UNIT – IV

Analytic functions

Notations:-

$z = x+iy$ – complex variable, where x and y are real variables.

For every z, define $w = f(z)$

$$= u(x,y) + iv(x,y)$$

which is called the func. of complex variable.

Limit of a function:-

Let $f(z)$ be a func. defined in a set D and z_0 be a limit point of D. Then A is said to be limit of $f(z)$ at z_0, if

$$\lim_{z \to z_0} f(z) = A.$$

Continuity of a function:-

Let $f(z)$ be a func. defined in a set D and z_0 be a limit point of D. If the limit of $f(z)$ at z_0 exists and if it is finite and is equal to $f(z_0)$, (i.e.) if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

then $f(z)$ is said to be continuous at z_0.

Derivative of a Complex fun:-

A fun. $f(z)$ is said to be differentiable at a point $z = z_0$ if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$ exists and is

the same in whatever way $\Delta z \to 0$. It is denoted by $f'(z_0)$.

(or) putting $z_0 + \Delta z = z \Rightarrow \Delta z = z - z_0$, we get, $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$.

(i) $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Analytic (or) Regular (or) Holomorphic function

A fun. defined at a point z_0 is said to be analytic at z_0, if it has a derivative at z_0 and at every point in the nghd. of z_0.

If it's analytic at every point in a region R then it is said to be analytic in the region R.

Necessary condi. for the fun. to be analytic.

The necess. cond. for the fun. $f(z) = u + iv$ to be analytic are the cauchy-~~Reiemann~~ Riemann eqn. (C.R eqns.)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Sufficient Condition for f(z) to be analytic in D.

(1) u and v are diffb. in D and $u_x = v_y$ & $u_y = -v_x$.

(2) the partial derivatives u_x, u_y, v_x, v_y are all cts. in D.

Polar form of CR eqns:

Let $f(z) = u + iv$ where $z = re^{i\theta}$.

$f(re^{i\theta}) = u + iv \longrightarrow (1)$

Diffb. (1) p.w.r.to 'r',

$$f'(re^{i\theta}) \cdot e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \longrightarrow (2)$$

Diffb. (1) p.w.r.to 'θ',

$$f'(re^{i\theta}) \cdot (ire^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

$$\Rightarrow f'(re^{i\theta}) e^{i\theta} = \frac{1}{ir}\left[\quad " \quad \right]$$

$$= -i\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} \longrightarrow (3)$$

Comparing (2) & (3),

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

– C.R eqns. in polar form.

$$\frac{e^{i\theta} - e^{-i\theta}}{2i}$$ //.

$e^{i\theta} = \cancel{\cos\theta} + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore \cos\theta = e^{i\theta} +$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\sin iy = \frac{e^{-y} - e^{y}}{2i} = -i\left(\frac{e^{-y} - e^{y}}{2i}\right) = i\left(\frac{e^{y} - e^{-y}}{2}\right).$$

$= i \sinh y.$

Analytic function.

1. Test the analyticity

   (i) $w = \bar{z}$ $= x - iy$ — Not analy. (ii) $w = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

2. P.T $f(z)$ is analytic.

   (i) $f(z) = \sin z$ (ii) $f(z) = z^n$. (n is positive int)

(2) (i) $f(z) = \sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$\therefore u = \sin x \cosh y$, $v = \cos x \sinh y$.

$u_x = \cos x \cosh y$, $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$, $v_y = \cos x \cosh y$.

$u_x = v_y$ & $u_y = -v_x$.

and all partial deriv. are cts.

$\therefore \sin z$ is analytic.

(ii) $f(z) = z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$= r^n(\cos n\theta + i \sin n\theta)$

$\therefore u = r^n \cos n\theta$, $v = r^n \sin n\theta$.

$u_r = nr^{n-1} \cos n\theta$, $v_r = nr^{n-1} \sin n\theta$

$u_\theta = -r^n n \sin n\theta$, $v_\theta = nr^n \cos n\theta$.

$\therefore u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$.

(iii) $f(z) = \log z$, $\rightarrow \log re^{i\theta} = u + iv$.

$\log r + \log e^{i\theta} = \log r + i\theta$.

$\therefore u = \log r$, $v = \theta$. $u_r = 1/r$ $v_r = 0$ C.R eqns. satisf.

(3). S.T $f(z)$ is discts. at $z=0$ given $f(z) = \frac{2xy^2}{x^2+3y^4}$ when $z \neq 0$ and $f(0)=0$.

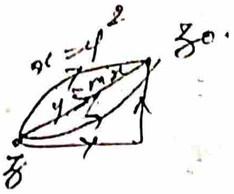
Soln For the path $y = mx$,

$$\lim_{z \to 0} f(z) = \lim_{\substack{y=mx \\ x \to 0}} f(z) = \lim_{x \to 0} \frac{2m^2x^3}{x^2+3m^4x^4}$$

$$= \lim_{x \to 0} \frac{2m^2 x}{1+3m^4x^2} = 0.$$

For the path $x = y^2$,

$$\lim_{\substack{x=y^2 \\ y \to 0}} f(z) = \lim_{y \to 0} \frac{2y^4}{y^4+3y^4} = \frac{2}{4} = \frac{1}{2} \neq 0.$$



$\lim_{z \to 0} f(z)$ does not exist.

$\therefore f(z)$ is discts.

(4). S.T $f(z)$ is discts. at $z=0$, given $f(z) = \frac{2xy}{x^2+y^2}$ and $f(0)=0$.

$z = x+iy$
$= x+imx$
$= x(1+im)$
$\Rightarrow x \to 0$

Soln: Along $y = mx$,

$$\lim_{z \to 0} f(z) = \lim_{\substack{y=mx \\ x \to 0}} f(z)$$

$$= \lim_{x \to 0} \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2} \neq f(0).$$

$\therefore f(z)$ is discts.

S.T the fun. $|z|^2$ is diffb. at $z=0$ but it is not analytic at any point.

Soln:- Let $z = x+iy$.

$$|z|^2 = z\bar{z} = (x+iy)(x-iy) = x^2+y^2.$$

$\Rightarrow u = x^2+y^2, \quad v = 0.$

$u_x = 2x, \quad u_y = 2y, \quad v_x = v_y = 0.$

C.R Eqns. not satisfied except at $z=0$.

$\therefore f(z)$ is diffb. only at $z=0$.

