right]=\frac{5}{s}.$$

$$L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]=\frac{1}{2}\left\{\frac{12}{s^2+144}+\frac{4}{s^2+16}\right\}+\frac{1}{4}\left\{\frac{s}{s^2+144}+\frac{3s}{s^2+16}\right\}+\frac{5}{s}.$$

6. **Find the Laplace transform of unit step function**

Solution: The Unit step function is $u_a(t)=\begin{cases}0, & t<a\\ 1, & t>a,\ a\geq 0\end{cases}$

The Laplace transform $L\left[f(t)\right]=\int_0^\infty e^{-st}f(t)\,dt=\int_a^\infty e^{-st}(1)\,dt=\left[\frac{e^{-st}}{-s}\right]_a^\infty=-\frac{1}{s}\left[e^{-\infty}-e^{-as}\right]=\frac{e^{-as}}{s}.$

7. **Find** $L\left[t^{3/2}\right]$.

Solution:

We know that $L\left[t^n\right]=\frac{\Gamma(n+1)}{s^{n+1}}$

$$L\left[t^{3/2}\right]=\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{\frac{5}{2}}}\ \because\Gamma(n+1)=n\Gamma(n)$$

$$=\frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}}=\frac{\frac{3}{2}.\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)}{S^{5/2}}$$

$$=\frac{3\sqrt{\pi}}{4s^{5/2}}\quad\left[\therefore\Gamma(1/2)=\sqrt{\pi}\right]$$

Problems based on First Shifting Property

8. **Find the Laplace transform of e^{-t} sin 2t.**

Solution:

$L[e^{-t}\sin 2t] = L[e^{-at} f(t)] = F(s + a) = F(s + 1)$

$F(s) = L[f(t)] = L(\sin 2t) = \frac{2}{s^2+4}$

$F(s + 1) = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$

9. **Find** $L\left[e^{-at}\cos bt\right]$.

Solution:

$$L\left[e^{-at}\cos bt\right] = \left[L(\cos bt)\right]_{s\to s+a}$$

$$= \left[\frac{s}{s^2+b^2}\right]_{s\to s+a}$$

$$= \left[\frac{s+a}{(s+a)^2+b^2}\right]$$

Problems based on Multiplication by *t*

10. **Find the Laplace transform of** $e^{-2t}t^{1/2}$.

Solution: $L\left[e^{-2t}t^{1/2}\right] = L\left[t^{1/2}\right]_{s\to s+2}$

$$\because \text{If } L[f(t)] = F(s), \text{then } L\left[e^{-at} f(t)\right] = F(s)\big|_{s\to s+a}$$

$$= \left[\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}\right]_{s\to s+2} = \left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}\right]_{s\to s+2}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{(s+2)^{3/2}} \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma n+1 = n\Gamma n\right)$$

11. **Obtain the Laplace transform of** $\sin 2t - 2t\cos 2t$.

Solution: $L[\sin 2t - 2t\cos 2t] = L[\sin 2t] - 2L[t\cos 2t] = L[\sin 2t] - 2\left(-\frac{d}{ds}L[\cos 2t]\right)$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4} + 2\left(\frac{\left(s^2+4\right)(1) - s(2s)}{\left(s^2+4\right)^2}\right)$$

$$= \frac{2\left(s^2+4\right) + 2\left(4-s^2\right)}{\left(s^2+4\right)^2}$$

$$\therefore L[\sin 2t - 2t\cos 2t] = \frac{16}{\left(s^2+4\right)^2}$$

12. **Find $L\left(t e^t\right)$.**

Solution

$$L\left(t\, f(t)\right) = -\frac{d}{ds} L\left(f(t)\right)$$

$$L\left(t e^t\right) = -\frac{d}{ds} L\left(e^t\right)$$

$$= -\frac{d}{ds} L\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

13. **Find $L\left(t \sin 2t\right)$.**

Solution

$$L\left(t\, f(t)\right) = -\frac{d}{ds} L\left(f(t)\right)$$

$$L\left(t \sin 2t\right) = -\frac{d}{ds} L(\sin 2t)$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{\left(s^2+4\right)^2}$$

14. **Find the Laplace transform of $f(t) = t^2 \cos t$.**

Solution

$$L\left[t^2 \cos t\right] = \left[\frac{d^2}{ds^2} L[\cos t]\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left(\frac{\left(s^2+1\right).1 - 1.2s.s}{\left(s^2+1\right)^2}\right) = \frac{d}{ds}\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)$$

$$= \frac{\left(s^2+1\right)^2(-2s) - \left(1-s^2\right)2\left(s^2+1\right)2s}{\left(s^2+1\right)^3} = \frac{-2s\left(3-s^2\right)}{\left(s^2+1\right)^3}$$

15. **Find the Laplace Transform of $f(t) = e^{-t} t\ cost.$**

Solution

$$L[e^{-t}tcost] = -\frac{d}{ds}L[\cos t]_{s\to s+1} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{((s+1)^2+1)^2} = \frac{s^2+2s}{(s^2+2s+2)^2}$$

$$= \frac{s(s+2)}{(s^2+2s+2)^2}$$

16. **Find the Laplace transform of $f(t) = te^{-3t}cos2t.$**

Solution

$$L[f(t)] = L[te^{-3t}cos2t] = -\frac{d}{ds}L[\cos 2t]_{s\to s+3} = -\frac{d}{ds}\left[\frac{s}{s^2+4}\right]_{s\to s+3}$$

$$= -\left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]_{s\to s+3} = \left[\frac{s^2-4}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \frac{(s+3)^2-4}{((s+3)^2+4)^2}$$

$$= \frac{s^2+6s+5}{(s^2+6s+13)^2}$$

17. **Find $L\left[t^2e^{-t}\cos t\right]$.**

Solution:

$$L\left[t^2e^{-t}\cos t\right]=L\left[t^2\cos t\right]_{s\to s+1}$$

$$=\left[(-1)^2\frac{d^2}{ds^2}L[\cos t]\right]_{s\to s+1}=\left[\frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]\right]_{s\to s+1}$$

$$=\left[\frac{d}{ds}\frac{\left(s^2+1\right)1-s.2s}{\left(s^2+1\right)^2}\right]_{s\to s+1}$$

$$=\left[\frac{d}{ds}\frac{1-s^2}{\left(s^2+1\right)^2}\right]_{s\to s+1}=\left[\frac{2s^3-6s}{\left(s^2+1\right)^3}\right]_{s\to s+1}=\frac{2(s+1)^3-6(s+1)}{\left((s+1)^2+1\right)^3}$$

18. **Find $L\left[t^2e^t\sin t\right]$**

Solution:

$$L\left[t^2e^t\sin t\right]=(-1)^2\frac{d^2}{ds^2}L\left[e^t\sin t\right] \quad \dots (1)$$

Now $L\left[e^t\sin t\right]=\left[L[\sin t]\right]_{s\to(s-1)}=\dfrac{1}{(s-1)^2+1}$ ... (2)

Substituting (2) in (1) we get

$$L\left[t^2e^t\sin t\right]=\frac{d}{ds}\left[\frac{0-2(s-1)}{\left((s-1)^2+1\right)^2}\right]=\frac{d}{ds}\left[\frac{-2(s-1)}{\left(s^2-2s+2\right)^2}\right]$$

$$=\frac{\left(s^2-2s+2\right)^2(-2)+2(s-1)2\left(s^2-2s+2\right)(2s-2)}{\left(s^2-2s+2\right)^4}$$

$$=\frac{2\left(s^2-2s+2\right)\left[-\left(s^2-2s+2\right)+4(s-1)^2\right]}{\left(s^2-2s+2\right)^4}$$

$$= \frac{2\left(s^2-2s+2\right)\left[-s^2+2s-2+4s^2+4-8s\right]}{\left(s^2-2s+2\right)^4}$$

$$\therefore F(s) = \frac{2\left(s^2-2s+2\right)\left[3s^2-6s+2\right]}{\left(s^2-2s+2\right)^4} = \frac{2\left(3s^2-6s+2\right)}{\left(s^2-2s+2\right)^3}$$

Problems based on Division by *t*

19. **Find $L\left[\frac{sint}{t}\right]$.**

Solution

$$L\left[\frac{sint}{t}\right]=L\left[\frac{f(t)}{t}\right]=\int_s^{\infty} F(s)ds$$

$$\text{F(s)} = \text{L } [sint]=\frac{1}{s^2+1^2}$$

$$\int_s^{\infty} F(s)ds=\int_s^{\infty}\frac{1}{s^2+1}ds=[tan^{-1}(s)]_s^{\infty}$$

$$=[tan^{-1}\infty - tan^{-1}s]=\left[\frac{\pi}{2}-tan^{-1}s\right] = cot^{-1}s$$

20. **Find the Laplace transform of $\frac{e^{-t}\sin t}{t}$.**

Solution:

$$L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^{\infty} L\left(e^{-t}\sin t\right)ds$$

$$= \int_s^{\infty} L\left(\sin t\right)_{s+1} ds = \int_s^{\infty}\left(\frac{1}{s^2+1}\right)_{s+1} ds = \int_s^{\infty}\frac{1}{(s+1)^2+1}ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

21. **Find $L\left[\frac{\sin^2 t}{t}\right]$.**

Solution:

$$L\left[\frac{\sin^2 t}{t}\right] = L\left[\frac{1-\cos 2t}{2t}\right] = \frac{1}{2}L\left[\frac{1-\cos 2t}{t}\right] = \frac{1}{2}\int_s^{\infty} L\left[1-\cos 2t\right]\ ds$$

$$= \frac{1}{2}\int_s^\infty \left\{L[1] - L[\cos 2t]\right\} ds \; = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log\left(s^2+4\right)\right]_s^\infty \; = \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right]_s^\infty \; = \frac{1}{2}\left[\log 1 - \log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right] = \frac{1}{2}\left[0 - \log\frac{s}{\sqrt{s^2+4}}\right]$$

$$F(s) = \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)^{-1} \; = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

22. **Find the Laplace Transform of $f(t) = \frac{1-cost}{t}$.**

Solution

$$L[1-cost] = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[1-cost]ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}[\log(s^2+1) - logs^2]_s^\infty$$

$$= -\frac{1}{2}\left[log\frac{s^2+1}{s^2}\right]_s^\infty = -\frac{1}{2}\left[log\left(1+\frac{1}{s^2}\right)\right]_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}log\left[1+\frac{1}{s^2}\right] \; = \frac{1}{2}log\left(\frac{s^2+1}{s^2}\right)$$

23. **Find $L\left[\frac{cos\ at - cos\ bt}{t}\right]$.**

Solution

$$L\left[\frac{cosat - cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}log(s^2+a^2) - \frac{1}{2}log(s^2+b^2)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty}=\frac{1}{2}\left[\log\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log 1-\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]=\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

24. **Using Laplace transform, evaluate $\int_0^{\infty} t\,e^{-2t}\sin t\,dt$**

Solution: $\int_0^{\infty} e^{-2t} f(t)dt=\left[\int_0^{\infty} e^{-st} f(t)dt\right]_{s=2}$ $=\left[L[t\sin t]\right]_{s=2}=\left[-\frac{d}{ds}L[\sin t]\right]_{s=2}$

$$=-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=-\left(\frac{-2s}{\left(s^2+1\right)^2}\right)=\frac{4}{25}$$

Problems based on Convolution Theorem

25. **Evaluate $\int_0^t \sin u\cos(t-u)du$ using Laplace Transform.**

Solution: Let $L\left[\int_0^t \sin u\cos(t-u)du\right]=L[\sin t*\cos t]$

$= \mathrm{L}[\sin t]\;\; \mathrm{L}[\cos t]$ (by Convolution theorem)

$$=\frac{1}{\left(s^2+1\right)}\frac{s}{\left(s^2+1\right)}=\frac{s}{\left(s^2+1\right)^2}$$

$$\int_0^t \sin u\cos(t-u)du=L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]=\frac{1}{2}L^{-1}\left[\frac{2s}{\left(s^2+1\right)^2}\right]=\frac{t}{2}\sin t\;\left(\because L^{-1}\left[\frac{2s}{\left(s^2+a^2\right)^2}\right]=t\sin at\right)$$

26. **Find the Laplace transform of $\int_0^t t\,e^{-t}\sin t\,dt$**

Solution:

$$L[\sin t]=\frac{1}{s^2+1}$$

$$L\left[t\sin t\right]=-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=-\left(\frac{\left(s^2+1\right)0-2s}{\left(s^2+1\right)^2}\right)=\frac{2s}{\left(s^2+1\right)^2}$$

$$\therefore\ L\left[te^{-t}\ \sin t\right]=\left.\frac{2s}{\left(s^2+1\right)^2}\right|_{s\to s+1}=\frac{2(s+1)}{\left((s+1)^2+1\right)^2}=\frac{2(s+1)}{\left(s^2+2s+2\right)^2}$$

$$L\left[\int_0^t t\,e^{-t}\sin t\,dt\right]=\frac{1}{s}L\left[te^{-t}\sin t\right]$$

$$\therefore \qquad =\frac{1}{s}\frac{2(s+1)}{\left(s^2+2s+2\right)^2}$$

27. **Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$.**

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right]=\left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1}=\left[\frac{1}{s}L\left(t\cos t\right)\right]_{s\to(s+1)}$$

$$=\left[\frac{1}{s}\left(-\frac{d}{ds}L\left(\cos t\right)\right)\right]_{s\to(s+1)}=\left[-\frac{1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to(s+1)}$$

$$=\left[-\frac{1}{s}\left(\frac{s^2+1-2s^2}{\left(s^2+1\right)^2}\right)\right]_{s\to(s+1)}=\left[-\frac{1}{s}\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)\right]_{s\to(s+1)}$$

$$\therefore F(s)=\left[\frac{s^2-1}{s\left(s^2+1\right)^2}\right]_{s\to(s+1)}=\left[\frac{(s+1)^2-1}{(s+1)\left[(s+1)^2+1\right]^2}\right]=\frac{s^2+2s}{(s+1)\left(s^2+2s+2\right)^2}$$

28. **Find the Laplace transform of** $e^{-4t}\int_0^t t\ \sin 3tdt$.

Solution:

$$L[\sin 3t]=\frac{3}{s^2+9}$$

$$L\left[t\sin 3t\right] = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -\left(\frac{\left(s^2+9\right)0-3(2s)}{\left(s^2+9\right)^2}\right) = \frac{6s}{\left(s^2+9\right)^2}$$

$$L\left(\int_0^t t\sin 3t dt\right) = \frac{L(t\sin 3t)}{s} = \frac{6}{(s^2+9)^2}$$

$$L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = L\left(\int_0^t t\sin 3t dt\right)\Bigg|_{s\to s+4} = \frac{6}{\left((s+4)^2+9\right)^2} = \frac{6}{\left(s^2+8s+16+9\right)^2}$$

$$\therefore L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = \frac{6}{\left(s^2+8s+25\right)^2}$$

Problems based on Initial and Final Value Theorems

29. **Verify initial value theorem for the function *f* (*t*) = 2 – cos t.**

Solution

Initial value theorem states that $\lim_{t\to 0} f\left(t\right) = \lim_{s\to\infty} sF\left(s\right)$

L. H. S. = $\lim_{t\to 0} f(t) = 2-\cos 0 = 1$

R. H. S. = $\lim_{s\to\infty} s\,L(f(t)) = \lim_{s\to\infty} s\,L(2-\cos t)$

$$= \lim_{s\to\infty} s\left(2-\frac{s^2}{s^2+1}\right) == \lim_{s\to\infty} s\left(2-\frac{1}{1+\frac{1}{s^2}}\right) = 2-1 = 1$$

L.H.S=R.H.S

Initial value theorem verified.

30. **Verify initial and final value theorems for the function f(t) = 1 + e^{-t} (sin *t* + cos *t*).**

Solution:

Initial value theorem states that $\lim_{t\to 0} f\left(t\right) = \lim_{s\to\infty} sF\left(s\right)$

$$L\left[f\left(t\right)\right] = F\left(s\right)$$

$$= \frac{1}{s} + L\left[\sin t + \cos t\right]_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

$$\text{L.H.S.} = \lim_{t\to 0} f(t) = 1+1 = 2$$

$$\text{R.H.S} = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right] = \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right] = \lim_{s\to\infty}\left[1 + \frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right] = 1+1 = 2$$

L.H.S=R.H.S

Initial value theorem verified.

Final value theorem states that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\text{L.H.S.} = \lim_{t\to\infty}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + 0 = 1$$

$$\text{R.H.S} = \lim_{s\to 0}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right] = 1 + 0 = 1$$

L.H.S.=R.H.S Hence final value theorem verified.

Problems based on Periodic Functions

Periodic function

A function $f(t)$ is said to be **periodic function** if $f(t + p) = f(t)$ for all t. The least value of $p > 0$ is called the **period** of $f(t)$. For example, sin t and cos t are periodic functions with period 2π. tan t is a periodic function with period π.

31. **Find the Laplace transform of the square wave function defined by** $f(t) = \begin{cases} E, & 0 < t < \frac{a}{2} \\ -E, & \frac{a}{2} < t < a \end{cases}$ **&** $f(t+a) = f(t)$.

Solution:

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}\ f(t)\,dt + \int_{a/2}^{a} e^{-st}\ f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} Ee^{-st}\ dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right] = \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-\left(e^{-\frac{as}{2}}-1\right)+\left(e^{-as}-e^{-\frac{as}{2}}\right)\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-e^{-\frac{as}{2}}+1+e^{-as}-e^{-\frac{as}{2}}\right]$$

$$= \frac{E}{s\left(1-e^{-\frac{as}{2}}\right)\left(1+e^{-\frac{as}{2}}\right)}\left(1-e^{-\frac{as}{2}}\right)^2 = \frac{E}{s}\left(\frac{1-e^{-\frac{as}{2}}}{1+e^{-\frac{as}{2}}}\right)$$

$$\therefore F(s) = \frac{E}{s}\left[\frac{e^{sa/4}-e^{-sa/4}}{e^{sa/4}+e^{-sa/4}}\right] = \frac{E}{s}\tanh\left(\frac{sa}{4}\right)$$

32. **Find the Laplace transform of the rectangular wave given by** $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$.

Solution:

Given $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

This function is periodic in the interval $(0, 2b)$ with period $2b$.

$$L\left[f(t)\right] = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st}\ f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}\ f(t)\,dt + \int_b^{2b} e^{-st}\ f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}\ dt + \int_b^{2b} e^{-st}(-1)\,dt\right] = \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{b} - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{s\left(1-e^{-2bs}\right)}\left[-\left(e^{-bs}-1\right)+\left(e^{-2bs}-e^{-bs}\right)\right]$$

$$= \frac{1}{s\left(1-e^{-2bs}\right)}\left[-e^{-bs}+1+\left(e^{-bs}\right)^2-e^{-bs}\right]$$

$$= \frac{1}{s\left(1-e^{-bs}\right)\left(1+e^{-bs}\right)}\left(1-e^{-bs}\right)^2 = \frac{1}{s}\left(\frac{1-e^{-bs}}{1+e^{-bs}}\right)$$

$$\therefore F(s) = \frac{1}{s}\left[\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}\right] = \frac{1}{s}\tanh\left(\frac{sb}{2}\right)$$

33. **Find the Laplace transform of f(t) =** $\begin{cases} t, & 0 \le t \le a \\ 2a-t, & a \le t \le 2a \end{cases}$ **and** $f(t+2a) = f(t)$ **for all** t**.**

Solution:

$$L\left[f(t)\right] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st}\,t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[-t\left(\frac{e^{-st}}{s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[-(2a-t)\left(\frac{e^{-st}}{s}\right)+\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[\left(-a\frac{e^{-as}}{s}-\frac{e^{-as}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]+\left[\frac{e^{-2as}}{s^2}-\left(-\frac{ae^{-as}}{s}+\frac{e^{-as}}{s^2}\right)\right]\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right] = \frac{\left(1-e^{-sa}\right)^2}{s^2\left(1-e^{-as}\right)\left(1+e^{-as}\right)}$$

$$\therefore\ F(s) = \frac{1-e^{-sa}}{s^2\left(1+e^{-as}\right)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

34. **Find the Laplace transform of the rectangular wave given by** $f(t) = \begin{cases} \sin\omega t, & 0 < t < \dfrac{\pi}{\omega} \\ 0, & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$.

Solution:

This function is periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$.

$$L\left[f(t)\right] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}\ f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}[-s\sin\omega t - \omega\cos\omega t]\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega + \omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)} = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

INVERSE LAPLACE TRANSFORMS

Inverse Laplace transform for some basic functions:

S. No.	F(s)	$f(t)=L^{-1}(F(s))$
1	$\frac{1}{s-a},\ s-a>0$	e^{at}
2	$\frac{1}{s+a},\ s+a>0$	e^{-at}
3	$\frac{a}{s^2+a^2},\ s>0$	$\sin at$
4	$\frac{s}{s^2+a^2},\ s>0$	$\cos at$
5	$\frac{a}{s^2-a^2},\ s>\lvert a\rvert$	$\sinh at$
6	$\frac{s}{s^2-a^2},\ s>\lvert a\rvert$	$\cosh at$
7	$\frac{1}{s}$	1
8	$\frac{1}{s^2}$	t
9	$\frac{n!}{s^{n+1}}$	t^n
10	$\frac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
11	$\frac{1}{(s-a)^2+b^2}$	$e^{at}\frac{\sin bt}{b}$
12	$\frac{s-a}{(s-a)^2-b^2}$	$e^{at}\cosh bt$
13	$\frac{1}{(s-a)^2-b^2}$	$e^{at}\frac{\sinh bt}{b}$
14	$\frac{1}{(s-a)^2}$	$t\,e^{at}$
15	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
16	$\frac{s}{(s^2+a^2)^2}$	$\frac{t\sin at}{2a}$

Properties of Inverse Laplace transforms:

S. No.	Property	Laplace Transform
1	Linear Property	$L^{-1}\left[a\,F(s)+b\,G(s)\right]=a\,L^{-1}[F(s)]+b\,L^{-1}[G(s)]$
2	First shifting theorem	$L^{-1}\left[F(s-a)\right]=e^{a\,t}\ f(t)$ $L^{-1}\left[F(s+a)\right]=e^{-a\,t}\ f(t)$
3	Second shifting theorem	$L^{-1}[e^{-a\,s}F(s)]=\begin{cases} f(t-a), t>a \\ 0, t<a \end{cases}$
4	Change of scale property	$L^{-1}\left[F(a\,s)\right]=\frac{1}{a}\,f\left(\frac{t}{a}\right)$
5	Multiplication by *s*	$L^{-1}\left[s\,F(s)\right]=f'(t)$
6	Division by *s*	$L^{-1}\left[\frac{F(s)}{s}\right]=\int_0^t f(t)dt$
7	Inverse Laplace Transforms of integrals	$L^{-1}\left[\int_s^\infty F(s)\,ds\right]=\frac{f(t)}{t}$
8	Inverse Laplace Transforms of derivatives	$L^{-1}\left[F(s)\right]=-\frac{1}{t}L^{-1}[F'(s)]$
9	**Convolution theorem for Inverse Laplace Transforms:** $L^{-1}\left[F(s)\bullet G(s)\right]=f(t)*g(t)$	

35. **Find** $L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)$.

Solution:

$$L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)=e^{3t}+1+\cosh 3t$$

36. **Find** $L^{-1}\left(\dfrac{s}{(s+2)^2}\right)$**.**

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)=L^{-1}\left(\frac{1}{(s+2)}\right)-2L^{-1}\left(\frac{1}{(s+2)^2}\right)=e^{-2t}-2te^{-2t}$$

37. **Find** $L^{-1}\left(\dfrac{1}{s^2+2s+5}\right)$**.**

Solution:

$$L^{-1}\left(\frac{1}{s^2+2s+5}\right)=L^{-1}\left(\frac{1}{(s+1)^2+4}\right)=\frac{e^{-t}\sin 2t}{2}$$

38. **Find** $\boldsymbol{L^{-1}\left[\frac{1}{s^2+6s+13}\right]}$**.**

Solution:

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right]=L^{-1}\left[\frac{1}{(s+3)^2+4}\right]=L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$
$$=\frac{1}{2}L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]=\frac{1}{2}\mathrm{e}^{-3\mathrm{t}}\sin 2t.$$

39. **Find** $\boldsymbol{L^{-1}\left(\frac{s}{s^2+4s+5}\right)}$**.**

Solution:

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right)=L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right)=e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$
$$=e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right)-2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$
$$=e^{-2t}\left[\cos t-2\sin t\right]$$

40. **Find** $\boldsymbol{L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]}$**.**

Solution: $L^{-1}\left[\dfrac{s+2}{s^2+2s+2}\right]=L^{-1}\left[\dfrac{(s+1)+1}{(s+1)^2+1}\right]\quad\because\ L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$\therefore L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = e^{-t}(\cos t + \sin t)$$

Problems based on Multiplication by s

41. **Find the inverse Laplace transform of $\frac{s}{(s+2)^2}$.**

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right) = \frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= \frac{d}{dt}\left(e^{-2t}\,t\right) = e^{-2t} + t\left(-2e^{-2t}\right) = e^{-2t}(1-2t)$$

42. **Find $L^{-1}\left(\frac{s}{(s+2)^3}\right)$.**

Solution: $L^{-1}\left(\frac{s}{(s+2)^3}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^3}\right)$

$$= L^{-1}\left(\frac{1}{(s+2)^2}\right) - 2\,L^{-1}\left(\frac{1}{(s+2)^3}\right)$$

$$= e^{-2t}L^{-1}\left(\frac{1}{s^2}\right) - e^{-2t}L^{-1}\left(\frac{2}{s^3}\right)$$

$$= e^{-2t}\left(t-t^2\right).$$

43. **Find $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$.**

Solution: Let F(s) = $\tan^{-1}\left(\frac{1}{s}\right)$

$$F'(s) = \frac{1}{1+(1/s)^2}\left(\frac{-1}{s^2}\right) = \frac{-1}{s^2+1}$$

By property $L^{-1}\left[F'(s)\right] = -L^{-1}\left[\frac{1}{s^2+1}\right] = -\sin t$

$$\therefore L^{-1}(F'(s)) = -\sin t\,;\; L^{-1}(F(s)) = \frac{-1}{t}L^{-1}\left[F'(s)\right]$$

$$\therefore L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = \frac{\sin t}{t}$$

44. **Find $L^{-1}\left[\cot^{-1}(s+1)\right]$.**

Solution: Let $L^{-1}\left[\cot^{-1}(s+1)\right] = f(t)$

$$\therefore L\left[f(t)\right] = \cot^{-1}(s+1)$$

$$L\left[tf(t)\right] = -\frac{d}{ds}\left[\cot^{-1}(s+1)\right] = \frac{1}{(s+1)^2+1}$$

$$tf(t) = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{-t}\sin t$$

$$\therefore f(t) = \frac{e^{-t}\sin t}{t}$$

45. **Find the inverse Laplace transform of $\log\left(\frac{1+s}{s^2}\right)$.**

Solution:

Let $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right] = f(t)$

$$\therefore L\left[f(t)\right] = \log\left(\frac{1+s}{s^2}\right)$$

$$L\left[t\,f(t)\right]=\frac{-d}{ds}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{-d}{ds}\left[\log(1+s)-\log\left(s^2\right)\right]=-\frac{1}{1+s}+\frac{1}{s^2}2s$$

$$L\left[t\,f(t)\right]=\frac{2}{s}-\frac{1}{s+1}$$

$$t\,f(t)=L^{-1}\left[\frac{2}{s}-\frac{1}{s+1}\right]=2L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+1}\right]=2(1)-e^{-t}$$

$$\therefore\ f(t)=\frac{2-e^{-t}}{t}$$

$$\therefore\ L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{2-e^{-t}}{t}$$

Problems based on Partial Fractions

46. **Find** $L^{-1}\left(\dfrac{s-5}{s^2-3s+2}\right)$**.**

Solution:

$$L^{-1}\left(\frac{s-5}{s^2-3s+2}\right)=L^{-1}\left(\frac{A}{s-1}+\frac{B}{s-2}\right)=L^{-1}\left(\frac{4}{s-1}\right)+L^{-1}\left(\frac{-3}{s-2}\right)=4e^{t}-3e^{2t}$$

47. **Find** $\boldsymbol{L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right]}$**.**

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}$$

$$5s^2-15s-11=A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)$$

Put $s=-1 \Rightarrow \boxed{A=-\frac{1}{3}}$

Equating the coefficients of $s^3 \Rightarrow \boxed{B=\frac{1}{3}}$

Put $s=2 \Rightarrow \boxed{D=-7}$

Put $s=0 \Rightarrow \boxed{C=4}$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{-1/3}{s+1}+\frac{1/3}{s-2}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=-\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2}\right]-7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}\,t-\frac{7}{2}e^{2t}L^{-1}\left[\frac{2}{s^3}\right]$$

$$\therefore f(t)=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}\,t-\frac{7}{2}e^{2t}t^2$$

Problems based on Convolution Theorem

48. **Using Convolution theorem, find** $\boldsymbol{L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]}$**.**

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos at*\frac{1}{a}\sin at=\frac{1}{a}\left[\cos at*\sin at\right]$$

$$=\frac{1}{a}\int_0^t \cos au\ \sin a(t-u)\ du=\frac{1}{a}\int_0^t \sin(at-au)\ \cos au\ \ du$$

$$=\frac{1}{a}\int_0^t \frac{\sin(at-au+au)+\sin(at-au-au)}{2}\ du$$

$$=\frac{1}{2a}\int_0^t \left[\sin at+\sin a(t-2u)\right]\ du$$

$$= \frac{1}{2a}\left[\sin at\ u + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[u\ \sin at + \left(\frac{\cos a(t-2u)}{2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[t\ \sin at + \left(\frac{\cos at}{2a}\right) - \left(0 + \frac{\cos at}{2a}\right)\right]$$

$$f(t) = \frac{1}{2a}\left[t \sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right] = \frac{1}{2a} t \sin at$$

49. **Find the inverse Laplace transform of $\frac{s}{(s^2+a^2)(s^2+b^2)}$ using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$\therefore\ L^{-1}\left[\frac{s}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b}\cos at * \sin bt$$

$$= \frac{1}{b}\int_0^t \cos au\ \ \sin b(t-u)\ du$$

$$= \frac{1}{2b}\int_0^t \left[\sin(au+bt-bu) - \sin(au-bt+bu)\right]\ du$$

$$= \frac{1}{2b}\int_0^t \left[\sin((a-b)u+bt) - \sin((a+b)u-bt)\right]\ du$$

$$= \frac{1}{2b}\left[\frac{-\cos(bt+(a-b)u)}{a-b} + \frac{\cos((a+b)u-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2b}\left[\left(\frac{-\cos(bt+at-bt)}{a-b} + \frac{\cos(at+bt-bt)}{a+b}\right) - \left(\frac{-\cos bt}{a-b} + \frac{\cos bt}{a+b}\right)\right]$$

$$= \frac{1}{2b}\left[\left(\frac{-\cos(at)}{a-b}+\frac{\cos(at)}{a+b}\right)-\left(\frac{-\cos bt}{a-b}+\frac{\cos bt}{a+b}\right)\right]$$

$$= \frac{1}{2b}\left(\frac{-2b\cos at}{a^2-b^2}+\frac{2b\cos bt}{a^2-\mathrm{b}^2}\right)$$

$$f(t) = \frac{\cos bt-\cos at}{a^2-b^2}$$

50. **Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos at * \cos bt$$

$$= \int_0^t \cos au \;\; \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu)+\cos(au-bt+bu)\right]\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos((a-b)u+bt)+\cos((a+b)u-bt)\right]\, du$$

$$= \frac{1}{2}\left[\frac{\sin(bt+(a-b)u)}{a-b}+\frac{\sin((a+b)u-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin(bt+at-bt)}{a-b}+\frac{\sin(at+bt-bt)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(at)}{a-b}+\frac{\sin(at)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2a\sin(at)}{a^2-b^2}-\frac{2b\sin(bt)}{a^2-\mathrm{b}^2}\right)$$

$$f(t) = \frac{a\sin(at)-b\sin(bt)}{a^2-b^2}$$

51. **Find the inverse Laplace transform of** $\dfrac{s}{(s^2+1)(s^2+4)}$**.**

Solution:

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]=L^{-1}\left[\frac{s}{s^2+1}\ \frac{1}{s^2+4}\right]=L^{-1}\left[\frac{s}{s^2+1}\right]*\frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=\frac{1}{2}\cos t*\sin 2t$$

$$=\frac{1}{2}\int_0^t \cos u\ \sin 2(t-u)\,du$$

$$=\frac{1}{4}\int_0^t\left[\sin(u+2t-2u)-\sin(u-2t+2u)\right]du \quad [2\cos A \sin B=\sin(A+B)-\sin(A-B)]$$

$$=\frac{1}{4}\int_0^t\left[\sin(2t-u)-\sin(u-2t)\right]du$$

$$=\frac{1}{4}\left[\frac{-\cos(2t-u)}{-1}+\frac{\cos(u-2t)}{1}\right]_0^t$$

$$=\frac{1}{4}\left[\cos t-\cos 2t+\cos t-\cos 2t\right]$$

$$=\frac{1}{4}\left[2\cos t-2\cos 2t\right]$$

$$\therefore\ f(t)=\frac{1}{2}\left[\cos t-\cos 2t\right]$$

52. **Using Convolution theorem, find the inverse Laplace transform of** $\dfrac{2}{(s+1)(s^2+4)}$**.**

