# Research Statement

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## 1. Summary

My current research areas are mainly classified according to the following themes:

(1) Theory of differential invariants and the equivalence problem in PDE's.
(2) Classification of CR manifolds in low dimensions, and the study of homogeneous models.
(3) Foliations by holomorphic subvarieties in CR manifolds.
(4) Analytic methods in CR geometry involving the study of the $\overline{\partial}_b$-operator and the Szegö kernels.

During my formative years as a Ph.D. student at the Paris-Saclay University, and as a post-doctoral researcher at the Chinese Academy of Sciences in Beijing, I collaborated with my Ph.D. supervisor Professor Joël Merker and some of his students, leading to the following list of preprints and publications.

## References

**[a]** Chen, Zhangchi; **Foo, Wei Guo**; Merker, Joël; Ta The-Anh: *Normal Forms for Rigid $\mathfrak{C}_{2,1}$ Hypersurfaces $M^5 \subset \mathbb{C}^3$.* Taiwanese J. Math. 25(2): 333-364 (April, 2021). DOI: 10.11650/tjm/200903.

**[b]** Chen, Zhangchi; **Foo, Wei Guo**; Merker, Joël; Ta The-Anh: *Lie-Cartan Differential Invariants and Poincaré-Moser Normal Forms: Confluences.* Submitted.

**[c]** **Foo, Wei Guo**; Merker, Joël: *Differential $\{e\}$-structures for equivalences of 2-nondegenerate Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^2$.* ArXiv: 1901.02028. Submitted.

**[d]** **Foo, Wei Guo**; Merker, Joël: *Holomorphic immersion of bi-disks into 9 dimensional real hypersurface of Levi signature (2,2).* In International Mathematics Research Notices. https://doi.org/10.1093/imrn/rnaa266.

**[e]** **Foo, Wei Guo**; Merker, Joël; Ta, The-Anh: *Parametric CR-umbilical locus of ellipsoids in $\mathbb{C}^2$.* C. R. Math. Acad. Sci. Paris 356 (2018), no. 2, 214–221. https://doi.org/10.1016/j.crma.2017.11.019.

**[f]** **Foo, Wei Guo**; Merker, Joël; Ta, The-Anh: *Nonvanishing of Cartan CR curvature on boundaries of Grauert tubes around hyperbolic surfaces.* ArXiv: 1904.10203.

**[g]** **Foo, Wei Guo**; Merker, Joël; Ta, The-Anh: *Rigid equivalences of 5-dimensional 2-nondegenerate rigid real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1.* Arxiv preprint 1904.02562. Accepted by Michigan Mathematical Journal.

**[h]** **Foo, Wei Guo**; Merker, Joël; Nurowski, Paweł; Ta, The-Anh: *Homogeneous $\mathfrak{C}_{2,1}$ models.* ArXiv: 2104.09608. Submitted.

**[i]** **Foo, Wei Guo**; Merker, Joël; Ta, The-Anh: *On Convergent Poincaré-Moser Reduction for Levi Degenerate Embedded 5-Dimensional CR Manifolds.* ArXiv: 2003.01952. Submitted.

These articles and manuscripts constitute my academic contributions in a very active research area known as the Cartan's equivalence problem (henceforth the "equivalence problem"), which is of great historic and scientific value in mathematical sciences. It is for this reason that a bulk of this document is devoted to the history and the implication of the equivalence problem, starting with the classical results of Sophus Lie in section 3.2, followed by their generalisations by Élie Cartan in section 3.3. The Erlangen Programme is introduced in section 3.4 which provides a formal framework for my current and future research programmes. Recently, my collaborators and I are looking at the equivalence problem from a new perspective coming from dynamical systems and celestial mechanics, and this will be discussed in section 3.5, ending with its relations to the results of Olver et. al. on generating higher order differential invariants, potentially answering some of the questions in planar shapes recognition [1]. We see the applications of Lie and Cartan theories in the classification of second and third order ordinary differential equations (ODEs) in section 4; and later in the classification of CR manifolds in section 5 where I present some of my results on a certain class of Levi degenerate CR hypersurfaces [a, b, c, e, g, h, i]. In section 6, I apply the same theories to generalise a result of Robert Bryant (ex-chair of the American Mathematical Society) on holomorphic immersions of complex manifolds into CR hypersurfaces, leading to the publication [d]. Finally, the research statement is concluded with descriptions of several ongoing projects with Professor Chin-Yu Hsiao during my current stay at Academia Sinica in Taipei.

Singapore's interest in developing a digital economy has tremendously been intensified recently, and the move towards this goal is starting to gain traction and pace. Therefore in the foreword and in the Introduction 3.1, I have endeavoured to explain the relevance of equivalence problems in applied sciences and applied mathematics, such as computer vision and robotics to name a few. *It is my intention to develop my competencies in some of these areas using the computational techniques that I have acquired during numerous exchanges with my Ph.D. supervisor Joël Merker on diverse problems*, ranging from Algebraic Geometry, Differential Geometry to Coding (such as circulant matrices in one of his student's papers [21]). As a result, I managed to handle some of the huge computations arising in my works[2], *and my future research directions will include finding applications of these techniques to handle large geometric data in applied mathematics and sciences*. In order to achieve this, I am also starting to take courses in AI-Singapore [3] to get a sense of how applied geometry might be needed in Artificial Intelligence and Machine Learning.

## 2. Foreword

The profound impact of equivalence problems in mathematical sciences can hardly be overstated, ranging from differential geometry to computer vision [4]. It is for this reason that despite its long history, it remains a very active research area in scientific and mathematical community. For instance a working research group devoted to such a study "Symmetry, Curvature Reduction, EquivAlence Methods" (going by the acronym "SCREAM" [81])

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[1] Alfred M. Bruckstein, "Differential Invariants and their use in Recognizing Planar Shapes: `https://freddy.cs.technion.ac.il/wp-content/uploads/2018/02/Differential-Invariants.pdf`

[2] cf. `https://fooweiguo.github.io`, Computations section

[3] AI-Singapore `https://aisingapore.org`

[4] Alfred M. Bruckstein, "Geometric Invariants and Applications": `https://freddy.cs.technion.ac.il/wp-content/uploads/2018/02/GeometricInvariants.pdf`

was formed in 2019, gathering physicists, and pure and applied mathematicians from several countries across few continents (Europe, US, Australia, New Zealand) to hold conferences and engage in academic collaborations.

A huge part of this research is to find all homogeneous models given an equivalence problem, and there are several approaches. Classically they can be found using Lie theory, and Cartan's curvature reduction method; while recently, in several joint-works with Joël Merker (who incidentally is also an affiliate of the SCREAM group) and a few of his students, we are exploring a more efficient approach with normal forms. A difficult question is to realise abstract homogeneous spaces into concrete objects. This means either constructing explicit vector fields that generate a Lie algebra in Lie theory, or constructing an explicit system of 1-forms satisfying Cartan's structure equations in the curvature reduction method, and both methods are two sides of a same coin. Then integration techniques may subsequently be applied to them to obtain explicit defining functions whose symmetry is isomorphic to the given Lie algebra. Interested readers may refer to the 1932 article of Élie Cartan [12], or the works of Lie on the classification of second order ODE's [42].

A main contribution of Élie Cartan is the Cartan-Killing's classification of simple Lie groups [8–11]. An important example which is still under research today is the equivalence of $(2,3,5)$-distribution leading to the first exceptional Lie group $\mathsf{G}_2$ [13]. It is here we see an alternative explicit description of the Lie algebra. Few years ago, Bor-Montgomery [4] and Bor-Nurowski [5] showed that this simple Lie group can be described as the symmetry of equations describing a ball rolling on top of the other without slipping. This idea leads to a further question of mechanic realisation of simple or semi-simple Lie groups.

Indeed, over the past few years, Gil Bor and Paweł Nurowski are investigating the motions of planar robots [5]. These are objects made of edges and vertices, and to each of some of the edges attached a pair of wheels. The angles of the adjacent edges can be varied, and the goal is to find a mechanical configuration whose velocity distribution corresponds to a given Lie algebra as its symmetry. Future projects could include generalising this by computing invariants that describe the obstruction of the planar robots to being homogeneous models; as well as investigating how certain parameters such as the length of the edges, or the position of the wheels, may affect the symmetry of its movements. On the other hand, Paweł Nurowski and some of his collaborators Michael Eastwood [27, 28], C. Denson Hill [50], and Andrei Agarchev [1], studied the aerobatics and aerodynamics of the flying saucers, movement of cars, and the movement of three ants, as alternate physical realisations of the Lie algebra.