Now for $z \neq 0$,

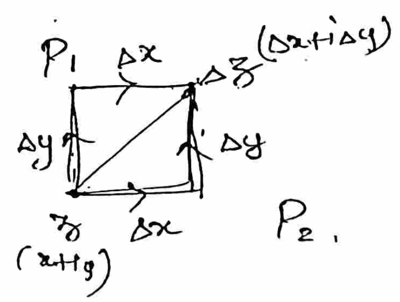
$$\lim_{\Delta z\to 0}\left(\frac{f(z+\Delta z)-f(z)}{\Delta z}\right) = \lim_{\Delta z\to 0}\left\{\frac{|z+\Delta z|^2-|z|^2}{\Delta z}\right\}.$$

$$= \lim_{\Delta z\to 0}\frac{(z+\Delta z)(\bar{z}+\Delta\bar{z}) - z\bar{z}}{\Delta z}.$$

$$= \lim_{\Delta z\to 0}\frac{\cancel{z\bar{z}}+\Delta z\,\bar{z}+z\,\Delta\bar{z}+\Delta z\,\Delta\bar{z}-\cancel{z\bar{z}}}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\left(\bar{z}+\frac{z\,\Delta\bar{z}}{\Delta z}+\Delta\bar{z}\right).$$

$$= \lim_{\Delta z\to 0}\left\{\frac{(x+iy)(\Delta x-i\Delta y)}{(\Delta x+i\Delta y)}+(x-iy)+(\Delta x-i\Delta y)\right\}.$$



Along P_1,

$$= \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \left[(x+iy)\frac{(\Delta x - i\Delta y)}{(\Delta x + i\Delta y)} + (x - iy) + \Delta x - i\Delta y) \right]$$

$$= \lim_{\Delta x \to 0} \left[(x+iy) + (x - iy) + \underset{(0)}{\Delta x} \right]. = 2x$$

Along P_2,

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\qquad \text{same.} \qquad \right]$$

$$= \lim_{\Delta y \to 0} \left[-(\cancel{x}+iy) + (\cancel{x}-iy) - \underset{(0)}{i\Delta y} \right] = -2iy.$$

$\therefore P_1 \neq P_2 \;\; \forall \; x$ and y. (ie) $z \neq 0$.

$\therefore f(z)$ is not diff. at any pt. $z \neq 0$.

(ie) it is not diff. in the neighbo. of $z = 0$.

$\therefore f(z)$ is not analytic at any point.

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(2) S.T the fun. $f(z) = \sqrt{|xy|}$ is not analytic at the origin although C-R Eqns. are satisfied at that point.

Soln. $f(z) = \sqrt{|xy|} \Rightarrow u = \sqrt{|xy|}$, $v = 0$.

S.T an analytic fun. with (i) constant R.P is const. and (ii) const. modulus is const.

Soln: (1) Given $u = c$, $\Rightarrow$ By C-R Eqns.

$v = c'$ $\quad u_x = 0 \quad u_y = 0$
$\Rightarrow v_x = 0 \quad v_y = 0$

$\therefore f(z) = c + ic' = \text{constant}.$

(2) $|f(z)| = \sqrt{u^2+v^2} = c \Rightarrow u^2+v^2 = k.$

Diffn. w.r.to x & y,

$2u\,u_x + 2v\,v_x = 0 \quad , \quad 2u\,u_y + 2v\,v_y = 0.$

By C R eqns.

$u^2 u_x - uv\,u_y = 0 \quad , \quad vu\,u_y + v^2 u_x = 0.$

adding $\rightarrow (u^2+v^2)u_x = 0 \Rightarrow u_x = 0$

solving similarly, $u_x = u_y = v_x = v_y = 0.$

$f'(z) = u_x + i v_x = 0 + i0 = 0$

$f(z)$ is constant.

$u v_y - v u_y =$

$u^2 u_x - uv\,\cancel{u_y} = 0.$
$v^2 u_x + uv\,\cancel{u_y} = 0$

$(u^2+v^2)u_x = 0.$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{|(x+\Delta x)y|} - \sqrt{|xy|}}{\Delta x}$$

At $(0,0)$, $u_x = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0.$

$|||^{ly}$, $u_y = 0$, $v_x = v_y = 0.$

$\therefore$ C.R eqn. satisfied at the origin.

Now $$f'(0) = \lim_{\Delta z \to 0.} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\sqrt{|\Delta x \Delta y|} - 0}{\Delta x + i\Delta y.}$$

$$= \lim_{\substack{\Delta y \to m\Delta x \\ \Delta x \to 0}} \frac{\sqrt{|m|\Delta x^2}}{\Delta x(1+im)} = \lim_{\Delta x \to 0} \frac{\sqrt{|m|}}{(1+im)} \quad \frac{\sqrt{|m|}}{(1+i}$$

$\because$ limit depends upon 'm', the limit is not unique

$\therefore f(z)$ is not analytic at the origin.

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(3) If $f(z)$ & $\overline{f(z)}$ are analytic fun. of z,

P.T $f(z)$ is constant.

<u>Soln:</u> $f(z) = u + iv$, $\overline{f(z)} = u - iv.$

Any. $f(z)$ & $\overline{f(z)}$ are analytic,

$$\left.\begin{aligned} u_x &= v_y, \quad u_y = -v_x \\ u_x &= -v_y, \quad u_y = v_x \end{aligned}\right\} \xRightarrow{\text{solving}} \begin{aligned} u_x &= 0, \ u_y = 0 \\ v_x &= 0, \ v_y = 0 \end{aligned}$$

$\Rightarrow v = c_1, \ u = c_2 \quad \Rightarrow f(z)$ is constant.

Definition

Harmonic functions.

Any function which has cts. second order partial derivatives and which satisfies the Laplace Eqn. is called harmonic fun. (ie) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, then u is said to be a harmonic fun.

Problems :-

1. Verify that the families of curve $u = c_1$ and $v = c_2$ cut each other orthogonally when $w = z^3$.

Soln :- $f(z) = z^3 = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\Rightarrow u = x^3 - 3xy^2 = c_1$, $v = 3x^2y - y^3 = c_2$

$3x^2 - 3\left[y^2 + 2xy\frac{dy}{dx}\right] = 0$, $3x^2\frac{dy}{dx} + 6xy - 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = m_1$, $\frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = m_2$.

$\Rightarrow m_1 m_2 = -1$.

2. Examine whether the fun. xy^2 can be the real part of an analytic fun.

$$u = xy^2 \qquad \frac{\partial u}{\partial x} = y^2 \qquad \frac{\partial^2 u}{\partial x^2} = 0.$$

$$\frac{\partial u}{\partial y} = 2yx. \quad \frac{\partial^2 u}{\partial y^2} = 2x.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x \neq 0$$ does not satisfies Laplace eqn..

$\therefore$ it cannot be a real part of an analytic fun..

Result:

(1) $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}.$$

$$\frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial x}{\partial \bar{z}} = 1/2, \quad \frac{\partial y}{\partial z} = \frac{1}{2i}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{\partial}{\partial x}\frac{1}{2} + \frac{\partial}{\partial y}\frac{1}{2i} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{1}{2} + \frac{\partial}{\partial y}\left(-\frac{1}{2i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) //$$

Harmonic fcns

Properties :- of analytic functions :-

(1) The real & Imaginary parts of an analytic fn.. are harmonic fcns.. assuming 2nd order partial derivatives are cts.

Proof:- Let $f(z) = u(x,y) + iv(x,y)$ be an analytic fn. Then u & v have continuous partial derivatives of first order which satisfy the C.R eqn.. are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(OR) $\left[\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} , \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \Rightarrow \nabla^2 u = 0 \right]$

Further,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}.$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ Thus, u is a harmonic fn.