Solution:

$$L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right]=L^{-1}\left[\frac{1}{s+1}\ \frac{2}{s^2+4}\right]=L^{-1}\left[\frac{1}{s+1}\right]*L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=e^{-t}*\sin 2t$$

$$=\int_0^t e^{-u}\ \sin 2(t-u)\,du$$

$$= \int_0^t e^{-u} \sin(2t-2u)\,du$$

$$= \int_0^t e^{-u} \left[\sin 2t \cos 2u - \cos 2t \sin 2u\right] du$$

$$= \int_0^t e^{-u} \sin 2t \cos 2u \; du - \int_0^t e^{-u} \cos 2t \sin 2u \; du$$

$$= \sin 2t \int_0^t e^{-u} \cos 2u \; du - \cos 2t \int_0^t e^{-u} \sin 2u \; du$$

$$= \sin 2t \left[\frac{e^{-u}}{1+4}(-\cos 2u + 2\sin 2u)\right]_0^t - \cos 2t \left[\frac{e^{-u}}{1+4}(-\sin 2u - 2\cos 2u)\right]_0^t$$

$$= \sin 2t \left[\left(\frac{e^{-t}}{5}(-\cos 2t + 2\sin 2t)\right) - \left(\frac{1}{5}(-1)\right)\right] - \cos 2t \left[\frac{e^{-t}}{5}(-\sin 2t - 2\cos 2t) - \left(\frac{1}{5}(-2)\right)\right]$$

$$= \sin 2t \left[\frac{e^{-t}}{5}(-\cos 2t + 2\sin 2t) + \frac{1}{5}\right] - \cos 2t \left[\frac{e^{-t}}{5}\left(-\sin 2t - 2\cos 2t + \frac{2}{5}\right)\right]$$

$$= \frac{e^{-t}}{5}\left[-\sin 2t \cos 2t + 2\sin^2 2t + \sin 2t \cos 2t + 2\cos^2 2t\right] + \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

$$= \frac{e^{-t}}{5}\left[2(1)\right] + \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

$$f(t) = \frac{1}{5}\left[2e^{-t} + \sin 2t - 2\cos 2t\right]$$

53. **Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+1)(s^2+4)}$.**

Solution:

$$L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$\therefore\; L^{-1}\left[\frac{s^2}{(s^2+1^2)(s^2+2^2)}\right] = L^{-1}\left[\frac{s}{s^2+1^2}\right] * L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= \cos t * \cos 2t$$

$$= \int_0^t \cos u \ \cos 2(t-u)\ du$$

$$= \frac{1}{2}\int_0^t \left[\cos(u+2t-2u)+\cos(u-2t+2u)\right]\ du$$

$$= \frac{1}{2}\int_0^t \left[\cos(-u+2t)+\cos(3u-2t)\right]\ du$$

$$= \frac{1}{2}\left[\frac{\sin(2t-u)}{-1}+\frac{\sin(3u-2t)}{3}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin t}{-1}+\frac{\sin t}{3}\right)-\left(\frac{\sin 2t}{-1}-\frac{\sin 2t}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2\sin t}{-3}-\frac{4\sin 2t}{-3}\right)$$

$$f(t) = \frac{\sin t - 2\sin 2t}{-3}$$

54. Find $L^{-1}\left(\dfrac{e^{-2s}}{\left(s^2+s+1\right)^2}\right)$.

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{\left(s^2+s+1\right)^2}\right) = L^{-1}\left(\frac{e^{-s}}{s^2+s+1}\ \frac{e^{-s}}{s^2+s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1} * L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1}$$

$$= L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1} * L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1}$$

$$= e^{-t/2} L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1} * e^{-t/2} L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1}$$

$$= \left[e^{-t/2}\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}} * e^{-t/2}\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}\right]_{t\to t-1}$$

$$= \frac{2}{\sqrt{3}} e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right) * \frac{2}{\sqrt{3}} e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)$$

$$= \frac{4}{3}\left[e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right) * e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\right]$$

$$= \frac{4}{3}\int_0^t e^{-\frac{u-1}{2}} e^{-\frac{t-u-1}{2}} \sin\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}\right)\ \sin\left(\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}\right) du$$

$$= \frac{4}{3}\int_0^t e^{-\left(\frac{t}{2}-1\right)} \frac{1}{2}\ \cos\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}t\right) - \cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right) du$$

$$= \frac{2}{3} e^{-\left(\frac{t-2}{2}\right)}\left[\frac{\sin\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}} - \cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right)u\right]_0^t$$

$$= e^{-\left(\frac{t-2}{2}\right)}\left[\frac{4}{3\sqrt{3}}\sin\frac{\sqrt{3}}{2}t - \frac{2}{3}t\cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right)\right]$$

Problems based on solving differential equations

55. **Solve using Laplace transform $\frac{dy}{dt} + y = e^{-t}$ given that y(0) = 0.**

Solution: Taking L.T. on both sides, we get $L\left[y'(t)\right] + L\left[y(t)\right] = L\left[e^{-t}\right]$

$$sL\left[y(t)\right] - y(0) + L\left[y(t)\right] = L\left[e^{-t}\right]$$

$$sL\left[y(t)\right]-0+L\left[y(t)\right]=\frac{1}{s+1}$$

$$(s+1)\ L[y(t)]=\frac{1}{s+1}$$

$$L[y(t)]=\frac{1}{(s+1)^2}$$

$$\therefore y(t)\ =L^{-1}\left(\frac{1}{(s+1)^2}\right)=e^{-t}L\left(\frac{1}{s^2}\right)=\ e^{-t}\ t \qquad \left(\because\ L\left[e^{-at}f(t)\right]=F(s+a)\right)$$

56. **Using Laplace transform to solve the differential equation** $\boldsymbol{y'' + y' = t^2 + 2t}$**, given** $\boldsymbol{y = 4,\ y' = -2}$ **when** $\boldsymbol{t = 0}$

Solution:

Given $y'' + y' = t^2 + 2t$

$$L\left[y''+y'\right]=L\left[t^2+2t\right]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]+\left[sL\left[y(t)\right]-y(0)\right]=\frac{2}{s^3}+\frac{2}{s^2}$$

$$L\left[y(t)\right]\left(s^2+s\right)=\frac{2}{s^3}+\frac{2}{s^2}+4s-2+4$$

$$L\left[y(t)\right]s(s+1)=\frac{2}{s^3}+\frac{2}{s^2}+4s+2$$

$$L\left[y(t)\right]=\frac{2+2s+4s^4+2s^3}{s^4(s+1)}$$

$$L\left[y(t)\right]=\frac{2}{s}+\frac{2}{s^4}+\frac{2}{s+1}$$

$$y(t)=L^{-1}\left[\frac{2}{s}+\frac{2}{s^4}+\frac{2}{s+1}\right]$$

$$=2+2\frac{t^3}{6}+2e^{-t}$$

$$y(t)=2+\frac{t^3}{3}+2e^{-t}$$

57. **Solve $\left(D^2+3D+2\right)y=e^{-3t}$, given $y(0)=1,$ and $y'(0)=-1$ using Laplace Transforms.**

Solution:

Given $y''+3y'+2y=e^{-3t}$

Taking Laplace transforms on both sides.

$$L\left(y''+3y'+2y\right)=L\left(e^{-3t}\right)$$

$$L\left[y''(t)\right]+3L\left[y'(t)\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]+3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-s(1)-(-1)\right]+3\left[sL\left[y(t)\right]-1\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$L\left[y(t)\right]\left[s^2+3s+2\right]=\frac{1}{s+3}+s+2$$

$$L\left[y(t)\right]=\frac{s^2+5s+7}{(s+3)\left(s^2+3s+2\right)},\ y(t)=L^{-1}\left[\frac{s^2+5s+7}{(s+1)(s+2)(s+3)}\right]$$

$$y(t)=L^{-1}\left[\frac{3/2}{s+1}-\frac{1}{s+2}+\frac{1/2}{s+3}\right]$$

$$y(t)=\frac{3}{2}L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{s+2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$y(t)=\frac{3}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t}$$

58. **Solve $y''+2y'-3y=\sin t$, given $y(0)=0,\ y'(0)=0$.**

Solution:

Given $y''+2y'-3y=\sin t$

$$L\left[y''(t)+2y'(t)-3y(t)\right]=L[\sin t]$$

$$L\left[y''(t)\right]+2L\left[y'(t)\right]-3L\left[y(t)\right]=L[\sin t]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]+2\left[sL\left[y(t)\right]-y(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$\left[s^2L\left[y(t)\right]-s(0)-0\right]+2\left[sL\left[y(t)\right]-(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$s^2L\left[y(t)\right]+2sL\left[y(t)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]\left(s^2+2s-3\right)=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]=\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}$$

$$y(t)=L^{-1}\left[\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}\right]==L^{-1}\left[\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right]$$

Now

$$\frac{1}{(s-1)(s+3)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{\left(s^2+1\right)}$$

$$1=A(s+3)\left(s^2+1\right)+B(s-1)\left(s^2+1\right)+(Cs+D)(s-1)(s+3)$$

Put $s=1\Rightarrow \boxed{A=\frac{1}{8}}$

Put $s=-3\Rightarrow \boxed{B=\frac{-1}{40}}$

Equating coeff. of $s^3\Rightarrow \boxed{C=\frac{-1}{10}}$

Equating the constant terms $\Rightarrow \boxed{D=\frac{-1}{5}}$

$$\therefore\ \frac{1}{(s-1)(s+3)\left(s^2+1\right)}=\frac{1/8}{s-1}+\frac{-1/40}{s+3}+\frac{(-1/10)s-1/5}{\left(s^2+1\right)}$$

$$L^{-1}\left[\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right]=L^{-1}\left[\frac{1/8}{s-1}+\frac{-1/40}{s+3}+\frac{(-1/10)s-1/5}{\left(s^2+1\right)}\right]$$

$$= \frac{1}{8} L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40} L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10} L^{-1}\left[\frac{s+2}{s^2+1}\right]$$

$$= \frac{1}{8} e^{t} - \frac{1}{40} e^{-3t} - \frac{1}{10}\left[L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{2}{s^2+1}\right]\right]$$

$$= \frac{1}{8} e^{t} - \frac{1}{40} e^{-3t} - \frac{1}{10}\left[\cos t + 2\sin t\right]$$

59. **Solve the equation y″+ 9y = cos2t with y(0) =1, y $\left(\frac{\pi}{2}\right)$ = –1.**

Solution:

Given $\left(D^2+9\right) y = \cos 2t$

Taking Laplace transforms on both sides

$$L\left[y''(t)\right] + 9L\left[y(t)\right] = L\left[\cos 2t\right]$$

$$s^2 L\left[y(t)\right] - sy(0) - y'(0) + 9L\left[y(t)\right] = \frac{s}{s^2+4}$$

Using the initial conditions

$y(0) = 1,$ and taking $y'(0) = k$

We have

$$s^2 L\left[y(t)\right] - (s)(1) - k + 9L\left[y(t)\right] = \frac{s}{s^2+4}$$

$$\Rightarrow L\left[y(t)\right] = \frac{s}{\left(s^2+4\right)\left(s^2+9\right)} + \frac{s+k}{s^2+9}$$

$$= \frac{s}{5\left(s^2+4\right)} - \frac{s}{5\left(s^2+9\right)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$\therefore\ y(t) = \frac{1}{5} L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{5} L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + k\, L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{k}{3}\sin 3t$$

Put $t = \frac{\pi}{2}$ we get $y\left(\frac{\pi}{2}\right) = \frac{1}{5}(-1) - \frac{1}{5}(0) + 0 + \frac{k}{3}(-1) = -\frac{1}{5} - \frac{k}{3}$

But given $y\left(\frac{\pi}{2}\right)=-1$

$\therefore\ -1=-\frac{1}{5}-\frac{k}{3}$

$\Rightarrow k=\frac{12}{5}$

$\therefore\ y(t)=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{4}{5}\sin 3t$

$y(t)=\frac{4}{5}\left[\cos 3t+\sin 3t\right]+\frac{1}{5}\cos 2t$

60. **Solve $x''+2x'+5x=e^{-t}\sin t$, where $x(0)=0,\ x'(0)=1$ using Laplace Transforms.**

Solution:

Given $x''+2x'+5x=e^{-t}\sin t$

Taking Laplace transforms on both side

$$L\left[x''+2x'+5x\right]=L\left[e^{-t}\sin t\right]$$

$$L\left[x''(t)\right]+2L\left[x'(t)\right]+5L\left[x(t)\right]=\frac{1}{s^2+2s+2}$$

$$\left[s^2L\left[x(t)\right]-sx(0)-x'(0)\right]+2\left[sL\left[x(t)\right]-x(0)\right]+5L\left[x(t)\right]=\frac{1}{s^2+2s+2}$$

$$\left[s^2L\left[x(t)\right]-s(0)-1\right]+2\left[sL\left[x(t)\right]-(0)\right]+5L\left[x(t)\right]=\frac{1}{s^2+2s+2}$$

$$L\left[x(t)\right]\left[s^2+2s+5\right]=\frac{1}{s^2+2s+2}+1$$

$$L\left[x(t)\right]\left[s^2+2s+5\right]=\frac{s^2+2s+3}{s^2+2s+2}$$

$$L\left[x(t)\right]=\frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}=\frac{(s+1)^2+2}{\left((s+1)^2+1\right)\left((s+1)^2+4\right)}$$

$$x(t) = L^{-1}\left[\frac{(s+1)^2+2}{\left((s+1)^2+1\right)\left((s+1)^2+4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{s^2+2}{\left(s^2+1\right)\left(s^2+4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{1/3}{s^2+1}+\frac{2/3}{s^2+4}\right]$$

$$= e^{-t}\left[\frac{1}{3}\sin t+\frac{1}{3}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}\left[\sin t+\sin 2t\right]$$

61. **Using Laplace transform to solve the differential equation**

$$\boldsymbol{y'' - 3y' + 2y = 4t + e^{3t}}, \textbf{ where } \boldsymbol{y(0) = 1,\ y'(0) = -1}$$

Solution:

Given $y'' - 3y' + 2y = 4t + 3e^{t}$

$$L\left[y'' - 3y' + 2y\right] = L\left[4t + 3e^{t}\right]$$

$$L\left[y''(t)\right] - 3L\left[y'(t)\right] + 2L\left[y(t)\right] = 4L[t] + 3L\left[e^{3t}\right]$$

$$\left[s^2L\left[y(t)\right] - sy(0) - y'(0)\right] - 3\left[sL\left[y(t)\right] - y(0)\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2L\left[y(t)\right] - s(1) - (-1)\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2L\left[y(t)\right] - s + 1\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2-3s+2\right)=s-4+\frac{4}{s^2}+\frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2-3s+2\right)=\frac{(s-4)s^2(s-3)+4(s-4)+3s^2}{s^2(s-3)}$$

$$L\left[y(t)\right]=\frac{s^4-7s^3+13s^2+4s-12}{\left(s^2-3s+2\right)s^2(s-3)}$$

$$L\left[y(t)\right]=\frac{s^4-7s^3+13s^2+4s-12}{(s-2)(s-1)s^2(s-3)}$$

$$y(t)=L^{-1}\left[\frac{s^4-7s^3+13s^2+4s-12}{(s-2)(s-1)s^2(s-3)}\right]$$

$$=L^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{s-2}+\frac{E}{s-3}\right]$$

$$=L^{-1}\left[\frac{3}{s}+\frac{2}{s^2}+\frac{-1/2}{s-1}+\frac{-2}{s-2}+\frac{1/2}{s-3}\right]$$

$$y(t)=3+2t-\frac{1}{2}e^t-2e^{2t}+\frac{1}{2}e^{3t}$$

62. **Solve $y''-3y'+2y=e^{2t}$, $y(0)=-3$, $y'(0)=5$.**

Solution:

Given $y''-3y'+2y=e^{2t}$

$$L\left[y''-3y'+2y\right]=L\left[e^{2t}\right]$$

$$L\left[y''\right]-3L\left[y'\right]+2L[y]=L\left[e^{2t}\right]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]-3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$\left[s^2L\left[y(t)\right]-s(-3)-5\right]-3\left[sL\left[y(t)\right]-(-3)\right]+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$s^2L\left[y(t)\right]+3s-5-3sL\left[y(t)\right]-9+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$L\left[y(t)\right]\left[s^2-3s+2\right]+3s-14=\frac{1}{s-2}$$

$$\therefore \quad L\left[y(t)\right]\left[s^2-3s+2\right]=\frac{1}{s-2}-3s+14$$

$$\therefore \quad L\left[y(t)\right]=\frac{-3s^2+20s-27}{(s-2)\left(s^2-3s+2\right)}$$

$$y(t)=L^{-1}\left[\frac{-3s^2+20s-27}{(s-2)\left(s^2-3s+2\right)}\right]$$

$$y(t)=L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]$$

$$\frac{-3s^2+20s-27}{(s-1)(s-2)^2}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$-3s^2+20s-27=A(s-2)^2+B(s-1)(s-2)+C(s-1)$$

Put $s=1 \Rightarrow \boxed{A=-10}$

Put $s=2 \Rightarrow \boxed{C=1}$

Equating the coeff. of $s^2 \Rightarrow \boxed{B=7}$

$$\therefore \quad \frac{-3s^2+20s-27}{(s-1)(s-2)^2}=\frac{-10}{s-1}+\frac{7}{s-2}+\frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]=L^{-1}\left[\frac{-10}{s-1}\right]+L^{-1}\left[\frac{7}{s-2}\right]+L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$=-10e^t+7e^{2t}+e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=-10e^t+7e^{2t}+te^{2t}$$

63. **Use Laplace Transform to solve** $(D^2-3D+2)y=e^{3t}$ **with** $y(0)=1$ **and** $y'(0)=0$**.**

Solution:

$$y''-3y'+2y=e^{3t} \text{ -----------(1)}$$

$$L(y'')-3L(y')+2L(y)=L(e^{3t})$$

$$\left(s^2L(y)-sy(0)-y'(0)\right)-3(sL(y)-y(0))+2L(y)=\frac{1}{s-3}$$

$$\left(s^2-3s+2\right)L(y)-s+3=\frac{1}{s-3}$$

$$(s-1)(s-2)L(y)=\frac{1}{s-3}+s-3$$

$$L(y)=\frac{s^2-6s+10}{(s-1)(s-2)(s-3)}$$

$$y(t)=L^{-1}\left[\frac{s^2-6s+10}{(s-1)(s-2)(s-3)}\right]$$

Consider $\frac{s^2-6s+10}{(s-1)(s-2)(s-3)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$

$$s^2-6s+10=A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$$

Put $s=1,\ A=\frac{5}{2}$, put $s=2$, $B=-2$ and for $s=3,\ C=\frac{1}{2}$

$$y(t)=L^{-1}\left[\frac{5/2}{s-1}+\frac{-2}{s-2}+\frac{1/2}{s-3}\right]$$

$$y(t)=\frac{5}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}$$

64. **Using Lapalce Transform, Solve** $\frac{d^2y}{dt^2}+4y=\sin 2t$**, given** $y(0)=3\ \&\ y'(0)=4$**.**

Solution:

$$y''+4y=\sin 2t$$

$$L(y'')+4L(y)=L(\sin 2t)$$

$$\left(s^2L(y)-sy(0)-y'(0)\right)+4L(y)=\frac{2}{s^2+4}$$

$$\left(s^2+4\right)L(y)-3s-4=\frac{2}{s^2+4}$$

$$\left(s^2+4\right)L(y)=\frac{2}{s^2+4}+3s+4$$

`

$$L(y)=\frac{3s^3+4s^2+12s+18}{(s^2+4)^2}$$

Consider $\frac{3s^3+4s^2+12s+18}{(s^2+4)^2}=\frac{(As+B)}{s^2+4}+\frac{(Cs+D)}{(s^2+4)^2}$

$$3s^3+4s^2+12s+18=(As+B)\left(s^2+4\right)+(Cs+D)$$

Comparing the co.eff of s^3, $A=3$

Comparing the co.eff of s^2, $B=4$

Comparing the co.eff of s, $C=0$

Comparing the constant term $D=2$

$$y(t) = L^{-1}\left[\frac{(3s+4)}{s^2+4} + \frac{(0.s+2)}{(s^2+4)^2}\right]$$

$$= \quad 3L^{-1}\left(\frac{s}{s^2+4}\right) + 2L^{-1}\left(\frac{2}{s^2+4}\right) + L^{-1}\left(\frac{2}{(s^2+4)^2}\right)$$

$$= 3\cos 2t + 2\sin 2t + \frac{t^3 e^{-2t}}{6}$$

65. **Solve $\frac{dx}{dt} - 2x + 3y = 0;\ \frac{dy}{dt} - y + 2x = 0$ with $x(0) = 8,\ y(0) = 3$.**

The given differential equation canbe written as

$$x'(t) - 2x + 3y = 0 \quad y'(t) - y + 2x = 0$$

Taking Laplace transforms weget,

$$L\left[x'(t) - 2x + 3y\right] = L[0]$$

$$sL[x(t)] - x(0) - 2L[x(t)] + 3L[y(t)] = 0$$

$$sL[x(t)] - 8 - 2L[x(t)] + 3L[y(t)] = 0$$

$$L[x(t)](s-2) + 3L[y(t)] = 8 \qquad (1)$$

And $L\left[y'(t) - y + 2x\right] = L[0]$

$$sL[y(t)] - y(0) - L[y(t)] + 2L[x(t)] = 0$$

$$sL[y(t)] - 3 - L[y(t)] + 2L[x(t)] = 0$$

$$2L[x(t)] + (s-1)L[y(t)] = 3 \qquad (2)$$

Solving (1) and (2) we get,

$$L[x(t)] = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4},$$

$$\therefore x(t) = L^{-1}\left[\frac{5}{s+1} + \frac{3}{s-4}\right],$$

$$\boxed{x(t) = 5e^{-t} + 3e^{4t}}$$

and $$L\left[y(t)\right]=\frac{3s-22}{(s+1)(s-4)}=\frac{5}{s+1}-\frac{2}{s-4}$$

$$y(t)=L^{-1}\left[\frac{5}{s+1}-\frac{2}{s-4}\right]=5e^{-t}-2e^{4t}$$

66. **Determine y which satisfies the equation** $\boldsymbol{\frac{dy}{dt}+2y+\int_0^t y\,dt=2\cos t,\ y(0)=1}$

Solution:

Given $y'(t)+2y(t)+\int_0^t y(t)\,dt=2\cos t,\ y(0)=1$

$$L\left[y'(t)\right]+2L\left[y(t)\right]+L\left[\int_0^t y(t)\,dt\right]=L\left[2\cos t\right]$$

$$sL\left[y(t)\right]-y(0)+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$sL\left[y(t)\right]-1+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$L\left[y(t)\right]=\frac{s}{s^2+1}$$

$$y(t)=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$

* * * * *

Module – 4 Analytic Functions

Definition of Analytic Function – Cauchy Riemann equations – Properties of analytic functions – Determination of analytic function using Milne Thomson's method – Conformal mappings: Magnification, Rotation, Inversion, Reflection – Bilinear Transformation – Cauchy's integral theorem (without proof) – Cauchy's integral theorem applications – Application of Bilinear transformation and Cauchy's Integral in Engineering.

Analytic function (or) Holomorphic function (or) Regular function.

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire (or) an Integral function.

A function which is analytic everywhere in the finite plane except at z = ∞ is called an entire function.
Example: e^z, sinz, coshz.

Necessary conditions for f(z) to be analytic.

The necessary conditions for a complex function f(z) = u(x,y) + i v(x,y) to be analytic in a region R are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \quad \text{(i.e) C – R equations.}$$

Sufficient conditions for f (z) to be analytic.

If the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial v}{\partial y},\frac{\partial v}{\partial x},\frac{\partial u}{\partial y}$ exist and continuous in D and satisfies the conditions

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$. Then the function f(z) is analytic in a domain D.

Harmonic function.

Any function which possess continuous second order partial derivatives and which satisfies Laplace equation is called a harmonic function. (i.e) If $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, then f is harmonic then

Show that the function u = 2x – x^3 + $3xy^2$ is harmonic.

Solution: Given u = 2x – x^3 + $3xy^2$

$u_x = 2 - 3x^2 + 3y^2 \qquad u_y = 6xy$

$u_{xx} = -6x \qquad u_{yy} = 6x$

$u_{xx}+u_{yy} = -6x + 6x = 0$

Hence u is harmonic

Show that the function $u=\frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find f(z).

Given $u=\frac{1}{2}\log\left(x^2+y^2\right)$

$$\frac{\partial u}{\partial x}=\frac{1}{2}.\frac{1}{x^2+y^2}(2x)=\frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{2}.\frac{1}{x^2+y^2}(2y)=\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\left(x^2+y^2\right)(1)-2y^2}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2}=0$$

Hence u is harmonic function

To find conjugate of u

$$\phi_1(x,y)=\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}$$

$$\phi_1(z,o)=\frac{1}{z}$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$$

$$\phi_2(z,o)=0$$

By Milne Thomson Methods

$$f'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

$$\int f'(z)\,dz=\int\frac{1}{z}dz+0$$

$$=\log z+c$$

$$f(z)=\log re^{i\theta}$$

$$f(z)=u+iv=\log r+i\theta$$

$u=\log r$, $v=\theta$

$u=\log\sqrt{x^2+y^2}$ $\left[\therefore r^2=x^2+y^2,\ \ \theta=\tan^{-1}\left(\frac{y}{x}\right)\right]$

$v=\tan^{-1}\left(\frac{y}{x}\right)$ $\therefore$ Conjugate of u is $\tan^{-1}\left(\frac{y}{x}\right)$.

Conformal transformation.

A mapping or transformation which preserves angles in magnitude and in direction between every pair of curves through a point is said to be conformal transformation.

Isogonal transformation.

A transformation under which angles between every pair of curves through a point are preserved in magnitude but altered in sense is said to be isogonal at that point.

Bilinear transformation (or) Mobius transformation (or) linear fractional transformation.

The transformation $w=\frac{az+b}{cz+d}$, ad – bc $\neq 0$ where a, b,c,d are complex numbers is called a bilinear transformation. This is also called as Mobius or linear fractional transformation.

Cross Ratio.

The cross ratio of four points z_1, z_2, z_3, z_4 is given by $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$.

Show that f (z) = $|z|^2$ is differentiable at z = 0 but not analytic at z = 0.

Solution: Let z = x + iy and $\bar{z}=x-iy$

$$|z|^2 = z\bar{z} = x^2+y^2$$

$$f(z)=|z|^2=(x^2+y^2)+i0$$

$$u=x^2+y^2 \quad , \quad v=0$$

$$u_x=2x \quad , \quad v_x=0$$

$$u_y=2y \quad , \quad v_y=0$$

So the C-R equations $u_x = v_y$ and $u_y = -v_x$ are not satisfied everywhere except at z = 0.

So f (z) may be differentiable only at z = 0. Now $u_x = 2x$, $v_y = 0$ and $u_y = 2y$, $v_x = 0$ are continuous everywhere and in particular at (0, 0). So f (z) is differentiable at z = 0 only and not analytic.

Obtain the invariant points of the transformation $w=\frac{z-1}{z+1}$

Solution: Given: $w=\frac{z-1}{z+1}$

The invariant points are obtained by replacing w by z.

$$\text{i.e, } z=\frac{z-1}{z+1} \Rightarrow z^2+1=0 \;\therefore z=\pm i$$

Can v = $\tan^{-1}\left(\frac{y}{x}\right)$ be the imaginary part of an analytic function? If so construct an analytic function f(z) = u + iv, taking v as the imaginary part and hence find u.

Solution:

Let $v=\tan^{-1}\left(\frac{y}{x}\right)$

$$v_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2};\ v_{xx} = -\left(\frac{(x^2+y^2).0 - y(2x)}{\left(x^2+y^2\right)^2}\right) = \frac{2xy}{\left(x^2+y^2\right)^2}$$

$$v_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2};\quad v_{yy} = -\left(\frac{(x^2+y^2).0 - x(2y)}{\left(x^2+y^2\right)^2}\right) = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

$v_{xx} + v_{yy} = 0 \Rightarrow v$ *is harmonic and hence v can be the imaginary part of an analytic function.*

By Milne's method, $f(z) = \int \left\{v_y(z,0) + i v_x(z,0)\right\} dz + c$

$$v_x = \frac{-y}{x^2+y^2};\ v_x(z,0) = 0;$$

$$v_y = \frac{x}{x^2+y^2};\ v_y(z,0) = \frac{1}{z}$$

$$f(z) = \int \frac{dz}{z} + c = \log z + c = \log r + i\theta + c_1 + ic_2 \quad (\because z = re^{i\theta})$$

$$= \underbrace{\left(\frac{1}{2}\log(x^2+y^2) + c_1\right)}_{u} + i \underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{v} \qquad \left(\because r = \sqrt{x^2+y^2}\ \&\ \theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$(c_2 = 0)$$

$$\therefore u = \frac{1}{2}\log(x^2+y^2) + c_1$$

Prove that $u = x^2 - y^2$ & $v = \dfrac{-y}{x^2+y^2}$ are harmonic functions but not harmonic conjugate.

Solution:

$u = x^2 - y^2$	$v = \dfrac{-y}{x^2+y^2}$
$u_x = 2x$	$v_x = \dfrac{2xy}{\left(x^2+y^2\right)^2}$
$u_y = -2y$	$v_y = \dfrac{y^2 - x^2}{\left(x^2+y^2\right)^2}$
$u_{xx} = 2$	$v_{xx} = \dfrac{2y(y^2 - 3x^2)}{\left(x^2+y^2\right)^3}$
$u_{yy} = -2$	$v_{yy} = \dfrac{2y(3x^2 - y^2)}{\left(x^2+y^2\right)^3}$
$\therefore u_{xx} + u_{yy} = 0$	$v_{xx} + v_{yy} = 0$

Hence u and v are harmonic.

But $u_x \neq v_y$ & $v_x \neq -u_y$

C-R equations are not satisfied. Hence u+iv are not an analytic function. So they are not harmonic conjugate.

Prove that $\mathbf{w = \dfrac{z}{z+a}}$ where $\mathbf{a \neq 0}$ is analytic whereas $\mathbf{w = \dfrac{\bar{z}}{\bar{z}+a}}$ is not analytic.

Solution:

$$w = \frac{z}{z+a} = \frac{x+iy}{x+iy+a} = \frac{x+iy}{(x+a)+iy} = \frac{x+iy}{(x+a)+iy}\left(\frac{(x+a)-iy}{(x+a)-iy}\right)$$

$$= \frac{(x+iy)((x+a)-iy)}{(x+a)^2+y^2} = \frac{x(x+a)+y^2}{(x+a)^2+y^2} + i\,\frac{(x+a)y-xy}{(x+a)^2+y^2}$$

$$w = \underbrace{\frac{x(x+a)+y^2}{(x+a)^2+y^2}}_{u} + i\underbrace{\frac{ay}{(x+a)^2+y^2}}_{v}$$

$$u = \frac{x(x+a)+y^2}{(x+a)^2+y^2};$$

$$u_x = \frac{((x+a)^2+y^2)(2x+a)-(x(x+a)+y^2)(2(x+a))}{((x+a)^2+y^2)^2}$$

$$= \frac{2x(x+a)+2xy^2-2x^2(x+a)-2xy^2-2ax(x+a)-2ay^2}{((x+a)^2+y^2)^2}$$

$$= \frac{(x+a)(2x^2+2ax+ax+a^2-2x^2-2ax)-ay^2}{((x+a)^2+y^2)^2}$$

$$u_x = \frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\ldots(1)$$

$$u_y = \frac{((x+a)^2+y^2)(2y)-(x(x+a)+y^2)(2y)}{((x+a)^2+y^2)^2}$$

$$= \frac{2y((x+a)^2+y^2-(x(x+a)+y^2))}{((x+a)^2+y^2)^2}$$

$$= \frac{2y(x^2+ax+a^2+y^2-x^2-ax-y^2)}{((x+a)^2+y^2)^2}$$

$$u_y = \frac{2ay(x+a)}{((x+a)^2+y^2)^2}\ldots(2)$$

$$v = \frac{ay}{(x+a)^2 + y^2};$$

$$v_x = \frac{((x+a)^2 + y^2)(0) - (ay)(2(x+a))}{((x+a)^2 + y^2)^2}$$

$$v_x = \frac{-2ay(x+a)}{((x+a)^2 + y^2)^2} \ldots (3)$$

$$v_y = \frac{((x+a)^2 + y^2)(a) - (ay)(2y)}{((x+a)^2 + y^2)^2}$$

$$= \frac{a((x+a)^2 + y^2 - 2y^2)}{((x+a)^2 + y^2)^2}$$

$$v_y = \frac{a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \ldots (4)$$

From (1) and (4), $u_x = v_y$

From (2) and (3), $u_y = -v_x$

Also u_x, u_y, v_x, v_y *are continuous functions in* x *and* y.

Hence $w = \dfrac{z}{z+a}$ *is analytic*.

$$Now\ \ w = \frac{\bar{z}}{\bar{z}+a} = \frac{x-iy}{x-iy+a} = \frac{x-iy}{(x+a)-iy} = \frac{x-iy}{(x+a)-iy}\left(\frac{(x+a)+iy}{(x+a)+iy}\right)$$

$$= \frac{(x-iy)((x+a)+iy)}{(x+a)^2 + y^2} = \frac{x(x+a)+y^2}{(x+a)^2 + y^2} + i\,\frac{(-(x+a)y + xy)}{(x+a)^2 + y^2}$$

$$w = \underbrace{\frac{x(x+a)+y^2}{(x+a)^2 + y^2}}_{u} + i\,\underbrace{\frac{-ay}{(x+a)^2 + y^2}}_{v}$$

$$u = \frac{x(x+a)+y^2}{(x+a)^2 + y^2};$$

$$u_x = \frac{a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \ldots (5)$$

$$u_y = \frac{2ay(x+a)}{((x+a)^2 + y^2)^2} \ldots (6)$$

$v = \frac{-ay}{(x+a)^2 + y^2}$;

$v_x = \frac{2ay(x+a)}{((x+a)^2 + y^2)^2} \ldots (7)$

$v_y = \frac{-a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \ldots (8)$

From (5) and (8), $u_x \neq v_y$

From (6) and (7), $u_y \neq -v_x$

Hence $w = \frac{\bar{z}}{\bar{z}+a}$ *is not analytic.*

Properties of Analytic function

Property : 1

The function f(z) = u + iv is analytic, show that u = constant and v = constant are orthogonal

Proof:

If $f(z) = u + iv$ is an analytic function of z, then it satisfies C-R equations

$u_x = v_y,\ u_y = -v_x$

Given $u(x,y) = C_1$.............(1)

$v(x,y) = C_2$.............(2)

By total differentiation

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

Differentiate equation (1) & (2) we get $du = 0$, $dv = 0$

$\therefore \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$ and $\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-\partial u / \partial x}{\partial u / \partial y} = m_1 (say)$

$\frac{dy}{dx} = \frac{-\partial v / \partial x}{\partial v / \partial y} = m_2 (say)$

$\therefore m_1\, m_2 = -\frac{-\partial u / \partial x}{\partial u / \partial y} \times \frac{-\partial v / \partial x}{\partial v / \partial y} \quad (\because u_x = v_y \quad u_y = -v_x)$

$\therefore m_1\, m_2 = -1$

The curves $u(x,y) = C_1$ and $v(x,y) = C_2$ cut orthogonally.

Property : 2

Prove that an analytic function with constant modulus is constant.