There is also another aspect of the equivalence problem related to the complexity of computations. It is well-known that computations in Cartan's method of moving frames can be extremely complicated. The invariants of Lie groups for even low dimensional manifolds can contain up to million terms, and hence computation by hand can lead to a lot of errors. It is therefore reasonable to find a way to automate the process using computer codings. One of the difficulties is that each time there is a group action on the torsions, a decision has to be made to normalise some of them. This is a decision problem for the programme to decide whether a certain choice (in this case normalisation) needs to be made. As far as equivalence problem is concerned, there are MAPLE and MATHEMATICA programmes coded by academics, such as the DIFFERENTIALGEOMETRY package created by Ian Anderson, Florin Catrina, Sydney Chamberlain, Cinnamon Hillyard, Jeff

[5]See `https://www.fuw.edu.pl/~nurowski/SCREAM_web_page.pdf`

Humphries, Jamie Jorgensen, Charles Miller, and Charles Torre, [6] to facilitate certain parts of calculations. However, to streamline various processes to solve the equivalence problem in a single strike still remains a faraway goal, and might involve machine learning to adapt computations to different geometric problems. Future projects might include creating similar packages for other computer algebra systems, such as SAGE, which could be useful for other programmers who are savvy in other programming languages.

## 3. Theory of Differential Invariants and the Equivalence Problem

3.1. **Introduction.** About 200 years after Leibniz's and Newton's discovery of the concept of derivative and the integral of a function, there was an interest in solving ordinary differential equations (ODE) that appear in physical sciences. There were *ad hoc* techniques to solve such equations, but a common theme involves finding a "good" change of variables that simplifies ODE into a form that can readily be solved. The description of the *equivalence problem* is as follows:

**Problem 3.1.** Let $\omega(x, y, y', \dots, y^{(n)}) = 0$ and $\theta(x, y, y', \dots, y^{(n)}) = 0$ be two n-th order ODE's. Determine if there is a change of variables $(x, y)$ that transforms one into the other.

This formulation can of course be generalised to class of PDE's and diverse manifolds. The main tools for my research consist of classical Lie theory, Élie Cartan's method of moving frames, and the theory of normal forms that is developed by Tresse and Poincaré. They will be described in more detail in the next few sections.

The foundation for solving the equivalence problem was laid down by Sophus Lie in his three monumental works *Theorie der Transformationsgruppen* I (in 1888), II (in 1890) and III (in 1893). He observed that such groups or pseudogroups of transformations that change variables to simplify ODE's, a delicate operation which is highly non-linear and are often difficult to understand, can equivalently be described by its linear counterpart consisting of infinitesimal symmetry of the equations. This profound and far-reaching discovery unifies and significantly extends the integration techniques, which explains its wide-ranging and everlasting impact on most areas of mathematics, including algebraic topology differential geometry, invariant theory, bifurcation theory, special functions, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics and so on [74].

Unfortunately Lie's theory for solving ODE has been forgotten after his death, and most textbooks on differential equations fail to mention his name. Anyone who is familiar with modern abstract theory of Lie groups might therefore be surprised to learn that its original source of inspiration comes from differential equations. My research will deal with equations whose symmetry groups are not elegant in the sense that they might not be semi-simple, nor solvable, nor belong to any special class of Lie groups that are fashionable in modern mathematics.

3.2. **S. Lie's solution to the Equivalence Problem, The Homogeneous Case and Klein Geometries.** The solution to the equivalence problem of S. Lie for Lie groups is summarised in his three main theorems and their converses [82]. Let $\mathsf{G}$ be an $r$-dimensional

[6] `https://www.maplesoft.com/support/help/Maple/view.aspx?path=DifferentialGeometry`

local parameter group which locally is diffeomorphic to $\mathbb{R}^r$, with the origin $0 \in \mathbb{R}^r$ functioning as the identity map $\mathsf{Id} \in \mathsf{G}$. The group has obvious action on itself by right multiplication, which is denoted by $\gamma$:

$$\mathbb{R}^r \ni x \longmapsto x \cdot b =: \gamma(x; b) \qquad \text{for } b \in \mathbb{R}^r.$$

Given any two points $x,\ y \in \mathsf{G}$ the element $b \in \mathbb{R}^r$ satisfying $x \cdot b = y$ can be solved as integration constants in the course of solving PDE's. For each fixed $b$, let

$$z^i = \gamma^i(x; b).$$

Introduce the new notation:

$$\frac{\partial z^i}{\partial x_j} = \frac{\partial \gamma^i}{\partial x_j}(x; b) =: \phi^i_j(x \cdot b; x).$$

When $x = 0$, the left transformation functions are:

$${}_L\psi^i_j(b) := \phi^i_j(b; e).$$

The matrix $({}_L\psi^i_j)_{(i,j)}$ is invertible, and let ${}_L\psi^{-1}{}^i_j$ be the $(i, j)$-th component of its inverse. A fundamental result shows that:

$$\frac{\partial z^i}{\partial x^j} = \sum_{k=1}^{r} {}_L\psi^i_k(z)\, {}_L\psi^{-1}{}^k_j(x) \qquad \text{for } 1 \leqslant i, j \leqslant r.$$

Directly from above, one achieves

$$\sum_{k=1}^{r} {}_L\psi^{-1}{}^i_k(z) \frac{\partial z^k}{\partial x^j} = {}_L\psi^{-1}{}^i_j(x). \tag{3.2}$$

Using

$$dz^k = \sum_{j=1}^{n} \frac{\partial z^k}{\partial x^j}\, dx^j,$$

multiplying both sides of equation (3.2) and taking summation over $j$:

$$\sum_{k=1}^{r} {}_L\psi^{-1}{}^i_k(z)\, dz^k = \sum_{j=1}^{r} {}_L\psi^{-1}{}^i_j(x)\, dx^j.$$

Since the $z^i$'s are the coordinates of the target point after action on $x$ via right multiplication by $b$, it follows that this 1-form enjoys an important property of being invariant under right multiplication. Let

$$\omega^i := \sum_{j=1}^{r} {}_L\psi^{-1}{}^i_j(x)\, dx^j. \tag{3.3}$$

**Définition 3.4.** The 1-forms in equation (3.3) are called the Maurer-Cartan forms for the local parameter group $\mathsf{G}$.

From the discussion above, one has

**Theorem 3.5** (S. Lie, The First Main Theorem)**.** *The Maurer-Cartan 1-forms $\omega^i$ are invariant with respect to the right multiplication $x \longmapsto z = x \cdot b$. Dually, the same conclusion holds for the following vector fields*

$$\mathsf{X}_j := \sum_{k=1}^{r} {}_L\psi^k_j(x) \frac{\partial}{\partial x^k}.$$

Since the matrix $({}_L\psi^i_j)_{(i,j)}$ is invertible, applying exterior derivative to $\omega^i$, and one has the following

**Theorem 3.6** (S. Lie, The Second Main Theorem)**.** *The Maurer-Cartan forms $\omega^1, \cdots, \omega^r$ satisfy the structure equations*

$$d\omega^i = \sum_{1 \leqslant j < k \leqslant r} c^i_{\ jk}\, \omega^j \wedge \omega^k \qquad \textit{for } i = 1, \cdots, r,$$

*where the $c^i_{\ jk}$ are the structure constants of the parameter group.*

Due to the anti-symmetry of the wedge product (dually the anti-symmetry of the Lie bracket), as well as the identity $d^2 \equiv 0$ (dually the Jacobi's identities), one has

**Theorem 3.7** (S. Lie, The Third Main Theorem)**.** *The structure constants $c^i_{\ jk}$ of a local parameter group satisfy*

- $c^i_{\ jk} = -c^i_{\ kj}$,
- $\sum_{s=1}^r (c^i_{\ js}c^s_{\ kl} + c^i_{\ ks}c^s_{\ lj} + c^i_{\ ls}c^s_{\ jk}) = 0.$

The three main theorems are necessary consequences of the action of local parameter group $G$, if ever such a group exists. The last piece of puzzle of the equivalence problem is due to the next two theorems:

**Theorem 3.8** (S. Lie, Converse to the Second Main Theorem)**.** *Let $\{\omega^1, \cdots, \omega^r\}$ be a local coframe on $\mathbb{R}^r$ satisfying the structure equations*

$$d\omega^i = \sum_{1 \leqslant j < k \leqslant r} C^i_{\ jk}\, \omega^j \wedge \omega^k \qquad \textit{for } i = 1, \cdots, r,$$

*with* ***constants*** *$C^i_{\ jk}$. Then the family of local diffeomorphisms leaving each of the $\omega^i$ invariant forms an $r$-parameter local Lie group.*

