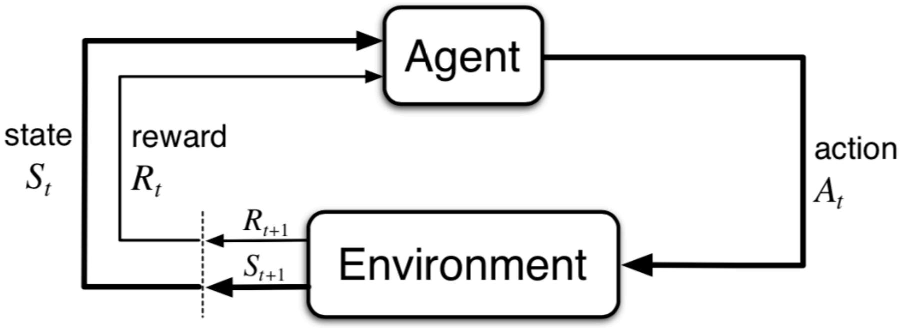
Chapter 3: Finite Markov Decision Process

From k-armed bandit problem: single state $\Rightarrow$ multiple state.

3.1 The Agent-Environment Interface

- **Agent**: The learner and decision maker
- **Environment**: The thing it interacts with, comprising everything outside the agent



The **trajectory**:

$$S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2, R_3, \ldots$$

Dynamics of MDP

Function p:

$$p(s', r \mid s, a) \doteq \Pr\{S_t = s', R_t = r \mid S_{t-1} = s, A_{t-1} = a\}$$

defines the *dynamics* of MDP. The conditional probability notation here is just a reminder of

$$\sum_{s' \in \mathcal{S}} \sum_{r \in \mathcal{R}} p(s', r \mid s, a) = 1, \text{for all } s \in \mathcal{S}, a \in \mathcal{A}(s)$$

This *dynamics* is not a constraint on the decision process but rather a constraint on states, which means the states must contains all informations of past states and actions. This property is called a *Markov property*.

The four-argument dynamics function p has some other useful forms.

- *state-transition probabilities*:
$$p(s' \mid s, a) = \Pr\{S_t = s' \mid S_{t-1} = s, A_{t-1} = a\} = \sum_{r \in \mathcal{R}} p(s', r \mid s, a)$$
- expected reward:
$$r(s, a) = \mathbb{E}[R_t \mid S_{t-1} = s, A_{t-1} = a] = \sum_{r \in \mathcal{R}} r \sum_{s \in \mathcal{S}} p(s', r \mid s, a)$$

$$r(s, a, s') = \mathbb{E}[R_t \mid S_{t-1} = s, A_{t-1} = a, S_t = s] = \sum_{r \in \mathcal{R}} r \frac{p(s', r \mid s, a)}{p(s' \mid s, a)}$$

The agent-environment boundary

- The boundary represents the limit of the agent's *absolute control*, not of its knowledge.
- The boundary is determined once one has selected particular states, actions, and rewards, and thus has identified a specific decision making task of interest

3.2 Goals and Rewards

Reward hypothesis: That all of what we mean by goals and purposes can be well thought of as the maximization of the expected value of the cumulative sum of a received scalar signal (called reward).

The reward signal is your way of communicating to the robot *what* you want it to achieve, not *how* you want it achieved.

3.3 Returns and Episodes

- **Return**:

  $$G_t \doteq R_{t+1} + R_{t+2} + R_{t+3} + \cdots + R_T$$

  Where T is a final time step.

- **Episode**: the agent–environment interaction breaks naturally into subsequences
- **Episodic tasks**: Tasks with episodes (finite time step)
- **Continuous tasks**: Final stime step $T = \infty$, the return form above may easily result in inifity reward.
- **Discounted return**:

  $$\begin{aligned} G_t &\doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

3.5 Policies and Value Functions

- **Policy**:

  $$\pi(a|s) = \Pr[A_t = a \mid S_t = s]$$

- **State-value function for policy π**:

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \middle| S_t = s\right], \text{for all } s \in \mathcal{S}$$

- **State-value function for policy** π:

$$q_\pi(s,a) \doteq \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \middle| S_t = s, A_t = a\right]$$

- **Monte Carlo methods**: fix π, averaging over random samples of v_π or q_π.

When state space is too large, we can maintain v_π and q_π as parameterized functions (with fewer parameters than states).

Value functions also have recursive relationships, thus can be applied in dynamic programming.

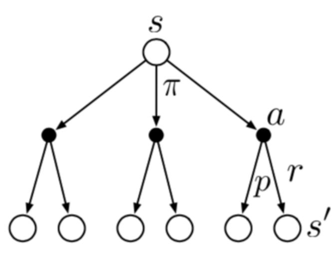
- **Bellman Equation**

$$\begin{aligned} v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] \\ &= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \mid S_t = t] \\ &= \sum_a \pi(a \mid s) \sum_{s'} \sum_r p(s', r \mid s, a)[r + \gamma \mathbb{E}_\pi[G_{t+1} \mid S_{t+1} = s']] \\ &= \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_\pi(s')] \end{aligned}$$

$$\begin{aligned} q_\pi(s,a) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] \\ &= \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma \sum_{a'} \pi(a' \mid s') q_\pi(s', a')\right] \end{aligned}$$

- **Backup (update) diagram**

  Backup operations: transfer value information back to a state (or a state–action pair) from its successor states (or state–action pairs)



Backup diagram for v_π

3.6 Optimal Policies and Optimal Value Functions

Frist define a *partial order* between policies π and π':

$$\pi \geq \pi' \text{ if and only if } v_\pi(s) \geq v_{\pi'}(s) \text{ for all } s \in \mathcal{S}$$

There is always *at least one* policy that is better than or equal to all other policies. They all are **optimal policy** π_*. They share the same state-value function, called the **optimal state-value function** v_*:

$$v_*(s) \doteq \max_\pi v_\pi(s), \text{ for all } s \in \mathcal{S}$$

and **optimal action-value function** q_*:

$$q_*(s,a) \doteq \max_\pi q_\pi(s,a), \text{for all } s \in \mathcal{S} \text{ and } a \in \mathcal{A}(s)$$

Bellman optimality equation:

$$\begin{aligned}
v_*(s) &= \max_{a \in \mathcal{A}(s)} q_*(s,a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_*(s')]
\end{aligned}$$

$$\begin{aligned}
q_*(s,a) &= \mathbb{E}[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \mid S_t = s, A_t = a] \\
&= \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma \max_{a'} q_*(s', a') \right]
\end{aligned}$$

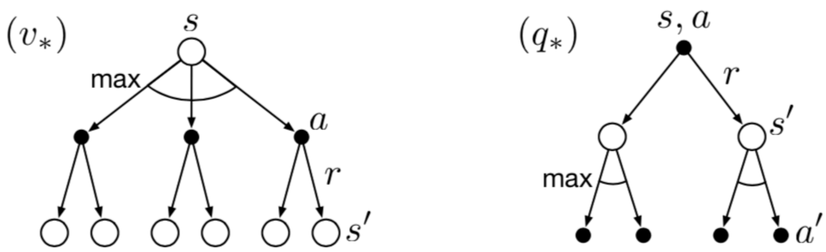


Figure 3.4: Backup diagrams for v_* and q_*

3.7 Optimality and Approximation

Solving optimality Bellman equation to solve RL problem is **impractical**, because

1. We have to accurately know environment dynamics ($p(s', r \mid s, a)$).
2. The computational cost & memory required is massive.
3. The true Markov property is rare.

So *approximations* are often necessary.

1. parameterized function representation
2. focues on frequently encountered states