Proof:

Let $f\left(z\right)=u+iv$ be analytic

By C.R equations satisfied

i.e., $u_x = v_y,\ u_y = -v_x$

$\therefore f\left(z\right)=u+iv$

$\Rightarrow \left|f\left(z\right)\right| = \sqrt{u^2+v^2} = C \Rightarrow \left|f\left(z\right)\right|^2 = u^2+v^2 = C^2$

$u^2+v^2 = C^2$...............(1)

Diff (1) with respect to x

$2u\dfrac{\partial u}{\partial x}+2v\dfrac{\partial v}{\partial x}=0$

$uu_x + vv_x = 0$.............(2)

Diff (1) with respect to y

$2u\dfrac{\partial u}{\partial y}+2v\dfrac{\partial v}{\partial y}=0$

$-uv_x + vu_x = 0$............(3)

$(2)\times u+(3)\times v \Rightarrow \left(u^2+v^2\right)u_x = 0$

$\Rightarrow u_x = 0$

$(2)\times v-(3)\times u \Rightarrow \left(u^2+v^2\right)v_x = 0$

$\Rightarrow v_x = 0$

W.K.T $f'\left(z\right)=u_x + iv_x = 0$

$f'\left(z\right)=0$ Integrate w.r.to z

$f\left(z\right)=C$

Property : 3

8. Prove that the real and imaginary parts of an analytic function are harmonic function.

Proof:

Let f (z) = u + iv be an analytic function of z. Then by C- R equations we have,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\text{.......... .(1)}\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\text{.........(2)}$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}\text{.......... .(3)}$$

Differentiating (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} \quad ....(4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$\therefore$ u satisfies the Laplace equation.

Similarly

Differentiating (1) partially with respect to y, we get

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \quad ....(5)$$

Differentiating (2) partially with respect to x, we get

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \quad ....(6)$$

Adding (5) and (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$\therefore$ v satisfies the Laplace equation.

Hence the real and imaginary parts of an analytic function are harmonic function.

Property : 4

9. The real part of an analytic function f(z) is constant, prove that f(z) is a constant function.

Proof:

Let f(z) = u + iv

Given u = constant. $\Rightarrow u_x = 0$ and $u_y = 0$

by C-R equations, $u_x = 0 \Rightarrow v_y = 0$ and $u_y = 0 \Rightarrow v_x = 0$

$f'(z) = u_x + iv_x = 0 + i0 = 0$

Integrating, f (z) = c (where c is a constant)

10. If f(z) is an analytic function, prove that $\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}|f(z)|^2 = 4|f'(z)|^2$

Proof:

Let f(z) = u + iv be analytic.

Then $u_x = v_y$ and $u_y = -v_x$ (1)

Also $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)

Now $|f(z)|^2 = u^2 + v^2$ and $f'(z) = u_x + iv_x$

$$\therefore \frac{\partial}{\partial x}|f(z)|^2 = 2u.u_x + 2v.v_x$$

and $$\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right] \qquad (3)$$

Similarly $$\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right] \qquad (4)$$

Adding (3) and (4)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u_x^2 + u_y^2 + u(u_{xx} + u_{yy}) + v_x^2 + v_y^2 + v(v_{xx} + v_{yy})\right]$$

$$= 2\left[u_x^2 + v_x^2 + u(0) + v_x^2 + u_x^2 + v(0)\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

11. Find the map of the circle (i) $|z| = 3$ under the transformation w = 2z

(ii) $|z| = 1$ by the transformation w = z + 2 + 4i

Solution (i) : Given w = 2z **,** $|z| = 3$

$|w| = 2|z|$

$|w| = 2(3) = 6$

Hence the image of the circle $|z| = 3$ in the z-plane maps to the circle $|w| = 6$ in the w-plane.

Solution (ii) :

Given: w = z + 2 + 4i

u + iv = x + iy + 2 + 4i = (x + 2) + i (y + 4)

u = x + 2, v = y + 4

⇨ x = u – 2, y = v – 4

⇨ $|z| = 1$

$x^2 + y^2 = 1$ Hence $(u - 2)^2 + (v - 4)^2 = 1$.

$\therefore$ The circle in the z-plane is mapped into the circle in the w -plane with centre (2, 4) and radius 1.

Find the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Solution:

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$z=\dfrac{1}{u+iv}=\dfrac{u-iv}{u^2+v^2} \Rightarrow x=\dfrac{u}{u^2+v^2}$ ________(1) $y=-\dfrac{v}{u^2+v^2}$ ____________________(2)

Given strip is $\dfrac{1}{4}<y<\dfrac{1}{2}$ when $y=\dfrac{1}{4}$

$\dfrac{1}{4}=-\dfrac{v}{u^2+v^2}$ $(by\ 2)$

$u^2+(v+2)^2=4$..........(3)

which is a circle whose centre is at $(0,-2)$ in the w-plane and radius 2.

When $y=\dfrac{1}{2}$

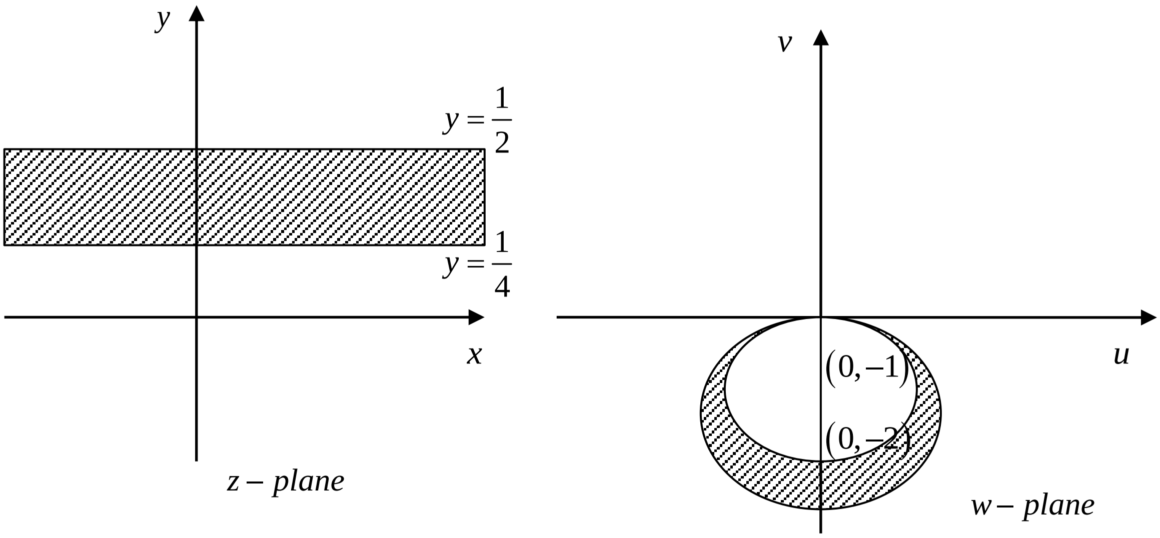
$\dfrac{1}{2}=\dfrac{-v}{u^2+v^2}$ $(by\ 2)$

$u^2+v^2+2v=0$

$u^2+(v+1)^2=1$..........(4)

which is a circle whose centre is at $(0,-1)$ and radius is 1 in the w-plane.

Hence the infinite strip $\dfrac{1}{4}<y<\dfrac{1}{2}$ is transformed into the region between circles $u^2+(v+1)^2=1$ and $u^2+(v+2)^2=4$ in the w-plane.



Find the image of $|z-2i|=2$ under the transformation $w=\dfrac{1}{z}$

Solution:

Given $w=\dfrac{1}{z}\Rightarrow z=\dfrac{1}{w}$

Now $w=u+iv$

$z=\dfrac{1}{w}=\dfrac{1}{u+iv}=\dfrac{u-iv}{(u+iv)(u-iv)}=\dfrac{u-iv}{u^2+v^2}$

i.e., $x+iy=\dfrac{u-iv}{u^2+v^2}$ $\therefore$ $x=\dfrac{u}{u^2+v^2}$..........(1) $y=\dfrac{-v}{u^2+v^2}$..........(2)

Given $|z-2i|=2$

$|x+iy-2i|=2 \Rightarrow |x+i(y-2)|=2$

$x^2+(y-2)^2=4 \Rightarrow x^2+y^2-4y=0$..................(3)

Sub (1) and (2) in (3)

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left[\frac{-v}{u^2+v^2}\right]=0$$

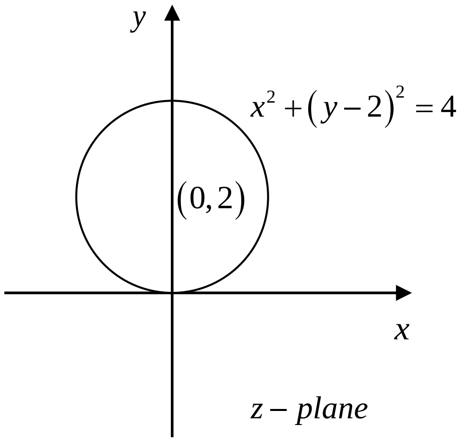
$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\left[\frac{4v}{u^2+v^2}\right]=0$$

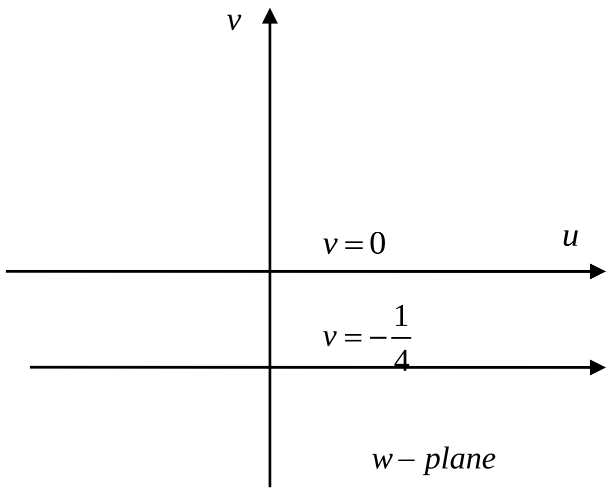
$$\frac{(u^2+v^2)+4v(u^2+v^2)}{(u^2+v^2)^2}=0$$

$$\frac{(1+4v)(u^2+v^2)}{(u^2+v^2)^2}=0$$

$1+4v=0 \Rightarrow v=-\frac{1}{4} \quad (\because u^2+v^2\neq 0)$

which is a straight line in w-plane.





Show that the transformation $w=\frac{1}{z}$ transforms in general, circles and straight lines into circles and straight lines.

Solution:

$w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$\Rightarrow x+iy=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$

$\Rightarrow x=\frac{u}{u^2+v^2}$ *and* $y=\frac{-v}{u^2+v^2}$

Consider the equation $a\,(x^2 + y^2) + bx + cy + d = 0$ ------------(1)
This equation represents a circle if $a \neq 0$ and a straight line if $a = 0$

Under the transformation $w = \dfrac{1}{z}$ equation (1) becomes

$d\,(u^2 + v^2) + bu - cv + a = 0$ --------------(2)

This equation represents a circle if $d \neq 0$ and a straight line if $d = 0$

Value of a & d	Equation (1) and (2)	Conclusion
$a \neq 0,\ d \neq 0$	Equation (1) and (2) represents a circle, not passing through the origin, in the z–plane and w–plane	The transformation maps a circle not passing through the origin in z–plane into a circle not passing through the origin in w–plane
$a \neq 0,\ d = 0$	Equation (1) represents a circle passing through the origin in the z–plane and equation (2) represents a straight line not passing through the origin in w–plane	The transformation maps a circle passing through the origin in z–plane into a straight line not passing through the origin in w–plane
$a = 0,\ d \neq 0$	Equation (1) represents a straight line not passing through the origin in the z–plane and equation (2) represents a circle passing through the origin in w–plane	The transformation maps a straight line not passing through the origin in the z–plane into a circle passing through the origin in w–plane
$a = 0,\ d = 0$	Equation (1) and (2) represents a straight line passing through the origin in the z–plane and w–plane	The transformation maps represents a straight line passing through the origin in z–plane into a straight line passing through the origin in w–plane

Thus the transformation $w = \dfrac{1}{z}$ maps the totality of circles and straight lines as circles or straight lines.

Find the image of the circle $|z-1| = 1$ under the transformation $w = z^2$

Solution:

In polar form $z = r\,e^{i\theta}$, $w = \mathrm{R}\,e^{i\phi}$

Given

$$|z-1|=1$$

$$|re^{i\theta}-1|=1$$

$$|r\cos\theta+i\,r\sin\theta-1|=1$$

$$|(r\cos\theta-1)+i\,r\sin\theta|=1$$

$$(r\cos\theta-1)^2+(r\sin\theta)^2=1^2$$

$$r^2-2r\cos\theta=0$$

$$r=2\cos\theta----(1)$$

Now, we have

$$w=z^2$$

$$\mathrm{R}\,e^{i\phi}=\left(r\,e^{i\theta}\right)^2$$

$$\mathrm{R}\,e^{i\phi}=r^2\,e^{i2\theta}$$

$$\mathrm{R}=r^2,\qquad \phi=2\theta$$

$$(1)\ \Rightarrow$$

$$r^2=\left(2\cos\theta\right)^2$$

$$=4\cos^2\theta$$

$$=4\left[\frac{1+\cos2\theta}{2}\right]$$

$$r^2=2(1+\cos2\theta)$$

$$R=2(1+\cos\phi)$$

Find the bilinear transformation of the points –1,0,1 in z- plane onto the points 0,*i*,3*i* in w- plane.

Solution:

Given $z_1=-1,\ w_1=0$

$z_2=0,\ w_2=i$

$z_3=i,\ w_3=3i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)}=\frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)}=\frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{2w}{w-3i}=\frac{z+1}{z-1}$$
$$2wz-2w=wz+w-3iz-3i$$
$$w(2z-2-z-1)=-3i(z+1)$$
$$w(z-3)=-3i(z+1)$$
$$\therefore w=-3i\frac{(z+1)}{(z-3)}$$

Find the bilinear transformation which maps the points $z=\infty, i, 0$ into $w=0, i, \infty$ respectively.

Solution:

Given $z_1=\infty,\ w_1=0$
$z_2=i,\ w_2=i$
$z_3=0,\ w_3=\infty$
Cross-ratio
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)z_1\left(\frac{z_2}{z_1}-z_1\right)}$$
$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$
$$\frac{(w-0)(0-1)}{(0-1)(i-0)}=\frac{(0-1)(i-0)}{(z-0)(0-1)}$$
$$\frac{w}{i}=\frac{i}{z},\qquad w=\frac{i^2}{z},\qquad \therefore w=-\frac{1}{z}$$

Find the bilinear transformation which maps the points z = 1, i, -1 into the points

w = 0, 1, ∞ .

Solution:

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Here, $w_3=\infty$

$$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)w_3\left(1-\frac{w}{w_3}\right)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{\infty}-1\right)}{(w_1-w_2)\left(1-\frac{w}{\infty}\right)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)(-1)}{(w_1-w_2)(1)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)}{(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)}{(1-0)}=\frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$w=\frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w=\frac{(z-1)(i+1)}{(z+1)(i-1)}\frac{(i+1)}{(i+1)}=i\frac{(z-1)}{(z+1)}$$

Find the bilinear transformation which maps the points $0,1,\infty$ in z-plane into itself in w-plane.

Solution:

Given $z_1=0,\ w_1=0,\quad z_2=1,\ w_2=1,\quad z_3=\infty,\ w_3=\infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{(z-z_1)z_3\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)}=\frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$w=z$$

Find the bilinear transformation which maps the points z=1,i,-1 into the points w= i,0,-i. Hence find the image of |z| < 1

Solution:

We know that

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-i)(0+i)}{(i-0)(-i-w)}=\frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\Rightarrow \frac{(w-i)(i)}{-(i)(w+i)}=\frac{(z-1)(1+i)}{-(1-i)(z+1)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{(z-1)(1+i)}{(z+1)(1-i)}\times\frac{(1+i)}{(1+i)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{(z-1)}{(z+1)}\times\frac{(1-1+2i)}{(1+1)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{(z-1)}{(z+1)}\times\frac{(2i)}{(2)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{i(z-1)}{(z+1)}\text{........(1)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{iz-i}{z+1}$$

Applying componendo and dividendo rule, we get

$$\Rightarrow \frac{w-i+w+i}{w-i-w-i}=\frac{iz-i+z+1}{iz-i-z-1}\Rightarrow \quad \frac{2w}{-2i}=\frac{iz-i+z+1}{iz-i-z-1}$$

$$\Rightarrow \boxed{w=-i\left[\frac{(1+i)z+1-i}{(i-1)z-1-i}\right]=\frac{(1-i)z-1-i}{(i-1)z-1-i}}$$

To find the image of $|z|<1$

From (1),

$$\frac{(z-1)}{(z+1)}=-i\frac{(w-i)}{(w+i)}=\frac{-iw-1}{w+i}$$

Applying Componendo and dividendo rule, we get

$$\frac{z-1+z+1}{z-1-z-1}=\frac{-iw-1+w+i}{-iw-1-w-i}$$

$$\Rightarrow \frac{2z}{-2} = \frac{(1-i)w+i-1}{(-1-i)w-1-i}$$

$$\Rightarrow \boxed{z = \frac{(1-i)w+i-1}{(1+i)w+1+i}}$$

Now $|z|<1 \Rightarrow \left|\frac{(1-i)w+i-1}{(1+i)w+1+i}\right|<1$

$$\Rightarrow |(1-i)w+i-1| < |(1+i)w+1+i|$$

$$\Rightarrow |(1-i)(u+iv)+i-1| < |(1+i)(u+iv)+1+i|$$

$$\Rightarrow |u+iv-iu+v+i-1| < |u+iv+iu-v+1+i|$$

$$\Rightarrow |u+v-1+i(1-u+v)| < |u-v+1+i(1+u+v)|$$

$$\Rightarrow \sqrt{(u+v-1)^2+(1-u+v)^2} < \sqrt{(u-v+1)^2+(1+u+v)^2}$$

$$\Rightarrow u^2+v^2+1-2u-2v+2uv+1+u^2+v^2-2u+2v-2uv <$$
$$u^2+v^2+1+2u-2v-2uv+1+u^2+v^2+2u+2v+2uv$$

$$\Rightarrow -2u-2u < 2u+2u$$

$$\Rightarrow -4u < 4u$$

$$\Rightarrow -8u < 0$$

$$\Rightarrow \boxed{u>0}$$

$\therefore$ the image of $|z|<1$ in z-plane is right half of w-plane $u>0$.

21. Prove that $w = \dfrac{z}{1-z}$ maps the upper half of the z-plane into the upper half of the w-plane.

What is the image of the circle $|z|=1$ under this transformation?

Solution:

$$w = \frac{z}{1-z} \Rightarrow w(1-z) = z$$

$$w - wz = z$$

$$w = (1+w)z$$

$$z = \frac{w}{1+w} \text{.............}(1)$$

$put \quad z = x+iy, \quad w = u+iv$

$$x+iy=\frac{u+iv}{1+u+iv}=\frac{(u+iv)}{(1+u)+iv}\;\frac{(1+u)-iv}{(1+u)-iv}$$

$$=\frac{u(1+u)-iuv+iv(1+u)+v^2}{(1+u)^2+v^2}$$

$$=\frac{(u+u^2+v^2)+iv}{(1+u)^2+v^2}$$

Equating real and imaginary parts

$$x=\frac{(u+u^2+v^2)}{(1+u)^2+v^2},\; y=\frac{v}{(1+u)^2+v^2}$$

$$y=0\Rightarrow\frac{v}{(1+u)^2+v^2}=0$$

$$y>0\Rightarrow\frac{v}{(1+u)^2+v^2}>0$$

$$\Rightarrow v>0$$

Thus the upper half of the z- plane is mapped onto the upper half of the w- plane.

Image of $|z|=1$:

by (1)

$$|z|=1\Rightarrow\left|\frac{w}{1+w}\right|=1$$

$$\Rightarrow\frac{|w|}{|1+w|}=1$$

$$\Rightarrow|w|=|1+w|$$

$$\Rightarrow|u+iv|=|1+u+iv|$$

$$\Rightarrow\sqrt{u^2+v^2}=\sqrt{(1+u)^2+v^2}$$

$$\Rightarrow u^2+v^2=(1+u)^2+v^2$$

$$\Rightarrow u^2=1+u^2+2u$$

$$\Rightarrow 2u+1=0$$

$$\Rightarrow u=\frac{-1}{2}$$

22. Determine the analytic function whose real part is $\boldsymbol{\dfrac{\sin 2x}{\cosh 2y-\cos 2x}}$

Solution:

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z} = -\frac{1}{\sin^2 z} = -\cos ec^2 z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = 0$$

By Milne's Thomson method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$
$$= \int -\cos ec^2 z\,dz - i0$$
$$= \cot z + c$$

23. If $f(z) = u + iv$ is an analytic function and $u - v = e^x(\cos y - \sin y)$ find $f(z)$ interms of z

Solution:

$$f(z) = u + iv \qquad (1)$$

$$if(z) = iu - v \qquad (2)$$

$$\therefore (1+i)f(z) = (u-v) + i(u+v)$$

$F(z) = U + iV$, where $F(z) = (1+i)f(z)$, $U = u - v$, $V = u + v$

$$\therefore U = u - v = e^x(\cos y - \sin y)$$

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x[\cos y - \sin y]$$

$$\phi_1(z,0) = e^z$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x[-\sin y - \cos y]$$

$\phi_2(z,0) = e^z(-1) = -e^z$

By Milne's Thomson Method

$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

$= \int e^z\,dz - i\int -e^z\,dz = e^z + ie^z$

$= (1+i)e^z$

$(1+i)f(z) = (1+i)e^z + C_1$

$f(z) = e^z + C$

24. Find the regular function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$

Solution:

$v = e^{-x}(x\cos y + y\sin y)$

$\phi_2(x,y) = \frac{\partial v}{\partial x} = e^{-x}[\cos y] + (x\cos y + y\sin y)(-e^{-x})$

$\phi_2(z,0) = e^{-z} + (z)(-e^{-z}) = e^{-z} - ze^{-z} = e^{-z}(1-z)$

$\phi_1(x,y) = \frac{\partial u}{\partial y} = e^{-x}[-x\sin y + y\cos y + \sin y(1)]$

$\phi_1(z,0) = e^{-z}[0+0+0] = 0$

By Milne's Thomson Method

$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$

$= \int 0\,dz + i\int (1-z)e^{-z}\,dz$

$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right] + C$

$= i\left[-(1-z)e^{-z} + e^{-z}\right] + C$

$= i\left[-e^{-z} + ze^{-z} + e^{-z}\right] + C = i\left[ze^{-z}\right] + C$

25. Determine the analytic function w = u + iv if u = e^{2x} (x cos2 y – y sin2y).
Solution:

$Given\ u = e^{2x}[x\cos 2y - y\sin 2y]$

$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^{2x}\cos 2y + [x\cos 2y - y\sin 2y]2e^{2x}$

$\therefore \phi_1(z,0) = e^{2z} + 2ze^{2z}$ ....(1)

$\phi_2(x,y) = \frac{\partial u}{\partial y} = e^{2x}[-x2\sin 2y - 2y\cos 2y - \sin 2y]$

$\therefore \phi_2(z,0) = 0$..................................(2)

By Milne Thomson method

$$F'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

From (1) & (2)

$$\int F'(z)\,dz=\int (e^{2z}+2ze^{2z})dz$$

$$=\frac{e^{2z}}{2}+2\left[z\frac{e^{2z}}{2}-(1)\frac{e^{2z}}{4}\right]$$

$$=\frac{e^{2z}}{2}+ze^{2z}-\frac{e^{2z}}{2}, \qquad \therefore F(z)=ze^{2z}$$

26. Construct the analytic function f (z) = u+iv given that 2u+3v = e^x (cos y – sin y).

Solution:

$$2u+3v=e^x[\cos y-\sin y]$$

$$f(z)=u+iv\text{...............}(1)$$

$$3if(z)=3iu-3v\text{...............}(2)$$

$$(1)\times 2\Rightarrow 2f(z)=2u+i2v\text{...........}(3)$$

$$(3)-(2)\Rightarrow (2-3i)f(z)=(2u+3v)+i(2v-3u)\text{............}(4)$$

$$F(z)=U+iV$$

$$\therefore 2u+3v=U=e^x[\cos y-\sin y]$$

$$\phi_1(x,y)=\frac{\partial U}{\partial x}=e^x\cos y-e^x\sin y$$

$$\phi_1(z,o)=e^z$$

$$\phi_2(x,y)=\frac{\partial U}{\partial y}=-e^x\sin y-e^x\cos y$$

$$\phi_2(z,o)=-e^z$$

By Milne Thomson method

$$F'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

$$\int F'(z)\,dz=\int e^z dz-i\int -e^z dz$$

$$F(z)=(1+i)e^z+C-----(5)$$

From (4) & (5)

$$(1+i)e^{z}+C=(2-3i)\,f(z)$$

$$f(z)=\frac{1+i}{2-3i}e^{z}+\frac{C}{2-3i}$$

$$f(z)=\frac{-1+5i}{13}e^{z}+\frac{C}{2-3i}$$

Module – 5 Complex Integration

Cauchy's integral formulae – Problems – Taylor's expansions with simple problems – Laurent's expansions with simple problems – Singularities – Types of Poles and Residues – Cauchy's residue theorem (without proof) – Contour integration: Unit circle, semicircular contour – Application of Contour integration in Engineering.

Cauchy's Integral Theorem

If $f(z)$ is analytic at every point of the region R bounded by a simple closed curve C and if $f'(z)$ is continuous at all points inside and on C, then $\int_C f(z)\,dz = 0$

Cauchy's integral formula for n^{th} derivative

If $f(z)$ is analytic inside and on a simple closed curve C and $z = a$ is any interior point of the region R enclosed by C, then $f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

$$(i.e.) \quad \boxed{\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(a)}$$

Taylor's series

If $f(z)$ is analytic inside a circle C with centre at a then Taylor's series about $z = a$ is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots$$

Laurent's series

If C_1, C_2 are two concentric circles with centre at z = a and radii r_1 and r_2 ($r_1 < r_2$) and if f(z) is analytic inside and on the circles and within the annular region between C_1 andC_2, then for any z in the annular region, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{n+1}}dz$ and $b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{-n+1}}dz$

Cauchy's Residue theorem

If f(z) is analytic inside a closed curve C except at a finite number of isolated singular points $a_1, a_2, \ldots a_n$ inside C, then

$\int_C f(z)dz = 2\pi i \times$(sum of the residues of f(z) at these singular points).

Contour Integration

Type I:

$$\boxed{\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta}$$

Let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta = izd\theta \Rightarrow d\theta = \frac{dz}{iz}$

Then we have

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$; $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$\cos 2\theta =$ Real part of z^2 ; $\cos n\theta =$ Real part of z^n

$\sin 2\theta =$ Im part of z^2; $\sin n\theta =$ Im part of z^n

$\cos^2\theta = \frac{1+\cos 2\theta}{2} =$ Real part of $\left[\frac{1+z^2}{2}\right]$;

$\sin^2\theta = \frac{1-\cos 2\theta}{2} =$ Real part of $\left[\frac{1-z^2}{2}\right]$

$\therefore$

$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_C f(z)\, dz$, where C is $|z| = 1$ and solve by known method.

Type II:

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx}$$

Using Cauchy's integral formula, find $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is $|z+1-i| = 2$

Solution:

$|z+1-i| = 2$

$|x+iy+1-i| = 2$

$|(x+1)+i(y-1)| = 2$, $\sqrt{(x+1)^2+(y-1)^2} = 2$

Squaring on both sides,

$$(x+1)^2+(y-1)^2=4$$

This is equation of circle with centre $(-1,1)$ and radius 2.

$$z^2+2z+5=0$$

$$z=\frac{-2\pm\sqrt{4-4(1)(5)}}{2(1)}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\int_C \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}\,dz$$

Here –1+2i lies inside the circle c and –1–2i lies outside the circle c.

Let $a=-1+2i$

By Cauchy's integral formula, $f(a)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-a}\,dz$

Substituting for a, $f(-1+2i)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-(-1+2i)}\,dz$ ......(1)

Comparing equation (1) with given problem,

$$f(z)=\frac{z+4}{z-(-1-2i)}$$

$$f(-1+2i)=\frac{-1+2i+4}{-1+2i-(-1-2i)}=\frac{2i+3}{-1+2i+1+2i}=\frac{2i+3}{4i}$$

Substituting for $f(-1+2i)$ in (1)

$$\frac{2i+3}{4i}=\frac{1}{2\pi i}\int_C \frac{z+4}{z^2+2z+5}\,dz$$

Cross multiplying

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\frac{(2i+3)(2\pi i)}{4i}=\frac{\pi}{2}(3+2i)$$

Using Cauchy's integral formula, evaluate $\displaystyle\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)\ (z-1)}dz$, where C is $|z|=3$

Solution:

We know that, Cauchy's integral formula is $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

$(i.e)$ $2\pi i\, f(a)=\int_C \frac{f(z)}{z-a}\,dz$

Given: $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$ Here, $f(z)=\sin\pi z^2+\cos\pi z^2$

The points $a_1=1, a_2=2$ lies inside $|z|=3$

Now, $\frac{1}{(z-1)(z-2)}=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$ (by Partial fraction method)

$$\therefore \int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz=-\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)}dz+\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}dz$$
$$=-2\pi i\, f(1)+2\pi i\, f(2)$$

$$f(z)=\sin\pi z^2+\cos\pi z^2$$
$$f(1)=\sin\pi+\cos\pi=-1 \text{ and } f(2)=\sin 4\pi+\cos 4\pi=1$$

$$\therefore \int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz=-2\pi i(-1)+2\pi i(1)=4\pi i$$

Using Cauchy's integral formula, evaluate $\int_C \frac{1}{(z-2)\ (z+1)^2}dz$, where C is $|z|=\frac{3}{2}$

Solution:

Here $z=-1$ is a pole lies inside the circle
$z=2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{(z+1)^2(z-2)}=\int \frac{\frac{1}{z-2}}{(z+1)^2}\,dz$$

Here $f(z)=\frac{1}{z-2}$, $f'(z)=-\frac{1}{(z-2)^2}$

Hence by Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz=\frac{2\pi i}{n!}f^n(a)$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2}\,dz = \frac{2\pi i}{1!}\,f'(-1)$$

$$= 2\pi i\left[\frac{-1}{(-1-2)^2}\right] \quad \left(\because f'(z) = \frac{-1}{(z-2)^2}\right) = 2\pi i\left[\frac{-1}{9}\right]$$

$$\int_C \frac{1}{(z-2)\ \ (z+1)^2}dz = \frac{-2}{9}\pi i.$$

Using Cauchy's integral formula, evaluate $\int_c \frac{z}{z^2+1}dz$ where C is $|z+i|=1$.

Solution:

Consider the curve

$|z+i|=1 \Rightarrow |x+iy+i|=1$
$|x+i(y+1)|=1 \Rightarrow x^2+(y+1)^2=1$

Which is a circle with centre (0,–1) and radius 1

The poles are obtained by $z^2+1=0$

$\Rightarrow z=i$ is a simple pole which lies outside C.

$z=-i$ is a simple pole which lies inside C.

$$\int_c \frac{z}{z^2+1}dz = \int_c \frac{z}{(z+i)(z-i)}dz = \int_c \frac{\frac{z}{(z-i)}}{(z+i)} = 2\pi i f(-i)\ldots(1)$$

$$f(z) = \frac{z}{(z-i)},\ f(-i) = \frac{-i}{(-i-i)} = \frac{-i}{-2i} = \frac{1}{2}$$

$$(1) \Rightarrow \int_c \frac{z}{z^2+1}dz = 2\pi i f(-i) = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

Expand $f(z)=\log(1+z)$ in Taylor's series about $z=0$

Solution: Let $f(z)=\log(1+z)$ $f(0)=\log 1=0$

$$f'(z) = \frac{1}{1+z} \quad f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \qquad f''(0) = -1$$

$$f'''(z)=\frac{2}{(1+z)^3} \quad f'''(0)=2$$

$$f^{iv}(z)=\frac{-6}{(1+z)^4} \qquad f^{iv}(0)=-6$$

$$\log(1+z)= f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\ldots= z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\ldots.$$

Find the Taylor's series expansion of f(z) = $\frac{z}{(z+1)(z-3)}$, in the region $|z|<1$

Solution:

Splitting f(z) into partial fractions, we have

$$f(z)=\frac{z}{(z+1)(z-3)}=\frac{A}{(z+1)}+\frac{B}{(z-3)}$$

$$\Rightarrow z=A(z-3)+B(z+1)$$

$$\text{put } z=-1, \text{ we get } A=\frac{1}{4}$$

$$\text{put } z=3, \text{ we get } B=\frac{3}{4}$$

$$\therefore \quad f(z)=\frac{1}{4}\left(\frac{1}{z+1}\right)+\frac{3}{4}\left(\frac{1}{z-3}\right)=\frac{1}{4}\left(\frac{1}{1+z}\right)+\frac{3}{4}\left(\frac{1}{-3}\right)\left(\frac{1}{1-\frac{z}{3}}\right)$$

$$=\frac{1}{4}\left[(1+z)^{-1}-\left(1-\frac{z}{3}\right)^{-1}\right]$$

$$=\frac{1}{4}\left[\left(1-z+z^2-\ldots.\right)-\left(1+\frac{z}{3}+\frac{z^2}{9}+\ldots\right)\right]$$

$$=\frac{1}{4}\left[\left((-1)-\frac{1}{3}\right)z+\left((-1)^2-\left(\frac{1}{3}\right)^2\right)z^2+\ldots\right]$$

$$\therefore f(z)=\frac{1}{4}\sum_{n=1}^{\infty}\left((-1)^n-\left(\frac{1}{3}\right)^n\right)z^n$$

Obtain Taylor's Series to represent the function $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ in the region | z | < 2

Solution:

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$$

Since the degree of the numerator and denominator are same we have to divide and apply partial fractions.