**Theorem 3.9** (S. Lie, Converse to the Third Main Theorem)**.** *let $\{c^i_{\ jk}\}^r_{i,j,k=1}$ be a collection of numbers satisfying*

- $c^i_{\ jk} = -c^i_{\ kj}$,
- $\sum_{s=1}^r (c^i_{\ js}c^s_{\ kl} + c^i_{\ ks}c^s_{\ lj} + c^i_{\ ls}c^s_{\ jk}) = 0.$

*Let $(\alpha^1, \cdots, \alpha^r)$ be a constant vector, and consider the initial value problem:*

$$\begin{aligned} &\frac{\partial f^i_j}{dt}(t;\alpha) = \delta^i_j + \sum_{p,q=1}^r c^i_{\ pq}\alpha^p f^q_j(t;\alpha), \\ &f^i_j(0;\alpha) = 0. \end{aligned} \tag{3.10}$$

*Then the functions $\chi^i_j(x) := f^i_j(1;x)$ have the following properties:*

$$\begin{aligned} &\sum_{j=1}^r \chi^i_j(x)x^j = x^j, \\ &\frac{\partial \chi^i_k}{\partial x^j} - \frac{\partial \chi^i_j}{\partial x^k} = \sum_{p,q=1}^r c^i_{\ pq}\chi^p_j(x)\chi^q_k(x), \\ &\chi^i_j(x=0) = \delta^i_j. \end{aligned} \tag{3.11}$$

*Therefore the structure equations for the 1-forms*

$$\omega^i := \sum_{j=1}^r \chi^i_j(x)\, dx^j$$

*are*

$$d\omega^i = \frac{1}{2}\sum_{j,k=1}^{r} c^i_{\ jk}\, \omega^i \wedge \omega^j \qquad \textit{for } i = 1, \cdots, r.$$

*Finally the converse of the second fundamental theorem shows that these one-forms define a local parameter group with the* $c^i_{\ jk}$ *as constants.*

The two converse theorems say that if two homogenous spaces $\mathsf{S}_1$, $\mathsf{S}_2$ (such as in some cases of PDE's or manifolds) produce two sets of Maurer-Cartan 1-forms with the same structure constants, then there is an invertible group transformation $\varphi : \mathsf{S}_1 \to \mathsf{S}_2$ sending one into the other. This answers a part of the equivalence problem.

When the spaces are not homogeneous, the most general way to proceed, which S. Lie has been insisting in many of his works and articles, is to compute differential invariants, or at least to understand them. In fact, the structure constants are just special cases of invariants of the parameter group $\mathsf{G}$ in question. Evidently he was upset when he learned that Felix Klein omitted this important point during the original formulation of the Erlangen Programme. In the foreword of Lie (cf. [57, page 10], [42, page 19]), he wrote:

*F. Klein, whom I kept abreast of all my ideas during these years, was occasioned to develop similar viewpoints for discontinuous groups. In his Erlangen Program, where he reports on his and on my ideas, he, in addition, talks about groups which, according to my terminology, are neither continuous or discontinuous. For example, he speaks of the group of all Cremona transformations and of the group of distortions. The fact that there is an essential difference between these types of groups and the groups which I have called continuous (given the fact that my continuous groups can be defined with the help of differential equations) is something that has apparently escaped him. Also, there is almost no mention of the important concept of a differential invariant in Klein's program. Klein shares no credit for this concept, upon which a general invariant theory can be built, and it was from me that he learned that each and every group defined by differential equations determines differential invariants which can be found through integration of complete systems.*

In many instances, explicitly calculating them can prove to be almost impossible tasks even for powerful machines. Nonetheless, this remains a very important and crucial step in the equivalence problem, and it is best to address this point directly as suggested by Lie, rather than to avoid it. This explains why the theory of differential invariants is included in my current research project.

3.3. **É. Cartan's solution to Equivalence Problem and the Cartan Geometries.** In this subsection we will describe how Lie's solution to equivalence problems can be generalised using Cartan's equivalence method. Consider a coframe $\{\omega^1, \cdots, \omega^m\}$ on an $m$-dimensional manifold $M$, and a $\mathsf{G}$-valued equivalence problem (or also known as the $\mathsf{G}$-structure) by looking at the lifted coframe:

$$\theta^i = \sum_{j=1}^{m} \mathsf{g}^i_j\, \omega^j.$$

Applying exterior differentiation to both sides:

$$d\theta^i = \sum_{j=1}^{m} \gamma^i_j \wedge \theta^j + \sum_{\substack{j,k=1\\ j<k}}^{m} T^i_{jk}(x, \mathsf{g})\, \theta^i \wedge \theta^j,$$

where the $T^i_{jk}(x,\mathsf{g})$ are called *torsions*, and the following differential 1-forms:

$$\gamma^i_j = \sum_{k=1}^{m} d\mathsf{g}^i_k (\mathsf{g}^{-1})^k_j,$$

are called the Maurer-Cartan forms with values in the Lie algebra $\mathfrak{g}$ of $\mathsf{G}$. If $\{\alpha^1,\dots,\alpha^r\}$ is a basis for the Maurer-Cartan forms, then the $\gamma^i_j$ can be written as linear combinations of the basis 1-forms

$$\gamma^i_j = \sum_{k=1}^{r} c^i_{jk}\alpha^k \qquad (i,j=1,\dots,m),$$

where $c^i_{jk}$ are structure constants of the Lie group. The Cartan's method of moving frames consists of three main steps: absorption, normalisation and prolongation. The main goal is to reduce the $\mathsf{G}$-structure to the identity by finding a suitable system of partial differential equations which $\alpha^i$ satisfy.

To this effect, introducing new variables $v^i_j$, we begin with the process of absorbing the torsions into $\alpha^i$ by modifying the Maurer-Cartan forms:

$$\pi^k := \alpha^k - \sum_{i=1}^{m} v^k_i \theta^i.$$

This gives a system of linear equations to be solved:

$$\sum_{l=1}^{r} c^i_{jl} v^l_k - c^i_{kl} v^l_j = -T^i_{jk}.$$

The resulting structure equations become

$$d\theta^i = \sum_{k=1}^{r}\sum_{j=1}^{m} c^i_{jk}\,\pi^k\wedge\theta^j + \sum_{\substack{j,k=1\\ j<k}}^{m} \gamma^i_{jk}(x,\mathsf{g})\,\theta^j\wedge\theta^k,$$

where $\gamma^i_{jk}$ are the unabsorbed torsions. Some of them may be normalised to $0$ or $1$ by choosing suitable values for $\mathsf{g}^i_j$, thus reducing the dimension of the $\mathsf{G}$-structure, and the process is restarted till it is reduced to the identity. When the system of absorption equations is underdetermined, prolongation of the system is necessary by introducing new variables and new equations in order for the system of 1-forms to be in involution.

We will now formalise these steps following the expositions of [75, page 304] and [44, Lecture 4]. Let $V$ be an $m$-dimensional vector space with basis $\{\mathsf{e}_i\}$, and let $\{\mathsf{f}^i\}$ be the dual basis of the dual vector space $V^*$. For a Lie group $\mathsf{G}$, let $T_e\mathsf{G} = \mathfrak{g}$ be its Lie algebra with basis $\{\varepsilon_1,\cdots,\varepsilon_r\}$, and let $\{\pi^1,\cdots,\pi^r\}$ be its dual basis. By the identification $\mathfrak{g}\subset \mathrm{Hom}(V,V)$, we write

$$\varepsilon_p = \sum c^j_{ip}\,\mathsf{e}_j\otimes\mathsf{f}^i.$$

Define the absorption operator

$$\begin{aligned} \boldsymbol{L}:\ \mathfrak{g}\otimes V^* &\longrightarrow V\otimes\wedge^2 V^* \\ \sum v^p_k\,\varepsilon_p\otimes\mathsf{f}^k &\longmapsto -\frac{1}{2}\sum\left(c^i_{jp}v^p_k - c^i_{kp}v^p_j\right)\mathsf{e}_i\otimes\mathsf{f}^j\wedge\mathsf{f}^k, \end{aligned} \tag{3.12}$$

and thus one has the following exact sequence of vector spaces:

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \mathfrak{g}\otimes V^* \xrightarrow{L} V\otimes\wedge^2 V^* \longrightarrow \Pi_{\mathfrak{g}} \longrightarrow 0, \tag{3.13}$$

where

$$\begin{aligned} \Pi_{\mathfrak{g}} &= V \otimes \wedge^2 V^* / \mathsf{Im}(\boldsymbol{L}), \\ \mathfrak{g}^{(1)} &= \mathsf{ker}(\boldsymbol{L}). \end{aligned} \tag{3.14}$$