Similarly, v is a harmonic fn..

Defn:- Let $f = u+iv$ be an analytic fn.. in a region D. Then v is said to be a harmonic conjugate of u.

(2) Let $f = u+iv$ be an analytic fn.. in a region D. Then v is a harmonic conjugate of u if and only if u is a harmonic conjugate of $-v$.

Proof:- Let v be a harmonic conjugate of u. Then $f = u+iv$ is analytic.

$\therefore if = iu - v$ is also analytic.

Hence u is a harmonic conjugate of $-v$.

Conversely, suppose u is a harm. Conj.. of $-v$, then $g = -v+iu$ is analytic.

$\therefore -ig = vi + u = u+iv$ is analytic.

(3) Any two harmonic conjugates of a given harmonic fun. u in a region differ by a constant.

Proof:- Let u be a harmonic fn.

Let v and v' be two harmonic conjugates of u.

$\therefore$ $u+iv$ and $u+iv'$ are analytic in D.

By C.R eqn.. we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v'}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial v'}{\partial x}.$$

$$\Rightarrow \frac{\partial v}{\partial y} - \frac{\partial v'}{\partial y} = 0 \qquad \& \qquad \frac{\partial v}{\partial x} - \frac{\partial v'}{\partial x} = 0.$$

$$\Rightarrow \frac{\partial}{\partial y}(v - v') = 0 \qquad \frac{\partial}{\partial x}(v - v') = 0.$$

$$v - v' = \text{Constant}.$$

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(4) If $f(z) = u+iv$ be an analytic fn., then the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ where c_1 & c_2 are constants cut each other orthogonally.

(or)

The real and imaginary parts of an analytic fn. form an orthogonal system.

Proof:- Let $u(x,y) = c_1$ and $v(x,y) = c_2$.

$\Rightarrow du = 0.$ $\qquad \Rightarrow dv = 0.$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0. \qquad \Rightarrow \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \qquad \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} \text{ (by C.R Eqn)} = m_2.$$

$$\therefore m_1 m_2 = -1.$$

$\therefore$ the curves cut each other orthogonally.

Construction of an analytic fun..
(when its real (or) Img.. part is known).

Milne - Thompson method :-

$z = x+iy$
$\therefore x = \frac{z+\bar{z}}{2}$
$y = \frac{z-\bar{z}}{2i}$

Let $f(z) = u(x,y) + iv(x,y)$.

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Assume $\bar{z} = z$ (is possible only if $y=0$)

$$f(z) = u(z,0) + iv(z,0).$$

W.K.T, $f'(z) = u_x(x,y) + iv_x(x,y)$
$= u_x - iu_y$ (by C.R eqn)

$$f'(z) = u_x(z,0) - iu_y(z,0).$$

Intg.. $f(z) = \int (u_x(z,0) - iu_y(z,0))\,dz + c$

(u is given).

Similarly,

$$f(z) = \int \left(V_y(z,0) + iV_x(z,0)\right) dz + c$$

(v is given)

) S.T $u = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic and determine harmonic conjugate and also find $f(z)$.

Sol. $u_x = 3x^2 - 3y^2 + 6x$, $u_y = -6xy - 6y$.

$u_{xx} = 6x + 6$, $u_{yy} = -6x - 6$.

$\Rightarrow u_{xx} + u_{yy} = 0$. , it satisfies Laplace eqn..

$\therefore u$ is harmonic.

$$f(z) = \int (3z^2+6z - i.0)\,dz = \cancel{3}\frac{z^3}{\cancel{3}} + 6z^2 + c$$

$$f(z) = z^3 + 3z^2 + c$$

is an analytic fun.

To find harmonic conjugate:

Acc. w.r.t $u_x = 3x^2 - 3y^2 + 6x = v_y$.

$$\Rightarrow v = 3x^2y - \cancel{3}\frac{y^3}{\cancel{3}} + 6xy + f(x)$$

$$\Rightarrow v_x = 6xy + 6y + f'(x) = -u_y = 6xy + 6y$$

$$\Rightarrow f'(x) = 0$$

$$\Rightarrow f(x) = c.$$

$\therefore v = 3x^2y - y^3 + 6xy + c$ // is the harmonic conjugate.

(2) Verify whether the fun. $\log\sqrt{x^2+y^2}$ (or) $\frac{1}{2}\log(x^2+y^2)$ is harmonic. Find the harmonic conjugate and also find $f(z)$.

Sol Let $u = \log\sqrt{x^2+y^2} \Rightarrow u = \frac{1}{2}\log(x^2+y^2)$.

$$u_x = \frac{1}{\cancel{2}} \frac{\cancel{2}x}{x^2+y^2}, \quad u_y = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}y}{x^2+y^2}$$

$$\Rightarrow u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad u_{yy} = \frac{-(y^2-x^2)}{(x^2+y^2)^2} \Rightarrow \nabla^2 u = 0$$

$\Rightarrow$ " is harmonic.

$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$ 6

$$f(z) = \int\left(\frac{z}{z^2} - i0\right)dz = \log z + c.$$

To find conjugate:

$$u_x = \frac{x}{x^2+y^2} = v_y \Rightarrow v = \cancel{x}\cdot\frac{1}{\cancel{x}}\tan^{-1}(y/x) + f(x)$$

$$v_x = \frac{1}{1+(y/x)^2}\cdot\left(\frac{-y}{x^2}\right) + f'(x) = \frac{-y}{x^2+y^2} + f'(x) = -u_y.$$

$$\Rightarrow f'(x) = 0 \Rightarrow f(x) = c.$$

$$\Rightarrow v = \tan^{-1}(y/x) + c //.$$

3) If $u+v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$

Find $f(z)$ interms of z.

Soln. $f(z) = u+iv$, $if(z) = iu - v$.

$$(1+i)f(z) = (u-v) + i(u+v) = U + iV. = F(z).$$

$$V = u+v = (x-y)(x^2+4xy+y^2).$$

$$V_x = 3x^2+6xy-3y^2, \quad V_y = 3x^2-6xy-3y^2.$$

$$F(z) = \int(3z^2 + i3z^2)dz = 3(1+i)\int z^2 dz$$

$$(1+i)f(z) = \cancel{3}(1+i)\frac{z^3}{\cancel{3}} + c$$

$$\therefore f(z) = z^3 + k //.$$

(4) Find the analytic form of $f(z) = u + iv$ if:
$-2V = e^x (\cos y - \sin y)$.

Soln: $f(z) = u + iv$, $\quad i f(z) = ui - v$.

$2i f(z) = 2iu - 2V$.

$= U + iV = F(z)$.

$U = -2V = e^x (\cos y - \sin y)$.

$U_x = e^x (\cos y - \sin y)$, $U_y = e^x(-\sin y - \cos y)$

$F(z) = \int e^z - i(-e^z)\, dz = \int (1+i) e^z dz$.

$2i f(z) = (1+i) e^z + c$.