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

$|z|<2 \Rightarrow \frac{|z|}{2}<1$ and $\therefore \frac{|z|}{3}<1$

Consider

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{3}{z+2}-\frac{8}{z+3}=\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}=\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

$$\therefore \frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Find the Laurent's series expansion of $\frac{1}{(z-2)(z-1)}$ valid in the regions $|z|>2$ and $0<|z-1|<1$

Solution:

$$f(z)=\frac{1}{(z-2)(z-1)}=\frac{A}{(z-1)}+\frac{B}{(z-2)}=\frac{A(z-2)+B(z-1)}{(z-2)(z-1)}$$

$$\Rightarrow 1=A(z-2)+B(z-1)$$

Put $z=1,\ A=-1$

$z=2,\ B=1$

$$\therefore f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$$

Region1:

$$|z|>2\Rightarrow 2<|z|$$

$$\Rightarrow\left|\frac{2}{z}\right|<1$$

$$f(z)=\frac{-1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$=-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=-\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^{2}+\dots\right)+\frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^{2}+\dots\right)$$

$$=-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}+\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n}$$

$$=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{2^{n}}{z^{n+1}}$$

Region2:

Put $z-1=t\Rightarrow z=1+t$

$$0<|z-1|<1\Rightarrow 0<|t|<1$$

$$\Rightarrow|t|<1$$

$$f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$$

$$=\frac{-1}{t}+\frac{1}{t-1}$$

$$=\frac{-1}{t}+\frac{1}{-(1-t)}$$

$$=\frac{-1}{t}-(1-t)^{-1}$$

$$=\frac{-1}{t}-\left(1+t+t^{2}+\dots\right)$$

$$=\frac{-1}{(z-1)}-\left(1+(z-1)+(z-1)^2+\dots\right)$$

$$=\frac{-1}{(z-1)}-\sum_{n=0}^{\infty}(z-1)^n$$

Expand f(z) = $\frac{z^2-1}{z^2+5z+6}$ in a Laurent's series expansion for $|z|>3$ and $2<|z|<3$

Solution:

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

Consider $\frac{-5z-7}{(z+3)(z+2)}$

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{A}{z+2}+\frac{B}{z+3}=\frac{A(z+3)+B(z+2)}{(z+3)(z+2)}$$

$$-5z-7=A(z+3)+B(z+2)$$

Put $z=-2$ then $A=3$

Put $z=-3$ then $B=-8$

Substituting we get, $\frac{-5z-7}{(z+3)(z+2)}=\frac{3}{z+2}-\frac{8}{z+3}$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}$$

(i) **Given** $|z|>3 \quad \Rightarrow \quad \frac{3}{|z|}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\dots\right)-\frac{8}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\dots\right)$$

(ii) **Given** $2<|z|<3\Rightarrow \quad \frac{2}{|z|}<1$ and $\frac{|z|}{3}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-...\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Obtain the Laurent's series expansion for the function f(z) = $\frac{4z}{\left(z^2-1\right)(z-4)}$ in

$|z-1|>4$ **and** $2<|z-1|<3$

Solution:

Put $z-1=u \Rightarrow z=u+1$

Now, $f(z)=\frac{4z}{\left(z^2-1\right)(z-4)}=\frac{4z}{(z-1)(z+1)(z-4)}$

Hence $f(u)=\frac{4(u+1)}{u(u+2)(u-3)}$

$$\frac{4(u+1)}{u(u+2)(u-3)}=\frac{A}{u}+\frac{B}{u+2}+\frac{C}{u-3}=\frac{A(u+2)(u-3)+Bu(u-3)+Cu(u+2)}{u(u+2)(u-3)}$$

$$4(u+1)=A(u+2)(u-3)+Bu(u-3)+Cu(u+2)$$

Put $u=0$ then $A=\frac{-2}{3}$

Put $u=-2$ then $B=\frac{-2}{5}$

Put $u=3$ then $C=\frac{16}{15}$

$$f(u)=\frac{4(u+1)}{u(u+2)(u-3)}=\frac{-2/3}{u}+\frac{-2/5}{u+2}+\frac{16/15}{u-3}$$

(i) $|u|>4 \quad \Rightarrow \quad \frac{4}{|u|}<1$

$$f(u)=\frac{-2/3}{u}-\frac{2/5}{u+2}+\frac{16/15}{u-3}$$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{u\left(1-\frac{3}{u}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} + \frac{16}{15}\left(\frac{1}{u}\right)\left(1-\frac{3}{u}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1 - \frac{2}{u} + \frac{4}{u^2} - ....\right) + \frac{16}{15}\left(1 + \frac{3}{u} + \frac{9}{u^2} + ...\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1 - \frac{2}{(z-1)} + \frac{4}{(z-1)^2} - ....\right) + \frac{16}{15}\left(1 + \frac{3}{(z-1)} + \frac{9}{(z-1)^2} + ...\right)\right]$$

(ii) $2 < |u| < 3 \Rightarrow \quad \frac{2}{|u|} < 1 \text{ and } \frac{|u|}{3} < 1$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{-3\left(1-\frac{u}{3}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} - \frac{16}{45}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1 - \frac{2}{u} + \frac{4}{u^2} - ....\right) - \frac{16}{45}\left(1 + \frac{u}{3} + \frac{u^2}{9} + ...\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1 - \frac{2}{(z-1)} + \frac{4}{(z-1)^2} - ....\right) - \frac{16}{45}\left(1 + \frac{(z-1)}{3} + \frac{(z-1)^2}{9} + ...\right)\right]$$

Find the Laurent's series expansion of $\boldsymbol{f(z) = \frac{7z-2}{z(z-2)(z+1)}}$ **in 1 < | z + 1| < 3**

Solution:

The singular points are z = 0, z = 2, z = –1

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$\Rightarrow$ 7z – 2 = A(z – 2) (z + 1) + B z (z + 1) + C z (z – 2)

Put z = 0, –2 = A(–2) $\Rightarrow$ A = 1

z = 2, 14 – 2 = B 2(2 + 1) $\Rightarrow$ B = 2

$z = -1, \quad -7-2 = C(-1)(-1-2) \Rightarrow C = -3$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Put t = z + 1 $\Rightarrow$ z = t – 1

$\therefore 1 < |t| < 3$

$1 < |t| \Rightarrow \left|\frac{1}{t}\right| < 1$ and $\left|\frac{t}{3}\right| < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$
$$= \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t}$$
$$= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{2}{(-3)\left(1-\frac{t}{3}\right)} - \frac{3}{t}$$
$$= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t}$$
$$= \frac{1}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots\right] - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots\right] - \frac{3}{t}$$
$$= -\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots\right]$$
$$= -2(z+1)^{-1}+(z+1)^{-2}+(z+1)^{-3}+\ldots-\frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\ldots\right]$$

Evaluate $\int_C \frac{z\ dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2| = \frac{1}{2}$ by Cauchy Residue theorem.

Solution:

The poles are obtained by $(z-1)(z-2)^2 = 0$

$\Rightarrow$ z = 1 is a simple pole and z = 2 is a pole of order 2.

C is the circle $|z-2| = \frac{1}{2}$

Here z = 1 lies outside C and z = 2 lies inside C.

Residue at z=2: (Pole of order 2)

$$\text{Res } f(z) = \underset{z\to 2}{lt}\frac{d}{dz}(z-2)^2\frac{z}{(z-1)(z-2)^2} = \underset{z\to 2}{lt}\frac{z-1-z}{(z-1)^2} = -1$$

By Cauchy Residue theorem,

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i(-1) = -2\pi i$$

Using Cauchy's residue theorem evaluate $\displaystyle\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$ **, where C is** $|z| = 2$

Solution:

$|z| = 2$ is the equation of the circle with centre at origin and radius 2.

$$(z^2-1)(z-3) = 0$$

$$(z^2-1) = 0, \quad (z-3) = 0$$

$$z^2 = 1, \quad z = 3$$

$$z = \pm 1, \quad z = 3$$

$z = 1, -1$ lies inside the circle and $z = 3$ lies outside the circle

Residue at $z = 1$ is

$$= Lt_{z\to 1}\left((z-1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to 1}\left(\frac{3z^2+z-1}{(z+1)(z-3)}\right) = -\frac{3}{4}$$

Residue at $z = -1$ is

$$= Lt_{z\to -1}\left((z+1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to -1}\left(\frac{3z^2+z-1}{(z-1)(z-3)}\right) = \frac{1}{8}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{3z^2+z-1}{\left(z^2-1\right)(z-3)}dz = 2\pi i\left(\frac{1}{8}-\frac{3}{4}\right) = -\frac{5\pi i}{4}$$

Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ **, where C is** $|z-i| = 2$ **using Cauchy's residue theorem**

Solution:

Let $f(z) = \dfrac{z-1}{(z+1)^2(z-2)}$

poles of $f(z)$ are z = - 1 (pole of order 2) and z = 2 (simple pole)

Given: $|z-i| = 2$

$$|x+iy-i| = 2 \Rightarrow |x+i(y-1)| = 2$$

Squaring on both sides $\sqrt{x^2+(y-1)^2} = 2 \Rightarrow x^2+(y-1)^2 = 4$

This is equation of circle with centre $(0,1)$ and radius 2

Hence, The pole $z=2$ lies outside C and $z=-1$ lies inside C

Residue of f(z) at $z=-1$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left((z+1)^2\frac{(z-1)}{(z+1)^2(z-2)}\right)$$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left(\frac{(z-1)}{(z-2)}\right) = Lt_{z\to -1}\left(\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right)$$

$$= Lt_{z\to -1}\left(\frac{-1}{(z-2)^2}\right) = -\frac{1}{9}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{(z-1)}{(z+1)^2(z-2)}\,dz = 2\pi i\left(0-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Using Cauchy's residue theorem, find $\displaystyle\int_C \frac{z+1}{(z-3)(z-1)}\,dz$ **, where C is** $|z| = 2$

Solution:

The singular points are given by (z – 1) (z – 3) = 0 $\Rightarrow$ z = 1 , 3

Given C is | z | = 2

If z = 1 then | z | = | 1 | = 1 < 2

If z = 3 then | z | = | 3 | = 3 > 2

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

Residue at z=1:

$$\text{Res}\Big|_{z=1} = \lim_{z\to 1}(z-1)\,f(z) = \lim_{z\to 1}(z-1)\frac{z+1}{(z-3)(z-1)} = -1$$

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)}\,dz = 2\pi i(-1) = -2\pi i$$

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **by using Contour integration.**

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\dfrac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz},\ \sin\theta = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+5\frac{(z^2-1)}{2iz}} = \int_C \frac{\frac{dz}{iz}}{\frac{26iz+5z^2-5}{2iz}} = 2\int_C \frac{dz}{5z^2+26iz-5}$$

Let $f(z) = \dfrac{1}{5z^2+26iz-5}$ $\quad\quad \therefore I = 2\int_C f(z)dz$

The poles of f(z) are given by $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i \pm \sqrt{(26i)^2 - 4 \ . \ 5(-5)}}{10} = \frac{-26i \pm \sqrt{-676+100}}{10} = \frac{-26i \pm \sqrt{-576}}{10} = \frac{-26i \pm 24i}{10}$$

$$z = -\frac{i}{5} \ , \ -5i$$

which are simple poles.

Now $5z^2 + 26iz - 5 = 5\left(z+\frac{i}{5}\right)(z+5i)$

Since $\left|\frac{-i}{5}\right| = \frac{1}{5} < 1$, the pole $z = \frac{-i}{5}$ lies inside C

and $|-5i| = 5 > 1$, $\therefore$ the pole $z = -5i$ lies outside C.

Now $R\left(-\frac{i}{5}\right) = \lim\limits_{z\to\frac{-i}{5}} \left(z+\frac{i}{5}\right) f(z) = \lim\limits_{z\to\frac{-i}{5}} \left(z+\frac{i}{5}\right) \dfrac{1}{5\left(z+\frac{i}{5}\right)(z+5i)} = \lim\limits_{z\to\frac{-i}{5}} \dfrac{1}{5(z+5i)}$

$$= \lim_{z\to\frac{-i}{5}} \frac{1}{5\left(-\frac{i}{5}+5i\right)} = \frac{1}{24i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\therefore I = 2.\frac{\pi}{12} = \frac{\pi}{6}$$

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$ by using Contour integration.

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\dfrac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+12\frac{(z^2+1)}{2z}} = \int_C \frac{dz}{\frac{iz(13z+6z^2+6)}{z}} = \int_C \frac{dz}{i(6z^2+13z+6)} = \frac{1}{i6}\int_C \frac{dz}{(z^2+\frac{13}{6}z+1)}$$

$$Let \quad f(z) = \int_C \frac{dz}{(z^2+\frac{13}{6}z+1)} \qquad \therefore I = \frac{1}{6i}\int_C f(z)dz$$

The poles of f(z) are given by $z^2 + \frac{13}{6}z + 1 = 0$

By solving we get $z = -\frac{2}{3}, \ -\frac{3}{2}$

which are simple poles.

Now $z^2 + \frac{13}{6}z + 1 = \left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)$

Since $\left|\frac{-2}{3}\right| = \frac{2}{3} < 1$, the pole $z = \frac{-2}{3}$ lies inside C

and $\left|\frac{-3}{2}\right| = 1.5 > 1$, $\therefore$ the pole $z = \frac{-3}{2}$ lies outside C.

$$Now \quad R\left(-\frac{2}{3}\right) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)f(z) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)\frac{1}{\left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)} = \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)}$$

$$= \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)} = \frac{6}{5}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{6}{5}\right) = \frac{12\pi i}{5}, \qquad \therefore I = \frac{1}{6i}\times\left(\frac{12\pi i}{5}\right) = \frac{2\pi}{5}.$$

Evaluate $\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$ by using Contour integration

Solution:

Consider the unit circle | z | = 1 as contour C.

$$Put\ z = e^{i\theta},\ then\ \frac{1}{z} = e^{-i\theta}$$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

cos3θ=R.P. of $e^{i3\theta}$ =R.P. of $(e^{i\theta})^3$ =R.P. of z^3

$$\therefore \qquad I = \int_C \frac{R.P. of\ z^3 \dfrac{dz}{iz}}{5 - 4\dfrac{(z^2+1)}{2z}} = R.P.\ of\ \int_C \frac{z^3 dz}{\dfrac{iz(5z-2z^2-2)}{z}}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{i(-2z^2+5z-2)}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{-i(2z^2-5z+2)}$$

$$= R.P. of\ \frac{-1}{2i}\int_C \frac{z^3 dz}{(2z-1)(z-2)}$$

Let $\int_C f(z)dz = \int_C \frac{z^3 dz}{(2z-1)(z-2)}$ $\qquad \therefore I = R.P.$ of $\frac{-1}{2i}\int_C f(z)dz$

The poles of f(z) are given by

$$(2z-1)(z-2) = 0$$

$$z = \frac{1}{2},\ z = 2$$

$$z = \frac{1}{2},\ z = 2 \ \text{(simple poles)}$$

$z = \frac{1}{2}$ is a pole lies inside c.

$z = 2$ is a pole lies outside c.

$$\text{Now} \quad \operatorname{Res}\left(z = \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z - \frac{1}{2}\right) f(z) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{\left(z-\frac{1}{2}\right)\ (z-2)} = \frac{-1}{12}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \ \left(\frac{-1}{12}\right) = \frac{-\pi i}{6}$$

$$\therefore I = R.P.of\ \frac{-1}{2i}\ .\frac{-\pi i}{6} = R.P.of\ \frac{\pi}{12} = \frac{\pi}{12}$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2}$ **, | p | < 1**

Solution: Let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{z^2-1}{2iz}$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \int_C \frac{\left(dz/iz\right)}{1-2p\left(\frac{z^2-1}{2iz}\right)+p^2} \quad , C \text{ is } |z|=1$$

$$= \int_C \frac{dz}{iz-p(z^2-1)+izp^2} = -\int_C \frac{dz}{pz^2-iz(p^2+1)-p} = -\frac{1}{p}\int_C \frac{dz}{z^2-iz\left(p+\frac{1}{p}\right)-1}$$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} \quad ....(1)$$

The poles are given by $z = ip \;\&\; z = \frac{i}{p}$

| z | = | i p | = p < 1. $\therefore$ z= ip lies inside C and $z = \frac{i}{p}$ lies outside C.

$$\therefore \left[\text{Res of } f(z)\right]_{z=ip} = \underset{z\to ip}{Lt}\,(z-ip)\left[\frac{1}{(z-ip)\left(z-\frac{i}{p}\right)}\right] = \underset{z\to ip}{Lt}\left(\frac{1}{z-\frac{i}{p}}\right) = \frac{1}{i\left(p-\frac{1}{p}\right)} = \frac{ip}{1-p^2}$$

By Cauchy Residue Theorem $\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} = 2\pi i\left(\frac{ip}{1-p^2}\right) = \frac{-2\pi p}{1-p^2}$

From (1) $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\left(-\frac{2\pi p}{1-p^2}\right) = \frac{2\pi}{1-p^2}$

Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}, (a>0)$ **using contour integration**

Solution:

Let $f(z)=\dfrac{1}{(z^2+a^2)^2}$. Consider $\int\limits_c f(z)dz$

where C is the contour consists of the upper half circle c_1 of $|z|=R$ & the real axix from –R to R.

$$\therefore \int\limits_c f(z)dz=\int\limits_{c_1} f(z)dz+\int\limits_{-R}^{R} f(z)dz\text{.......... ....}\ (1)$$

The poles of $f(z)$ are given by $(z^2+a^2)^2=0 \Rightarrow z=\pm ai$ (twice)

z = ai is a pole of order 2 & lies inside C

z = -ai is a pole of order 2 & lies outside C

$$\mathrm{Res}\left[f(z),ai\right]=\lim_{z\to ai}\frac{d}{dz}\left[(z-ai)^2\frac{1}{(z+ai)^2(z-ai)^2}\right]=\lim_{z\to ai}\frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]=\frac{-2}{(2ai)^3}=\frac{1}{4a^3i}$$

By Cauchy's Residue Theorem $\int f(z)dz=2\pi i(\dfrac{1}{4a^3i})=\dfrac{\pi}{2a^3}$

In (1) $R\to\infty$, then $\int\limits_{c_1} f(z)dz=0$

$$\therefore (1)\Rightarrow \int\limits_c f(z)dz=\int\limits_{-\infty}^{\infty} f(x)dx$$

$$=\int\limits_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{2a^3}$$

$$=2\int\limits_{0}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{2a^3}$$

$$\int\limits_{0}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{4a^3}$$

Evaluate $\displaystyle\int\limits_0^\infty \frac{\cos ax\,dx}{x^2+1}$, $a>0$, using contour integration.

Solution:

$$\int\limits_0^\infty \frac{\cos ax\ dx}{1+x^2}=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\cos ax\ dx}{1+x^2}$$

Now $\displaystyle\int\limits_{-\infty}^{\infty}\frac{\cos ax\ dx}{1+x^2}=\int\limits_{-\infty}^{\infty}\frac{\text{RP of } e^{iax}}{1+x^2}dx$ $\quad\left\{\because e^{i\theta}=\cos\theta+i\sin\theta\right\}$

Consider $\int_c f(z)\,dz = \text{R.P}\int_c \frac{e^{iaz}}{1+z^2}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [–R, R]. By Cauchy's residue theorem, we have

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

The poles of f(z) are at $1+z^2=0$

$z^2=-1 \Rightarrow z=\pm i$

The point $z=i$ lies inside the semi-circle and the point $z=-i$ lies outside the semi-circle

Residue at $z=i$ is given by

$$Lt_{z\to i}(z-i)\,f(z) = Lt_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)}$$

$$=Lt_{z\to i}\frac{e^{iaz}}{(z+i)} = \frac{e^{ia(i)}}{i+i} = \frac{e^{ai^2}}{2i} = \frac{e^{-a}}{2i}$$

By Cauchy Residue theorem,

$$R.P\int_c \frac{e^{iaz}\,dz}{1+z^2} = \text{R.P of } 2\pi i\left(\frac{e^{-a}}{2i}\right) = \text{R.P of } \pi e^{-a} = \pi e^{-a}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \pi e^{-a}$$

If $R\to\infty$, then $\int_{\Gamma} f(z)\,dz \to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-a}$

$$\int_0^{\infty}\frac{\cos ax\,dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax\,dx}{1+x^2} = \frac{\pi e^{-a}}{2}$$

Evaluate $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}\,dx$**, using contour integration.**

Solution:

Let $f(z) = \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}$

Consider $\int_c f(z)\,dz = \int_c \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [-R, R]. By Cauchy's residue theorem, we have

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

The poles f(z) are at $z^4 + 10z^2 + 9 = 0$

$$(z^2+1)(z^2+9) = 0$$

$$z^2 = -1; \qquad z^2 = -9$$

$$z = \pm i; \qquad z = \pm 3i$$

The poles are at $3i,\ -3i,\ i,\ -i$

Here the poles $3i$ and i lie inside the semi-circle.

Residue at $z = 3i$ is given by

$$= Lt_{z\to 3i}(z-3i)\,f(z)$$

$$= Lt_{z\to 3i}(z-3i)\frac{z^2 - z + 2}{(z^2+9)(z^2+1)}$$

$$= Lt_{z\to 3i}(z-3i)\frac{z^2 - z + 2}{(z-3i)(z+3i)(z^2+1)}$$

$$= Lt_{z\to 3i}\frac{z^2 - z + 2}{(z+3i)(z^2+1)} = \frac{7+3i}{48i}$$

Residue at $z = i$ is given by

$$= Lt_{z\to i}(z-i)\,f(z)$$

$$= Lt_{z\to i}(z-i)\frac{z^2 - z + 2}{(z^2+9)(z^2+1)}$$

$$= Lt_{z\to i}(z-i)\frac{z^2 - z + 2}{(z-i)(z+i)(z^2+9)}$$

$$= Lt_{z\to i}\frac{z^2-z+2}{(z+i)(z^2+9)}=\frac{1-i}{16i}$$

By Cauchy Residue theorem,

$$\int_c \frac{z^2-z+2}{z^4+10z^2+9}dz = 2\pi i\left[\frac{7+3i}{48i}+\frac{1-i}{16i}\right]=2\pi i\left[\frac{7+3i+3-3i}{48i}\right]=2\pi i\left[\frac{10}{48i}\right]=\frac{5\pi}{12}$$

$$\therefore \int_{-R}^{R} f(x)\,dx+\int_{\Gamma} f(z)\,dz=\frac{5\pi}{12}$$

If $R\to\infty$, then $\int_{\Gamma} f(z)\,dz\to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx=\frac{5\pi}{12}$

$$\therefore \int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx=\frac{5\pi}{12}$$

Evaluate $\int_0^{\infty}\frac{x\sin mx\ \ dx}{(x^2+a^2)}$, where a > 0 , m > 0

Solution:

Let $f(z)=\int_0^{\infty}\frac{x\sin mxdx}{(x^2+a^2)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin mxdx}{(x^2+a^2)}$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin mxdx}{(x^2+a^2)}=\frac{1}{2}\text{IP}\int_{-\infty}^{\infty}\frac{xe^{imx}dx}{(x^2+a^2)}=\frac{1}{2}\text{IP}(\text{I}_1)$$

$$\text{I}_1=\int_{-\infty}^{\infty}\frac{xe^{imx}}{x^2+a^2}dx=\int_{-\infty}^{\infty}F(x)dx$$

Here $F(x)=\frac{xe^{imx}}{x^2+a^2}$ let $F(z)=\frac{ze^{imx}}{z^2+a^2}$

The poles of F(z) are given by

$\Rightarrow$ z = ±ia are poles of order 1

$\Rightarrow$ z = ia lies inside C

Consider $\int_C f(z)\ dz$ where C is the contour consists of the upper half circle C, of | z| = R. and the real axis from –R to R.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad ----(1)$$

$$\therefore \left[\text{Res of } f(z)\right]_{z=ai} = \underset{z\to ia}{\text{Lt}} (z-ia)\,\frac{ze^{imz}}{(z+ib)(z-ib)}$$

$$= \frac{e^{-ma}(ia)}{2ia} = \frac{e^{-ma}}{2}$$

$$I_1 = 2\pi i\left(\frac{e^{-ma}}{2}\right) + \pi i(0) = i\pi e^{-ma}$$

$$I = \frac{1}{2}IP(I_1) = \frac{1}{2}IP(i\pi e^{-ma}) = \frac{\pi e^{-ma}}{2}$$

By Cauchy's Residue Theorem

$$\therefore (1) \Rightarrow \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \qquad \because \int_C f(z)\,dz \to 0 \text{ as } R \to \infty$$

$$\Rightarrow \int_0^{\infty} f(x)\,dx = \frac{\pi e^{-ma}}{2}$$

Evaluate $\int_0^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$**, a > 0 , b > 0**

Solution:

Let $f(z) = \text{Real Part of } \frac{e^{iz}}{(z^2+a^2)\ (z^2+b^2)}$

Consider $\int_C f(z)\,dz$ where C is the contour consists of the upper half circle C, of | z| = R. and the real axis from –R to R.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad ----(1)$$

The poles of $f(z)$ are given by $(z^2+a^2)(z^2+b^2) = 0$

$\Rightarrow z = \pm ia, \pm ib$

$\Rightarrow z = ia, ib$ lies inside C and $z = -ia, -ib$ lies in lower half plane

$$\therefore \left[\text{Res of } f(z)\right]_{z=ai} = \underset{z\to ia}{\text{Lt}}\ (z-ia)\ \frac{e^{iz}}{(z+ia)(z-ia)(z^2+b^2)}$$

$$= \frac{e^{-a}}{2ia(b^2-a^2)}$$

$$\left[\text{Res of } f(z)\right]_{z=bi} = \underset{z\to ia}{\text{Lt}}\ (z-ib)\ \frac{e^{iz}}{(z+ib)(z-ib)(z^2+a^2)}$$

$$= \frac{e^{-a}}{2ib(a^2-b^2)}$$

By Cauchy's Residue Theorem

$$\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz = 2\pi i\left[\frac{e^{-a}}{2ia(b^2-a^2)} + \frac{e^{-b}}{2ib(a^2-b^2)}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

In (1) if $R\to\infty$, $\int_{C_1} f(z)\,dz \to 0$

$$\therefore (1) \Rightarrow \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \text{Real Part of } \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT I - MULTIPLE INTEGRALS

Part – A

1.	$\int_0^2 \int_0^2 dx\,dy =$ (A) 4 (B) 2 (C) 0 (D) 1	ANS **A**	(CLO-1, Apply)
2.	$\int_0^2 \int_0^2 e^{x+y}\,dx\,dy =$ (A) $(e-1)^2$ (B) $(e^2-1)^2$ (C) 1 (D) 0	ANS **B**	(CLO-1, Apply)
3.	$\int_1^2 \int_2^5 x\,y\,dx\,dy =$ (A) 1 (B) -1 (C) $\frac{63}{4}$ (D) $\frac{53}{4}$	ANS **C**	(CLO-1, Apply)
4.	$\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy =$ (A) 0 (B) 9 (C) $\frac{8}{3}$ (D) $-\frac{8}{3}$	ANS **C**	(CLO-1, Apply)
5.	$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} d\theta\,d\varphi =$ (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi^2}{4}$ (D) $\frac{\pi^2}{8}$	ANS **C**	(CLO-1, Apply)
6.	$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(\theta+\varphi)\,d\theta\,d\varphi =$ (A) 2 (B) 1 (C) 0 (D) -2	ANS **A**	(CLO-1, Apply)

7.	$\int_0^1 \int_0^x dy\,dx \;=$ (A) 1 (B) -1 (C) $\frac{1}{2}$ (D) $\frac{1}{3}$	ANS **C**	(CLO-1, Apply)
8.	$\int_0^{\pi} \int_0^{a\sin\theta} r\,dr\,d\theta =$ (A) $\pi\,a^2$ (B) $\frac{\pi}{4}\,a^2$ (C) $\frac{\pi}{4}\,a^3$ (D) $\frac{\pi}{6}\,a^2$	ANS **B**	(CLO-1, Apply)
9.	$\int_0^2 \int_1^2 \int_1^2 x\,y^2\,z\;dz\,dy\,dx \;=$ (A) 24 (B) 28 (C) 20 (D) 7	ANS **D**	(CLO-1, Apply)
10.	If R is the region bounded by $x = 0$, $y = 0$ and $x + y = 1$, then $\iint_R dx\,dy =$ (A) 1 (B) -1 (C) $\frac{1}{2}$ (D) $\frac{1}{3}$	ANS **C**	(CLO-1, Apply)
11.	The region of integration of the integral $\int_0^1 \int_0^x f(x,y)\;dy\,dx$ is (A) square (B) rectangle (C) triangle (D) circle	ANS **C**	(CLO-1, Apply)
12.	To change Cartesian into polar coordinates in double integration, the transformation used is (A) $x = r\cos\theta, y = r\sin\theta$ (B) $x = a\cos\theta, y = b\sin\theta$ (C) $x = r\sin\;\theta, y = r\cos\theta$ (D) $x = a\sec\theta, y = b\tan\theta$	ANS **A**	(CLO-1, Remember)
13.	Change the order of integration in $\int_0^a \int_x^a f(x,y)\,dy\,dx$ (A) $\int_0^a \int_x^a f(x,y)\,dy\,dx$ (B) $\int_0^a \int_0^y f(x,y)dx\,dy$ (C) $\int_0^a \int_0^{x^2} f(x,y)\,dy\,dx$ (D) $\int_0^a \int_x^{x^2} f(x,y)\,dy\,dx$	ANS **B**	(CLO-1, Apply)
14.	Area of a region R in Cartesian co-ordinates system is (A) $\iint_R dr\,d\theta$ (B) $\iint_R dy\,dx$ (C) $\iint_R x\,dx\,dy$ (D) $\iint_R x^2\,dx\,dy$	ANS **B**	(CLO-1, Remember)
15.	Volume of a region R is given by (A) $\iiint_R dv$ (B) $2\iint_R dx\,dy$ (C) $\iint_R dx\,dy\,dz$ (D) $\iint_R dx\,dy$	ANS **A**	(CLO-1, Remember)

16.	$\int_0^1 \int_0^2 \int_0^3 dx\, dy\, dz \quad =$ (A) 3 (B) 4 (C) 2 (D) 6	ANS **D**	(CLO-1, Apply)
17.	$\int_1^a \int_1^b \frac{dx\, dy}{x\, y} \quad =$ (A) $\log a + \log b$ (B) $\log a$ (C) $\log b$ (D) $\log a \log b$	ANS **D**	(CLO-1, Apply)
18.	$\int_0^{\pi/2} \int_0^{\sin\theta} dr\, d\theta =$ (A) 1 (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$	ANS **A**	(CLO-1, Apply)
19.	Area of the region R in polar coordinates is (A) $\iint_R dr\, d\theta$ (B) $\iint_R r^2\, dr\, d\theta$ (C) $\iint_R r\, dr\, d\theta$ (D) $\iint_R (r+1)\, dr\, d\theta$	ANS **C**	(CLO-1, Remember)
20.	Area of an ellipse is (A) $\pi\, r^2$ (B) $\pi\, a^2\, b$ (C) $\pi\, a\, b^2$ (D) $\pi\, a\, b$	ANS **D**	(CLO-1, Remember)
21.	$\int_0^2 \int_0^1 4\, x\, y\, dx\, dy \quad =$ (A) 4 (B) 3 (C) 2 (D) 1	ANS **A**	(CLO-1, Apply)
22.	$\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta =$ (A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}\, a^2$	ANS **C**	(CLO-1, Apply)
23.	$\int_0^1 \int_0^2 \int_1^2 x^2\, y\;\; z\;\; dz\, dy\, dx \quad =$ (A) 2 (B) 4 (C) 3 (D) 1	ANS **D**	(CLO-1, Apply)
24.	Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$. (A) $\int_0^a \int_x^a \frac{x}{x^2+y^2}\, dy\, dx$ (B) $\int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$ (C) $\int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$ (D) $\int_0^a \int_x^{x^2} \frac{x}{x^2+y^2}\, dy\, dx$	ANS **B**	(CLO-1, Apply)

25.	Change the order of integration in $\int_0^1 \int_0^x dy\, dx$. (A) $\int_0^1 \int_1^y dx\, dy$ (B) $\int_0^1 \int_0^x dx\, dy$ (C) $\int_0^1 \int_0^y dy\, dx$ (D) $\int_0^1 \int_y^1 dx\, dy$	ANS **D**	(CLO-1, Apply)
26.	In double integration, the transformation used to change Cartesian into polar coordinates is (A) $dx\, dy = dr\, d\theta$ (B) $dx\, dy = \lvert J \rvert\, dr\, d\theta$ (C) $dx\, dy = -J\, dr\, d\theta$ (D) $dx\, dy = \lvert J \rvert^2 dr\, d\theta$	ANS **B**	(CLO-1, Remember)
27.	$\int_0^\pi \int_0^\pi d\theta\, d\varphi \;=$ (A) 1 (B) 0 (C) $\frac{\pi}{2}$ (D) π^2	ANS **D**	(CLO-1, Apply)
28.	$\int_0^1 \int_0^1 \int_0^1 dx\, dy\, dz \;=$ (A) 3 (B) 0 (C) 2 (D) 1	ANS **D**	(CLO-1, Apply)
29.	$\int_0^\pi \int_0^x \sin y\; dy\, dx \;=$ (A) π (B) $2\,\pi$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$	ANS **A**	(CLO-1, Apply)
30.	$\int_0^{1/2} \int_1^2 x\, dx\, dy \;=$ (A) 3 (B) $\frac{3}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$	ANS **D**	(CLO-1, Apply)
31.	$\iiint_R dx\, dy\, dz$ over the volume of the sphere of radius 'a' is (A) $4\,\pi\, a^3$ (B) $2\,\pi\, a^3$ (C) $\frac{2}{3}\,\pi\, a^3$ (D) $\frac{4}{3}\,\pi\, a^3$	ANS **D**	(CLO-1, Remember)
32.	$\int_0^2 \int_0^1 x\, y\;\, dx\, dy \;=$ (A) 1 (B) 2 (C) 3 (D) 4	ANS **A**	(CLO-1, Apply)
33.	$\int_0^1 \int_0^1 (x+y)\, dx\, dy \;=$ (A) 1 (B) 2 (C) 3 (D) 4	ANS **A**	(CLO-1, Apply)

34.	The region of integration of the integral $\int_{-b}^{b} \int_{-a}^{a} f(x,y)\ dx\ dy$ is (A) square (B) rectangle (C) triangle (D) circle	ANS **B**	(CLO-1, Apply)
35.	$\int_{0}^{2} \int_{0}^{1} y\, dx\, dy \ =$ (A) 4 (B) 2 (C) 0 (D) 1	ANS **B**	(CLO-1, Apply)
36.	$\int_{0}^{\infty} \int_{0}^{y} \frac{e^{-y}}{y}\, dx\, dy \ =$ (A) 4 (B) 2 (C) 0 (D) 1	ANS **D**	(CLO-1, Apply)

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 1 – Multiple Integrals

Part – B (Each question carries 3 Marks)

1. Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\, dx\, dy$.

Solution

$$\int_2^3 \int_1^2 \frac{1}{xy}\, dx\, dy = \left[\int_2^3 \frac{1}{y} dy\right]\left[\int_1^2 \frac{1}{x} dx\right] = [\log y]_2^3 [\log x]_1^2$$

$$= (\log 3 - \log 2)(\log 2 - \log 1) = \left(\log \frac{3}{2}\right)(\log 2)$$

2. Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\, dr\, d\theta$.