The space $\mathfrak{g}^{(1)}$ is called the first prolongation of $\mathfrak{g}$, and as has been mentioned earlier, arises from the ambiguity of the absorption. Define the torsion map:

$$\begin{aligned} \mathsf{T}:\ M \times \mathsf{G} &\longrightarrow V \otimes \wedge^2 V^*, \\ (x, \mathsf{g}) &\longmapsto \sum T^i_{jk}(x, \mathsf{g})\ \mathsf{e}_i \otimes \mathsf{f}^j \wedge \mathsf{f}^k. \end{aligned} \tag{3.15}$$

The absorbed torsions $\tau_M$ can then be obtained by composing $\mathsf{T}$ with the projection map:

$$\tau_M :\ M \times \mathsf{G} \xrightarrow{\mathsf{T}} V \otimes \wedge^2 V^* \longrightarrow \Pi_{\mathfrak{g}},$$

giving the following set of well-defined structure equations:

$$d\omega^i = \sum c^i_{jp}\ \pi^p \wedge \omega^j + \frac{1}{2} \sum \gamma^i_{jk}(x, \mathsf{g})\ \omega^j \wedge \omega^k.$$

A natural question arises: *when do we have* $\Pi_{\mathfrak{g}} = 0$? This holds for several well-known situations such as Riemannian and Hermitian geometry, where respectively the Levi-Civita connection and the Hermitian connection are known to be torsion-free. However as we shall see, this does not hold even for equivalence problems of second order ODE [75, Page 395]. Moreover the Cartan connections in CR geometry is seldom torsionless, such as [12, 14, 22] to name a few.

Returning to the Cartan's method, the normalisation process comes from the fact that equation (3.13) is not only an exact sequence of vector spaces, but also an exact sequence of $\mathsf{G}$-*modules*. To study the group actions on the absorbed torsions $\tau_M$, a most general way is to use representation theory of Lie groups to decompose

$$\Pi_{\mathfrak{g}} = \bigoplus_{\alpha} W_\alpha$$

into irreducible representation of $\mathsf{G}$, and it remains to study $\tau_M$ in each of the simpler subspaces $W_\alpha$.

**Problem 3.16.** To apply results in representation theory to understand how these absorbed torsions can be normalised to solve complicated equivalence problems.

Irrespective of which point of view to look at the Cartan's equivalence method, the goal is to show that given two manifolds $M_1$, $M_2$ with the same set of invariants, they belong to the same equivalence class. Indeed, following the argument of É. Cartan (cf. [44, Lecture 6]), let $\{\omega^1, \cdots, \omega^n\}$ and $\{\eta^1, \cdots, \eta^n\}$ be respective coframes of $M_1$ and $M_2$, satisfying the Cartan's structure equations

$$\begin{aligned} d\omega^i &= \sum C^i_{jk}\ \omega^j \wedge \omega^k, \\ d\eta^i &= \sum C^i_{jk}\ \eta^j \wedge \eta^k, \end{aligned} \tag{3.17}$$

with $C^i_{jk}$ being invariants as opposed to constants, in contrast to Lie's solution. Our aim is to show that there is a group transformation $\varphi : M_1 \to M_2$ sending one onto the other. In fact let $M_1 \times M_2$ be the product space of these two manifolds, and let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be projection maps. Then the pull-back of these one-forms $\{\pi_1^* \omega^i, \pi_2^* \eta^j\}$ constitute a coframe on $M_1 \times M_2$, and equation (3.17) shows that the system

$$\Theta := \{\theta^i := \pi_1^* \omega^i - \pi_2^* \eta^i\}$$

is involutive. Frobenius theorem tells us that $M_1 \times M_2$ is foliated by $n$-dimensional leaves $\mathscr{L}$ over an $n$-dimensional leaf-space. Given any point $(x, y) \in M_1 \times M_2$, we let $\mathscr{L}_{(x,y)}$ be one of the leaves passing through it. There is an isomorphism of the tangent bundles

$$\pi_{1*} : T\mathscr{L}_{(x,y)} \longrightarrow TM_1,$$

and hence local inversion theorem implies that the projection map

$$\pi_1\big|_{\mathscr{L}_{(x,y)}} : \mathscr{L}_{(x,y)} \longrightarrow M_1$$

is a local diffeomorphism. The local map $\varphi : M_1 \to M_2$ sending $x$ to $y$ can then be found by factoring through $\mathscr{L}_{(x,y)}$ so that the following diagram commutes:

$$\begin{tikzcd} & \mathscr{L}_{(x,y)} \arrow[dr, "\pi_2"] & \\ M_1 \arrow[ur, "\pi_1^{-1}"] \arrow[rr, "\varphi"'] & & M_2 \end{tikzcd}$$

It remains to show that $M_1$ and $M_2$ belong to the same equivalence class in this equivalence problem setting by verifying that $\varphi$ is a group transformation, but this is immediate since it satisfies $\varphi^*\eta^i = \omega^i$ (cf. converse to Lie's second main theorem). This way Elie Cartan has solved the *local* equivalence problem in the most general setting. The *global* equivalence problem will depend on the topological nature of the leaves $\mathscr{L}$, and will be left for future considerations.

**Problem 3.18.** To understand the global foliation in the context of equivalence problem as mentioned above.

3.4. **The Erlangen Programme: Klein and Cartan geometries.** Following the discussion, we are now poised to reinterpret the equivalence problem in the form of the following well-known:

**Définition 3.19** (The Erlangen Programme, 1872 (cf. Guggenheimer [49])**.** Klein geometry is the theory of geometric invariants of transitive transformation group.

In modern formalism, the widely accepted definition of the Klein and Cartan geometry can be found in Sharpe [79, Chapters 4, 5]:

**Définition 3.20** (Klein Geometries)**.** Let $\mathsf{G}$ be a Lie group and $\mathsf{H} \subset \mathsf{G}$ be its closed Lie subgroup. A Klein geometry $(G, H)$ is given by

- the smooth manifold $M = G/H$,
- the principal $H$-bundle $H \subset G \to G/H$,
- the Maurer-Cartan form $\omega_G : TG \to \mathfrak{g}$ satisfying 1) $\omega_G$ is a linear isomorphism on each fibre, 2) $R_h^*\omega_G = \mathrm{Ad}(h^{-1})\omega_G$ for all $h \in H$, 3) $\omega_G(X^+) = X$ for all $X \in \mathfrak{h}$.

Then $d\omega_G + \frac{1}{2}[\omega_G \wedge \omega_G] = 0$ is the Cartan's structure equation for curvature.

There are several philosophical justifications for finding homogeneous spaces. Historically one of the well-known examples is Gauss' famous result *Theorema Egregium* which deals with the equivalence problem of determining whether two surfaces can be brought from one onto the other by isometry. He solved this problem by discovering the curvature formula that is named after him. His original approach was to use sphere as the model, by infinitesimally comparing the area of a small patch of a surface with the area of another small patch on the sphere via what is now known as the Gauss map. Centuries after Gauss' discovery, Cartan's method of moving frame [75, page 377] provides the most general algorithmic and algebraic approach of producing the homogeneous model $\mathsf{SO}(3)/\mathsf{SO}(2)$ which is isomorphic to $\mathbb{S}^2$. Cartan's method has an added advantage of being applicable to other geometric problems, especially in higher dimensional situations, where geometric intuitions fail.

**Problem 3.21.** Given any equivalence problem, can we find all possible suitable homogeneous models like what Gauss did for Theorema Egregium, so that we can use them to approximate geometric objects and obtain information about their curvatures?

**Définition 3.22** (Cartan Geometries)**.** The Cartan geometry $(P, \omega)$ on $M$ modelled on $(\mathfrak{g}, \mathfrak{h})$ with the Lie groups $H \subset G$ consists of

- the smooth manifold $M$,
- the principal $H$-bundle $H \subset G \to G/H$,
- the Maurer-Cartan form $\omega : TG \to \mathfrak{g}$ satisfying 1) $\omega$ is a linear isomorphism on each fibre, 2) $R_h^* \omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in H$, 3) $\omega(X^+) = X$ for all $X \in \mathfrak{h}$.

Now $M$ is no longer required to be homogeneous, but a perturbation of the homogeneous model. The 2-form

$$\Omega := d\omega + \tfrac{1}{2}[\omega \wedge \omega]$$

is the Cartan's structure equation for curvature.