$f(z) = \frac{1}{2}(1-i) e^z + c$

— x —

If $f(z)$ is an analytic function of z,

P.T $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Proof:- Let $f(z) = u + iv$. , $|f'(z)|^2 = u_x^2 + v_x^2$.

Since $f(z)$ is an analytic function.

$u_x = v_y$ and $u_y = -v_x$ (CR Eqns.)

$u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$ (Laplace Eqn.)

Consider,

$$\frac{\partial}{\partial x}(u^2) = 2u u_x.$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2(u u_{xx} + u_x^2).$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2(u u_{yy} + u_y^2)$.

$$\frac{\partial^2}{\partial x^2}(v^2) = 2(v v_{xx} + v_x^2)$$

$$\frac{\partial^2}{\partial y^2}(v^2) = 2(v v_{yy} + v_y^2).$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v^2.$$

$$= 2\left(u(u_{xx} + u_{yy}) + u_x^2 + u_y^2\right) + 2\left(v(v_{xx} + v_{yy}) + v_x^2 + v_y^2\right)$$

$$= 2\left(u_x^2 + (-v_x)^2 + v_x^2 + (u_x^2) \right)$$

$$= 2\left(2u_x^2 + 2v_x^2 \right)$$

$$= 4\left(u_x^2 + v_x^2 \right)$$

$$= 4\, |f'(z)|^2 \;//$$

1. If $f(z) = u + iv$ is an analytic fun.. of z, P.T

(i) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2.$$

(ii) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f(z)| = 0. \text{ (or) } \nabla^2 \log |f(z)| = 0.$$

Proof:-

(i)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = 4 \frac{\partial^2}{\partial z \partial \bar{z}} |f(z)|^p.$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} f(z)^{p/2} f(\bar{z})^{p/2}.$$

$|z|^2 = z\bar{z}$

$|z| = z^{1/2} \bar{z}^{1/2}.$

$$= 4 \frac{\partial}{\partial z} f(z)^{p/2}. \frac{\partial}{\partial \bar{z}} f(\bar{z})^{p/2}.$$

$$= 4 \frac{p}{2} f(z)^{\frac{p}{2}-1} f'(z). \frac{p}{2} f(\bar{z})^{\frac{p}{2}-1}. f'(\bar{z}).$$

$$= \cancel{4} \frac{p^2}{\cancel{4}} |f(z)|^{p-2} |f'(z)|^2.$$

$$= p^2 |f(z)|^{p-2} |f'(z)|^2.$$

(ii) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f(z)| = 4 \frac{\partial^2}{\partial z \partial \bar{z}} \log \left(f(z)^{1/2} f(\bar{z})^{1/2}\right)$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left[\log f(z)^{1/2} + \log f(\bar{z})^{1/2}\right.$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \frac{1}{2}\left[\log f(z) + \log f(\bar{z})\right]$$

$$= 2 \frac{\partial}{\partial z}\left[\frac{1}{f(\bar{z})} f'(\bar{z})\right]$$

$$= 0.$$

Construction of an Analytic function

Milne Thomson Method.

u given.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

$$= \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0).$$

$$\therefore f(z) = \int\left[\frac{\partial}{\partial x}u(z,0) - i\frac{\partial}{\partial y}u(z,0)\right]dz + c.$$

v given.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}.$$

$$= \frac{\partial v}{\partial y}(z,0) + i\frac{\partial}{\partial x}v(z,0).$$

$$\therefore f(z) = \int\left[\frac{\partial}{\partial y}v(z,0) + i\frac{\partial}{\partial x}v(z,0)\right]dz + c.$$

—— x ——

(23)

Result: $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=4\frac{\partial^2}{\partial z\partial \bar{z}}$.

W.K.T, $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$

$$\frac{\partial x}{\partial z}=\frac{1}{2},\quad \frac{\partial x}{\partial \bar{z}}=\frac{1}{2},\quad \frac{\partial y}{\partial z}=\frac{1}{2i},\quad \frac{\partial y}{\partial \bar{z}}=\frac{-1}{2i}$$

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right).$$

$$\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

$$\frac{\partial^2}{\partial z\partial \bar{z}}=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

---

(1). If $f(z)$ is an analytic function of z, Prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2.$$

Solution:-

W.K.T $|z|^2=z\bar{z} \Rightarrow |f(z)|^2=f(z)f(\bar{z})$.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\cdot\frac{\partial^2}{\partial z\partial \bar{z}}|f(z)|^2=4\frac{\partial^2}{\partial z\partial \bar{z}}f(z)f(\bar{z})$$

$$=4\cdot f'(z)\cdot f'(\bar{z})=4|f'(z)|^2.$$

Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Solution :-

$$u + iv = (x + ay) + i(bx + cy).$$

$\Rightarrow$ $u = x + ay$, $v = bx + cy$.

$u_x = 1$ $\quad$ $u_y = a$ $\quad$ $v_x = b$ $\quad$ $v_y = c$.

By C.R eqn.

$u_x = v_y$. $\qquad$ $u_y = -v_x$.

$\boxed{c = 1}$ $\qquad$ $\boxed{a = -b}$.

— x —

Conformal mapping.

Definition: (1) The function $w=f(z)$ is called the mapping or transformation function.

A transformation under which angles between every pair of curves through a point are preserved in both magnitude and sense (direction) is said to be conformal at that point.

Definition: (2) A transformation under which angles between every pair of curves through a point are preserved in magnitude, but not in sense is said to be isogonal at that point.

Types of Transformation :-

(1) Magnification :-

The transformation $w=az$ where 'a' is a real constant, represents magnification.

(2) Magnification and Rotation :-

The transformation $w=az$ where 'a' is a complex constant, represents both magnificatio and rotation.

(3) Inversion and Reflection :-

The Transformation $w=1/z$ represents inversion with respect to the unit circle $|z|=1$ followed by reflection in the real axis.

(4) Translation : The transformation $w=z+c$, where c is a complex constant, represents a translatio Here any fig. shifted by a dist. $|c|=\sqrt{a^2+b^2}$ in the direction of C. Also the corresp. regions in the z, w planes will have the same shape, size & orientation

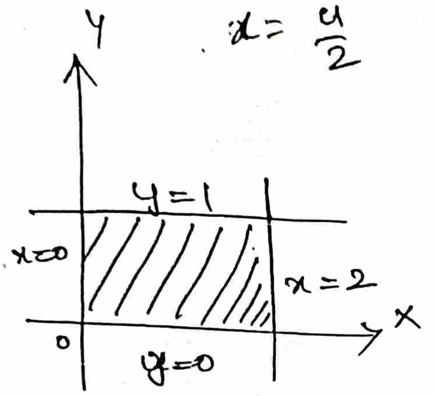
Problems:-

(1) Find the image of the rectangular region in the z-plane bounded by the lines $x=0$, $y=0$, $x=2$, and $y=1$ under the transformation $w=2z$.