Solution

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{(\sin\theta)^2}{2}\right] d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta \quad = \frac{1}{2} * \frac{1}{2} * \frac{\pi}{2} = \frac{\pi}{8}$$

3. Evaluate $\int_0^2 \int_0^2 dx\, dy$.

Solution

$$\int_0^2 \int_0^2 dx\, dy = \int_0^2 [x]_0^2 dy = \int_0^2 [2-0] dy = [2\,y]_0^2 = (2)(2) - 0 = 4$$

4. Evaluate $\int_0^3 \int_0^2 (x^2 + y^2) dx\, dy$.

Solution

$$I = \int_0^3 \int_0^2 (x^2 + y^2) dxdy = \int_0^3 \left[\left(\frac{x^3}{3}\right) + xy^2\right]_0^2 dy = \int_0^2 \left[\frac{8}{3} + 2y^2\right] dy$$

$$= \left[\frac{8\mathrm{y}}{3} + \frac{2\mathrm{y}^3}{3}\right]_0^3 = \frac{8 * 3}{3} + \frac{2 * 3^3}{3} = 8 + 18 = 26$$

5. Evaluate $\int_0^a \int_0^b \int_0^c dx\, dy\, dz$.

Solution

$$\int_0^a \int_0^b \int_0^c dx\, dy\, dz = \int_0^a \int_0^b (x)_0^c\, dy\, dz = \int_0^a \int_0^b (c - 0) dy\, dz$$

$$= c \int_0^a (y)_0^b\, \mathrm{dz} = c \int_0^a (b - 0) dz = b\, c \int_0^a dz = bc\, (z)_0^a$$

$$= \mathrm{bc(a - 0) = abc}$$

6. Evaluate $\int_{\mathbf{0}}^{\boldsymbol{\pi}} \int_0^a r\, dr\, d\theta$.

Solution

$$\int_0^\pi \int_0^a r\, dr\, d\theta = \int_0^\pi \left(\frac{r^2}{2}\right)_0^a d\theta = \int_0^\pi \left[\frac{(a)^2}{2} - 0\right] d\theta = \frac{a^2}{2} \int_0^\pi d\theta = \frac{a^2}{2} (\theta)_0^\pi$$

$$= \frac{a^2}{2} (\pi - 0) = \frac{\pi a^2}{2}$$

7. Evaluate $\int_0^2 \int_0^2 e^{x+y}\, dx\, dy$.

Solution

$$\int_0^2 \int_0^2 e^{x+y} dxdy = \int_0^2 e^x dx \int_0^2 e^y dy = [e^x]_0^2 [e^y]_0^2$$

$$= (e^2 - e^0)(e^2 - e^0) = (e^2 - 1)^2$$

8. Evaluate $\int_1^2 \int_0^{2-y} xy\, dx\, dy.$

Solution

$$\int_1^2 \int_0^{2-y} xy\, dx\, dy = \int_1^2 \left(y\left(\frac{x^2}{2}\right)\right)_0^{2-y} dy = \left(\frac{1}{2}\right) \int_1^2 (y(2-y)^2 dy$$

$$= \frac{1}{2}\int_1^2 y(4 + y^2 - 4y)dy = \frac{1}{2}\int_1^2 (4y + y^3 - 4y^2)dy$$

$$= \frac{1}{2}\left[4\left(\frac{y^2}{2}\right) + \left(\frac{y^4}{4}\right) - 4\left(\frac{y^3}{3}\right)\right]_1^2$$

$$= \frac{1}{2}\left\{\left[4\left(\frac{4}{2}\right) + \left(\frac{16}{4}\right) - 4\left(\frac{8}{3}\right)\right] - \left[4\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right) - 4\left(\frac{1}{3}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{5}{12}\right\} = \frac{5}{24}$$

9. Evaluate $\int_0^1 \int_y^1 \frac{x}{x^2+y^2} dx\, dy.$

Solution

$$\int_0^1 \int_y^1 \frac{x}{x^2 + y^2} dxdy = \int_0^1 \int_0^x \frac{x}{x^2 + y^2} dydx = \int_0^1 \left(tan^{-1}\left(\frac{y}{x}\right)_{y=0}^{y=x}\right) dx$$

$$= \int_0^1 (tan^{-1}(1) - tan^{-1}(0))\; dx$$

$$= \int_0^1 \left(\frac{\pi}{4} - 0\right) dx = \frac{\pi}{4}\int_0^1 dx = \frac{\pi}{4}(x)_0^1 = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$$

10. Evaluate $\int_0^3 \int_0^2 x\,y(x+y)\,dy\,dx$.

Solution

$$\int_0^3 \int_0^2 x\,y(x+y)\,dy\,dx = \int_0^3 \int_0^2 \left(x^2\,y + x\,y^2\right) dy\,dx$$

$$= \int_0^3 \left(\frac{x^2\,y^2}{2} + x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_0^3 \left(2\,x^2 + \frac{8}{3}x\right) dx$$

$$= \left(2\frac{x^3}{3} + \frac{8}{3}\frac{x^2}{2}\right)_0^3 = 30$$

11. Evaluate $\int_0^1 \int_0^1 (x+y)\,dx\,dy$.

Solution

$$\int_0^1 \int_0^1 (x+y)\,dx\,dy \quad = \int_0^1 \left[\left(\frac{x^2}{2} + xy\right)\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} + y\right) dy$$

$$= \left(\frac{y}{2} + \frac{y^2}{2}\right)_0^1$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right) - (0+0)$$

$$= 1$$

12. Find the value of $\int_0^\pi \int_0^1 (x^2 \sin y)\,dx\,dy$.

Solution

$$\int_0^\pi \int_0^1 (x^2 \sin y)\,dx\,dy \quad = \int_0^1 x^2\,dx \quad \int_0^\pi \sin y\,dy$$

$$= \left(\frac{x^3}{3}\right)_0^1 (-\cos y)_0^\pi$$

$$= \left(\frac{1}{3} - 0\right)(-\cos\pi + \cos 0)$$

$$= \left(\frac{1}{3} - 0\right)(1 + 1)$$

$$= \frac{2}{3}$$

13. Evaluate $\int_0^c \int_0^b \int_0^a (x + y + z)\, dx\, dy\, dz$.

Solution

$$\int_0^c \int_0^b \int_0^a (x + y + z)\, dx\, dy\, dz = \int_0^c \int_0^b \left(\frac{x^2}{2} + x\,y + x\,z\right)_0^a dy\, dz$$

$$= \int_0^c \int_0^b \left(\frac{a^2}{2} + a\,y + a\,z\right) dy\, dz$$

$$= \int_0^c \left(\frac{a^2}{2} b + a\frac{b^2}{2} + a\,z\,b\right) dz$$

$$= \int_0^c \left(\frac{a^2}{2} y + a\frac{y^2}{2} + a\,z\,y\right)_0^b dz$$

$$= \left(\frac{a^2}{2} b\,z + a\frac{b^2}{2} z + a\,b\frac{z^2}{2}\right)_0^c$$

$$= \frac{a\,b\,c\,(a + b + c)}{2}$$

14. Evaluate $\int_0^4 \int_0^x \int_0^{\sqrt{x+y}} z\; dx\, dy\, dz$.

Solution

$$\mathrm{I} = \int_{x=0}^{4} \int_{y=0}^{x} \int_{z=0}^{\sqrt{x+y}} z\; dz\, dy\, dx$$

$$= \int_0^4 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dydx$$

$$= \frac{1}{2}\int_0^4\int_0^x (x+y)\,dy\,dx$$

$$= \frac{1}{2}\int_0^4 \left(xy + \frac{y^2}{2} \right)_0^x dx \quad = \frac{1}{2}\int_0^4 \left(x^2 + \frac{x^2}{2} \right) dx \quad = \frac{3}{4}\int_0^4 x^2 dx = \frac{3}{4}\left(\frac{x^3}{3} \right)_0^4 = 16$$

15. Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{1+x^2+y^2}$.

Solution

$$I = \int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{1+x^2+y^2}$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+y^2}} \tan^{-1}\left(\frac{x}{\sqrt{1+y^2}} \right) \right)_0^{\sqrt{1+y^2}} dy$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+y^2}} \left(\tan^{-1}(1) - \tan^{-1}(0) \right) \right) dy$$

$$= \int_0^1 \frac{\pi}{4} \frac{dy}{\sqrt{1+y^2}} = \frac{\pi}{4} \log(1+\sqrt{2})$$

16. Evaluate $\int_0^a \int_0^{\sqrt{a\,y}} x\,y\;dx\,dy$.

Solution

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y\;dy\,dx = \int_0^a y\left(\frac{x^2}{2} \right)_0^{\sqrt{a\,y}} dy$$

$$= \frac{1}{2}\int_0^a y\,a\,y\;dy = \frac{a^4}{6}$$

17. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \dfrac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$.

Solution

Let $I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \dfrac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

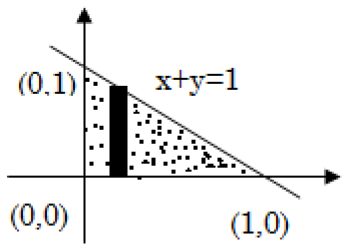
$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1)-\sin^{-1}(0)\right] dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{\pi}{2}-0\right] dy\,dx = \frac{\pi}{2}\int_0^a [y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a = \frac{\pi}{2}\left[\left(0+\frac{a^2}{2}\frac{\pi}{2}\right)-(0+0)\right] = \frac{\pi^2 a^2}{8}$$

18. Evaluate $\iint_R (\mathbf{x}^2+\mathbf{y}^2)\,\mathbf{dy\,dx}$ over the region R for which $x, y \ge 0,\ x+y \le 1$.

Solution

The region of integration is the triangle bounded by the lines $x=0,\ y=0,\ x+y=1$



Limits of y : 0 to $1-x$; Limits of x : 0 to 1

$$\iint_R (x^2+y^2)\,dy\,dx = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x)+\frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{(1-x)^4}{12}\right]_0^1$$

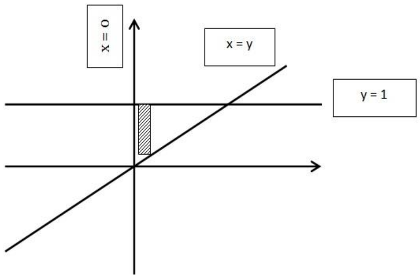
$$= \int_0^1 \left[x^2(1-x)+\frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{(1-x)^4}{12}\right]_0^1$$

$$=\frac{1}{3}-\frac{1}{4}+\frac{1}{12}$$
$$=\frac{1}{6}$$

19. Find the area bounded by the lines $x=0,\ y=1$ and $y=x$ using double integration.

Solution



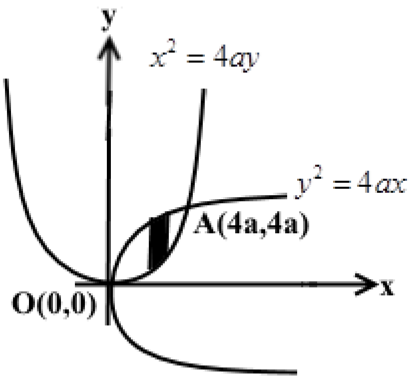
Given $x=0,\ y=1$ *and* $y=x$.

Hence x *varies from* 0 *to* 1 *and* y *varies from* x *to* 1.

$$I=\int_0^1\int_x^1 dydx=\int_0^1[y]_x^1dx=\int_0^1(1-x)dx=\left[x-\frac{x^2}{2}\right]_0^1=1-\frac{1}{2}=\frac{1}{2}$$

20. **Find by double integration the area between the parabolas $y^2=4ax$ and $x^2=4ay$.**

Solution



$$\therefore \text{Area }=\int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}}dydx=\int_0^{4a}[y]_{\frac{x^2}{4a}}^{\sqrt{4ax}}dx=\int_0^{4a}\left[\sqrt{4ax}-\frac{x^2}{4a}\right]dx$$

$$=\int_0^{4a}\left[2\sqrt{a}\,x^{1/2}-\frac{1}{4a}x^2\right]dx=\left[2\sqrt{a}\,\frac{x^{3/2}}{3/2}-\frac{1}{4a}\frac{x^3}{3}\right]_0^{4a}$$

$$=\frac{4\sqrt{a}}{3}(4a)^{3/2}-\frac{1}{12a}(4a)^3$$

$$=\frac{4\sqrt{a}}{3}(4)^{3/2}(a)^{3/2}-\frac{1}{12a}64a^3=\frac{4^{5/2}}{3}a^{4/2}-\frac{1}{12a}64a^3$$

$$=\frac{(2^2)^{5/2}}{3}a^2-\frac{16}{3}a^2=\frac{32}{3}a^2-\frac{16}{3}a^2$$

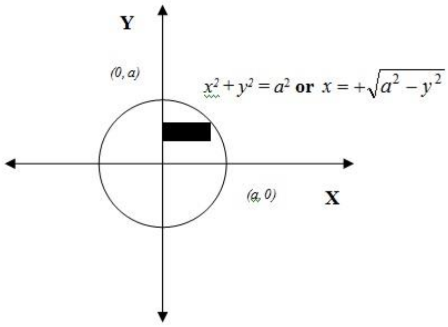
$$=\frac{16}{3}a^2$$

21. **Find the area of the circle $x^2 + y^2 = a^2$ using double integration.**

Solution

Area of circle = 4 × Area in first quadrant

$$= 4\int_0^a \int_0^{\sqrt{a^2-y^2}} dx\,dy$$



$$= 4\int_0^a (x)_0^{\sqrt{a^2-y^2}}\, dy$$

$$= 4\int_0^a \sqrt{a^2-y^2}\, dy$$

$$= 4\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right]_0^a$$

$$= 4\left[\frac{a^2}{2}\frac{\pi}{2}\right] = \pi a^2$$

22. Find the area of the circle $x^2 + y^2 = a^2$ using polar coordinates.

Solution

$x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2$

$r^2 = a^2$

$$\text{Area} = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{a^2}{2}\,d\theta$$

$$= \pi a^2$$

23. Find the area of the cardioid $\mathbf{r = a(1+\cos\theta)}$ by using double integration.

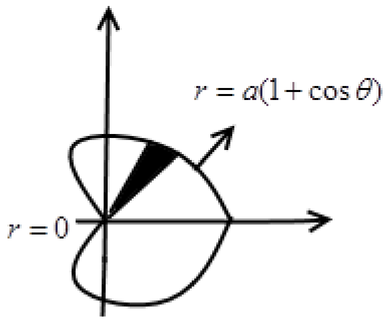
Solution

Given the curve in polar co ordinates $\mathbf{r = a(1+\cos\theta)}$

$\therefore$ **Area of the cardioid = 2(Area above the initial line)**

θ varies from 0 to π

r varies from 0 to $r = a(1+\cos\theta)$



$$\text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left[1+2\cos\theta+\left(\frac{1+\cos\theta}{2}\right)\right]d\theta \qquad = a^2\int_0^{\pi}\left[\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right]d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\pi} \qquad \because \sin n\pi = 0,\ \forall n$$

$$= a^2\left[\frac{3}{2}\pi\right] \qquad = \frac{3\pi a^2}{2}$$

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT II - VECTOR CALCULUS

Part – A

1.	If $\varphi(x,y,z) = xyz$, then $\nabla\varphi =$ (A) $yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ (B) $xy\,\vec{i} + yz\,\vec{j} + xz\,\vec{k}$ (C) $xz\,\vec{i} + zy\,\vec{j} + xy\,\vec{k}$ (D) $x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$	ANS **A**	(CLO-2, Apply)
2.	Curl ($grad\ \varphi$) = (A) $\vec{0}$ (B) 1 (C) 2 (D) – 1	ANS **A**	(CLO-2, Remember)
3.	The maximum directional derivative of $\varphi(x,y,z) = x^2 + y^2 + z^2$ at $(1,1,1)$ is (A) 0 (B) 3 (C) 2 (D) $2\sqrt{3}$	ANS **D**	(CLO-2, Apply)
4.	If $\vec{r}$ is the position vector of the point (x,y,z) with respect to the origin, then $\nabla \bullet \vec{r} =$ (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Apply)
5.	If $\vec{u}$ and $\vec{v}$ are irrotational, then $\vec{u} \times \vec{v}$ is (A) irrotational (B) solenoidal (C) zero vector (D) constant	ANS **B**	(CLO-2, Remember)
6.	The condition for $\vec{F}$ to be conservative is (A) $\nabla \bullet \vec{F} = 0$ (B) 0 (C) $\nabla \times \vec{F} = \vec{0}$ (D) 1	ANS **C**	(CLO-2, Remember)

7.	The relation between the surface integral and the volume integral is given by (A) Green's theorem (B) Stoke's theorem (C) Gauss Divergence theorem (D) Cauchy's theorem	ANS **C**	(CLO-2, Remember)
8.	By Stoke's theorem, $\int_C \vec{F} \bullet d\vec{r} =$ (A) $\iint_S \nabla \times \vec{F}\, dS$ (B) $\iint_S \nabla \bullet \vec{F}\, dS$ (C) $\iint_S (\nabla \bullet \vec{F})\, \hat{n}\, dS$ (D) $\iint_S (\nabla \times \vec{F}) \bullet \hat{n}\, dS$	ANS **D**	(CLO-2, Remember)
9.	The maximum value of the directional derivative is (A) $\lvert\nabla \varphi\rvert$ (B) curl φ (C) grad φ (D) $\lvert\nabla \times \varphi\rvert$	ANS **A**	(CLO-2, Remember)
10.	If $\vec{F}$ is irrotational, then $Curl\, \vec{F} =$ (A) 1 (B) 2 (C) 3 (D) $\vec{0}$	ANS **D**	(CLO-2, Apply)
11.	If the divergence of the vector is zero, then the vector is said to be (A) irrotational vector (B) constant vector (C) zero vector (D) solenoidal vector	ANS **D**	(CLO-2, Remember)
12.	The unit normal vector to the surface $x^2\, y + 2\, x\, z = 4$ at the point $(2, -2, 3)$ is (A) $-\frac{1}{3}\,\vec{\imath} + \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (B) $\frac{1}{3}\,\vec{\imath} + \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (C) $-\frac{1}{3}\,\vec{\imath} - \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (D) $\frac{1}{3}\,\vec{\imath} - \frac{2}{3}\,\vec{\jmath} - \frac{2}{3}\,\vec{k}$	ANS **A**	(CLO-2, Apply)
13.	If the vector $\vec{F} = (x + 3y)\,\vec{\imath} + (y - 2z)\,\vec{\jmath} + (x + a\,z)\,\vec{k}$ is solenoidal, then $a =$ (A) 2 (B) 0 (C) -2 (D) -1	ANS **C**	(CLO-2, Apply)
14.	The work done by the conservative force when it moves a particle around a closed curve is (A) $\nabla \bullet \vec{F} = 0$ (B) 0 (C) $\nabla \times \vec{F} = 0$ (D) $\nabla \bullet (\nabla \times \vec{F}) = 0$	ANS **C**	(CLO-2, Remember)

15.	The value of $\int_C x\,dy - y\,dx$ around the circle $x^2 + y^2 = 1$ is (A) π (B) 2π (C) 3π (D) 0	ANS **B**	(CLO-2, Apply)
16.	By Green's theorem, the area bounded by a simple closed curve is (A) $\int_C x\,dy - y\,dx$ (B) $\int_C x\,dy + y\,dx$ (C) $\int_C y\,dx - x\,dy$ (D) $\frac{1}{2}\left(\int_C x\,dy - y\,dx\right)$	ANS **D**	(CLO-2, Apply)
17.	To be conservative, $\vec{F}$ should be (A) solenoidal (B) irrotational (C) rotational (D) constant vector	ANS **B**	(CLO-2, Remember)
18.	The unit normal vector to the surface $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$ is (A) $\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$ (B) $\frac{\vec{i}+\vec{j}-\vec{k}}{\sqrt{3}}$ (C) $\frac{\vec{i}-\vec{j}-\vec{k}}{\sqrt{3}}$ (D) $\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{2}}$	ANS **B**	(CLO-2, Apply)
19.	If $\vec{r}$ is the position vector of the point (x, y, z) with respect to the origin, then $div\,\vec{r} =$ (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Remember)
20.	If φ is a scalar function, then $\nabla \times \nabla\varphi =$ (A) $\vec{0}$ (B) solenoidal (C) irrotational (D) constant	ANS **A**	(CLO-2, Remember)
21.	The value of line integral $\int_C \vec{F} \bullet d\vec{r}$ where C is the line $y = x$ in XY plane from (1, 1) to (2, 2) is (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Apply)
22.	Angle between two level surfaces $\varphi_1 = C$ and $\varphi_2 = C$ is given by (A) $\sin\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (B) $\cos\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (C) $\tan\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (D) $\tan\theta = \frac{\nabla\varphi_1 \times \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$	ANS **B**	(CLO-2, Apply)

23.	The condition for a vector $\vec{r}$ to be solenoidal is (A) $div\ \vec{r} = 0$ (B) $curl\ \vec{r} = 0$ (C) $div\ \vec{r} \neq 0$ (D) $curl\ \vec{r} \neq 0$	ANS **A**	(CLO-2, Remember)
24.	The unit normal vector to the surface $x^2 + 2\,y^2 + z^2 = 7$ at the point $(1, -1, 2)$ is (A) $\frac{\vec{\imath}-2\,\vec{\jmath}-2\,\vec{k}}{3}$ (B) $\frac{\vec{\imath}-2\,\vec{\jmath}+2\,\vec{k}}{3}$ (C) $\frac{\vec{\imath}+2\,\vec{\jmath}+2\,\vec{k}}{3}$ (D) $\frac{\vec{\imath}-2\,\vec{\jmath}+2\,\vec{k}}{3}$	ANS **D**	(CLO-2, Apply)
25.	If the integral $\int_A^B \vec{F} \bullet d\vec{r}$ depends only on the end points but not on the path *C*, then $\vec{F}$ is (A) neither solenoidal nor irrotational (B) solenoidal (C) irrotational (D) conservative	ANS **D**	(CLO-2, Remember)
26.	According to Gauss divergence theorem, $\int_C\ (P\,dx + Q\,dy) =$ (A) $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ (B) $\iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dx\,dy$ (C) $\iint_R \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy$ (D) $\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$	ANS **A**	(CLO-2, Apply)
27.	By Green's theorem, $\frac{1}{2}\left(\int_C\ x\,dy - y\,dx\right) =$ (A) Area of a closed curve (B) 2 × Area of a closed curve (C) Volume of a closed curve (D) 3 × Volume of a closed curve	ANS **A**	(CLO-2, Apply)
28.	The value of $\iint_S\ \vec{r} \bullet \vec{n}\,dS$ where *S* is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ is (A) $2\,\pi\,a^3$ (B) $3\,\pi\,a^3$ (C) $4\,\pi\,a^3$ (D) $5\,\pi\,a^3$	ANS **C**	(CLO-2, Apply)
29.	The maximum directional derivative of $\varphi\ (x, y, z) = xyz^2$ at (1, 0, 3) is (A) 9 (B) 1 (C) – 9 (D) 0	ANS **A**	(CLO-2, Apply)
30.	The relation between line integral and double integral is given by (A) Gauss divergence theorem (B) Cauchy's theorem (C) Green's theorem (D) Convolution theorem	ANS **C**	(CLO-2, Remember)

31.	If $\varphi\,(x, y, z) = x^2 + y^2 + z^2$, then $\nabla\varphi$ at $(1, 1, 1)$ = (A) $2\,\vec{\imath} + 2\,\vec{\jmath} + 2\,\vec{k}$ (B) $2\,\vec{\imath} - 2\,\vec{\jmath} + \vec{k}$ (C) $\vec{\imath} + \vec{\jmath} + \vec{k}$ (D) $2\,\vec{\imath} - 2\,\vec{\jmath} - 2\,\vec{k}$	ANS **A**	(CLO-2, Apply)
32.	If $\varphi\,(x, y, z) = xyz$, then $\nabla\varphi$ at $(1, 1, 1)$ is (A) $\vec{\imath} + \vec{\jmath} + \vec{k}$ (B) $2\,\vec{\imath} + 2\,\vec{\jmath} + 2\,\vec{k}$ (C) $2\,\vec{\imath} - 2\,\vec{\jmath} + \vec{k}$ (D) $2\,\vec{\imath} - 2\,\vec{\jmath} - 2\,\vec{k}$	ANS **A**	(CLO-2, Apply)
33.	The unit normal vector to the surface $\varphi = xy - yz - zx$ at the point (– 1, 1, 1) is (A) $-2\,\vec{\jmath}$ (B) $-\vec{\jmath}$ (C) $3\,\vec{\imath}$ (D) $4\,\vec{\imath}$	ANS **B**	(CLO-2, Apply)
34.	. $\nabla\, r^n =$ (A) $n\,\vec{r}$ (B) $n\,(n-1)\,\vec{r}$ (C) $n\,r^{n-2}\,\vec{r}$ (D) $n\,r^{n+2}\,\vec{r}$	ANS **C**	(CLO-2, Apply)
35.	The directional derivative of $\varphi = 2xy + z^2$ at (1, –1, 3) in the direction of $\vec{\imath} + 2\,\vec{\jmath} + 2\,\vec{k}$ is (A) $\frac{14}{3}$ (B) $-\frac{14}{3}$ (C) $\frac{4}{3}$ (D) $\frac{3}{14}$	ANS **A**	(CLO-2, Apply)
36.	If $\vec{F} = (3x - 2y + z)\,\vec{\imath} + (4x + ay - 2)\,\vec{\jmath} + (x - y + 2)\,\vec{k}$ is solenoidal, then $a =$ (A) 3 (B) 0 (C) -3 (D) -1	ANS **C**	(CLO-2, Apply)

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 2 – Vector Calculus

Part – B (Each question carries 3 Marks)

1. Find $\nabla\phi$ if $\phi = \log\left(x^2+y^2+z^2\right)$.

Solution

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right) + \vec{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$$

$$= \vec{i}\frac{2x}{(x^2+y^2+z^2)} + \vec{j}\frac{2y}{(x^2+y^2+z^2)} + \vec{k}\frac{2z}{(x^2+y^2+z^2)}$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{i}+y\vec{j}+z\vec{k}\right) = \frac{2\vec{r}}{r^2} \qquad \because (\vec{r} = x\vec{i}+y\vec{j}+z\vec{k} \;\&\; r^2 = x^2+y^2+z^2)$$

2. Find the unit normal vector to the surface $x^2+y^2=z$ at the point $(1,-2,5)$.

Solution

Given

$$\phi = x^2+y^2-z$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi \text{ at } (1,-2,5) = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = 3$$

Unit Normal vector is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{3}$$

3. Prove that $\boldsymbol{curl(grad\phi) = 0}$.

Solution

$$grad\phi = \nabla\phi = \vec{\imath}\frac{\partial\phi}{\partial x} + \vec{\jmath}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$Curl\,(grad\,\varphi) = \nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right) - \vec{j}\left(\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial x}\right) + \vec{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right)$$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$ **(Since mixed partial derivatives are equal.)**

4. Find $\boldsymbol{curl\vec{F}}$ if $\boldsymbol{\vec{F} = xy\vec{\imath} + yz\vec{\jmath} + zx\vec{k}}$.

Solution

Given $\vec{F} = xy\vec{\imath} + yz\vec{\jmath} + zx\vec{k}$

$$curl\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{\imath}\,(0-y) - \vec{\jmath}(z-0) + \vec{k}\,(0-x)$$

$$= -y\,\vec{\imath} - z\,\vec{\jmath} - x\,\vec{k}$$

5. In what direction from $\boldsymbol{(3,1,-2)}$ is the directional derivative of $\boldsymbol{\phi = x^2y^2z^4}$ maximum? Find also the magnitude of this maximum.

Solution

Given $\phi = x^2y^2z^4$

$$\nabla\phi = \vec{\imath}\frac{\partial\phi}{\partial x} + \vec{\jmath}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xy^2z^4\vec{\imath} + 2x^2yz^4\vec{\jmath} + 4x^2y^2z^3\vec{k}$$

$$\nabla\phi\ at\ (3,1,-2) = 92\vec{\imath} + 144\vec{\jmath} - 92\vec{k}$$

$$|\nabla\phi| = \sqrt{92^2 + 144^2 + 92^2} = \sqrt{37664}$$

The directional derivative is maximum in the direction $\nabla\phi$ and the magnitude of this maximum is $|\nabla\phi| = \sqrt{37664}$.

6. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, – 2 , – 1) in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution

Given $\phi = x^2yz + 4xz^2$,

$$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k},\ |\vec{a}| = \sqrt{4+1+4} = 3$$

$$\nabla\phi = (2xyz + 4z^2)\vec{i} + x^2z\vec{j} + (x^2y + 8xz)\vec{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\text{D.D.} = \nabla\phi.\frac{\vec{a}}{|\vec{a}|} = (8\vec{i} - \vec{j} - 10\vec{k}).\frac{2\vec{i} - \vec{j} - 2\vec{k}}{3} = \frac{37}{3}$$

7. Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at P (1, 2 , 3) in the direction of line PQ where Q is (5, 0, 4).

Solution

$$\nabla\varphi = grad\ \varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = grad\ \varphi = \vec{i}\ 2x + \vec{j}\ (-2y) + \vec{k}\ 4z$$

$$\nabla\varphi \text{ at } (1,2,3) = 2\vec{i} - 4\vec{j} + 12\vec{k}$$

$$\vec{a} = OQ - OP = (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$$

Directional derivative = $\nabla\varphi \bullet \dfrac{\vec{a}}{|\vec{a}|}$

$$= (2\vec{i} - 4\vec{j} + 12\vec{k}) \bullet \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

8. Find the angle between the normals to the surfaces $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).

Solution

Given $\varphi = x^2 - yz$

$$\nabla\varphi = 2x\vec{i} - z\vec{j} - y\vec{k}$$

$$\nabla\varphi_1 /(1,1,1) = 2\vec{i} - \vec{j} - \vec{k} \qquad \nabla\varphi_2 /(2,4,1) = 4\vec{i} - \vec{j} - 4\vec{k}$$

$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6}$ $\qquad$ $|\nabla\varphi_2| == \sqrt{16+1+16} = \sqrt{33}$

$$\cos\theta = \frac{\nabla\varphi_1 \circ \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|} = \frac{(2\vec{i} - \vec{j} - \vec{k}) \circ (4\vec{i} - \vec{j} - 4\vec{k})}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}.$$

9. Find a such that $\vec{F} = (3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$ is solenoidal.

Solution

Given $\nabla \circ \vec{F} = 0 \Rightarrow \frac{\partial}{\partial x}(3x-2y+z) + \frac{\partial}{\partial y}(4x+ay-z) + \frac{\partial}{\partial z}(x-y+2z) = 0$

$$3 + a + 2 = 0 \Rightarrow a + 5 = 0 \Rightarrow a = -5$$

10. Find the constant a, b, c so that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Solution

Given $\vec{F}$ is irrotational **i.e.,** $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\vec{i}\left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right)$$
$$+\vec{k}\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right) = \vec{0}$$

$= i.e., \quad \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$

$= \therefore c+1 = 0, 4-a = 0, and\, b-2 = 0$

$\Rightarrow a = 4, b = 2, c = -1$

11. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, then find div curl $\vec{F}$.

Solution $div\ curl\,\vec{F} = \nabla.\,(\nabla \times \vec{F})$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$$

$$\nabla \times \vec{F} = \vec{0}$$

$$\therefore \nabla.(\nabla \times \vec{F}) = 0$$

12. Prove that $div\,\vec{r} = 3$**.**

Solution

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$div\,\vec{r} = \nabla \bullet \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \bullet \left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

13. Show that the vector $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$ is irrotational.

Solution

Given $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$

$$curl\vec{F} = \nabla \times \vec{F} = \vec{0}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

14. If $F = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$. **Evaluate** $\int_C \vec{F} \bullet d\vec{r}$ **from (0,0,0) to (1,1,1) along the curve**

x = t, y = t^2, z = t^3.

Solution

The end points are (0,0,0) and (1,1,1).

These points correspond to t = 0 and t = 1.

$\therefore$ dx = dt, dy = 2t dt, dz = $3t^2$dt

$$\int_C \vec{F} \bullet d\vec{r} = \int_C (3x^2 + 6y)dx - 14yzdy + 20xz^2 dz$$

$$= \int_0^1 (3t^2 + 6t^2)dt - 14t^5(2tdt) + 20t^7(3t^2)dt = \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = 5$$

15. If $F = ax\vec{i} + by\vec{j} + cz\vec{k}$, **a, b, c are constants, show that** $\iint_S \vec{F} \bullet \hat{n}\, ds = \frac{4\pi}{3}(a+b+c)$ **where S is the surface of a unit sphere.**

Solution

W.K.T. Gauss's divergence theorem

$$\iint_S \vec{F} \bullet \hat{n}ds = \iiint_V \nabla \bullet \vec{F}dV = \iiint_V \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \right) dV$$

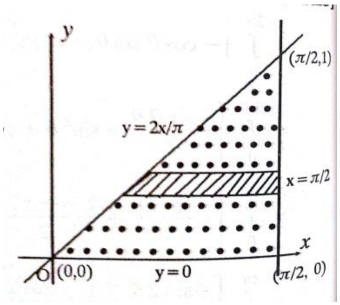
$$= \iiint_V (a+b+c)dV = (a+b+c)V = (a+b+c)\frac{4}{3}\pi(1)^3$$

$$\iint_S \vec{F} \bullet \hat{n}ds = \frac{4}{3}\pi(a+b+c)$$

16. Using Green's theorem, evaluate $\int_c (y-\sin x)\,dx+\cos x\,dy$ where c is the triangle formed by $y=0, x=\frac{\pi}{2}, y=\frac{2x}{\pi}$.

Solution

Using Green's theorem, we convert the line integral to double integral over the given region. $ie., \int_C u\,dx+v\,dy=\iint_R\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy$



$u=y-\sin x$ $\qquad v=\cos x$

$\frac{\partial u}{\partial y}=1$ $\qquad \frac{\partial v}{\partial x}=-\sin x$

Hence, $\int_C\{(y-\sin x)dx+\cos x dy\}=\iint_R(-\sin x-1)dxdy$

$$=\int_0^1\int_{\pi y/2}^{\pi/2}(-\sin x-1)dxdy=\int_0^1[\cos x-x]_{\pi y/2}^{\pi/2}$$

$$=\int_0^1\left(0-\frac{\pi}{2}-\cos\frac{\pi y}{2}+\frac{\pi y}{2}\right)dy$$

$$=\left[\frac{-\pi y}{2}-\frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}}+\frac{\pi}{2}.\frac{y^2}{2}\right]_0^1=\frac{-\pi}{2}-\frac{2}{\pi}+\frac{\pi}{4}$$

$$=\frac{-\pi^2-8}{4\pi}=-\left[\frac{\pi}{4}+\frac{2}{\pi}\right].$$

17. Using Green's theorem, evaluate $\int_c (3x^2-8y^2)\,dx+(4y-6xy)\,dy$ where c is the boundary of the triangle formed by the lines $x=0, y=0, x+y=1$ in the xy plane.