However a pertinent chicken-and-egg question of epistemological nature arises: given an equivalence problem, whether one should start with homogeneous spaces before computing the invariants, or one should use Cartan's reduction method to find the invariants first which will then be used to find the homogeneous models. In the first scenario, there are examples such as the classification of second order ODE [84] or Levi non-degenerate 3-dimensional CR manifolds [12, 14], where we are fortunate to have the Bianchi-Lie's classification of 3-dimensional Lie algebras at our disposal. This allows us to proceed with solving equivalence problems very quickly and efficiently.

But classifications of finite dimensional Lie algebra is complete up to dimension 4, and any classifications of 5-dimensional Lie algebras and above are at best incomplete, and this highlights some of the limitations of starting with Klein geometries first in higher dimensional differential geometry. For instance the classification of $\mathfrak{C}_{2,1}$ homogeneous models is complete only in 2008 by Fels-Kaup [31], while the classification of strictly pseudoconvex real hypersurfaces in $\mathbb{C}^3$ is complete only in 2020 by [60] and [26], and they are highly non-trivial results. Therefore any attempt to solve the equivalence problem by starting with complete classification of 5-dimensional Lie algebras might have been hopeless.

After having expounded much on solving equivalence problems using Lie and Cartan theory, in my recent joint works with Joël Merker, Paweł Nurowski, Julien Heyd, Zhangchi Chen, and The-Anh Ta, we are exploring a new technique coming from dynamical systems and celestial mechanics to achieve the same goals.

The next sub-section will briefly discuss about this aspect.

3.5. **Future Projects: Normal Forms.** The theory of normal forms first appeared in the works of A. Tresse [84] on the classification of second order ODE, as well as in the works of H. Poincaré on certain problems in dynamical systems and celestial mechanics. Loosely speaking (cf. [43]), Poincaré considered a system of first order ODE's in $\mathbb{R}^n$:

$$\dot{x} = Ax + \sum_{k=1}^{+\infty} f_k(x),$$

where $x = (x_1, \cdots, x_n)^t$, $x_i$ is a function of $t$, and $A$ is an $n \times n$-matrix with constant coefficients. Sometimes for simplicity, $A$ can be assumed to be semi-simple so that it is diagonalisable, but for now this assumption will not be used. The functions $f_k$ are homogeneous in the sense that for each $a \in \mathbb{R}$, one has $f_k(ax) = a^k f_k(x)$.

The process of obtaining the normal form consists of making a change of variables

$$x = y + h_m(x),$$

where $h_m(x)$ is homogeneous of order $m$. Then by chain rule:

$$\dot{x} = (\mathsf{Id} + H)\dot{y},$$

with $H$ being a matrix of derivatives of $h_m$. Thus

$$\begin{aligned}
\dot{y} &= (\mathsf{Id} + H)^{-1}\left(Ay + Ah_m + \sum_{k=1}^{\infty} f_k(y + h_m)\right) \\
&= Ay + Ah_m + \sum_{k=1}^{\infty} f_k(y + h_m) - H \cdot \left(Ay + Ah_m + \sum_{k=1}^{\infty} f_k(y + h_m)\right) + O(H^2) \\
&:= Ay + \sum_{k=1}^{\infty} g_k(y),
\end{aligned} \tag{3.23}$$

Comparing the $m$-th order term,

$$g_m(y) = f_m(y) - (HAy - Ah_m) + \cdots,$$

where the rest are non-linear terms consisting of homogeneous functions of lower order $f_{k \leqslant m-1}$ which have already been normalised during the previous $m-1$ steps. We have therefore the *homological equation*:

$$g_k(y) = f_m(y) - \mathscr{L}_0(h_m), \qquad \mathscr{L}_0(h_m) := HAy - Ah_m.$$

Thus the form

$$\dot{y} = Ay + \sum_{k=1}^{\infty} g_k(y)$$

is called *normal* if $g_k(y)$ lies in the complement space of the image of $\mathscr{L}_0$.

The same method could be applied not just to system of ODE's, but to manifolds such as in the works of Tresse [84], Chern-Moser [22], Moser-Webster [73], Fels-Olver-Pohjanpelto-Valiquette [32, 33, 76–78] etc. The idea is that given an equivalence problem, there is a series of Lie group actions on $\mathbb{R}^{n+1}$ transforming a manifold defined by

$$y = F(x_1, \cdots, x_n)$$

into the normal form

$$y = N(x_1, \cdots, x_n)$$

via the homological equation. Each of the Taylor coefficients of $N$ is an invariant of the Lie group evaluated at the origin. This solves the equivalence problem because if two manifolds $M_1$, $M_2$ are given respectively by $y = F_1(x_1, \cdots, x_n)$, $y = F_2(x_1, \cdots, x_n)$, it suffices to bring them to the normal forms $y = N_1(x_1, \cdots, x_n)$, $y = N_2(x_1, \cdots, x_n)$. If they are equivalent, they have the same set of invariants, and hence $N_1 = N_2$. Conversely, if both give the same normal form $N_1 = N_2$, the group transformation $M_1 \to M_2$ can be found by factoring through the normalisation maps and their inverses:

$$\begin{tikzcd}
& \{y = N(x_1, \cdots, x_n)\} \arrow[dr] & \\
M_1 \arrow[ur] \arrow[rr, "\varphi"'] & & M_2
\end{tikzcd}$$

This provides a dynamical system point of view of solving the equivalence problems.

It is reasonable to guess that the higher order Taylor coefficients of $N$ may be derived from lower order ones by applying invariant differentiation to them. Indeed recently Fels-Olver-Pohjanpelto-Valiquette [32, 33, 76–78] managed to explicitly write down this formula. Olver [77] outlined an algorithm for $(x, y) \in \mathbb{R}^2$ with $x$ being an independent variable, and $y$ being a dependent variable. In an equivalence problem, let $\mathsf{G}$ be a Lie group acting on $\mathbb{R}^2$, and

$$v_\sigma = \xi_\sigma(x, y)\frac{\partial}{\partial x} + \varphi_\sigma(x, y)\frac{\partial}{\partial y} \qquad (\sigma=1, \cdots, r),$$

be a basis for the Lie algebra $\mathfrak{g}$ of infinitesimal generators of $\mathsf{G}$ which are vector fields on $\mathbb{R}^2$. Let

$$v_\sigma^{(\infty)} := \xi_\sigma(x, y)\frac{\partial}{\partial x} + \sum_{k \geqslant 0} \varphi_\sigma^k(x, y, y_x, \cdots, y_{x^k})\frac{\partial}{\partial y_{x^k}}$$

be the infinite prolongation of $v_\sigma$ to infinite-order jet space $J^\infty(\mathbb{R}^2)$ with coordinates $(x, y, y_x, y_{xx}, \cdots)$. The prolongation formula [74, Theorem 2.3.6] states that

$$\varphi_\sigma^k(x, y, \cdots, y_{x^k}) = \mathsf{D}_x^k\big[\varphi_\sigma(x, y) - \xi_\sigma(x, y)y_x\big] + \xi_\sigma(x, y)y_{x^{k+1}},$$

where

$$\mathsf{D}_x = \frac{\partial}{\partial x} + y_x\frac{\partial}{\partial y} + y_{xx}\frac{\partial}{\partial y_{xx}} + \cdots$$

is the total derivative with respect to the independent variable $x$. Then there exists an invariant differential operator $\mathscr{D}$ such that given an invariant $I_k$ of order $k$, the invariant of the next order has a formula:

$$I_{k+1} = \mathscr{D}(I_k) - \sum_{\sigma=1}^{r} K_\sigma \cdot \iota\big[\varphi_\sigma^k(x, y, y_x, \cdots, y_{x^k})\big],$$

where $\iota$ is the invariantisation map, and $K_\sigma$ are Maurer-Cartan invariants.

In one of the examples he gave, he let $y = f(x)$ be a curve in $\mathbb{R}^2$ equipped with the Lie group action $\mathsf{G} = SE(2)$. Its normal form after transformation can be written as:

$$y(x) = \tfrac{1}{2}I_2\, x^2 + \tfrac{1}{6}I_3\, x^3 + \tfrac{1}{24}I_4\, x^4 + \cdots + \tfrac{1}{k!}I_k\, x^k + \cdots,$$

where

$$I_2 := \frac{f_{xx}}{(1 + f_x^2)^{3/2}}\bigg|_{x=0}$$

is the Eucliden curvature of the curve evaluated at the origin, and thanks to the recurrence formula above, the higher order coefficients $I_3$, $I_4$ etc. which are also invariants at the origin, can then be found. In 2020 and 2021, using the same technique, Zhangchi Chen and Joël Merker obtained complete set of differential invariants and homogenous models for parabolic surfaces, and for affine surfaces with Hessian rank 2 [19, 20].