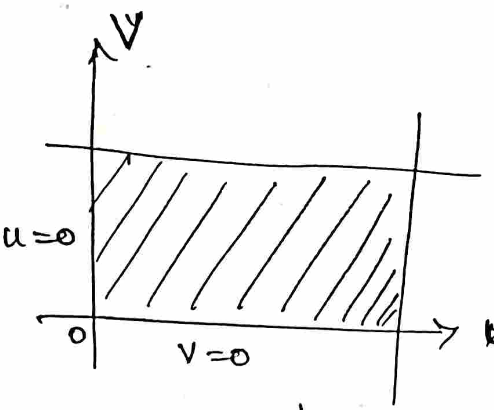
Solution:- Given $w=2z = 2(x+iy) = 2x+i2y$.

$\Rightarrow u=2x$, $V=2y$. $x=\frac{u}{2}$

when	
$x=0$	$u=0$
$x=2$	$u=4$
$y=0$	$V=0$
$y=1$	$V=2$.



z-plane



w-plane.

$\therefore$ the transformation in the z-plane is mapped into w-plane in magnification. (magnified twice.)

---

(2). Show that the transformation $w=\sin z$ transform the semi-infinite strip $0 \le x \le \pi/2$, $y \ge 0$ onto the upper w-plane.

Solution:- Given $w=\sin z = \sin(x+iy)$.

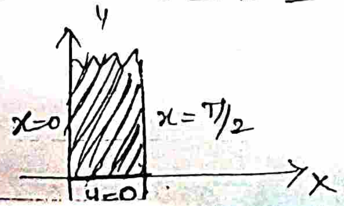
$= \sin x \cosh y + i\cos x \sinh y$.

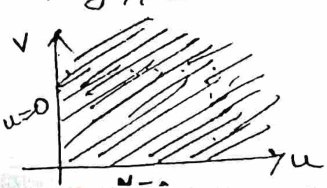
$\Rightarrow u=\sin x \cosh y$, $V=\cos x \sinh y$.

when $x=0$, $y \ge 0 \rightarrow u=0$, $V \ge 0$.

$x=\pi/2$, $y \ge 0 \Rightarrow u \ge 0$, $V=0$.

$\therefore$ when $0 \le x \le \pi/2$, $y \ge 0$ then $u \ge 0$, $V \ge 0$.





(3) Find the image of the rectangular region in the z-plane bounded by the lines $x=0$, $y=0$, $x=2$ and $y=1$ under the transformation $w=(1+2i)z+(1+i)$

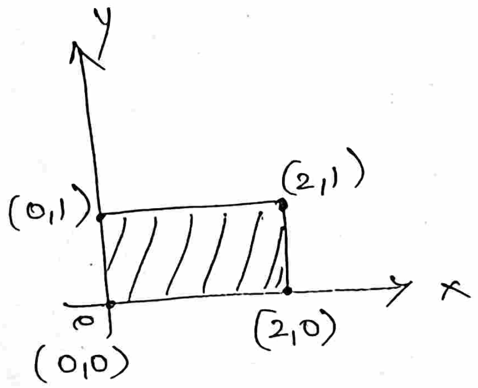
Solution:

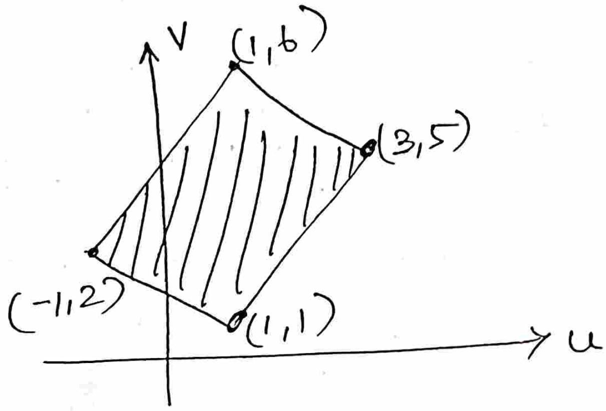
$$w = (1+2i)z+(1+i)$$
$$= (1+2i)(x+iy)+(1+i).$$
$$= (x-2y+1)+i(y+2x+1).$$

$\Rightarrow u = x-2y+1$, $v = y+2x+1$.

∴ the transformation is given by,

(x,y)	(u,v)
$(0,0)$	$(1,1)$
$(0,1)$	$(-1,2)$
$(2,0)$	$(3,5)$
$(2,1)$	$(1,6)$





(4) Find the images of the infinite strips (i) $1/4 < y < 1/2$ (ii) $0 < y < 1/2$ under the transformation $w = 1/z$.

solution:- $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} = \frac{1}{u+iv}$.

$$\therefore z = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} \quad \text{(taking conjugate)}.$$

$$= \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} + i\,\frac{(-v)}{u^2+v^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}.$$

(i) $1/4 < y < 1/2$.

when $y > 1/4 \Rightarrow \frac{-v}{u^2+v^2} > 1/4$

$\Rightarrow -4v > u^2+v^2$ (cross multipl.)

$\Rightarrow u^2+v^2+4v < 0$

$\Rightarrow u^2+\underbrace{v^2+4v+4}-4 < 0$

$\Rightarrow u^2+(v+2)^2 < 4.$

$\Rightarrow u^2+(v+2)^2 < 2^2.$

represents a circle in w-plane with centre $(0,-2)$ and Radius 2. (Interior portion).

when $y < 1/2$, $\Rightarrow \frac{-v}{u^2+v^2} < 1/2$.

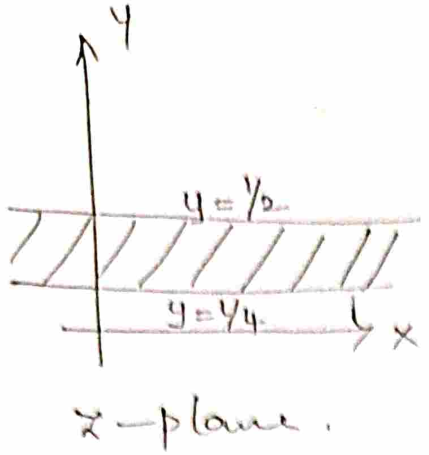
$\Rightarrow -2v < u^2+v^2$.

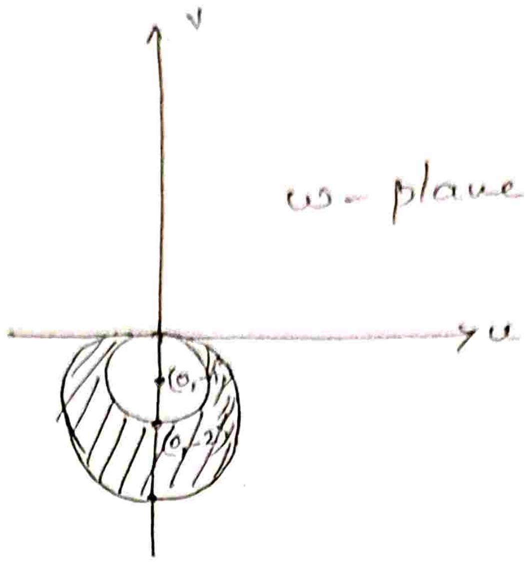
$\Rightarrow u^2+v^2+2v > 0$

$\Rightarrow u^2+v^2+2v+1 > +1$.