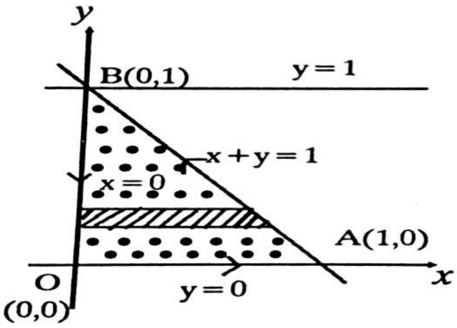
Solution

Using Green's theorem, we convert the line integral to double integral over the given region. $ie., \int_C u\,dx+v\,dy=\iint_R\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy$

$u = 3x - 8y^2$ $\qquad$ $v = 4y - 6xy$

$\frac{\partial u}{\partial y} = -16y$ $\qquad$ $\frac{\partial v}{\partial x} = -6y$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -6y + 16y = 10y$$



Hence, $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = \iint_R (10y) dxdy$

$$= 10\int_0^1 \int_0^{1-y} (y) dxdy = \int_0^1 y[x]_0^{1-y} dy$$

$$= 10\int_0^1 y(1-y)dy = 10\int_0^1 (y - y^2)dy$$

$$= 10\left(\frac{y^2}{2} - \frac{y^3}{3} \right)_0^1$$

$$= 10\left(\frac{1}{2} - \frac{1}{3} \right).$$

$$= 10\frac{3-2}{6} = \frac{10}{6} = \frac{5}{3}$$

18. Using Gauss divergence theorem evaluate $\iiint_V \nabla \circ \vec{F} dv$ **where** $\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ **taken over the cube bounded by the planes** $x = 0,\ x = 1, y = 0,\ y = 1, z = 0, z = 1.$

Solution

$$\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 4z - 2y + y = 4z - y$$

$$\iiint_V \nabla\circ\vec{F}dv=\int_0^1\int_0^1\int_0^1(4z-y)dxdydz=\int_0^1\int_0^1[4zx-yx]_0^1dydz=\int_0^1\int_0^1[4z-y]dydz$$

$$=\int_0^1\left[4zy-\frac{y^2}{2}\right]_0^1dz=\int_0^1\left[4z-\frac{1}{2}\right]dz=\left[4\frac{z^2}{2}-\frac{z}{2}\right]_0^1=\frac{4}{2}-\frac{1}{2}=\frac{3}{2}$$

19. Using Gauss divergence theorem theorem evaluate $\iiint_V \nabla\circ\vec{F}dv$ where $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$ taken over the cube bounded by the planes $x=0,\ x=1, y=0,\ y=1, z=0, z=1.$

Solution

$$\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$$

$$\nabla\circ\vec{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$\nabla\circ\vec{F}=2x+2y+2z=2(x+y+z)$$

$$\iiint_V \nabla\circ\vec{F}dv=2\int_0^1\int_0^1\int_0^1(x+y+z)dxdydz=2\int_0^1\int_0^1\left[\frac{x^2}{2}+xy+xz\right]_0^1dydz=2\int_0^1\int_0^1\left[\frac{1}{2}+y+z\right]dydz$$

$$=2\int_0^1\left[\frac{y}{2}+\frac{y^2}{2}+yz\right]_0^1dz=2\int_0^1\left[\frac{1}{2}+\frac{1}{2}+z\right]_0^1dz=2\int_0^1[1+z]_0^1dz=2\left[z+\frac{z^2}{2}\right]_0^1$$

$$=2\left(1+\frac{1}{2}\right)=2\left(\frac{3}{2}\right)=3$$

20. Using Stokes theorem find $\iint_S curl\,\vec{F}\,ds$ where $\vec{F}=(x^2-y^2)\vec{i}+2xy\vec{j}$ in the rectangular region of $x=0,\ y=0,\ x=a$ and $y=a.$

Solution Stokes theorem $\int_c \vec{F}.d\vec{r}=\iint_s curl\,\vec{F}.d\vec{s}$

Given $\vec{F}=(x^2-y^2)\vec{i}+2xy\vec{j}$

$$curl\,\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2-y^2 & 2xy & 0\end{vmatrix}=4y\,\vec{k}$$

Here $\hat{n}=\vec{k}$

$$\iint_S curl\,\vec{F}.\hat{n}ds = \iint_S 4ydxdy = \int_0^b\int_0^a 4ydxdy = 2ab^2$$

21. Prove that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\int_C (xdy - ydx).$$

Solution

W.K.T. Green's theorem

$$\int_C (udx + vdy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy \qquad \dots 1$$

Here $v = \frac{x}{2}$ $\qquad u = -\frac{y}{2}$

$\frac{\partial v}{\partial x} = \frac{1}{2}$ $\qquad \frac{\partial u}{\partial y} = -\frac{1}{2}$

$$(1) \Rightarrow \int_C \left(\frac{x}{2}dy - \frac{y}{2}dx\right) = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right)dxdy$$

$$\frac{1}{2}\int_C (xdy - ydx) = \iint_R dxdy$$

22. Find the area of the ellipse x = a cos θ, y = b sin θ using Green's theorem.

Solution

Given $\qquad x = a\cos\theta, \qquad y = b\sin\theta$

$dx = -a\sin\theta\, d\theta, \qquad dy = b\cos\theta\, d\theta$

θ varies from 0 to 2π.

Area of the ellipse $= \frac{1}{2}\oint_C xdy - ydx$

$$=\frac{1}{2}\int_{0}^{2\pi}(a\cos\theta)(-b\cos\theta d\theta)-(b\sin\theta)(-a\sin\theta d\theta)$$

$$=\frac{1}{2}\int_{0}^{2\pi}[ab\cos\theta\cos\theta+ab\sin\theta\sin\theta]d\theta$$

$$=\frac{ab}{2}\int_{0}^{2\pi}(\cos^{2}\theta+\sin^{2}\theta)d\theta=\frac{ab}{2}\int_{0}^{2\pi}d\theta=\frac{ab}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

Area of the ellipse $=\frac{ab}{2}[2\pi]=\pi ab$

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT III - LAPLACE TRANSFORMS

Part – A

1.	$L[t] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) s (D) s^2	ANS **B**	(CLO-3, Apply)
2.	$L[\cos t] =$ (A) $\frac{1}{s^2-1}$ (B) $\frac{1}{s^2+1}$ (C) $\frac{s}{s^2-1}$ (D) $\frac{s}{s^2+1}$	ANS **D**	(CLO-3, Apply)
3.	$L[e^{3t}] =$ (A) $\frac{1}{s-3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s-\log 9}$ (D) $\frac{9}{s}$	ANS **A**	(CLO-3, Apply)
4.	If $L[f(t)] = F(s)$, then $L[e^{at} f(t)] =$ (A) $F(s+a)$ (B) $F(s-a)$ (C) $e^{as}F(s)$ (D) $e^{-as}F(s)$	ANS **B**	(CLO-3, Remember)
5.	$L[f(t) * g(t)] =$ (A) $F(s) - G(s)$ (B) $F(s) + G(s)$ (C) $F(s)\ G(s)$ (D) $F(s) \div G(s)$	ANS **C**	(CLO-3, Remember)
6.	$L[\sin t] =$ (A) $\frac{1}{s^2-1}$ (B) $\frac{1}{s^2+1}$ (C) $\frac{s}{s^2-1}$ (D) $\frac{s}{s^2+1}$	ANS **B**	(CLO-3, Apply)

7.	$L[e^{-3\,t}] =$ (A) $\frac{1}{s+3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s-\log 3}$ (D) $\frac{3}{s}$	ANS **A**	(CLO-3, Apply)
8.	$L^{-1}\left[\frac{1}{s}\right] =$ (A) t (B) s (C) 1 (D) $\delta(t)$	ANS **C**	(CLO-3, Apply)
9.	$L^{-1}\left[\frac{1}{s^2+9}\right] =$ (A) $\frac{\cos 3t}{3}$ (B) $\frac{\sin 3t}{3}$ (C) $\sin 3t$ (D) $\cos 3t$	ANS **B**	(CLO-3, Apply)
10.	$L^{-1}\left[\frac{s}{s^2+9}\right] =$ (A) $\frac{\cos 3t}{3}$ (B) $\frac{\sin 3t}{3}$ (C) $\sin 3t$ (D) $\cos 3t$	ANS **D**	(CLO-3, Apply)
11.	If $L[f(t)] = F(s)$, then $L[e^{-at}\,f(t)] =$ (A) $F(s+a)$ (B) $F(s-a)$ (C) $e^{as}F(s)$ (D) $e^{-as}F(s)$	ANS **A**	(CLO-3, Remember)
12.	$L[t^2] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) $\frac{2}{s^3}$ (D) $\frac{1}{s^3}$	ANS **C**	(CLO-3, Apply)
13.	$L[1] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) $\frac{2}{s^3}$ (D) $\frac{1}{s^3}$	ANS **A**	(CLO-3, Apply)
14.	$L[e^{-2\,t}] =$ (A) $\frac{1}{s\,+2}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 4}$ (D) $\frac{4}{s}$	ANS **A**	(CLO-3, Apply)

15.	$L[\sin 3\,t] =$ (A) $\frac{1}{s^2-9}$ (B) $\frac{3}{s^2+9}$ (C) $\frac{s}{s^2-9}$ (D) $\frac{s}{s^2+9}$	ANS **B**	(CLO-3, Apply)
16.	$L[\sinh 2\,t] =$ (A) $\frac{2}{s^2-4}$ (B) $\frac{2}{s^2+4}$ (C) $\frac{1}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **A**	(CLO-3, Apply)
17.	$L[2^{\,t}] =$ (A) $\frac{1}{s-2}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 2}$ (D) $\frac{2}{s}$	ANS **C**	(CLO-3, Apply)
18.	$L[t\,e^{2t}] =$ (A) $\frac{1}{s-2}$ (B) $\frac{1}{(s-2)^2}$ (C) $\frac{2}{(s-2)^3}$ (D) $\frac{1}{s^3}$	ANS **B**	(CLO-3, Apply)
19.	If $L[f(t)] = F(s)$, then $L[\,f(at)] =$ (A) $\frac{1}{a}\,F\left(\frac{s}{a}\right)$ (B) $F\left(\frac{s}{a}\right)$ (C) $F(s+a)$ (D) $F(s-a)$	ANS **A**	(CLO-3, Remember)
20.	$L^{-1}\left[\frac{s-2}{s^2-4s+13}\right] =$ (A) $e^{-2t}\sin 3t$ (B) $e^{-2t}\cos 3t$ (C) $e^{2t}\sin 3t$ (D) $e^{2t}\cos 3t$	ANS **D**	(CLO-3, Apply)
21.	If $L[f(t)] = F(s)$, then $L\left[\int_0^t f(u)du\right] =$ (A) $\frac{F(s)}{s}$ (B) $F\left(\frac{s}{a}\right)$ (C) $\frac{f(t))}{t}$ (D) $F\,(u)$	ANS **A**	(CLO-3, Remember)
22.	$L^{-1}\,[1\,] =$ (A) $\frac{1}{s}$ (B) s (C) 1 (D) $\delta(t)$	ANS **D**	(CLO-3, Apply)

23.	$L^{-1}\left[\frac{s-3}{s^2-6s+13}\right] =$ (A) $e^{-3t}\cos\ 3t$ (B) $e^{2t}\cos 3t$ (C) $e^{3t}\cos 2t$ (D) $e^{-2t}\cos 2t$	ANS **C**	(CLO-3, Apply)
24.	$L[4^{t}] =$ (A) $\frac{1}{s-4}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 4}$ (D) $\frac{4}{s}$	ANS **C**	(CLO-3, Apply)
25.	$L[\cosh 3t] =$ (A) $\frac{s}{s^2+9}$ (B) $\frac{1}{s^2-9}$ (C) $\frac{s}{s^2-9}$ (D) $\frac{s}{s^2+9}$	ANS **C**	(CLO-3, Apply)
26.	$L[\mathrm{t}\cos at] =$ (A) $\frac{s^2+a^2}{(s^2-a^2)^2}$ (B) $\frac{s^2-a^2}{(s^2-a^2)^2}$ (C) $\frac{s^2-a^2}{(s^2+a^2)^2}$ (D) $\frac{s}{s^2+9}$	ANS **C**	(CLO-3, Apply)
27.	$L[t\sin 2t] =$ (A) $\frac{4s}{(s^2+4)^2}$ (B) $\frac{4s}{(s^2-4)^2}$ (C) $\frac{s}{(s^2+4)^2}$ (D) $\frac{4s}{(s^2-4)^2}$	ANS **A**	(CLO-3, Apply)
28.	$L[t\,e^{t}] =$ (A) $\frac{1}{s-1}$ (B) $\frac{1}{(s-2)^2}$ (C) $\frac{1}{(s-1)^2}$ (D) $\frac{1}{(s-1)^3}$	ANS **C**	(CLO-3, Apply)
29.	$L[2\,e^{-3\,t}] =$ (A) $\frac{2}{s+3}$ (B) $\frac{2}{(s-3)^2}$ (C) $\frac{1}{(s-3)^2}$ (D) $\frac{2}{(s-1)^3}$	ANS **A**	(CLO-3, Apply)
30.	$L[3] =$ (A) $\frac{1}{s-3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s+3}$ (D) $\frac{3}{s}$	ANS **D**	(CLO-3, Apply)

31.	$L[\sin 5t] =$ (A) $\frac{5}{s^2+29}$ (B) $\frac{5}{s^2+25}$ (C) $\frac{1}{s^2+29}$ (D) $\frac{s}{s^2+29}$	ANS **B**	(CLO-3, Apply)
32.	$L[\cos 2t] =$ (A) $\frac{1}{s^2-4}$ (B) $\frac{1}{s^2+4}$ (C) $\frac{s}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **D**	(CLO-3, Apply)
33.	$L[\cosh 2t] =$ (A) $\frac{s}{s^2+4}$ (B) $\frac{1}{s^2-4}$ (C) $\frac{s}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **C**	(CLO-3, Apply)
34.	$L^{-1}\left[\frac{1}{s-3}\right] =$ (A) e^{3t} (B) e^{-3t} (C) $\cos 3t$ (D) $\sin 3t$	ANS **A**	(CLO-3, Apply)
35.	$L^{-1}\left[\frac{s}{s^2-9}\right] =$ (A) $\cos 3t$ (B) $\sin 3t$ (C) $\cosh 3t$ (D) $\sinh 3t$	ANS **C**	(CLO-3, Apply)
36.	$L^{-1}\left[\frac{1}{(s-1)^2}\right] =$ (A) $t\,e^{t}$ (B) e^{t} (C) e^{-t} (D) $t\,e^{-t}$	ANS **A**	(CLO-3, Apply)

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 3 – Laplace Transforms

Part – B (Each question carries 3 Marks)

1. Find $L[2e^{-3t}]$.

Solution

$$\mathbf{L[e^{-at}] = \frac{1}{s+a}}$$

$$\mathbf{L[2e^{-3t}] = 2L[e^{-3t}] = 2\left(\frac{1}{s+3}\right)}$$

2. Find $L[e^{3t+5}]$.

Solution

$$\mathbf{L[e^{at}] = \frac{1}{s-a}}$$

$$\mathbf{L\left[e^{3t}.e^{5}\right] = e^{5}L[e^{3t}] = e^{5}\left(\frac{1}{s-3}\right)}$$

3. Find the Laplace transform of $f(t) = cos^2(3t)$.

Solution

$$\mathrm{L}[\cos^2 3t] = L\left[\frac{1+\cos 6t}{2}\right] = \frac{L(1)+L(\cos 6t)}{2} \qquad \because \cos^2 t = \frac{1+\cos 2t}{2}$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+36)} \qquad \because L(1) = \frac{1}{s}, L(\cos at) = \frac{s}{s^2+a^2}$$

$\therefore$ **L[cos² 3t]** $= \dfrac{s^2+18}{s(s^2+36)}$

4. Find $L\left(t^2 - 4\sin 2t + 2\cos 3t\right)$**.**

Solution

$$L\left(t^2 - 4\sin 2t + 2\cos 3t\right) = \frac{2}{s^3} - 4\left(\frac{2}{s^2+4}\right) + 2\left(\frac{s}{s^2+9}\right)$$

5. Find the Laplace transform of $e^{-t}\sin 2t$**.**

Solution

$$L[e^{-t}\sin 2t] = L[e^{-at} f(t)] = F(s+a) = F(s+1)$$

$$F(s) = L[f(t)] = L(\sin 2t) = \frac{2}{s^2+4}$$

$$F(s+1) = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$$

6. Obtain the Laplace transform of $\sin 2t - 2t\cos 2t$ **.**

Solution

$$L[\sin 2t - 2t\cos 2t] = L[\sin 2t] - 2L[t\cos 2t] = L[\sin 2t] - 2\left(-\frac{d}{ds}L[\cos 2t]\right)$$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4} + 2\left(\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right)$$

$$= \frac{2(s^2+4) + 2(4-s^2)}{(s^2+4)^2}$$

$$\therefore L[\sin 2t - 2t\cos 2t] = \frac{16}{(s^2+4)^2}$$

7. Find $L\left(t e^t\right)$**.**

Solution

$$L\left(t\, f(t)\right) = -\frac{d}{ds} L(f(t))$$

$$L\left(t\,e^{t}\right)=-\frac{d}{ds}L\left(e^{t}\right)$$

$$=-\frac{d}{ds}L\left(\frac{1}{s-1}\right)=\frac{1}{(s-1)^{2}}$$

8. Find $L\left(t\sin 2t\right)$**.**

Solution

$$L\left(t\,f(t)\right)=-\frac{d}{ds}L\left(f(t)\right)$$

$$L\left(t\sin 2t\right)=-\frac{d}{ds}L\left(\sin 2t\right)$$

$$=-\frac{d}{ds}\left(\frac{2}{s^{2}+4}\right)=\frac{4s}{\left(s^{2}+4\right)^{2}}$$

9. Find the Laplace transform of $f(t)=\boldsymbol{t^{2}\cos t}$.

Solution

$$L\left[t^{2}\cos t\right]=\left[\frac{d^{2}}{ds^{2}}L\left[\cos t\right]\right]=\frac{d^{2}}{ds^{2}}\left(\frac{s}{s^{2}+1}\right)$$

$$=\frac{d}{ds}\left(\frac{\left(s^{2}+1\right).1-1.2s.s}{\left(s^{2}+1\right)^{2}}\right)=\frac{d}{ds}\left(\frac{1-s^{2}}{\left(s^{2}+1\right)^{2}}\right)$$

$$=\frac{\left(s^{2}+1\right)^{2}(-2s)-\left(1-s^{2}\right)2\left(s^{2}+1\right)2s}{\left(s^{2}+1\right)^{3}}=\frac{-2s\left(3-s^{2}\right)}{\left(s^{2}+1\right)^{3}}$$

10. Find the Laplace transform of $\boldsymbol{f(t)=te^{-3t}cos2t}$

Solution

$$L[f(t)]=L[te^{-3t}cos2t]=-\frac{d}{ds}L[\cos 2t]_{s\to s+3}=-\frac{d}{ds}\left[\frac{s}{s^{2}+4}\right]_{s\to s+3}$$

$$=-\left[\frac{(s^{2}+4)(1)-s(2s)}{(s^{2}+4)^{2}}\right]_{s\to s+3}=\left[\frac{s^{2}-4}{(s^{2}+4)^{2}}\right]_{s\to s+3}$$

$$= \frac{(s+3)^2-4}{((s+3)^2+4)^2}$$

$$= \frac{s^2+6s+5}{(s^2+6s+13)^2}$$

11. Find the Laplace Transform of $f(t) = e^{-t} t\, cost.$

Solution

$$L[e^{-t}tcost] = -\frac{d}{ds}L[\cos t]_{s\to s+1} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s \to s+1}$$

$$= -\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]_{s \to s+1}$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s \to s+1}$$

$$= \frac{(s+1)^2-1}{((s+1)^2+1)^2} = \frac{s^2+2s}{(s^2+2s+2)^2}$$

$$= \frac{s(s+2)}{(s^2+2s+2)^2}$$

12. Find $L\left[\frac{sint}{t}\right]$.

Solution

$$L\left[\frac{sint}{t}\right]=L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(s)ds$$

$$\text{F(s)} = \text{L}\ [sint]=\frac{1}{s^2+1^2}$$

$$\int_s^\infty F(s)ds=\int_s^\infty \frac{1}{s^2+1}ds=[tan^{-1}(s)]_s^\infty$$

$$=[tan^{-1}\infty - tan^{-1}s]=\left[\frac{\pi}{2} - tan^{-1}s\right] = cot^{-1}s$$

13. Find the Laplace transform of $f(t) = \dfrac{e^{-t}\sin t}{t}$.

Solution

$$L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L\left(e^{-t}\sin t\right)ds$$

$$= \int_s^\infty L(\sin t)_{s+1}\, ds = \int_s^\infty \left(\frac{1}{s^2+1}\right)_{s+1} ds = \int_s^\infty \frac{1}{(s+1)^2+1}ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

14. Find the Laplace Transform of $f(t) = \frac{1-cost}{t}$.

Solution

$$L[1 - cost] = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[1 - cost]ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}[\log(s^2+1) - logs^2]_s^\infty$$

$$= -\frac{1}{2}\left[log\frac{s^2+1}{s^2}\right]_s^\infty = -\frac{1}{2}\left[log\left(1+\frac{1}{s^2}\right)\right]_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}log\left[1+\frac{1}{s^2}\right] \;= \frac{1}{2}log\left(\frac{s^2+1}{s^2}\right)$$

15. Find $L\left[\frac{cos\ at - cos\ bt}{t}\right]$.

Solution

$$L\left[\frac{cosat - cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}log(s^2+a^2) - \frac{1}{2}log(s^2+b^2)\right]_s^\infty$$

$$=\frac{1}{2}\left[log\,\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty=\frac{1}{2}\left[log\,\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$=\frac{1}{2}\left[log1-log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]=\frac{1}{2}\;log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

16. Evaluate $\int_0^\infty t\,e^{-2t}\,sin\,t\,dt$ using Laplace transform.

Solution

$$\int_0^\infty te^{-2t}\sin t\,dt=\int_0^\infty e^{-st}f(t)dt=F(s)\text{ Here s = 2.}$$

$$F(s)\;=\;L\,[f\;(t)],F(s)\;=\;L\,[t\sin t]$$

$$=-\frac{d}{ds}\left[\frac{1}{s^2+1}\right]=\frac{2s}{(s^2+1)^2}$$

$$\int_0^\infty \boldsymbol{t\,e^{-2t}\,sint\,dt}\;=[\boldsymbol{F(s)}]_{s\,=\,2}=\frac{\mathbf{4}}{(\mathbf{4+1})^2}=\frac{\mathbf{4}}{\mathbf{25}}$$

17. Verify initial value theorem for the function *f* (*t*) = 2 – cos t.

Solution

Initial value theorem states that $\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$

L. H. S. = $\lim_{t\to 0} f(t)=2-\cos 0=1$

R. H. S. = $\lim_{s\to\infty} s\,L(f(t))=\lim_{s\to\infty} s\,L(2-\cos t)$

$$=\lim_{s\to\infty} s\left(2-\frac{s^2}{s^2+1}\right)==\lim_{s\to\infty} s\left(2-\frac{1}{1+\frac{1}{s^2}}\right)=2-1=1$$

L.H.S=R.H.S

Initial value theorem verified.

18. Verify final value theorem for the function f(t) = 1 + e^{-t} (sin t + cos t).

Solution

$$L\left[f(t)\right] = F(s)$$

$$= \frac{1}{s} + L\left[\sin t + \cos t\right]_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

Final value theorem states that $\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} sF(s)$

$$\text{L.H.S.} = \lim_{t\to\infty}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + 0 = 1$$

$$\text{R. H. S.} = \lim_{s\to 0} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right] = \lim_{s\to 0}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1$$

L.H.S.=R.H.S

Hence final value theorem verified

19. Find $L^{-1}\left(\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-9}\right)$**.**

Solution

$$L^{-1}\left(\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-9}\right) = e^{3t} + 1 + \cosh 3t$$

20. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$**.**

Solution

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right) = L^{-1}\left(\frac{1}{(s+2)}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2}\right) = e^{-2t} - 2te^{-2t}$$

21. Find $L^{-1}\left(\dfrac{1}{s^2+2s+5}\right)$**.**

Solution

$$L^{-1}\left(\frac{1}{s^2+2s+5}\right)=L^{-1}\left(\frac{1}{(s+1)^2+4}\right)=\frac{e^{-t}\ \sin 2t}{2}$$

22. Find $\boldsymbol{L^{-1}\left(\dfrac{s}{s^2+4s+5}\right)}$**.**

Solution

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right)=L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right)=e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$

$$=e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right)-2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$=e^{-2t}\left[\cos t-2\sin t\right]$$

23. Find $L^{-1}\left(\dfrac{s-5}{s^2-3s+2}\right)$**.**

Solution:

$$L^{-1}\left(\frac{s-5}{s^2-3s+2}\right)=L^{-1}\left(\frac{A}{s-1}+\frac{B}{s-2}\right)=L^{-1}\left(\frac{4}{s-1}\right)+L^{-1}\left(\frac{-3}{s-2}\right)=4e^{t}-3e^{2t}$$

24. Find $\boldsymbol{L^{-1}\left[\dfrac{s+2}{s^2+2s+2}\right]}$**.**

Solution:

$$L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]\ =\ L^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right]\ \because\ L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]$$

$$=\ L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right]+L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{1}{s^2+1}\right]\right)=\mathrm{e}^{-t}\ (\cos t+\sin t)$$

25. Find $L^{-1}\left[\frac{1}{s^2+6s+13}\right]$.

Solution

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right]=L^{-1}\left[\frac{1}{(s+3)^2+4}\right]=L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$
$$=\frac{1}{2}L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]=\frac{1}{2}\mathrm{e}^{-3\mathrm{t}}\sin 2t.$$

26. Find $L^{-1}\left[\cot^{-1}(s+1)\right]$.

Solution:

Let $L^{-1}\left[\cot^{-1}(s+1)\right]=f(t)$

$$\therefore\ L\left[f(t)\right]=\cot^{-1}(s+1)$$

$$L\left[tf(t)\right]=-\frac{d}{ds}\left[\cot^{-1}(s+1)\right]=\frac{1}{(s+1)^2+1}$$

$$tf(t)=L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]=e^{-t}\sin t$$

$$\therefore\ f(t)=\frac{e^{-t}\sin t}{t}$$

27. Find the inverse Laplace transform of $\frac{s}{(s+2)^2}$.

Solution

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$
$$=\frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right)=\frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$
$$=\frac{d}{dt}\left(e^{-2t}\,t\right)=e^{-2t}+t\left(-2e^{-2t}\right)=e^{-2t}(1-2t)$$

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT IV - ANALYTIC FUNCTIONS

Part – A

1.	The critical point of the transformation $w = z^2$ is (A) $z = 0$ (B) $z = -i$ (C) $z = 1$ (D) $z = -1$	ANS **A**	(CLO-4, Apply)
2.	If $w = f(z) = u + iv$ is analytic, then the family of curves $u = C_1$ and $v = C_2$ (A) cut orthogonally (B) intersect each other (C) are parallel (D) coincide	ANS **A**	(CLO-4, Remember)
3.	If a function $u(x, y)$ satisfies the equation $u_{xx} + u_{yy} = 0$, then u is called (A) analytic function (B) harmonic function (C) differential function (D) continuous function	ANS **B**	(CLO-4, Remember)
4.	Cauchy-Riemann equations in Polar co-ordinates are (A) $u_r = \frac{1}{r} v_\theta,\ v_r = -\frac{1}{r} u_\theta$ (B) $u_r = -\frac{1}{r} v_\theta,\ v_r = \frac{1}{r} u_\theta$ (C) $u_r = -\frac{1}{r} v_\theta,\ v_r = -\frac{1}{r} u_\theta$ (D) $u_r = \frac{1}{r} v_\theta,\ v_r = \frac{1}{r} u_\theta$	ANS **A**	(CLO-4, Remember)
5.	The critical point of the transformation $w = z^4$ is (A) $z = 2$ (B) $z = -2$ (C) $z = 0$ (D) $z = 1$	ANS **C**	(CLO-4, Apply)
6.	If $w = f(z) = u + i\,v$ is an analytic function of z, then (A) u and v are not harmonic (B) u is not harmonic (C) both u and v are harmonic (D) u and v are constants	ANS **C**	(CLO-4, Remember)

7.	An analytic function with constant modulus is (A) zero (B) analytic (C) harmonic (D) constant	ANS **D**	(CLO-4, Remember)
8.	Cauchy – Riemann equation in Cartesian co-ordinates are (A) $u_x = v_y, u_y = -v_x$ (B) $u_x = -v_y, u_y = v_x$ (C) $u_x = v_y, u_y = v_x$ (D) $u_x = -v_y, u_y = -v_x$	ANS **A**	(CLO-4, Remember)
9.	The invariant point of the transformation $w = \frac{1}{z-2i}$ is (A) $z = 0$ (B) $z = 1$ (C) $z = -1$ (D) $z = i$	ANS **D**	(CLO-4, Apply)
10.	The transformation $w = a\,z$, where a is a real constant represents (A) magnification (B) rotation (C) reflection (D) inversion	ANS **A**	(CLO-4, Apply)
11.	The fixed points of the transformation $w = \frac{z-1}{z+1}$ are (A) $\pm i$ (B) ± 1 (C) ± 2 (D) ± 3	ANS **A**	(CLO-4, Apply)
12.	An analytic function with constant real part is (A) zero (B) analytic (C) harmonic (D) constant	ANS **D**	(CLO-4, Remember)
13.	An analytic function with constant imaginary part is (A) zero (B) analytic (C) harmonic (D) constant	ANS **D**	(CLO-4, Remember)
14.	The transformation $w = a\,z$, where a is a complex constant represents (A) magnification (B) reflection (C) magnification and rotation (D) inversion	ANS **C**	(CLO-4, Remember)
15.	If $f(z) = e^z$, then $f(z)$ is (A) zero function (B) analytic function (C) discontinuous function (D) constant function	ANS **B**	(CLO-4, Remember)

16.	$f(z) = \frac{1}{z^2+1}$ is analytic everywhere except at (A) $z = \pm i$ (B) $z = \pm 1$ (C) $z = \pm 2$ (D) $z = \pm 3$	ANS **A**	(CLO-4, Apply)
17.	The invariant points of the transformation $w = \frac{2z+6}{z+7}$ are (A) $6, -1$ (B) $3, 2$ (C) $-3, 2$ (D) $-6, 1$	ANS **D**	(CLO-4, Apply)
18.	The fixed points of the transformation $w = \frac{z-1}{z+1}$ are (A) $\pm i$ (B) ± 1 (C) ± 2 (D) ± 3	ANS **A**	(CLO-4, Apply)
19.	The image of $\lvert z - 2i \rvert = 2$ under the transformation $w = \frac{1}{z}$ is (A) $x^2 + y^2 = 0$ (B) $x^2 + y^2 + 4y = 0$ (C) $x^2 + y^2 - 4y = 0$ (D) $x^2 + y^2 + y = 0$	ANS **C**	(CLO-4, Apply)
20.	The image of $\lvert z \rvert = 2$ under the transformation $w = 3z$ is (A) $x^2 + y^2 = 0$ (B) $x^2 + y^2 = 4$ (C) $x^2 - y^2 = 0$ (D) $x^2 - y^2 = 4$	ANS **B**	(CLO-4, Apply)
21.	The image of $\lvert z + 1 \rvert = 1$ under the transformation $w = \frac{1}{z}$ is (A) $x^2 + y^2 + 2x = 0$ (B) $x^2 + y^2 + 2y = 0$ (C) $x^2 + y^2 - 2x = 0$ (D) $x^2 - y^2 - 2y = 0$	ANS **A**	(CLO-4, Apply)
22.	The transformation $w = \frac{1}{z}$ is known as (A) magnification (B) reflection (C) rotation (D) inversion	ANS **D**	(CLO-4, Remember)
23.	If the image of a point z under the transformation $w = f(z)$ is itself, then the point is called (A) fixed point (B) critical point (C) singular point (D) regular point	ANS **A**	(CLO-4, Remember)
24.	The function $f(z) = \bar{z}$ is (A) nowhere differentiable (B) analytic (C) constant (D) singular	ANS **A**	(CLO-4, Apply)

25.	The function $f(z) = \sin z$ is (A) nowhere differentiable (B) analytic (C) not analytic (D) constant	ANS **B**	(CLO-4, Apply)
26.	A mapping that preserves angles between oriented circles both in magnitude and in sense is called a ______________ mapping. (A) isogonal (B) conformal (C) regular (D) formal	ANS **B**	(CLO-4, Remember)
27.	A transformation that preserves angles between every pair of curves through a point only in magnitude, but not in direction is said to be ____________ at that point. (A) isogonal (B) conformal (C) regular (D) formal	ANS **A**	(CLO-4, Remember)
28.	The real part of $f(z) = e^{2z}$ is (A) $e^x \cos y$ (B) $e^x \sin y$ (C) $e^{2x} \cos 2y$ (D) $e^{2x} \sin 2y$	ANS **C**	(CLO-4, Apply)
29.	The points at which the function $f(z) = \frac{1}{z^2 - 1}$ fails to be analytic are (A) $z = \pm i$ (B) $z = \pm 1$ (C) $z = \pm 2$ (D) $z = \pm 3$	ANS **B**	(CLO-4, Apply)
30.	The transformation $w = z + a$, where a is a complex constant represents (A) magnification (B) reflection (C) translation (D) inversion	ANS **C**	(CLO-4, Remember)
31.	The fixed points of the transformation $w = \frac{5z+4}{z+5}$ are (A) $\pm i$ (B) ± 1 (C) ± 2 (D) ± 3	ANS **C**	(CLO-4, Apply)
32.	The harmonic conjugate of $u = e^x \cos y$ is (A) $e^x \sin y$ (B) $e^{2x} \sin y$ (C) $e^{2x} \cos 2y$ (D) $e^{2x} \sin 2y$	ANS **A**	(CLO-4, Apply)
33.	The invariant points of the transformation $w = \frac{1 - iz}{z - i}$ are (A) $\pm i$ (B) ± 1 (C) ± 2 (D) ± 3	ANS **B**	(CLO-4, Apply)

34.	The real part of $f(z) = \log z$ is (A) $u = \log r$ (B) $u = \log x$ (C) $u = \log y$ (D) $u = \log \theta$	ANS **A**	(CLO-4, Apply)
35.	If $f(z) = x + y + i\,(cy - x)$ is analytic, then the value of c is (A) $\pm i$ (B) 1 (C) 2 (D) -1	ANS **B**	(CLO-4, Apply)
36.	The critical points of the transformation $w = z + \frac{1}{z}$ are (A) $\pm i$ (B) ± 1 (C) ± 2 (D) ± 3	ANS **B**	(CLO-4, Apply)

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 4 – Analytic Functions

Part – B (Each question carries 3 Marks)

1. Test the analyticity of the function w = sin z.

Solution

$w = f(z) = \sin z$

$u + iv = \sin(x + iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$	$v_x = -\sin x \sinh y$
$u_y = \sin x \sinh y$	$v_y = \cos x \cosh y$

$u_x = v_y$ and $u_y = -v_x$

∴ C-R equations are satisfied.