**Problem 3.24.** To explore the use of normal forms and Olver's recurrence formula in other areas of mathematics (complex geometry, CR geometry, dynamical systems) and mathematical sciences (i.e. physics).

In author's view, this problem would be interesting, because while Cartan's method can produce homogeneous models, it is also known to be computationally heavy. Recently due to an observation of Joël Merker, it is discovered that the normal form method allows us to obtain invariants very quickly. and we are planning to use this new technique to address some of these difficulties.

**Problem 3.25.** To begin a programme which finds both homogeneous models and differential invariants in any given equivalence problem using the method of normal forms.

We will end this introductory section on the equivalence problem here, and the next few sections will cover some of the specific projects that I am working on, and what are the goals to achieve in the next few years.

## 4. Equivalence of ODE's

The geometric study of partial differential equations (PDE) is best formulated in the language of Jet spaces. Let $(x^1, \cdots, x^p, y^1, \cdots, y^q)$ be coordinates in the Euclidean space of dimension $p+q$, with the $x^i$'s seen as independent variables and the $y^j$'s as dependent variables. For a total space $E = \mathbb{R}^p \times \mathbb{R}^q$, the $n$-th order Jet space $J^n(E)$ is the Euclidean space of dimension $p+q\binom{p+n}{n}$ parametrised by $(x^1, \cdots, x^p, y^1, \cdots, y^q, \cdots y^\alpha_J, \cdots)$. Here

$$y^\alpha_J := \partial^{i_1}_{x^1} \cdots \partial^{i_p}_{x^p} y^\alpha \qquad \big(1\leqslant \alpha \leqslant q,\ J=(i_1,\cdots,i_p),\ i_1+\cdots+i_p\leqslant n\big).$$

In the case $p=q=1$, where there is only one independent variable $x$ and one dependent variable $y$, the $n$-th order Jet space $J^n(\mathbb{R}^2)$ has the coordinates $(x, y, y_x, \cdots, y_{x^n})$. Sophus Lie considered a special class of $n$-th order ODE of the form

$$y_{x^n} = \omega(x, y, \cdots, y_{x^{n-1}}), \tag{4.1}$$

which defines an $n-1$-dimensional submanifold $M$ in $J^n(\mathbb{R}^2)$. He observed that (cf. [74, Chapter 2]) to every change of variables:

$$\varphi:\ (x,y) \longmapsto (X(x,y), Y(x,y)),$$

that transforms the ODE (4.1) into the same form:

$$Y_{X^n} = \theta(X, Y, Y_X, \cdots, Y_{X^{n-1}}),$$

which defines another $n-1$-dimensional submanifold $M'$ in the target space $J^2(\mathbb{R}^2)$, there exists a unique lift $\varphi^{(n)}$ of $\varphi$ :

$$\begin{tikzcd} J^n(\mathbb{R}^2) \arrow[r, "\varphi^{(n)}"] \arrow[d] & J^n(\mathbb{R}^2) \arrow[d] \\ \mathbb{R}^2 \arrow[r, "\varphi"'] & \mathbb{R}^2 \end{tikzcd}$$

so that the diagram commutes, and $\varphi^{(n)}(M) \subset M'$.

Let $\hat{\mathsf{G}}$ be the Lie pseudo-group of point transformations $\varphi$. Seen as a manifold, it may be infinite-dimensional. However, a closed submanifold $\mathsf{G} \subset \hat{\mathsf{G}}$ consisting of transformations that preserves a certain structure, such as some fixed $n$-th order ODE $y_{x^n} = \omega(x, y, y_x, \ldots, y_{x^{n-1}})$, may be finite dimensional. Even so, the elements in $\mathsf{G}$

can be difficult to understand, and Sophus Lie observed that they can be better understood by their infinitesimal behaviours near the identity transformation $\mathsf{Id} \in \mathsf{G}$. These can be better described by its Lie algebra $\mathfrak{g}$:

**Theorem 4.2** (S. Lie, 1890, criteria for membership of $\mathfrak{g}$ (cf. [80, Page 199])). *Let* $\mathsf{X} := f(x,y)\partial_x + g(x,y)\partial_y$ *be a vector field on the Euclidean space* $\mathbb{R}^2$*, and let* $\mathsf{X}^{(n)}$ *be its prolongation to* $J^n(\mathbb{R}^2)$*. Then* $\mathsf{X}$ *is an infinitesimal symmetry of the ODE* $y_{x^n} = \omega(x, y, \cdots, y_{x^{n-1}})$ *if and only if on* $J^n(\mathbb{R}^2)$*:*

$$\mathsf{X}^{(n)}\big(y_{x^n} - \omega(x, y, \ldots, y_{x^{n-1}})\big)\big|_M \equiv 0. \quad \square$$

With this criteria, Sophus Lie proved the following

**Theorem 4.3** (S. Lie, 1893 (cf. [80, Page 199], or [61, Kapitel 12, Satz 3]) ). *Let* $y_{x^n} = \omega(x, y, \cdots y_{x^{n-1}})$ *be an* $n$*-th order ODE. Then* $\mathsf{dim}(\mathfrak{g}) \leqslant 8$ *for* $n = 2$*, and* $\mathsf{dim}(\mathfrak{g}) \leqslant n + 4$ *for* $n \geqslant 3$*.*

*In particular if* $\omega \equiv 0$*, then* $\mathsf{dim}(\mathfrak{g}) = 8$ *for* $n = 2$ *and* $\mathsf{dim}(\mathfrak{g}) = n + 4$ *for* $n \geqslant 3$*.*

Of particular interests are second and third order ODE.

4.1. **Second Order ODE.** For a general second order ODE of the form $y_{xx} = \omega(x, y, y_x)$, we summarise the following

**Theorem 4.4** (S. Lie, A. Tresse, E. Cartan (cf. [75, Pages 405, 475])). *The solution to the equivalence problem of classifying* $y_{xx} = \omega(x, y, y_x)$ *up to point transformation is given by 8 Maurer-Cartan 1-forms* $\{\theta^1,\ \theta^2,\ \theta^3,\ \pi^1,\ \pi^2,\ \pi^4,\ \pi^6, \rho\}$ *satisfying the following structure equations:*

$$\begin{aligned}
d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\
d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^6) \wedge \theta^2, \\
d\theta^3 &= \pi^4 \wedge \theta^1 + \pi^6 \wedge \theta^3, \\
d\pi^1 &= \rho \wedge \theta^1 + \pi^4 \wedge \theta^2 - \pi^2 \wedge \theta^3, \\
d\pi^2 &= \tfrac{1}{2}\rho \wedge \theta^2 + \pi^2 \wedge \pi^6 + K_2\theta^1 \wedge \theta^2, \\
d\pi^4 &= \tfrac{1}{2}\rho \wedge \theta^3 + \pi^4 \wedge (\pi^1 - \pi^6) + K_1\theta^1 \wedge \theta^3, \\
d\pi^6 &= \tfrac{1}{2}\rho \wedge \theta^1 - \pi^4 \wedge \theta^2 - 2\pi^2 \wedge \theta^3, \\
d\rho &= \rho \wedge \pi^1 - 2\pi^2 \wedge \pi^4 - K_3\theta^1 \wedge \theta^2 - K_4\theta^1 \wedge \theta^3.
\end{aligned} \tag{4.5}$$

*The primary relative invariants are:*

$$K_1 := \omega_{pppp}, \qquad K_2 := \widehat{\mathsf{D}}_x^2\omega_{pp} - 4\widehat{\mathsf{D}}_x\omega_{yp} - 6\omega_p\widehat{\mathsf{D}}_x(\omega_{pp})\omega_{yy} - 3\omega_y\omega_{pp} + 4\omega_p\omega_{yp},$$

*for some differential operator* $\widehat{\mathsf{D}}_x$ *on* $J^2(\mathbb{R}^2)$*, while* $K_3$ *and* $K_4$ *identically vanish if* $K_1$*,* $K_2$ *do, in which case the second order ODE is equivalent to* $y_{xx} \equiv 0$ *having 8-dimensional infinitesimal symmetry isomorphic to* $\mathfrak{sl}(3)$*.*

The $K_1$ and $K_2$ are relative invariants because they can be modified under the action of isotropy subgroup $\mathsf{H} \subset \mathsf{G}$. If $K_1K_2 \neq 0$, then at least one of them can be normalised to produce absolute invariants on $M$. These can be used to find out which of the second order ODE's $y_{xx} = \omega(x, y, y_x)$ admits free transitive group action, the dimension of which can be less than or equal to 8. In modern terms, this means finding homogeneous spaces coming from the equivalence problem. The following table is taken from [75, Page 476].