$\Rightarrow u^2+(v+1)^2 > 1^2$.

represents a circle with centre $(0,-1)$, $R=1$ (Exterior portion.).

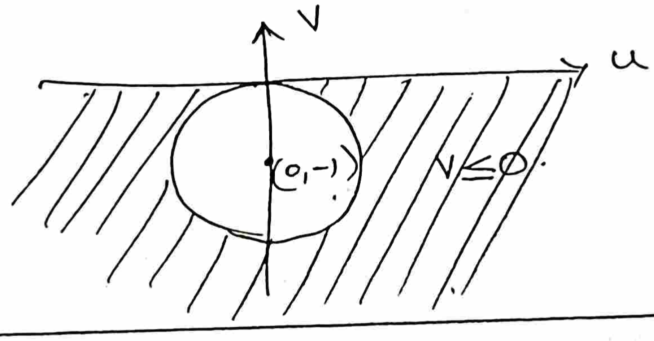




(ii) $0 < y < 1/2$.

From (i), $y < 1/2 \Rightarrow u^2 + (v+1)^2 > 1^2$.

when $y > 0$, $\frac{-v}{u^2+v^2} > 0 \Rightarrow -v > 0 \Rightarrow v \leq 0$.



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(5) Find the image of $|z-2i| = 2$ under the transformation. $w = 1/z$.

$z = \frac{1}{w} = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$

Solution:- Since $w = 1/z$, we know that

$x+iy = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$

$w = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} =$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

And $|z-2i| = 2 \Rightarrow |x + i(y-2)| = 2$

$\Rightarrow x^2 + (y-2)^2 = 2^2. \longrightarrow (1)$

it is a circle with Centre $(0,+2)$ & $R = 2$ in z-plane.

$$(1) \Rightarrow x^2+y^2-4y+4 = 4.$$

$$\Rightarrow x^2+y^2-4y = 0.$$

$$\Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0.$$

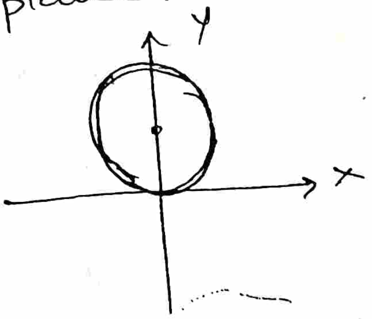
$$\Rightarrow (u^2+v^2) + 4v(u^2+v^2) = 0.$$

$$\Rightarrow (u^2+v^2)(1+4v) = 0$$

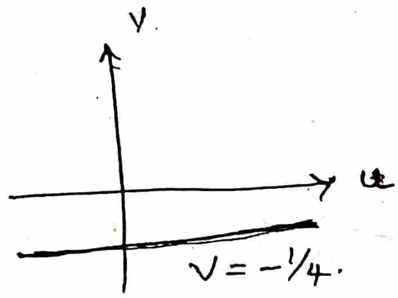
$$\Rightarrow 1+4v = 0.$$

$$\Rightarrow v = -1/4.$$ is a straight line in

ω-plane.



z-plane



ω-plane.

∴ circle is mapped onto a straight line.

---

(6) Find the image of the circle $|z|=2$ under $w=\sqrt{2}e^{i\pi/4}z$.

Solution: $\omega = \sqrt{2}\, e^{i\pi/4} z = \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)(x+iy)$

$$\omega = (1+i)(x+iy) = (x-y)+i(x+y)$$

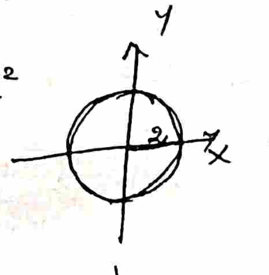
$$\Rightarrow u = x-y, \quad v = x+y \Rightarrow x = \frac{u+v}{2}, \; y = \frac{v-u}{2}.$$

Given $|z|=2 \Rightarrow x^2+y^2 = 2^2$

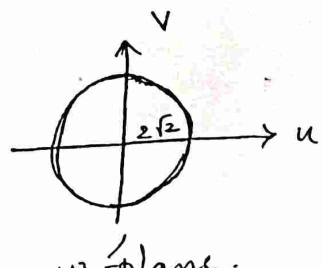
$$\Rightarrow \left(\frac{u+v}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 = 2^2$$

$$\Rightarrow u^2+v^2 = 8.$$

$$\Rightarrow u^2+v^2 = (2\sqrt{2})^2.$$



z-plane



w plane.

Bilinear Transformation.

The transformation $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants such that $ad - bc \neq 0$ is called a bilinear transformation. It is also called linear fractional or Mobius transformation.

Remark: If $ad - bc = 0$, every point of the z-plane becomes a critical point of the bilinear transformation.

Invariant point or Fixed point:

The image of a point z under a transformation $w = f(z)$ is itself, then the point is called fixed point or an invariant point of the transformation.

Cross-ratio of four points:-

If z_1, z_2, z_3 and z_4 are four points in the z-plane, then $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ is called the cross ratio of these points.

Bilinear Transformation.

Cross-Ratio Property of a B.T

The Cross-ratio of four points is invariant under a bilinear Transformation.

(ie) if w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively under a bilinear transformation, then

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Note:- (1) A mapping $w=f(z)$ is said to be conformal at $z=z_0$, if $f'(z_0) \neq 0$.

(2) A point z_0 at which the mapping $w=f(z)$ is not conformal (ie) if $\frac{dw}{dz}=0$ is called a critical point of the mapping.

Ex: $w=z^2$, $\frac{dw}{dz}=0$

$2z=0 \Rightarrow z=0$.

1. Find the B.T which maps the points $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ onto $w_1 = 0$, $w_2 = i$, $w_3 = \infty$ respectively.

<u>Solution:</u>

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

Assume , , $z_4 = z$, $w_4 = w$.