∴ The function is analytic.

2. Verify whether the function 2xy + i $(x^2 - y^2)$ is analytic or not.

Solution

$u = 2xy$	$v = x^2 - y^2$
$u_x = 2y$	$v_x = 2x$
$u_y = 2x$	$v_y = -2y$

$\therefore u_x \neq v_y$ and $u_y \neq -v_x$

∴ C-R equations are not satisfied.

∴ The function is not analytic.

3. Test the analyticity of the function f (z) = e^z.

Solution

f (z) $= e^z$

$u + iv = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

$u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = e^x \cos y$

$u_x = v_y$ and $u_y = -v_x$

∴ The function is analytic.

4. Verify whether w = z^3 is analytic or not.

Solution

Given $w = z^3 = (x+iy)^3 = x^3 + 3x^2 i y + 3xi^2y^2 + i^3y^3$

$= x^3 - 3xy^2 + i\,(3x^2y - y^3)$

$u = x^3 - 3xy^2 \qquad v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2 ; \qquad v_x = 6xy$

$u_y = -6xy; \qquad v_y = 3x^2 - 3y^2$

Now $u_x = v_y$ and $u_y = -v_x$

∴ w = z^3 is analytic.

5. Is the function f (z) = $\bar{z}$ analytic?

Solution

Given $u + iv = x - iy$

$u = x \qquad v = -y$

$u_x = 1 \qquad v_x = -1$

$u_y = 0 \qquad v_y = -1$

$u_x \neq v_y$

∴ C-R equations are not satisfied.

∴ f (z) = $\bar{z}$ is not analytic.

6. Find the invariant points of the transformation $f(z) = z^2$.

Solution

Put w = f(z) = z to find the invariant points.

$$z = z^2$$
$$z - z^2 = 0$$
$$z(1 - z) = 0$$
$$z = 0,1$$

7. Find the invariant points of the transformation $w = \frac{z-1}{z+1}$**.**

Solution

The fixed points of the transformation are obtained by replacing w by z.

$$z = \frac{z-1}{z+1}$$
$$z^2 + z - z + 1 = 0$$
$$z^2 + 1 = 0$$

$z = \pm\, i$ are called fixed points of the transformation.

8. Find the invariant points of the transformation $w = \frac{3z-5}{z+1}$**.**

Solution

To get the invariant points, put w = z

$$\therefore z = \frac{3z-5}{1+z}$$
$$z^2 - 2z + 5 = 0$$

Solving for z,

$$Z = \frac{2\pm\sqrt{4-20}}{2} =$$
$$= \frac{2\pm 4i}{2} = 1\pm 2i$$

$\therefore$ The invariant points are $z = 1\pm 2i$

9. Find the critical point of the transformation $w = z^2$**.**

Solution

Put $\frac{dw}{dz} = 0$

$2z = 0$

The critical point is z = 0.

10. Find the critical points of the transformation $w = z + \frac{1}{z}$.

Solution

Put $\frac{dw}{dz} = 0$

$1 - \frac{1}{z^2} = 0 \Rightarrow \frac{1}{z^2} = 1 \Rightarrow z^2 = 1$

The critical points are z = 1 or z = – 1.

11. Show that the function u = 2x – x³ + 3xy² is harmonic.

Solution Given u = 2x – x³ + 3xy²

$u_x = 2 - 3x^2 + 3y^2 \qquad u_y = 6xy$

$u_{xx} = -6x \qquad u_{yy} = 6x$

$u_{xx} + u_{yy} = -6x + 6x = 0$

Hence u is harmonic

12. Prove that the function u = $\boldsymbol{e^x(x\cos y - y\sin y)}$ **satisfies Laplace's equation.**

Solution

Given u $= e^x(x\cos y - y\sin y)$

$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$

$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$

$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x(\cos y) + e^x(\cos y)$

$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y - \cos y + ysin\ y)$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x(x\cos y - y\sin y + \cos y + \cos y - x\cos y - \cos y - \cos y + ysin\ y) \therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

∴u satisfies Laplace equation.

13. Prove that the function u = $\boldsymbol{x^3 - 3xy^2 + 3x^2 - 3y^2}$ **+ 1 satisfies Laplace's equation.**

Solution

Given u $= x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$

$\frac{\partial u}{\partial y} = 6xy - 6y$

$\frac{\partial^2 u}{\partial x^2} = 6x + 6$

$$\frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$u satisfies Laplace equation.

14. Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic.

Solution

Given $u = \frac{1}{2}\log\left(x^2 + y^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}.\frac{1}{x^2+y^2}\left(2x\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}.\frac{1}{x^2+y^2}\left(2y\right) = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2+y^2\right) - x\left(2x\right)}{\left(x^2+y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\left(x^2+y^2\right)\left(1\right) - 2y^2}{\left(x^2+y^2\right)^2} = \frac{x^2 - y^2}{\left(x^2+y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2 + x^2 - y^2}{\left(x^2+y^2\right)^2} = 0$$

Hence u is harmonic function.

15. Show that the function $u = e^x \cos y$ is harmonic.

Solution

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = e^x\,(-\sin\, y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = e^x\,(-\cos y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore u$ is harmonic.

16. Find the analytic function $f(z) = u + iv$ where $u = 3x^2\, y - y^3$.

Solution

$u = 3x^2\, y - y^3$

$u_x = 6xy$

$u_x\,(z,0) = 0$

$u_y = 3x^2 - 3y^2$

$u_y(z,0) = 3z^2$

Milne Thomson Method

$f(z) = \int [u_x(z,0) - iu_y(z,0)]dz + C$

$f(z) = \int -i3z^2\, dz + C$

$f(z) = -iz^3 + C$

17. Find the image of the circle $|z| = 3$ under the transformation w = 2z.

Solution

Method 1

Given w = 2z

$u + \mathrm{i}\, v = 2\,(x + i\,y)$

$x = \frac{u}{2}, y = \frac{v}{2}$

Given $|z| = 3$

$|x + i\,y| = 3$

$\sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9 \Rightarrow \left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = 9$

$u^2 + v^2 = 36$

which represents a circle with centre (0, 0) and radius 6.

(or) Method 2

w = 2 z

$|w| = 2|z|$

$|w| = 2(3) = 6$

Hence the image of the circle $|z| = 3$ in the z-plane maps to the circle $|w| = 6$ in the w-plane.

18. Find the image of the circle $|z| = 1$ by the transformation w = z + 2 + 4i.

Solution

Given: w = z + 2 + 4i

u + iv = x + iy + 2 + 4i = (x + 2) + i (y + 4)

u = x + 2, v = y + 4

⇨ x = u – 2, y = v – 4

⇨ $|z| = 1$

$x^2 + y^2 = 1$ Hence $(u - 2)^2 + (v - 4)^2 = 1$.

∴ The circle in the z-plane is mapped into the circle in the w -plane with centre (2, 4) and radius 1.

19. Find the image of $|z-2i|=2$ under the transformation $w=\dfrac{1}{z}$.

Solution

Given $w=\dfrac{1}{z} \Rightarrow z=\dfrac{1}{w}$

Now $w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$$

i.e., $x+iy=\dfrac{u-iv}{u^2+v^2}$

$\therefore\ x=\dfrac{u}{u^2+v^2}$ ....(1) $y=\dfrac{-v}{u^2+v^2}$ ....(2)

Given $|z-2i|=2$

$$|x+iy-2i|=2 \Rightarrow |x+i(y-2)|=2$$

$$x^2+(y-2)^2=4 \Rightarrow x^2+y^2-4y=0\text{...................(3)}$$

Sub (1) and (2) in (3)

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left[\frac{-v}{u^2+v^2}\right]=0$$

$$\frac{(u^2+v^2)+4v(u^2+v^2)}{(u^2+v^2)^2}=0$$

$$\frac{(1+4v)(u^2+v^2)}{(u^2+v^2)^2}=0$$

$1+4v=0 \Rightarrow v=-\dfrac{1}{4} \quad (\therefore u^2+v^2\neq 0)$

which is a straight line in w - plane.

20. Find the bilinear transformation of the points –1, 0, 1 in z - plane onto the points 0, *i*, 3*i* in w- plane.

Solution

Given $z_1=-1,\ w_1=0\ z_2=0,\ w_2=i\ z_3=i,\ w_3=3i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)}=\frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)}=\frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{2w}{w-3i}=\frac{z+1}{z-1}$$
$$2wz-2w=wz+w-3iz-3i$$
$$w(2z-2-z-1)=-3i(z+1)$$
$$w(z-3)=-3i(z+1)$$
$$\therefore w=-3i\,\frac{(z+1)}{(z-3)}$$

21. Find the bilinear transformation which maps the points $z=\infty, i, 0$ into $w=0, i, \infty$ respectively.

Solution

Given $z_1=\infty,\ w_1=0\ z_2=i,\ w_2=i\ z_3=0,\ w_3=\infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\,z_1\left(\frac{z_2}{z_1}-z_1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)}=\frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{w}{i}=\frac{i}{z},\qquad w=\frac{i^2}{z},\qquad \therefore w=-\frac{1}{z}$$

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT V - COMPLEX INTEGRATION

Part – A

1.	The point z_0 at which a function $f(z)$ is not analytic is known as (A) zeros (B) isolated singular point (C) singular point (D) removable singular point	ANS **C**	(CLO-5, Remember)
2.	The singular points of $f(z) = \frac{z+3}{(z-3)(z-2)}$ are (A) $z = 1, 3$ (B) $z = 1, 0$ (C) $z = 1, 2$ (D) $z = 2, 3$	ANS **D**	(CLO-5, Apply)
3.	The value of $\int_C \frac{dz}{z-1}$ where C is the circle $\mid z \mid = 2$ is (A) 0 (B) $2\pi i$ (C) $-2\pi i$ (D) 1	ANS **B**	(CLO-5, Apply)
4.	The residue of $f(z) = \frac{z}{z-1}$ at its pole is (A) 0 (B) 1 (C) -1 (D) $2\pi i$	ANS **B**	(CLO-5, Apply)
5.	The residue of $f(z) = \frac{z}{z+1}$ at its pole is (A) 0 (B) 2 (C) -1 (D) $2\pi i$	ANS **C**	(CLO-5, Apply)
6.	The singular points of $f(z) = \frac{z+3}{(z+1)(z+2)}$ are (A) $z = 1, 3$ (B) $z = 1, 0$ (C) $z = -1, -2$ (D) $z = 2, 3$	ANS **C**	(CLO-5, Apply)

7.	The value of $\int_C \frac{z}{z-2}\,dz$ where C is the circle $\lvert z\rvert = 3$ is (A) 0 (B) $4\,\pi\,i$ (C) $-2\pi i$ (D) 1	ANS **B**	(CLO-5, Apply)
8.	The residue of $f(z) = \frac{z}{(z-1)^2}$ at its pole is (A) 0 (B) 1 (C) -1 (D) $2\pi i$	ANS **B**	(CLO-5, Apply)
9.	The value of $\int_C \frac{e^{-z}}{z+1}\,dz$ is (A) 0 (B) $2\,\pi\,i\,e$ (C) $-2\pi i$ (D) 1	ANS **B**	(CLO-5, Apply)
10.	The singularity of $f(z) = \frac{z}{(z-2)^3}$ is (A) pole of order 2 (B) pole of order 3 (C) simple pole (D) pole of order *n*	ANS **B**	(CLO-5, Apply)
11.	If $f(z) = \frac{\sin z}{z}$, then $z = 0$ is (A) pole (B) removable singularity (C) essential singularity (D) isolated singularity	ANS **B**	(CLO-5, Apply)
12.	If $f(z) = \int_C e^z\,dz$, where C is $\lvert z\rvert = 1$, then $f(z) =$ (A) 0 (B) $\pi\,i$ (C) -1 (D) $2\pi i$	ANS **A**	(CLO-5, Apply)
13.	The value of $\int_C \frac{3\,z^2+5\,z+1}{z+1}\,dz$, where $C : \lvert z\rvert = \frac{1}{2}$ is (A) 0 (B) $2\,\pi\,i$ (C) $-2\pi i$ (D) 1	ANS **A**	(CLO-5, Apply)
14.	The value of $\int_C \frac{dz}{z-1}$ where C is the circle $\lvert\, z-1\,\rvert = 1$ is (A) 0 (B) $2\pi i$ (C) $-2\pi i$ (D) πi	ANS **B**	(CLO-5, Apply)
15.	The value of $\int_C \frac{z^2}{(z-1)^2(z+1)}\,dz$, where $C : \lvert z\rvert = \frac{1}{2}$ is (A) 0 (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$	ANS **A**	(CLO-5, Apply)

16.	The value of $\int_C \frac{z}{z-2}\,dz$ where C is the circle $\lvert z\rvert = 1$ is (A) 0 (B) $4\,\pi\,i$ (C) $-2\pi i$ (D) 1	ANS **A**	(CLO-5, Apply)
17.	The residue of $f(z) = \frac{z}{(z-1)^2}$ at its pole is (A) 0 (B) 1 (C) -1 (D) 2	ANS **D**	(CLO-5, Apply)
18.	A zero of an analytic function $f(z)$ is a value of z for which (A) $f(z) = 1$ (B) $f(z) \neq 1$ (C) $f(z) \neq 0$ (D) $f(z) = 0$	ANS **D**	(CLO-5, Apply)
19.	The annular region for the function $f(z) = \frac{1}{z\,(z-1)}$ is (A) $0 < \lvert z \rvert < 1$ (B) $1 < \lvert z \rvert < 2$ (C) $2 < \lvert z \rvert < 3$ (D) $\lvert z \rvert > 1$	ANS **A**	(CLO-5, Apply)
20.	If $f(z)$ is analytic and $f'(z)$ is continuous at all points in the region bounded by the simple closed curves C_1 and C_2, then (A) $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ (B) $\int_{C_1} f(z)\,dz \neq \int_{C_2} f(z)\,dz$ (C) $\int_{C_1} f'(z)\,dz = \int_{C_2} f'(z)\,dz$ (D) $\int_{C_1} f'(z)\,dz \neq \int_{C_2} f'(z)\,dz$	ANS **A**	(CLO-5, Remember)
21.	If $f(z)$ is analytic and $f'(z)$ is continuous at all points inside and on a simple closed curve C, then $\int_C f(z)\,dz =$ (A) 0 (B) $2\pi i$ (C) $-2\pi i$ (D) 1	ANS **A**	(CLO-5, Remember)
22.	If $f(z)$ is analytic inside and on C, then the value of $\oint_C \frac{f(z)}{z-a}\,dz$, where C is a simple closed curve and 'a' is any point within C is, (A) 0 (B) $2\pi i\,f(a)$ (C) $-2\pi i\,f(a)$ (D) 1	ANS **B**	(CLO-5, Remember)
23.	The annular region for the function $f(z) = \frac{1}{z^2-z-6}$ is (A) $0 < \lvert z \rvert < 1$ (B) $1 < \lvert z \rvert < 2$ (C) $2 < \lvert z \rvert < 3$ (D) $\lvert z \rvert < 3$	ANS **C**	(CLO-5, Apply)

24.	The annular region for the function $f(z) = \frac{1}{z^2 - 3z + 2}$ is (A) $0 < \mid z \mid < 1$ (B) $1 < \mid z \mid < 2$ (C) $1 < \mid z \mid < 0$ (D) $\mid z \mid < 1$	ANS **B**	(CLO-5, Apply)
25.	The value of $\int_C \frac{e^z}{(z-1)^3}\, dz$, where $C : \lvert z\rvert = \frac{1}{2}$ is (A) 0 (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$	ANS **A**	(CLO-5, Apply)
26.	The value of $\int_C \frac{1}{(z-1)^2(z-2)(z-3)}\, dz$, where $C : \lvert z\rvert = \frac{1}{2}$ is (A) 0 (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$	ANS **A**	(CLO-5, Apply)
27.	If C is a simple closed curve containing a and b, then $\int_C \frac{1}{(z-a)(z-b)}\, dz$ is (A) 0 (B) $2\pi i\, a$ (C) $2\pi i\, b$ (D) 1	ANS **A**	(CLO-5, Apply)
28.	$f(z) = \frac{z-2}{(z-1)(z+3)(z+2)}$ has a zero at (A) $z = 1$ (B) $z = 2$ (C) $z = -2$ (D) $z = -3$	ANS **B**	(CLO-5, Apply)
29.	$f(z) = \frac{z+2}{(z-1)^2(z-2)}$ has (A) poles at $z = 1, 2$ (B) a simple pole at $z = 1$ (C) essential singularity (D) no poles	ANS **A**	(CLO-5, Apply)
30.	The value of $\int_C \frac{z^2+1}{z^2-1}\, dz$ where C is the circle $\mid z - 1 \mid = 1$ is (A) 0 (B) $4\pi i$ (C) $2\pi i$ (D) 1	ANS **C**	(CLO-5, Apply)
31.	The residue of $f(z) = \frac{z-2}{z(z-1)}$ at $z = 0$ is (A) 0 (B) -2 (C) 2 (D) 1	ANS **C**	(CLO-5, Apply)

32.	If $f(z) = \frac{1}{(z^2+1)^2}$, then (A) $z = \pm i$ each simple pole (B) $z = \pm i$ each pole of order 2 (C) $z = \pm 1$ each simple pole (D) $z = i$ is not a pole	ANS **B**	(CLO-5, Apply)
33.	The value of $\int_C \frac{dz}{z-a}\, dz$, where $C : \mid z - a \mid = r$ is (A) 0 (B) $4\,\pi\, i$ (C) $2\pi i$ (D) 1	ANS **C**	(CLO-5, Apply)
34.	If $z = a$ is inside a simple closed curve C, then $\int_C \frac{dz}{(z-a)^2} =$ (A) 0 (B) $2\pi i$ (C) $-2\pi i$ (D) 1	ANS **A**	(CLO-5, Apply)
35.	Let $C_1: \lvert z-a\rvert = R_1$ and $C_2: \lvert z-a\rvert = R_2$ be two concentric circles with $R_2 < R_1$, the annular region is defined as (A) within C_1 (B) within C_2 (C) within C_2 and outside C_1 (D) within C_1 and outside C_2	ANS **D**	(CLO-5, Remember)
36.	The value of of $\int_C \frac{dz}{3z+1}\, dz$ where C is the circle $\mid z \mid = 1$ is (A) 0 (B) $\pi\, i$ (C) $\frac{2\pi i}{3}$ (D) 1	ANS **C**	(CLO-5, Apply)

* * * * *

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 5 – Complex Integration

Part – B (Each question carries 3 Marks)

1. Evaluate $\int_C e^{\frac{1}{z}}\, dz$ where C is | z – 2 |=1 by Cauchy's integral theorem.

(A) π i (B) 4 π i (C) 0 (D) 2 π i

Solution

$e^{\frac{1}{z}}$ is analytic inside and on C.

Hence by Cauchy's Integral theorem, $\int_C e^{\frac{1}{z}}\, dz = 0$.

Answer: (C)

2. Evaluate $\int_C \frac{1}{2z-3}\, dz$ where C is | z | = 1 by Cauchy's integral formula.

(A) 1 (B) 4 π i (C) 0 (D) 2 π i

Solution

Here $a = \frac{3}{2}$ lies outside | z | = 2.

By Cauchy's Integral formula,

$$\int_C \frac{1}{2z-3}\, dz = 0$$

Answer: (C)

3. Evaluate $\int_C \frac{1}{(z-3)^2} dz$ where C is | z | = 1 by Cauchy's integral formula.

(A) 1 (B) 4π i (C) 0 (D) 2π i

Solution

Here $a = 3$ lies outside | z | = 1.

By Cauchy's Integral formula,

$$\int_C \frac{1}{(z-3)^2} dz = 0$$

Answer: (C)

4. Evaluate $\int_C \frac{2z}{z-1} dz$ where C is | z | = 2 by Cauchy's integral formula.

(A) 1 (B) 4π i (C) 0 (D) 2π i

Solution

Here $f(z) = 2z$ and $a = 1$ lies inside | z | = 2.

By Cauchy's Integral formula,

$$\int_C \frac{2z}{z-1} dz = 2\pi i \; f(1) = 2\pi i(2) = 4\pi i$$

Answer: (B)

5. Evaluate $\int_C \frac{\cos \pi z}{z-1} dz$ where C is | z | = 3.

(A) -2π i (B) 4π i (C) 0 (D) 2π i

Solution

Here $f(z) = \cos \pi z$ and $a = 1$ lies inside | z | = 3.

By Cauchy's Integral formula,

$$\int_C \frac{\cos \pi z}{z-1} dz = 2\pi i \; f(1) = 2\pi i(-1) = -2\pi i$$

Answer: (A)

6. Evaluate $\int_C \frac{e^{-z}}{z+1} dz$ where C is | z | = 1.5.

(A) $-2\pi i e$ (B) 4π i (C) 0 (D) $2\pi i e$

Solution

Here $f(z) = e^{-z}$ and $a = -1$ lies inside | z | = 1.5.

By Cauchy's Integral formula,

$$\int_C \frac{e^{-z}}{z+1} dz = 2\pi i \; f(-1) = 2\pi i e$$

Answer: (D)

7. Evaluate $\int_C \frac{1}{z e^z} dz$ where C is | z | = 1.

(A) $-2\pi i e$ (B) 2π i (C) 0 (D) $2\pi i e$

Solution

Here $f(z) = \frac{1}{e^z}$ and $a = 0$ lies inside | z | = 1.

By Cauchy's Integral formula,

$$\int_C \frac{\frac{1}{e^z}}{z} dz = 2\pi i \; f(0) = 2\pi i 1 = 2\pi i$$

Answer: (B)

8. Evaluate $\int_C \frac{z+1}{z(z-2)} dz$ where C is | z | = 1.

(A) $-2\pi i e$ (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) $2\pi i e$

Solution

Here $f(z) = \frac{z+1}{z-2}$ and $a = 0$ lies inside | z | = 1.

By Cauchy's Integral formula,

$$\int_C \frac{\frac{z+1}{z-2}}{z}\,dz = 2\pi i\ f(0) = -\frac{1}{2}$$

Answer: (C)

9. Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$ where C is | z | = 1.5.

(A) 1 (B) 4π i (C) 0 (D) 2π i

Solution

Here $f(z) = \frac{\cos \pi z^2}{z-2}$ and $a = 1$ lies inside | z | = 1.5.

By Cauchy's Integral formula,

$$\int_C \frac{\frac{\cos \pi z^2}{z-2}}{z-1}\,dz = 2\pi i\ f(1) = 2\pi i\ \frac{\cos \pi}{1-2} = 2\pi i$$

Answer: (D)

10. Evaluate $\int_C \frac{1}{(z+1)(z-2)^2}\,dz$ where C is | z | = 1.5.

(A) 1 (B) $\frac{4\pi i}{9}$ (C) 0 (D) $\frac{2\pi i}{9}$

Solution

Here $f(z) = \frac{1}{(z-2)^2}$ and $a = -1$ lies inside | z | = 1.5.

By Cauchy's Integral formula,

$$\int_C \frac{\frac{1}{(z-2)^2}}{z+1}\,dz = 2\pi i\ f(-1) = 2\pi i\ \frac{1}{9} = \frac{2\pi i}{9}$$

Answer: (D)

11. Evaluate $\int_C \frac{z}{(z-1)^3}dz$ where C is | z | = 2 by Cauchy's integral formula for derivatives.

(A) 1 (B) 4π i (C) 0 (D) 2π i

Solution

Here $f(z) = z$ and $a = 1$ lies inside | z | = 2.

By Cauchy's Integral formula for derivatives,

$$\int_C \frac{z}{(z-1)^3}dz = \frac{2\pi i}{2!}\ f''(1) = \pi i(0) = 0$$

Answer: (C)

12. Calculate the residue at z = 0 for the function $f(z) = \frac{3-e^{2z}}{z}$.

(A) 1 (B) 2 (C) 3 (D) – 2

Solution

$$\operatorname{Re}s\left[f(z), a\right] = \lim_{z\to a}(z-a)\,f(z)$$

$$\operatorname{Re}s\left[f(z), 0\right] = \lim_{z\to 0}(z-0)\frac{(3-e^{2z})}{z} = 2$$

Answer: (B)

13. Calculate the residue at z = i for the function $f(z) = \frac{1}{z^2+1}$.

(A) 1 (B) 2 (C) $\frac{1}{2i}$ (D) – 2

Solution

$$\operatorname{Re}s\left[f(z), a\right] = \lim_{z\to a}(z-a)\,f(z)$$

$$\operatorname{Re}s\left[f(z), i\right] = \lim_{z\to i}(z-i)\frac{1}{(z+i)(z-i)} = \frac{1}{2i}$$

Answer: (C)

14. Calculate the residue at z = $-i$ for the function $f(z)=\dfrac{z}{z^2+1}$.

(A) 1 (B) 2 (C) 1/2 (D) – 2

Solution

$$\operatorname{Res}\left[f(z),a\right]=\lim_{z\to a}(z-a)\,f(z)$$

$$\operatorname{Res}\left[f(z),-i\right]=\lim_{z\to -i}\ (z+i)\frac{z}{(z+i)(z-i)}=\frac{1}{2}$$

Answer: (C)

15. Calculate the residue of the function $f(z)=\dfrac{e^{2z}}{(z+1)^2}$ at its pole.

(A) $2e$ (B)$3e$ (C) $2e^{-2}$ (D)$2e^2$

Solution

z = – 1 is a pole of order 2.

$$\operatorname{Res}\left[f(z),a\right]=\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}(z-a)^n\ f(z)$$

$$\operatorname{Res}\left[f(z),-1\right]=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d^{2-1}}{dz^{2-1}}(z+1)^2\ \frac{e^{2z}}{(z+1)^2}=\frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}e^{2z}\ =2e^{-2}$$

Answer: (C)

SRM OF INSTITUTE OF SCIENCE AND TECHNOLOGY
FACULTY OF ENGINEERING AND TECHNOLOGY
18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS
PART - A : MULTIPLE CHOICE QUESTIONS

UNIT – I: MULTIPLE INTEGRALS

1. Evaluation of $\int_0^1\int_0^1 dxdy$ is

   (a) 1 (b) 2 (c) 0 (d) 4

2. The curve $y^2 = 4x$ is a

   (a) parabola (b) hyperbola (c) straight line (d) ellipse

3. Evaluation of $\int_0^\pi\int_0^\pi d\theta d\phi$ is

   $a)\ 1 \quad b)\ 0 \quad c)\ \pi/2 \quad d)\ \pi^2$

4. The area of an ellipse is

   $a)\ \pi r^2 \quad b)\ \pi a^2 b \quad c)\ \pi ab^2 \quad d)\ \pi ab$

5. $\int_1^b\int_1^a \frac{dxdy}{xy}$ is equal to

   a) $\log a + \log b$ b) $\log a$ c) $\log b$ d) $\log a \log b$

6. $\int_0^1\int_0^x dxdy$ is equal to

   $a)\ 1 \quad b)\ 1/2 \quad c)\ 2 \quad d)\ 3$

7. $\int_0^1\int_0^2 dxdy$ is equal to

   $a)\ \int_0^2\int_0^1 dydx \quad b)\ -\int_0^1\int_0^2 dxdy \quad c)\ \int_2^0\int_0^1 dydx \quad d)\ \int_1^0\int_0^2 dydx$

8. If R is the region bounded $x = 0,\ y = 0,\ x + y = 1$ then $\iint_R dxdy$ is equal to

   $a)\ 1 \quad b)\ 1/2 \quad c)\ 1/3 \quad d)\ 2/3$

9. Area of the double integral in cartesian co-ordinate is equal to

   $a)\ \iint_R dydx \quad b)\ \iint_R rdrd\theta \quad c)\ \iint_R xdxdy \quad d)\ \iint_R x^2dxdy$

10. Change the order of integration in $\int_0^a \int_0^x dxdy$ is

$a)\ \int_0^a \int_0^x dxdy$ $\quad b)\ \int_0^a \int_0^x xdydx$ $\quad c)\ \int_0^a \int_y^a dxdy$ $\quad d)\ \int_0^a \int_0^y dxdy$

11. Area of the double integral in polar co-ordinate is equal to

$a)\ \iint_R drd\theta$ $\quad b)\ \iint_R r^2 drd\theta$ $\quad c)\ \iint_R (r+1)drd\theta$ $\quad d)\ \iint_R rdrd\theta$

12. $\int_0^1 \int_0^2 \int_0^3 dxdydz$ is equal to

$a)\ 3 \quad b)\ 4 \quad c)\ 2 \quad d)\ 6$

13. The name of the curve $r = a(1+\cos\theta)$ is

a) lemniscate b) cycloid c) cardioid d) hemicircle

14. The volume integral in cartesian coordinates is equal to

$a)\ \iiint_V dxdydz$ $\quad b)\ \iiint_V drd\theta d\phi$ $\quad c)\ \iint_R drd\theta$ $\quad d)\ \iint_R rdrd\theta$

15. $\int_0^1 \int_0^2 x^2 ydxdy$ is equal to

$a)\ \frac{2}{3} \quad b)\ \frac{1}{3} \quad c)\ \frac{4}{3} \quad d)\ \frac{8}{3}$

16. $\int_0^1 \int_0^1 (x+y)dxdy$ is equal to

$a)\ 1 \quad b)\ 2 \quad c)\ 3 \quad d)\ 4$

17. After changing the double integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$ into polar coordinates, we have

$a)\ \int_0^{\pi/2} \int_0^\infty e^{-r^2} drd\theta$ $\quad b)\ \int_0^{\pi/4} \int_0^\infty e^{-r} drd\theta$ $\quad c)\ \int_0^{\pi/2} \int_0^\infty e^{-r^2} rdrd\theta$ $\quad d)\ \int_0^{\pi/2} \int_0^\infty e^{-r} drd\theta$

18. $\int_0^\infty \int_0^y \frac{e^{-y}}{y} dxdy$ is equal to

$a)\ 1 \quad b)\ 0 \quad c)\ -1 \quad d)\ 2$

19. The value of the integral $\int_0^2 \int_0^1 xydxdy$ is

(a) 1 (b) 2 (c) 3 (d) 4

20. The value of the integral $\int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta+\phi)\,d\theta d\phi$

(a) 1 (b) 2 (c) 3 (d) 4

21. The region of integration of the integral $\int_{-b}^{b}\int_{-a}^{a} f(x,y)dxdy$ is

(a) square (b) circle (c) rectangle (d) triangle

22. The region of integration of the integral $\int_0^1\int_0^x f(x,y)dxdy$ is

(a) square (b) rectangle (c) triangle (d) circle

23. The limits of integration is the double integral $\iint_R f(x,y)dxdy$, where R is in the first quadrant and bounded by $x=0,\ y=0,\ x+y=1$ are

$(a)\ \int_{x=0}^{1}\int_{y=0}^{1-x} f(x,y)dydx$ $(b)\ \int_{y=1}^{2}\int_{x=0}^{1-y} f(x,y)dxdy$

$(c)\ \int_{y=0}^{1}\int_{x=1}^{y} f(x,y)dxdy$ $(d)\ \int_{y=0}^{2}\int_{x=0}^{1-y} f(x,y)dxdy$

ANSWERS:

1	a	6	b	11	d	16	a	21	c
2	a	7	a	12	d	17	c	22	c
3	d	8	b	13	c	18	a	23	a
4	d	9	a	14	a	19	a		
5	d	10	c	15	c	20	b		

UNIT – II: VECTOR CALCULUS

1. The directional derivative of $\phi = xy + yz + zx$ at the point (1,2,3) along x - axis is
   (a) 4 (b) 5 (c) 6 (d) 0

2. In what direction from (3, 1, -2) is the directional derivative of $\phi = x^2 y^2 z^4$ maximum?
   a) $\frac{1}{\sqrt{19}}(\vec{i} + 3\vec{j} - \vec{k})$ (b) $19(\vec{i} + 3\vec{j} - 3\vec{k})$
   (c) $96(\vec{i} + 3\vec{j} - 3\vec{k})$ d) $\frac{1}{\sqrt{19}}(3\vec{i} + 3\vec{j} - \vec{k})$

3. If $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin, then $\nabla \cdot \vec{r}$ is
   (a) 2 (b) 3 (c) 0 (d) 1

4. If $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin, then $\nabla \times \vec{r}$ is
   a) $\nabla \times \vec{r} = 0$ b) $x\vec{i} + y\vec{j} + z\vec{k} = 0$ c) $\nabla \times \vec{r} \neq 0$ d) $\vec{i} + \vec{j} + \vec{k} = 0$

5. The unit vector normal to the surface $x^2 + y^2 - z^2 = 1$ at (1, 1, 1) is
   a) $\frac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}}$ b) $\frac{2\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{2}}$ c) $\frac{3\vec{i} + 3\vec{j} - 3\vec{k}}{2\sqrt{3}}$ d) $\frac{\vec{i} + \vec{j} - \vec{k}}{3\sqrt{2}}$

6. If $\phi = xyz,$ then $\nabla\phi$ is
   a) $yz\vec{i} + zx\vec{j} + xy\vec{k}$ b) $xy\vec{i} + yz\vec{j} + zx\vec{k}$ c) $zx\vec{i} + xy\vec{j} + yz\vec{k}$ d) 0

7. If $\vec{F} = (x + 3y)\vec{i} + (y - 3z)\vec{j} + (x - 2z)\vec{k}$ then $\vec{F}$ is
   a) solenoidal b) irrotational c) constant vector
   d) both solenoidal and irrotational

8. If $\vec{F} = \left(axy - z^3\right)\vec{i} + (a-2)x^2\vec{j} + (1-a)xz^2\vec{k}$ is irrotational then the value of a is
   a) 0 b) 4 c) -1 d) 2

9. If $\vec{u}$ and $\vec{v}$ are irrotational then $\vec{u} \times \vec{v}$ is
   a) solenoidal b) irrotational c) constant vector d) zero vector

10. If ϕ and ψ are scalar functions then $\nabla\phi\times\nabla\psi$ is

a) solenoidal b) irrotational c) constant vector
d) both solenoidal and irrotational

11. If $\vec{F} = \left(y^2 - z^2 + 3yz - 2x\right)\vec{i} + \left(3xz + 2xy\right)\vec{j} + \left(3xy - 2xz + 2z\right)\vec{k}$ then $\vec{F}$ is

a) solenoidal b) irrotational c) both solenoidal and irrotational
d) neither solenoidal nor irrotational

12. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $grad(\vec{a}\cdot\vec{r})$ is

a) 0 b) 1 c) $\vec{a}$ d) $\vec{r}$

13. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $div(\vec{a}\times\vec{r})$ is

a) 0 b) 1 c) $\vec{a}$ d) $\vec{r}$

14. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $curl(\vec{a}\times\vec{r})$ is

a) 0 b) 1 c) $2\vec{a}$ d) $2\vec{r}$

15. If ϕ scalar functions then $curl(grad\phi)$ is

a) solenoidal b) irrotational c) constant vector d) 0

16. If the value of $\int_A^B \vec{F}\cdot d\vec{r}$ does not depend on the curve C, but only on the terminal points A and B then $\vec{F}$ is called

a) solenoidal vector b) irrotational vector c) conservative vector
d) neither conservative nor irrotational