**Symmetry Classification of**
**Second Order Ordinary Differential Equations**

| Symmetry Group | Dim | Type | Invariant Equation |
|---|---|---|---|
| $\partial_u$ | 1 | 3.1 | $u_{xx} = F(x, u_x)$ |
| $\partial_x, \partial_u$ | 2 | 1.5 | $u_{xx} = F(u_x)$ |
| $\partial_x, e^x\partial_u$ | 2 | 1.5 | $u_{xx} - u_x = F(u_x - u)$ |
| $\partial_x, x\partial_x - u\partial_u,$ $x^2\partial_x - 2xu\partial_u$ | 3 | 1.1 | $u_{xx} = \dfrac{3u_x^2}{2u} + cu^3$ |
| $\partial_x, x\partial_x - u\partial_u,$ $x^2\partial_x - (2xu+1)\partial_u$ | 3 | 1.2 | $u_{xx} = 6uu_x - 4u^3 + {} + c(u_x - u^2)^{3/2}$ |
| $\partial_x, \partial_u, x\partial_x + \alpha u\partial_u$ $\alpha \neq 0, \frac{1}{2}, 1, 2$ | 3 | 1.7 | $u_{xx} = cu_x^{\frac{\alpha-2}{\alpha-1}}$ |
| $\partial_x, \partial_u, x\partial_x + (x+u)\partial_u$ | 3 | 1.8 | $u_{xx} = ce^{-u_x}$ |
| $\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u,$ $x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u$ | 8 | 2.3 | $u_{xx} = 0$ |

In fact, the original proof of the classification list above follows directly from a more general result of classifying all homogeneous models of second order ODE's of the form $\omega(x, y, y_x, y_{xx}) = 0$, where S. Lie exhibited 27 distinct cases (cf. [80, Page 138]). We will not discuss about this in this research statement. In section 5, we will explain its link with several complex variables, notably with CR geometry.

4.2. **Third Order ODE.** The equivalence problem of classifying third order ODE's of the form $y_{xxx} = \omega(x, y, y_x, y_{xx})$ has interesting and profound consequences in other areas of mathematical sciences. We state the following result of Élie Cartan, redeveloped in recent years by Godliński-Nurowski:

**Theorem 4.6** (Élie Cartan [15], Godliński-Nurowski [47,48])**.** *A third order ODE $y_{xxx} = \omega(x, y, y_x, y_{xx})$ with its associated 1-forms*

$$\omega^1 = dy - p\,dx, \quad \omega^2 = dx, \quad \omega^3 = dy_x - y_{xx}\,dx, \quad \omega^4 = dy_{xx} - \omega(x, y, y_x, y_{xx})\,dx$$

*uniquely defines a 7-dimensional fibre bundle $P^7 \to J^2(\mathbb{R}^2)$ on the space of second order Jet space $J^2(\mathbb{R}^2) \ni (x, y, y_x, y_{xx})$ and a unique coframe $\{\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3\}$ on*

*$P^7$ satisfying the structure equations of the form*

$$
\begin{aligned}
d\theta^1 &= \Omega_1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\
d\theta^2 &= (\Omega_1 - \Omega_3) \wedge \theta^2 + \boldsymbol{B}_1\, \theta^1 \wedge \theta^3 - \boldsymbol{B}_2\, \theta^1 \wedge \theta^4, \\
d\theta^3 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^3 + \theta^2 \wedge \theta^4, \\
d\theta^4 &= (2\Omega_3 - \Omega_1) \wedge \theta^4 - \Omega_2 \wedge \theta^3 - \boldsymbol{A}_1\, \theta^1 \wedge \theta^2, \\
d\Omega_1 &= \Omega_2 \wedge \theta^2 + (\boldsymbol{A}_2 - 2\boldsymbol{C}_1)\, \theta^1 \wedge \theta^2 \\
&\qquad + (3\boldsymbol{B}_3 + \boldsymbol{E}_1)\, \theta^1 \wedge \theta^3 + (2\boldsymbol{B}_1 - 3\boldsymbol{B}_4)\, \theta^1 \wedge \theta^4 + \boldsymbol{B}_2\, \theta^3 \wedge \theta^4, \\
d\Omega_2 &= \Omega_2 \wedge (\Omega_1 - \Omega_3) - \boldsymbol{A}_3\, \theta^1 \wedge \theta^2 + \boldsymbol{E}_2\, \theta^1 \wedge \theta^3 \\
&\qquad - (\boldsymbol{B}_3 + \boldsymbol{E}_1)\, \theta^1 \wedge \theta^4 + \boldsymbol{C}_1\, \theta^2 \wedge \theta^3 + (\boldsymbol{B}_1 - 2\boldsymbol{B}_4)\, \theta^3 \wedge \theta^4, \\
d\Omega_3 &= -\boldsymbol{C}_1\, \theta^1 \wedge \theta^2 + (2\boldsymbol{B}_3 + \boldsymbol{E}_1)\, \theta^1 \wedge \theta^3 + 2(\boldsymbol{B}_1 - \boldsymbol{B}_4)\, \theta^1 \wedge \theta^4 + 2\boldsymbol{B}_2\, \theta^3 \wedge \theta^4.
\end{aligned}
$$

*Exactly 3 invariants are primary $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{C}_1$, while others are expressed in terms of them and their covariant derivatives. The Wünschmann and Cartan invariants are respectively $\boldsymbol{A}_1$ and $\boldsymbol{C}_1$.*

The structure equation provides the principal bundle $P^7$ a 4-dimensional distribution annihilating the three differential 1-forms $\theta^1$, $\theta^3$, $\theta^4$. The 3-dimensional leaf space $M^3$ is equipped with the Einstein-Weyl geometry if and only if both the Wünschmann and Cartan invariants vanish. It is described by the equivalence class $[(g, \Omega_3)]$, where $g$ is the Einstein's pseudometric for gravitation given by

$$
g = \theta^3 \otimes \theta^3 + \theta^1 \otimes \theta^4 + \theta^4 \otimes \theta^1,
$$

while the differential 1-form $\Omega_3$ describes Maxwell's theory of electromagnetism. A deep result by Élie Cartan shows this construction provides a one-to-one correspondence between third order ODE with vanishing Wünschmann and Cartan invariants, and the Einstein-Weyl geometries.

4.3. **Current and Future Research Projects.** In 1989, Hsu-Kamran [55] considered the following equivalence problem

**Problem 4.7.** Classify all second order ODE $y_{xx} = \omega(x, y, y_x)$ up to fibre-preserving maps

$$
(x, y) \longmapsto (X(x), Y(x, y)).
$$

Using Cartan's equivalence method, they showed that there are three relative invariants, namely

$$
\begin{aligned}
I_1 &= \omega_{ppp}, \\
I_2 &= \mathsf{D}_x(\omega_{pp}) - \omega_{py}, \\
I_3 &= \mathsf{D}_x(\omega_{yp}) - \omega_{pp}\omega_y - \omega_{py}\omega_p - 2\omega_{yy}.
\end{aligned}
\tag{4.8}
$$

In our current research project with Joël Merker and Julien Heyd, we plan to answer the following question:

**Problem 4.9** (Working Paper)**.** To bring a second order ODE $y_{xx} = \omega(x, y, y_x)$ to the normal form $y_{xx} = N(x, y, y_x)$ via fibre preserving maps. Find the relationship between the Taylor coefficients of $N$ and the three relative invariant. Calculate a formula to generate higher order invariants and relate them to higher order Taylor coefficients of $N$. Finally classify all homogeneous models using the normal forms.

4.3.1. *Future Research Projects.* Given the ubiquitous nature of ODE's in science and mathematics, it is expected that the tools and techniques used and developed may be extended to other areas of mathematics, both pure and applied. Here we list some of the problems for future research works:

**Problem 4.10.** To rework the article of Tresse [84] on the normal forms on second order ODE for general audience, and reinterpret the results in this setting.

**Problem 4.11.** Find the Poincaré normal forms of the third order ODE $y_{xxx} = \omega(x, y, y_x, y_{xx})$ up to point transformation. Find the possible branchings of the invariants and deduce the corresponding homogeneous models.

**Problem 4.12.** To find possible applications of the normal forms in other areas of mathematics and sciences using the results on third order ODE.