$$\frac{(w_1 - w_2)\left(1 - \frac{w_4}{w_3}\right)}{(w_1 - w_4)\left(1 - \frac{w_2}{w_3}\right)} = \frac{\left(1 - \frac{z_2}{z_1}\right)(z_3 - z_4)}{\left(1 - \frac{z_4}{z_1}\right)(z_3 - z_2)}$$

$$\frac{(0 - i)(1 - 0)}{(0 - w)(1 - 0)} = \frac{(1 - 0)(0 - z)}{(1 - 0)(0 - i)}$$

$$\frac{-i}{-w} = \frac{-z}{-i}$$

$w = f(z)$

$$\frac{i}{w} = \frac{z}{i} \Rightarrow \frac{i^2}{z} = w.$$

$$\Rightarrow \boxed{w = -\frac{1}{z}}$$

2. Find the B.T which maps the points $z=(1, i, -1)$ onto $w=(i, 0, -i)$ and hence find the image of $|z|<1$.

Solution: Let us consider

$z_1 = z, \quad z_2 = 1, \quad z_3 = i, \quad z_4 = -1$

$w_1 = w, \quad w_2 = i, \quad w_3 = 0, \quad w_4 = -i$

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}.$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)}{-(w+i)} = \frac{zi+z-i-1}{zi-z+i-1}.$$

$$(w-i)(zi-z+i-1) = -(w+i)(zi+z-i-1)$$

$$w(zi-z+i-1) - i(zi-z+i-1) = -w(zi+z-i-1) - i(zi+z-i-1).$$

$$w\left[zi - \cancel{z} + \cancel{i} - 1 + zi + \cancel{z} - \cancel{i} - 1\right] = i\left[-\cancel{zi} - z + i + \cancel{1} + \cancel{zi} - z + i - \cancel{1}\right]$$

$$w(2zi - 2) = i\left[-2z + 2i\right]$$

$$w = \frac{i(-z+i)}{(zi-1)} = \frac{(-zi-1)}{(zi-1)}$$

$$w = \frac{-(zi+1)}{(zi-1)} = \frac{(1+zi)}{(1-zi)}$$

$$\boxed{\therefore w = \frac{1+zi}{1-zi}} \longrightarrow (1)$$

To find the image of $|z| < 1$.

From (1), $(1-zi)w = (1+zi)$

$$w - ziw = 1 + zi$$

$$w - 1 = zi(1+w)$$

$$z = \frac{(w-1)}{i(1+w)} = \frac{-i(w-1)}{(1+w)} = \frac{i(1-w)}{(1+w)}$$

Given, $|z| < 1$.

$$\Rightarrow \left| \frac{i(1-w)}{(1+w)} \right| < 1.$$

$$\Rightarrow |i|\,|1-w| < |1+w|$$

$$\Rightarrow |1-(u+iv)| < |1+u+iv|$$

$$\Rightarrow |(1-u)-iv| < |(1+u)+iv|$$

$$\Rightarrow (1-u)^2+v^2 < (1+u)^2+v^2$$

$$\Rightarrow 1-u < 1+u$$

$$\Rightarrow 2u > 0$$

$$\Rightarrow u > 0$$

$x^2+y^2 < 1.$

$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} < 1.$

$\cancel{(u^2+v^2)} < (u^2+v^2)^2$

$0 <$.

Practice Problem.

3. Find the B.T which maps $z = (0, 1, \infty)$ onto $w = (i, -1, -i)$.

Ans: $$w = \frac{-(1+zi)}{z+i}$$

Eg. $w = z$, $\frac{d}{dz} = \ldots -1$, $2z = 0 \Rightarrow z = 0$.

Problem:-

1. Find the invariant points of the transformation

(i) $w = \dfrac{1}{z-2i}$, (ii) $w = \dfrac{z-2}{z+3}$.

(i) $z = \dfrac{1}{z-2i} \Rightarrow z^2 - 2iz - 1 = 0$

$$z = \frac{2i \pm \sqrt{-4+4}}{2} = i$$

(ii) $z = \dfrac{z-2}{z+3} \Rightarrow z^2 + 2z + 2 = 0$

$$z = -1 \pm i$$

2. Find the B.T which maps the points $z_1 = 1$, $z_2 = i$, $z_3 = -1$, into $w_1 = i$, $w_2 = 0$, $w_3 = -i$ and hence find the image $|z| < 1$.

Soln: Let $w_1 = w$, $w_2 = i$, $w_3 = 0$, $w_4 = -i$

$z_1 = z$, $z_2 = 1$, $z_3 = i$, $z_4 = -1$.

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \Rightarrow \frac{(w-i)}{-(w+i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}.$$

C/D rule $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a-b}{a+b} = \frac{c-d}{c+d}$.

$$\frac{(w-i)+(w+i)}{(w-i)-(w+i)} = \frac{(z-1)(i+1)-(z+1)(i-1)}{(z-1)(i+1)+(z+1)(i-1)}.$$

$$\frac{\cancel{2}w}{-\cancel{2}i} = \frac{z\cancel{i} - i + z - \cancel{1} - \cancel{zi} - i + z + \cancel{1}}{zi \cancel{-i} + \cancel{z} \cancel{-1} + zi + \cancel{i} - \cancel{z} - 1} = \frac{2z-2i}{2zi-2}$$

$$\Rightarrow \frac{w}{-i} = \frac{z-i}{zi-1} \rightarrow w = \frac{1+zi}{1-zi}$$

Conformal mapping

Definition:(1) The function $w = f(z)$ is called the mapping or transformation function.

A transformation under which angles between every pair of curves through a point are preserved in both magnitude and sense (direction) is said to be conformal at that point.

Definition: (2) A transformation under which angles between every pair of curves through a point are preserved in magnitude, but not in sense is said to be isogonal at that point.

Types of Transformation :-

(1) Magnification :-

The transformation $w = az$ where 'a' is a real constant, represents magnification.

(2) Magnification and Rotation :-

The transformation $w = az$ where 'a' is a complex constant, represents both magnification and rotation.

(3) Inversion and Reflection :-

The Transformation $w = 1/z$ represents inversion with respect to the unit circle $|z| = 1$ followed by reflection in the real axis.

(4) Translation :-

The transformation $w = z + c$, where c is a complex constant, represents a translation. Here any fig. shifted by a dist. $|c| = \sqrt{a^2+b^2}$ in direction a c. Also the corresp. regions in the z, w

To find the image,

$$(1-zi)\,\omega = 1+zi \Rightarrow \omega - zi\omega = 1+zi$$

$$\omega - 1 = zi + zi\omega$$

$$\omega - 1 = zi(1+\omega)$$

$$z = \frac{\omega-1}{i(1+\omega)} = \frac{i(1-\omega)}{(\omega+1)}$$

Given. $|z| < 1 \Rightarrow x^2+y^2 < 1.$ $\{|i| = \sqrt{0^2+1^2} = 1.\}$

$$\Rightarrow \left|\frac{1-\omega}{1+\omega}\right| < 1$$

$$\Rightarrow |1-(u+iv)| < |1+(u+iv)| \qquad \{\because \omega = u+iv\}$$

$$\Rightarrow |(1-u) - iv| < |(1+u)+iv|$$

$$\Rightarrow (1-u)^2 + v^2 < (1+u)^2 + v^2$$

$$\Rightarrow 1-u < 1+u$$

$$\Rightarrow 2u > 0$$

$$\Rightarrow u > 0 \,//.$$

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3. Find the B.T which maps the pts. $z_1 = \infty,\ z_2 = i,\ z_3 = 0$ onto $\omega_1 = 0,\ \omega_2 = i,\ \omega_3 = \infty$ respectively.