17. The condition for $\vec{F}$ to be Conservative is, $\vec{F}$ should be

a) solenoidal vector b) irrotational vector c) rotational
d) neither solenoidal nor irrotational

18. The value of $\int_C \vec{r} \cdot d\vec{r}$ where C is the line $y = x$ in the xy-plane from (1,1) to (2,2) is

a) 0 b) 1 c) 2 d) 3

19. The work done by the conservative force when it moves a particle around a closed curve is

a) $\nabla \cdot \vec{F} = 0$ b) $\nabla \times \vec{F} = 0$ c) 0 d) $\nabla \cdot (\nabla \times \vec{F}) = 0$

20. The connection between a line integral and a double integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) convolution theorem

21. The connection between a line integral and a surface integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) Residue theorem

22. The connection between a surface integral and a volume integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) Cauchy's theorem

23. Using Gauss divergence theorem, find the value of $\iint_S \vec{r}\,\vec{ds}$ where $\vec{r}$ is the position vector and V is the volume

a) $4V$ b) 0 c) $3V$ d) volume of the given surface

24. If S is any closed surface enclosing the volume V and if $\vec{F} = ax\,\vec{i} + by\,\vec{j} + cz\,\vec{k}$ then the value of $\iint_S \vec{F} \cdot \vec{n}\, dS$ is

a) $abcV$ b) $(a+b+c)V$ c) 0 d) $abc(a+b+c)V$

ANSWERS:

1	b	6	a	11	c	16	c	21	b
2	c	7	a	12	c	17	b	22	c
3	b	8	b	13	a	18	d	23	c
4	a	9	a	14	a	19	c	24	b
5	a	10	a	15	d	20	a		

UNIT-III LAPLACE TRANSFORMS

1. L(1) =

$(a)\ \frac{1}{s}\quad (b)\ \frac{1}{s^2}\quad (c)\ 1\quad (d)\ s$

2. $L(e^{3t}) =$

$(a)\ \frac{1}{s+3}\qquad (b)\ \frac{1}{s-3}\qquad (c)\ \frac{3}{s+3}\qquad (d)\ \frac{s}{s-3}$

3. $L(e^{-at}) =$

$(a)\ \frac{1}{s+1}\qquad (b)\ \frac{1}{s-1}\qquad (c)\ \frac{1}{s+a}\qquad (d)\ \frac{1}{s-a}$

4. $L(\cos 2t) =$

$(a)\ \frac{s}{s^2+4}\qquad (b)\ \frac{s}{s^2+2}\qquad (c)\ \frac{2}{s^2+2}\qquad (d)\ \frac{4}{s^2+4}$

5. $L(t^4) =$

$(a)\ \frac{4!}{s^5}\quad (b)\ \frac{3!}{s^4}\quad (c)\ \frac{4!}{s^4}\quad (d)\ \frac{5!}{s^4}$

6. $L(a^t) =$

$(a)\ \frac{1}{s-\log a}\qquad (b)\ \frac{1}{s+\log a}\qquad (c)\ \frac{1}{s-a}\qquad (d)\ \frac{1}{s+a}$

7. $L(\sinh \omega t) =$

$(a)\ \frac{s}{s^2+\omega^2}\qquad (b)\ \frac{\omega}{s^2+\omega^2}\qquad (c)\ \frac{s}{s^2-\omega^2}\qquad (d)\ \frac{\omega}{s^2-\omega^2}$

8. An example of a function for which the Laplace transforms does not exists is

$(a)\ f(t) = t^2\quad (b)\ f(t) = \tan t\quad (c)\ f(t) = \sin t\qquad (d)\ f(t) = e^{-at}$

9. If $L(f(t)) = F(s)$, then $L(e^{-at} f(t)) =$

$(a)\ F(s+a)\qquad (b)\ F(s-a)\qquad (c)\ F(s)\quad (d)\ \frac{1}{a}F\left(\frac{s}{a}\right)$

10. $L(e^{-at} \cos bt) =$

$(a)\ \frac{s+b}{(s+b)^2+a^2}\qquad (b)\ \frac{s+a}{(s+a)^2+b^2}\qquad (c)\ \frac{a}{s^2+a^2}\qquad (d)\ \frac{s}{s^2+b^2}$

11. $L(te^{t}) =$

$(a)\ \frac{1}{(s+1)^2}$ $\quad (b)\ \frac{1}{s+1}$ $\quad (c)\ \frac{1}{s-1}$ $\quad (d)\ \frac{1}{(s-1)^2}$

12. $L(t\sin at) =$

$(a)\ \frac{2as}{(s^2+a^2)^2}$ $\quad (b)\ \frac{2s}{(s^2+a^2)^2}$ $\quad (c)\ \frac{s^2-a^2}{(s^2+a^2)^2}$ $\quad (d)\ \frac{1}{s^2+a^2}$

13. $L(\sin 3t) =$

$(a)\ \frac{3}{s^2+3}$ $\quad (b)\ \frac{3}{s^2+9}$ $\quad (c)\ \frac{s}{s^2+3}$ $\quad (d)\ \frac{s}{s^2+9}$

14. $L(\cosh t) =$

$(a)\ \frac{s}{s^2+1}$ $\quad (b)\ \frac{s}{s^2-1}$ $\quad (c)\ \frac{1}{s^2+1}$ $\quad (d)\ \frac{1}{s^2-1}$

15. $L(t^{1/2}) =$

$(a)\ \frac{\Gamma(3/2)}{s^{1/2}}$ $\quad (b)\ \frac{\Gamma(1/2)}{s^{3/2}}$ $\quad (c)\ \frac{\Gamma(1/2)}{s^{1/2}}$ $\quad (d)\ \frac{\Gamma(3/2)}{s^{3/2}}$

16. $L(t^{-1/2}) =$

$(a)\ \sqrt{\frac{\pi}{s}}$ $\quad (b)\ \sqrt{\frac{\pi}{2s}}$ $\quad (c)\ \sqrt{\frac{1}{s}}$ $\ (d)\ \frac{1}{s}$

17. $L[te^{2t}] =$

$(a)\ \frac{1}{(s-2)^2}$ $\quad (b)\ -\frac{1}{(s-2)^2}$ $\quad (c)\ \frac{1}{(s-1)^2}$ $\quad (d)\ \frac{1}{(s+1)^2}$

18. If $L[f(t)] = F(s)$ then $L\left\{f\left(\frac{t}{a}\right)\right\}$ is

(a) $aF(as)$ $\quad$ (b) $\frac{1}{a}F\left(\frac{s}{a}\right)$ $\quad$ (c) $F(s+a)$ $\quad$ (d) $\frac{1}{a}F(as)$

19. $L\left(\int_0^t \sin t\,dt\right)$ is

(a) $\frac{1}{s^2+1}$ $\quad$ (b) $\frac{s}{s^2+1}$ $\quad$ (c) $\frac{1}{(s^2+1)^2}$ $\quad$ (d) $\frac{1}{s(s^2+1)}$

20. $L(\sin t \cos t)$ is

(a) $L(\sin t)..L(\cos t)$ (b) $L(\sin t) + L(\cos t)$ (c) $L(\sin t) - L(\cos t)$ (d) $\dfrac{L(\sin 2t)}{2}$

21. If $L[f(t)] = F[s]$ then $L[tf(t)] =$

$(a)\ \dfrac{d}{ds}F(s)$ $(b)\ -\dfrac{d}{ds}F(s)$ $(c)\ (-1)^n \dfrac{d}{ds}F(s)$ $(d)\ -\dfrac{d^2}{ds^2}F(s)$

22. If $L[f(t)] = F[s]$ then $L\left[\dfrac{f(t)}{t}\right] =$

$(a)\ \int_0^\infty F(s)\,ds$ $(b)\ \int_s^\infty F(s)\,ds$ $(c)\ \int_{-\infty}^\infty F(s)\,ds$ $(d)\ \int_a^\infty F(s)\,ds$

23. $L\left[\dfrac{\cos t}{t}\right] =$

$(a)\ \dfrac{s}{s^2+a^2}$ $(b)\ \dfrac{1}{s^2+a^2}$ (c) *does not exist* $(d)\ \dfrac{s^2-a^2}{(s^2+a^2)^2}$

24. If $L[f(t)] = F[s]$ then $L[t^n f(t)] =$

$(a)\ (-1)^n \dfrac{d^n}{ds^n}F(s)$ $(b)\ \dfrac{d^n}{ds^n}F(s)$ $(c)\ -\dfrac{d^n}{ds^n}F(s)$ $(d)\ (-1)^{n-1}\dfrac{d^n}{ds^n}F(s)$

25. $L\left[\dfrac{1-e^{-t}}{t}\right] =$

$(a)\ \log\left(\dfrac{s}{s-1}\right)$ $(b)\ \log\left(\dfrac{s}{s+1}\right)$ $(c)\ \log\left(\dfrac{s+1}{s}\right)$ $(d)\ \log\left(\dfrac{s-1}{s}\right)$

26. $L(u_a(t))$ is

(a) $\dfrac{e^{as}}{s}$ (b) $\dfrac{e^{-as}}{s}$ (c) $-\dfrac{e^{-as}}{s}$ (d) $-\dfrac{e^{as}}{s}$

27. If $L[f(t)] = F[s]$ then $L[f^{'}(t)] =$

(a) $sL[f(t)] - f(0)$ (b) $sL[f(t)] - sf(0)$ (c) $L[f(t)] - f(0)$ (d) $sL[f(t)] - f^{'}(0)$

28. Using the initial value theorem, find the value of the function $f(t) = ae^{-bt}$

$(a)\ a$ $(b)\ a^2$ $(c)\ ab$ $(d)\ 0$

29. Using the initial value theorem, find the value of $f(t) = e^{-2t}\sin t$

$(a)\ 0$ $(b)\ \infty$ $(c)\ 1$ $(d)\ 2$

30. Using the initial value theorem, find the value of the function $f(t) = \sin^2 t$

$(a)\ 0 \quad (b)\ \infty \quad (c)\ 1 \quad (d)\ 2$

31. Using the initial value theorem, find the value of the function $f(t) = 1 + e^{-t} + t^2$

$(a)\ 2 \quad (b)\ 1 \quad (c)\ 0 \quad (d)\ \infty$

32. Using the initial value theorem, find the value of the function $f(t) = 3 - 2\cos t$

$(a)\ 3 \quad (b)\ 2 \quad (c)\ 1 \quad (d)\ 0$

33. Using the final value theorem, find the value of the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

(a) 1 (b) 0 (c) ∞ (d) -2

34. Using the final value theorem, find the value of the function $f(t) = t^2 e^{-3t}$

(a) 0 (b) ∞ (c) 1 (d) -1

35. Using the final value theorem, find the value of the function $f(t) = 1 - e^{-at}$

$(a)\ 0 \quad (b)\ 1 \quad (c)\ 2 \quad (d)\ \infty$

36. The period of $\tan t$ is

(a) π (b) $\frac{\pi}{2}$ (c) 2π (d) $\frac{\pi}{4}$

37. The period of $|\sin \omega t|$ is

(a) $\frac{\pi}{\omega}$ (b) $\frac{2\pi}{\omega}$ (c) 2π (d) $2\pi\omega$

38. Inverse Laplace transform of $\frac{1}{(s-1)^2}$ is

(a) te^{-t} (b) te^t (c) $t^2 e^t$ (d) t

39. Inverse Laplace transform of $\frac{2}{s-b}$ is

(a) $2e^{-bt}$ (b) $2e^{bt}$ (c) $2te^{bt}$ (d) $2bt$

40. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{F(s)}{s}\right)$ is

(a) $\int_0^\infty f(t)dt$ (b) $\int_0^t f(t)dt$ (c) $\int_{-\infty}^\infty f(t)dt$ (d) $\int_{-a}^a f(t)dt$

41. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{1}{s^2+4}\right)$ is

(a) $\frac{\sin 2t}{2}$ (b) $\frac{\sin\sqrt{2}t}{\sqrt{2}}$ (c) $\sin 2t$ (d) $\sin\sqrt{2}t$

42. Inverse Laplace transform of $\frac{1}{s^2-a^2}$ is

(a) $\frac{\sin at}{a}$ (b) $\frac{\sinh at}{a}$ (c) $\sin at$ (d) $\sinh at$

43. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{1}{s^2}\right)$ is

(a) t (b) $2t$ (c) $3t$ (d) t^2

44. Inverse Laplace transform of $\frac{s}{s^2-9}$ is

(a) $\cos 9t$ (b) $\cos 3t$ (c) $\cosh 9t$ (d) $\cosh 3t$

45. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}(F(as))$ is

(a) $\frac{f(t)}{a}$ (b) $\frac{1}{a}f\left(\frac{t}{a}\right)$ (c) $f\left(\frac{t}{a}\right)$ (d) $f(at)$

46. Inverse Laplace transform of $\frac{1}{s^3}$ is

(a) $\frac{t}{2}$ (b) t (c) $\frac{t^2}{2}$ (d) t^2

47. Inverse Laplace transform of $\frac{s+3}{(s+3)^2+9}$ is

(a) $e^{3t}\cos 3t$ (b) $e^{-3t}\cos 3t$ (c) $e^{-3t}\cosh 3t$ (d) $e^{-3t}\cos 9t$

48. Inverse Laplace transform of $\frac{b}{s+a}$ is

(a) ae^{-bt} (b) be^{-bt} (c) ae^{bt} (d) be^{at}

49. The value of $e^{-t} * \sin t =$

$(a)\left(\frac{\sin t-\cos t}{2}\right)$ $(b)\left(\frac{\cos t-\sin t}{2}\right)$ $(c)\left(\frac{e^{-t}}{2}\right)+\left(\frac{\sin t-\cos t}{2}\right)$ $(d)\left(\frac{e^{-t}}{2}\right)$

50. The value of $1 * e^t$ is

(a) $e^t - 1$ (b) $e^t + 1$ (c) e^t (d) e

ANSWERS:

1	a	11	d	21	b	31	a	41	a
2	b	12	a	22	b	32	c	42	b
3	c	13	b	23	c	33	a	43	a
4	a	14	b	24	a	34	a	44	d
5	a	15	d	25	c	35	b	45	b
6	a	16	a	26	b	36	a	46	c
7	d	17	a	27	a	37	a	47	b
8	b	18	a	28	a	38	b	48	b
9	a	19	d	29	a	39	b	49	c
10	b	20	d	30	a	40	b	50	a

UNIT– IV: ANALYTIC FUNCTIONS

1. Cauchy – Riemann equation in polar co-ordinates are
   $(a)\ ru_r = v_\theta, u_\theta = -rv_r$ $(b)\ -ru_r = v_\theta, u_\theta = rv_r$
   $(c)\ -ru_r = v_\theta, u_\theta = rv_r$ $(d)\ u_r = rv_\theta, ru_\theta = v_r$

2. If $w = f(z)$ is analytic function of $z,$ then
   $(a)\ \frac{\partial w}{\partial z} = i\frac{\partial w}{\partial x}$ $(b)\ \frac{\partial w}{\partial z} = i\frac{\partial w}{\partial y}$ $(c)\ \frac{\partial^2 w}{\partial z \partial \overline{z}} = 0$ $(d)\ \frac{\partial w}{\partial \overline{z}} = 0$

3. The function $f(z) = u + iv$ is analytic if
   $(a)\ u_x = v_y, u_y = -v_x$ $(b)\ u_x = -v_y, u_y = v_x$
   $(c)\ u_x + v_y = 0, u_y - v_x = 0$ $(d)\ u_y = v_y, u_x = v_x$

4. The function $w = \sin x \cosh y + i \cos x \sinh y$ is
   (a) need not be analytic (b) analytic (c) discontinuous
   (d) differentiable only at origin

5. If u and v are harmonic, then $u + iv$ is
   (a) harmonic (b) need not be analytic (c) analytic (d) continuous

6. If a function $u(x, y)$ satisfies $u_{xx} + u_{yy} = 0,$ then u is
   (a) analytic (b) harmonic (c) differentiable (d) continuous

7. If $u + iv$ is analytic, then the curves $u = c_1$ and $v = c_2$ are
   (a) cut orthogonally (b) intersect each other (c) are parallel
   (d) coincides

8. The invariant point of the transformation $w = \frac{1}{z - 2i}$ is
   (a) $z = i$ $(b)\ z = -i$ $(c)\ z = 1$ $(d)\ z = -1$

9. The transformation $w = cz$ where c is real constant represents
   (a) rotation (b) reflection (c) magnification (d) magnification and rotation

10. The complex function $w = az$ where a is complex constant represents
    (a) rotation (b) magnification and rotation (c) translation (d) reflection

11. The values of $C_1 \& C_2$ such that the function $f(z) = C_1 xy + i[C_2 x^2 + y^2]$ is analytic are
    $(a)\ C_1 = 0, C_2 = 1$ $(b)\ C_1 = 2, C_2 = -1$
    $(c)\ C_1 = -2, C_2 = 1$ $(d)\ C_1 = -2, C_2 = 0$

12. The real part of $f(z)=e^{2z}$ is

$(a)\ e^x\cos y$ $(b)\ e^x\sin y$ $(c)\ e^{2x}\cos 2y$ $(d)\ e^{2x}\sin 2y$

13. If *f(z)* is analytic where $f(z)=r^2\cos 2\theta + ir^2\sin p\theta$, the value of p is

$(a)\ p=1$ $(b)\ p=-2$ $(c)\ p=-1$ $(d)\ p=2$

14. The points at which the function $f(z)=\dfrac{1}{z^2+1}$ fails to be analytic an

$(a)\ z=\pm 1$ $(b)\ z=\pm i$ $(c)\ z=0$ $(d)\ z=\pm 2$

15. The critical point of transformation $w=z^2$ is

$(a)\ z=2$ $(b)\ z=0$ $(c)\ z=1$ $(d)\ z=-2$

16. An analytic function with constant modulus is

(a) zero (b) analytic (c) constant (d) harmonic

17. The image of the rectangular region in the z-plane bounded by the lines $x=0,\ \ y=0,\ \ x=2$ and $y=1$ under the transformation $w=2z.$

(a) parabola (b) circle (c) straight line (d) rectangle is magnified twice

18. If $f(z)$ and $\overline{f(z)}$ are analytic function of $z,$ then $f(z)$ is

(a) analytic (b) zero (c) constant (d) discontinuous

19. The invariant points of the transformation $w=-\left(\dfrac{2z+4i}{iz+1}\right)$ are

$(a)\ z=4i,-i$ $(b)\ z=-4i,i$ $(c)\ z=2i,i$ $(d)\ z=-2i,i$

20. The function $|z|^2$ is

(a) differentiable at the origin (b) analytic (c) constant (d) differentiable everywhere

21. If f(z) is regular function of z then,

$(a)\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2=|f'(z)|^2$ $(b)\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$

$(c)\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}\right)|f(z)|^2=4|f'(z)|^2$ $(d)\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|$

22. The transformation $w=z+c$ where c is a complex constant represents

(a) rotation (b) magnification (c) translation (d) magnification & rotation

23. The mapping $w = \frac{1}{z}$ is

(a) conformal (b) not conformal at $z = 0$ (c) conformal every where
(d) orthogonal

24. The function $u + iv = \frac{x - iy}{x - iy + a}$ $(a \neq 0)$ is not analytic function of z where as $u - iv$ is

(a) need not be analytic (b) analytic at all points (c) analytic except at $z = a$
(d) continuous everywhere

25. If z_1, z_2, z_3, z_4 are four points in the z-plane then the cross-ratio of these point is

(a) $\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$ (b) $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$

(c) $\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_4)(z - z_3)}$ (d) $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_3 - z_2)}$

26. The invariant points of the transformation $w = \frac{1 - iz}{z - i}$

(a) 0 (b) $\pm i$ (c) ± 2 (d) ± 1

ANSWERS:

1	a	6	b	11	b	16	c	21	b	26	d
2	d	7	a	12	c	17	d	22	c		
3	a	8	a	13	d	18	c	23	b		
4	b	9	c	14	b	19	a	24	c		
5	b	10	b	15	b	20	a	25	b		

UNIT – V: COMPLEX INTEGRATION

1. A curve which does not cross itself is called a
   (a) curve (b) closed curve (c) simple closed curve (d) multiple curve

2. The value of $\int_c \frac{zdz}{z-2}$ where c is the circle $|z|=1$ is
   (a) 0 (b) $\frac{\pi}{2}i$ (c) $\frac{\pi}{2}$ (d) 2

3. The value of $\int_c \frac{z}{(z-1)^2}dz$ where c is the circle $|z|=2$ is
   (a) πi (b) $2\pi i$ (c) $4\pi i$ (d) 0

4. The value of $\int_c (z-2)^n dz;\ (n \neq 1)$ where c is the circle $|z-2|=4$ is
   a. 2^n (b) n^2 (c) 0 (c) n

5. The value of $\int_c \frac{1}{2z+1}dz$ where c is the circle $|z|=1$ is
   (a) 0 (b) πi (c) $\frac{\pi}{2}i$ (d) 2

6. The value of $\int_c \frac{1}{3z+1}dz$ where c is the circle $|z|=1$ is
   (a) 0 (b) πi (c) $\frac{2\pi}{3}i$ (d) 2

7. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{z-a}dz$, where c is the simple closed curve and a is any point within c, is
   (a) $f(a)$ (b) $2\pi i f(a)$ (c) $\pi i f(a)$ (d) 0

8. If $f(z)$ is analytic inside and on c, the value of $\int_c f(z)dz$, where c is the simple closed curve, is
   (a) $f(a)$ (b) $2\pi i f(a)$ (c) $\pi i f(a)$ (d) 0

9. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{(z-a)^2}dz$, where c is the simple closed curve and a is any point within c, is
   (a) $f'(a)$ (b) $2\pi i f'(a)$ (c) $\pi i f'(a)$ (d) 0

10. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{(z-a)^3}dz$, where c is the simple closed curve and a is any point within c, is

(a) $f''(a)$ (b) $2\pi i f''(a)$ (c) $\pi i f''(a)$ (d) 0

11. Let $C: |z-a| = r$ be a circle, the f(z) can be expanded as a Taylor's series if

(a) $f(z)$ is a defined function within c
(b) $f(z)$ is a analytic function within c
(c) $f(z)$ is not a analytic function within c
(d) $f(z)$ is a analytic function outside c

12. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$ be two concentric circles ($R_2 < R_1$), the $f(z)$ can be expanded as a Laurent's series if

(a) $f(z)$ is analytic within C_2
(b) $f(z)$ is not analytic within C_2
(c) $f(z)$ is analytic in the annular region
(d) $f(z)$ is not analytic in the annular region

13. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$ be two concentric circles ($R_2 < R_1$), the annular region is defined as

(a) within C_1 (b) within C_2
(c) within C_2 and outside C_1 (d) within C_1 and outside C_2

14. The part $\sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral powers of $(z-a)$ is called as

(a) The analytic part of the Laurent's series
(b) The principal part of the Laurent's series
(c) The real part of the Laurent's series
(d) The imaginary part of the Laurent's series

15. The part $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral powers of $(z-a)$ is called as

(a) The analytic part of the Laurent's series
(b) The principal part of the Laurent's series
(c) The real part of the Laurent's series
(d) The imaginary part of the Laurent's series

16. The annular region for the function $f(z) = \frac{1}{z(z-1)}$ is

(a) $0 < |z| < 1$ (b) $1 < |z| < 2$ (c) $1 < |z| < 0$ (d) $|z| < 1$

17. The annular region for the function $f(z) = \dfrac{1}{(z-1)(z-2)}$ is

(a) $0 < |z| < 1$ (b) $1 < |z| < 2$ (c) $1 < |z| < 0$ (d) $|z| < 1$

18. The annular region for the function $f(z) = \dfrac{1}{z^2 - z - 6}$ is

(a) $0 < |z| < 1$ (b) $1 < |z| < 2$ (c) $2 < |z| < 3$ (d) $|z| < 3$

19. If $f(z)$ is not analytic at $z = z_0$ and there exists a neighborhood of $z = z_0$ containing no other singularity, then

(a) The point $z = z_0$ is isolated singularity of $f(z)$

(b) The point $z = z_0$ is a zero point of $f(z)$

(c) The point $z = z_0$ is nonzero of $f(z)$

(d) The point $z = z_0$ is non isolated singularity of $f(z)$

20. If $f(z) = \dfrac{\sin z}{z}$, then

(a) $z = 0$ is a simple pole (b) $z = 0$ is a pole of order 2

(c) $z = 0$ is a removable singularity (d) $z = 0$ is a zero of $f(z)$

21. If $f(z) = \dfrac{\sin z - z}{z^3}$, then

(a) $z = 0$ is a simple pole (b) $z = 0$ is a pole of order 2

(c) $z = 0$ is a removable singularity (d) $z = 0$ is a zero of $f(z)$

22. If $\lim_{z \to a} (z - a)^n f(z) \neq 0$ then

(a) $z = a$ is a simple pole (b) $z = a$ is a pole of order n

(c) $z = a$ is a removable singularity (d) $z = a$ is a zero of $f(z)$

23. If $f(z) = \dfrac{1}{(z-4)^2 (z-3)^3 (z-1)}$, then

(a) 4 is a simple pole, 3 is a pole of order 3 and 1 is a pole of order 2

(b) 3 is a simple pole, 1 is a pole of order 3 and 4 is a pole of order 2

(c) 1 is a simple pole, 3 is a pole of order 3 and 4 is a pole of order 2

(d) 3 is a simple pole, 4 is a pole of order 1 and 4 is a pole of order 2

24. If $f(z) = e^{\frac{1}{z-4}}$ then

(a) $z = 4$ is removable singularity (b) $z = 4$ is pole of order 2

(c) $z = 4$ is an essential singularity (d) $z = 4$ is zero of $f(z)$

25. Let $z = a$ is a simple pole for $f(z)$ and $b = \lim_{z \to a}(z-a) f(z)$, then

(a) b is a simple pole (b) b is a residue at a

(c) b is removable singularity (d) b is a residue at a of order n

26. The residue of $f(z) = \dfrac{1-e^{2z}}{z^3}$ is

(a) 0 (b) 2 (c) -2 (d) 1

27. The residue of $f(z) = \dfrac{e^{2z}}{(z+1)^2}$ is

(a) e^{-2} (b) -2 e^{-2} (c) -1 (d) 2 e^{-2}

28. The residue of $f(z) = \cot z$ is

(a) π (b) 1 (c) -1 (d) 0

ANSWERS:

1	c	6	c	11	b	16	a	21	c	26	c
2	a	7	b	12	c	17	b	22	b	27	d
3	b	8	d	13	d	18	c	23	c	28	b
4	c	9	b	14	a	19	a	24	c		
5	b	10	b	15	b	20	c	25	b		

SRM University
Department of Mathematics
Complex Integration- Multiple Choice questions
UNIT V

Slot-C

1. A contour integral is an integral along a ---------------- curve.
    a. Open Curve
    b. Closed curve
    c. Simple closed curve
    d. Multiple curve

    Answer: c. Simple closed curve

2. If f(z) is analytic inside and on C , the value of $\oint_C f(z)\, dz$,where C is the simple closed curve is
    a. f(a)
    b. $2\pi i$f(a)
    c. πif(a)
    d. 0

    Answer: d. 0

3. If f(z) is analytic inside and on C , the value of $\oint_C \frac{f(z)}{(z-a)^n}\, dz$,where C is the simple closed curve and a is any point within c is
    a. $2\pi i\, \frac{f^n(a)}{n!}$
    b. $2\pi i\, f(a)$
    c. $2\pi i\, \frac{f^{n-1}(a)}{(n-1)!}$
    d. 0

    Answer: c. $2\pi i\, \frac{f^{n-1}(a)}{(n-1)!}$

4. The value of $\oint_C \frac{\sin z}{z+1}\, dz$ where C is the circle $|z| = \frac{1}{3}$ *is*
    a. 0

b. $2\pi i$
c. $\frac{\pi}{2} i$
d. πi

Answer: a. 0

5. The value of $\oint_C \frac{e^z}{(z-2)^2} dz$ where C is the circle $|z| = 3$ is
   a. 0
   b. $2\pi i e^{-2}$
   c. $2\pi i e^{2}$
   d. $4\pi i e^{-2}$

   Answer: c. $2\pi i e^2$

6. The value of $\oint_C \frac{z}{2z-1} dz$ where C is the circle $|z| = 1$ is
   a. 0
   b. $2\pi i$
   c. $\frac{\pi}{2} i$
   d. πi

   Answer: c. $\frac{\pi}{2} i$

7. The value of $\oint_C \frac{1}{(z-3)^2} dz$ where C is the circle $|z| = 1$ is
   a. 0
   b. $2\pi i$
   c. $\frac{\pi}{2} i$
   d. πi

   Answer: a. 0

8. Let $C_1: |z - a| = R_1$ and $C_2: |z - a| = R_2$be two concentric circles($R_2 > R_1$), the annular region is defined as
   a. Within C_1
   b. Within C_2

c. Within C_2 and outside C_1
d. Within C_1 and outside C_2

Answer: c. Within C_2 and outside C_1

9. $The\ part\ \sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral powers of (z - a) is called as
   a. The analytic part of the Laurent’s series
   b. The principal part of the Laurent’s series
   c. The real part of the Laurent’s series
   d. The imaginary part of the Laurent’s series

   Answer: a. The analytic part of the Laurent’s series

10. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$be two concentric circles ($R_2 < R_1$), the f(z) can be expanded as a Laurent’s series if
    a. f(z) is analytic within C_2
    b. f(z) is not analytic within C_2
    c. f(z) is analytic in the annular region
    d. f(z) is not analytic in the annular region

    Answer: c. f(z) is analytic in the annular region

11. Expansion of $\frac{1-\cos z}{z}$ in Laurent’s series about $z = 0$ is
    a. $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots$
    b. $\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots$
    c. $\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$
    d. $\frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots$

    Answer: a. $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots$

12. The annular region for the function f(z) = $\frac{1}{z^2-3z+2}$ is

   a. $0 < |z| < 1$
   b. $1 < |z| < 2$
   c. $2 < |z| < 3$
   d. $|z| < 3$

   Answer :b. $1 < |z| < 2$

13. The Laurent's series expansion $1 + \frac{3}{z}\sum\frac{(-1)^n 2^n}{z^n} - \sum\frac{(-1)^n 3^n}{z^n}$ for the function $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ is valid in the region

   a. $|z| < 3$
   b. $|z| < 2$
   c. $2 < |z| < 3$
   d. $|z| > 3$

   Answer :d. $|z| > 3$

14. If f(z) is not analytic at $z = z_o$ and there exists a neighborhood of $z = z_o$ containing no other singularity, then

   a. The point $z = z_o$ is isolated singularity of f(z)
   b. The point $z = z_o$ is a zero point of f(z)
   c. The point $z = z_o$ is nonzero of f(z)
   d. The point $z = z_o$ is non isolated singularity of f(z)

   Answer : a. The point $z = z_o$ is isolated singularity of f(z)

15. If f(z) = $e^{\frac{1}{z+1}}$ then

   a. z = -1 is removable singularity
   b. z = -1 is pole of order 2
   c. z = -1 is an essential singularity
   d. z = -1 is zero of f(z)

   Answer : c. z = -1 is an essential singularity

16. Let z = a is a simple pole for $f(z) = \frac{P(z)}{Q(z)}$, then the Residue of $f(z)$ is

a. $\frac{P'(a)}{Q(a)}$
b. $\frac{P(a)}{Q(a)}$
c. $\frac{P'(a)}{Q'(a)}$
d. $\frac{P(a)}{Q'(a)}$

Answer : d. $\frac{P(a)}{Q'(a)}$

17. Let z = a is a pole of order 3 for f(z), then the residue is

a. $\lim_{z \to a} [(z - a)f(z)]$
b. $\lim_{z \to a} [(z - a)f''(z)]$
c. $\lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} [(z - a)^3 f(z)]$
d. $\lim_{z \to a} \frac{1}{3!} \frac{d^3}{dz^3} [(z - a)^3 f(z)]$

Answer: c. $\lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} [(z - a)^3 f(z)]$

18. The residue of $f(z) = \frac{z}{(z-2)}$ is

a. $2\pi i$
b. 1
c. 2
d. 0

Answer: c. 2

19. The residue of $f(z) = \frac{1}{(z^2+1)^2}$ at $z = i$ is

a. 4i
b. 1/4i
c. 0
d. 1/2i

Answer :b. 1/4i

20. If f(z) = $\frac{sinz-z}{z^3}$, then

a. z= 0 is a simple pole
b. z= 0 is a pole of order 2
c. z= 0 is a removable singularity
d. z= 0 is a zero of f(z)

Answer: c. z= 0 is a removable singularity

21. The value of the integral $\oint_C \frac{1}{ze^z} dz$ where $|z| = 1$ is

a. $2\pi i$
b. $\frac{\pi}{2} i$
c. πi
d. 0

Answer: a. $2\pi i$

22. If $f(z) = \frac{1}{z} + [2 + 3z + 4z^2 + \cdots]$ then the residue of $f(z)$ at z=0 is

a. 1
b. -1
c. 0
d. -2

Answer: a. 1

23. If the integral $\oint_0^{2\pi} \frac{d\theta}{13+5\cos\theta} = \oint_C f(z)dz$, C is $|z| = 1$, then
(A) $z = -i/5$ lies inside C and
(B) $z = -5i$ lies outside C. Which of the following is true.

a. Both A and B
b. Only A
c. Only B
d. Neither A nor B

Answer: a. Both A and B

24. If the integral $\oint_{-\infty}^{\infty} \frac{\cos mx}{(x^2+1)^2} dx \ , m > 0$, then
(A) $z = i$ double pole lies in the upper half of the z-plane and
(B) $z = -i$ double pole does not lie in the upper half of the z-plane.
Which of the following is true.

a. Both A and B
b. Only A
c. Only B
d. Neither A nor B

Answer: a. Both A and B

25. If $f(z)$ be continuous function such that $|f(z)| \to 0$ as $|z| \to \infty$, for C is the semicircle $|z| = R$ above the real axis, then

a. $\oint_C e^{-imz} f(z)dz \ \to \infty \ as \ R \ \to \infty$.
b. $\oint_C e^{imz} f(z)dz \ \to 0 \ as \ R \ \to \infty$.
c. $\oint_C e^{imz} f(z)dz \ \to 0 \ as \ R \ \to 0$.
d. $\oint_C f(z)dz \ \to \infty \ as \ R \ \to 0$.

Answer : b. $\oint_C e^{imz} f(z)dz \ \to 0 \ as \ R \ \to \infty$.