Soln:
$$\frac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_1-\omega_4)(\omega_3-\omega_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}.$$

$(w, 0, i, \infty)$ $\quad$ $(z, \infty, i, 0)$.

$$\frac{(w_1 - w_2)\left(\frac{w_3}{w_4} - 1\right)}{\left(\frac{w_1}{w_4} - 1\right)(w_3 - w_2)} = \frac{\left(\frac{z_1}{z_2} - 1\right)(z_3 - z_4)}{(z_1 - z_4)\left(\frac{z_3}{z_2} - 1\right)}.$$

$$\frac{(w-0)(-1)}{(-1)(i-0)} = \frac{(-1)(i-0)}{(z-0)(-1)}$$

$$\frac{w}{i} = \frac{i}{z} \Rightarrow w = -\frac{1}{z} \quad //$$

(4) $z = 0, 1, \infty$ onto $w = i, -1, -i$.

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

$$\frac{(w-i)(-1+i)}{(w+i)(-1-i)} = \frac{(z-0)\left(\frac{z_3}{z_4} - 1\right)}{\left(\frac{z_1}{z_4} - 1\right)(1-0)}.$$

$$\frac{(w-i)(-1+i)}{(w+i)(-1-i)} = \frac{-z}{-1} = z.$$

$$\frac{(w-i)(1-i)}{(w+i)(1+i)} = z. \Rightarrow \frac{(w-i)(1-i)}{(w+i)(1+i)} \times \frac{(1-i)}{(1-i)} = z.$$

$$\cancel{\frac{w - iw - i - 1}{w + wi + i - 1} = z.} \Rightarrow \frac{(w-i)(-2i)}{(w+i)\,2} = z$$

$$w = \frac{-(1+zi)}{z+i}$$

$$\frac{\cancel{w}-iw-i-\cancel{1}-\cancel{w}+wi-\cancel{i}+\cancel{1}}{w-i\cancel{w}-\cancel{i}-1+w+\cancel{wi}+\cancel{i}-1}=\frac{z-1}{z+1}$$

$$\frac{-\cancel{2}i(w+1)}{\cancel{2}(w-1)}=\frac{z-1}{z+1}$$

$$(w+1)(z+1)=i(w-1)(z-1).$$

$$w(z+1)+(z+1)=i\,w(z-1)-i(z-1).$$

$$w[(z+1)-i(z-1)]=-i(z-1)-(z+1)$$

$$w(z(1-i)+(1+i))=-zi-z+i-1.$$

$$w=\frac{-(1-i)\;(i-1)-z(1+i)}{(i+1)+z(1-i)}\times\frac{(1+i)}{(1+i)}=\frac{-\cancel{2}-z\cancel{2}i}{\cancel{2}i+\cancel{2}z}$$

$$=\frac{-1-zi}{z+i}$$

$$=\frac{-(zi+1)}{(z+i)}$$

$$\frac{(w-i)}{w+i}=\frac{z}{-i}.$$

c-d rule

$$\frac{\cancel{w}-i-\cancel{w}-i}{w-\cancel{i}+w+\cancel{i}}=\frac{z+i}{z-i}.$$

$$\frac{-\cancel{2}i}{\cancel{2}w}=\frac{z+i}{z-i}.$$

$$w=\frac{-i(z-i)}{z+i}=\frac{-(zi+1)}{z+i}$$

$\longrightarrow\alpha$ ——

5) $(z, 0, 1, \infty)$ onto $(w, i, 1, -i)$

Ans:- $w = \dfrac{z+i}{zi+1}$

85, 98, 101, 104, 117, 118, 124, 145, 153, 156.

$$\frac{w-i}{w+i} = \frac{z}{-i}$$

$$\frac{(w-i)+(w+i)}{(w-i)-(w+i)} = \frac{(z-i)}{z+i}$$

$$\frac{2w}{-2i} = \frac{z-i}{z+i}$$

$$w = -i\frac{(z-i)}{z+i}$$

$$w = \frac{-(zi+1)}{(z+i)}$$

To find the image,

$(1-zi)\omega = 1+zi \Rightarrow \omega - zi\omega = 1+zi$

$\omega - 1 = zi + zi\omega$

$\omega - 1 = zi(1+\omega)$

$z = \dfrac{\omega-1}{i(1+\omega)} = \dfrac{i(1-\omega)}{(\omega+1)}$

Given. $|z| < 1 \Rightarrow x^2+y^2 < 1.$

$\left\{ |i| = \sqrt{0^2+1^2} = 1. \right\}$

$\Rightarrow \left| \dfrac{1-\omega}{1+\omega} \right| < 1$

$\Rightarrow |1-(u+iv)| < |1+(u+iv)|$ $\{\because \omega = u+iv\}$

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Soln: $\dfrac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_1-\omega_4)(\omega_3-\omega_2)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}.$

$(w, 0, i, \infty) \quad (z, \infty, i, 0).$

$$\frac{(w_1 - w_2)\left(\frac{w_3}{w_4} - 1\right)}{\left(\frac{w_1}{w_4} - 1\right)(w_3 - w_2)} = \frac{\left(\frac{z_1}{z_2} - 1\right)(z_3 - z_4)}{(z_1 - z_4)\left(\frac{z_3}{z_2} - 1\right)}.$$

$$\frac{(w-0)(-1)}{(-1)(i-0)} = \frac{(-1)(i-0)}{(z-0)(-1)}$$

$$\frac{w}{i} = \frac{i}{z} \Rightarrow w = -1/z \,//.$$

— x —

(4) $z = 0, 1, \infty$ onto $w = i, -1, -i$.

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

$$\frac{(w - i)(-1 + i)}{(w + i)(-1 - i)} = \frac{(z - 0)\left(\frac{z_3}{z_4} - 1\right)}{\left(\frac{z_1}{z_4} - 1\right)(1 - 0)}.$$

$$\frac{(w - i)(-1 + i)}{(w + i)(-1 - i)} = \frac{-z}{-1} = z.$$

$$\frac{(w - i)(1 - i)}{(w + i)(1 + i)} = z. \Rightarrow \frac{(w - i)(1 - i)}{(w + i)(1 + i)} \times \frac{(1 - i)}{(1 - i)} = z.$$

$$\cancel{\frac{w - iw - i - 1}{w + wi + i - 1} = z.} \Rightarrow \frac{(w - i)(-\cancel{2}i)}{(w + i)\cancel{2}} = z$$

Cross-ratio property of a b.T :-

The cross-ratio of four points is invariant under a b.T.

(i) if w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 resp.. under a b.T, then

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}.$$

Proof:- Let the b.T be $w = \dfrac{az+b}{cz+d}$.

Then $$w_i - w_j = \frac{az_i+b}{cz_i+d} - \frac{az_j+b}{cz_j+d}.$$

$$= \frac{(ad-bc)(z_i-z_j)}{(cz_i+d)(cz_j+d)}$$

$$\therefore (w_1-w_2)(w_3-w_4) = \frac{(ad-bc)^2(z_1-z_2)(z_3-z_4)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)}.$$

and similarly,

$$(w_1-w_4)(w_3-w_2) = \frac{(ad-bc)^2(z_1-z_4)(z_3-z_2)}{(cz_1+d)(cz_4+d)(cz_3+d)(cz_2+d)}$$

$$\therefore \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} //.$$