## 5. Classification of CR manifolds and their Homogeneous Models

5.1. **The Levi non-degenerate cases.** The classification of CR geometry started with H. Poincaré in 1907 when he showed that two real hypersurfaces in $\mathbb{C}^2$ are in general locally biholomorphically inequivalent (cf. [58, Page ix]). This gives a heuristic proof of the failure of the Riemann mapping theorem in $\mathbb{C}^2$.

Let $(z, w = u + iv) \in \mathbb{C}^2$ be holomorphic coordinates, and consider a Levi non-degenerate real hypersurface $M \subset \mathbb{C}^2$ given by the following real-analytic defining function

$$v = F(z, \overline{z}, u).$$

The equivalence problem asks between two such hypersurfaces, whether there exists a local biholomorphism $\mathbb{C}^2 \to \mathbb{C}^2$ that sends one into the other. Élie Cartan showed the following

**Theorem 5.1** (É. Cartan, 1932, [12, 14])**.** *The solution is given by a complex-valued relative invariant $I$, and 8 Maurer-Cartan 1-forms $\{\rho, \zeta, \alpha, \beta, \overline{\zeta}, \overline{\alpha}, \overline{\beta}, \delta\}$ that satisfy the following structure equations:*

$$\begin{aligned}
d\rho &= \alpha \wedge \rho + \overline{\alpha} \wedge \rho + i\zeta \wedge \overline{\zeta}, \\
d\zeta &= \beta \wedge \rho + \alpha \wedge \zeta, \\
d\alpha &= \delta \wedge \rho + 2i\zeta \wedge \overline{\beta} + i\overline{\zeta} \wedge \beta, \\
d\beta &= \delta \wedge \zeta + \beta \wedge \overline{\alpha} + I\, \overline{\zeta} \wedge \beta, \\
d\overline{\zeta} &= \overline{\beta} \wedge \rho + \overline{\alpha} \wedge \overline{\zeta}, \\
d\overline{\alpha} &= \delta \wedge \rho - 2i\overline{\zeta} \wedge \beta - i\zeta \wedge \overline{\beta}, \\
d\overline{\beta} &= \delta \wedge \overline{\zeta} + \overline{\beta} \wedge \alpha + \overline{I}\, \zeta \wedge \overline{\beta}, \\
d\delta &= \delta \wedge \alpha + \delta \wedge \overline{\alpha} + i\beta \wedge \overline{\beta} + I\, \rho \wedge \zeta + \overline{I}\, \rho \wedge \overline{\zeta}. \quad \square
\end{aligned} \tag{5.2}$$

When $I \equiv 0$, the resulting Lie algebra is isomorphic to $\mathfrak{su}(2,1)$ (cf. [22]), and the manifold is CR equivalent to the hyperquadric $v = z\overline{z}$. There is a 5-dimensional isotropy Lie subgroup $\mathsf{H}_0 \subset \mathsf{SU}(2,1)$ such that the hyperquadric $v = z\overline{z}$ can be identified with the homogenous space $\mathsf{SU}(2,1)/\mathsf{H}_0$. Since $I$ is a relative invariant, if $I \neq 0$, further reduction should be done to obtain all possible homogeneous models. In the same year, Élie Cartan provided all the possible list:

**Theorem 5.3** (E. Cartan, 1932, [12])**.** *If a hypersurface admitting a holomorphic transitive group is locally equivalent to a hypersphere, it is globally equivalent to one of the following hypersurfaces*

- $z\overline{z} + w\overline{w} = 1$,
- $v = z\overline{z}$,
- $w\overline{w} = e^{z\overline{z}}$,
- $v = \mathsf{exp}\big(\frac{z-\overline{z}}{2i}\big)$,
- $z\overline{z} + (w\overline{w})^m - 1$, *where* $w \neq 0$, *and* $m > 0$,
- $w\overline{w}(1 + z\overline{z})^n = 0$, *where* $n > 0$ *is a positive integer,*
- $v = \big(\frac{z-\overline{z}}{2i}\big)^2$, *with* $\frac{z-\overline{z}}{2i} > 0$,
- *hypersurface with a few points removed, or one of its covering.*

*If a hypersurface with transitive holomorphic group action is not equivalent to the hypersphere, then it is globally equivalent to one of the following hypersurfaces or its coverings*

- $v = \big(\frac{z-\overline{z}}{2i}\big)^m$, *with* $\frac{z-\overline{z}}{2i} > 0$, $|m| \geqslant 1$, $m \neq 1,\ 2$,
- $v = \mathsf{exp}\big(\frac{z-\overline{z}}{w-\overline{w}}\big)$,
- $(z-\overline{z})^2 + (w-\overline{w})^2 + 4\mathsf{exp}\big(2m \cdot \mathsf{arctan}\frac{z-\overline{z}}{w-\overline{w}}\big) = 0$,
- $1 + z\overline{z} - w\overline{w} = \mu|1 + z^2 - w^2|$ *with* $\frac{1}{i}\big(z(1+\overline{w}) - \overline{z}(1+w)\big) > 0$, $\mu > 1$,
- $z\overline{z} + w\overline{w} - 1 = \mu|z^2 + w^2 - 1|$ *except real point* $(|\mu| < 1,\ \mu \neq 0)$,
- $x_1\overline{x}_1 + x_2\overline{x}_2 + x_3\overline{x}_3 = \mu|x_1\overline{x}_1 + x_2\overline{x}_2 + x_3\overline{x}_3|$, *where* $\mu > 1$.

In 1974, Chern-Moser generalised Cartan's results to higher dimensional Levi nondegenerate CR manifolds of hypersurface type using Cartan's method of moving frame, as well as Moser's normal form. In $\mathbb{C}^2$, the Moser normal form obtained is given by:

$$v = z\overline{z} + c_{42}(u)z^4\overline{z}^2 + c_{24}(u)z^2\overline{z}^4 + \sum_{\substack{j+k\geqslant 7\\ j,k\geqslant 2}} c_{jk}(u)z^j\overline{z}^k,$$

where $c_{42}(0)$, $c_{24}(0)$ are Cartan tensors $I$, $\overline{I}$ evaluated at the origin. This form is unique up to the following isotropic transformation

$$z \longmapsto \frac{\lambda(z+aw)}{1 - 2i\delta + (r + i|a|^2 w)}, \qquad w \longmapsto \frac{\lambda\overline{\lambda}w}{1 - 2i\delta + (r + i|a|^2 w)},$$

where $\lambda,\ a \in \mathbb{C}$, $r \in \mathbb{R}$. This last step will normalise the Cartan tensor $c_{42}(0)$, $c_{24}(0)$, provided that the origin lies outside of the umbilical locus $\{x \in M : I(0) = 0\}$. The resulting hypersurface can be osculated up to order 7 by

$$v = z\overline{z} + 2\mathsf{Re}\big((1 + jz + iku)z^4\overline{z}^2\big),$$

where $j \in \mathbb{C}$, $k \in \mathbb{R}$, and $j^2$, $k$ are invariants at the origin. This partly answers the question posed in Problem (3.21).

Due to recently acquired deeper understanding of Cartan's method of moving frames, as well as the Moser's theory of normal forms, over the next few years, in collaboration with Joël Merker, we plan to address the following

**Problem 5.4.** To make the 1974 article of Chern-Moser accessible to readers, especially with regards to Cartan's reduction algorithm, and the link between formal and existence theory of Moser's normal forms.

**Problem 5.5.** To obtain the same Cartan's list of homogeneous models using Moser's normal form.

On the other hand, the study of the umbilical locus remains an open problem.

**Problem 5.6.** To study the nature of the umbilical locus in CR geometry.

Let us digress for the moment to give a remark about the computational difficulty of calculating the Cartan's invariant $I$. Let $\rho(z, w, \overline{z}, \overline{w})$ be a defining function of a Levi non-degenerate real hypersurface in $\mathbb{C}^2$.

Its Levi determinant $\mathrm{L}(\rho)$ has the following explicit expression:

$$\mathrm{L}(\rho) = \det \begin{pmatrix} 0 & \rho_z & \rho_w \\ \rho_{\bar{z}} & \rho_{z\bar{z}} & \rho_{w\bar{z}} \\ \rho_{\bar{w}} & \rho_{z\bar{w}} & \rho_{w\bar{w}} \end{pmatrix}$$

and let

$$H(\rho) := \rho_z\rho_z\rho_{ww} - 2\rho_z\rho_w\rho_{zw} + \rho_w\rho_w\rho_{zz}.$$

By Merker-Sabzevari [71, 72], there exists an explicit vector field $L$ such that the Cartan's invariant is given by

$$\begin{aligned} I_{[w]} \quad = \quad & \left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^3 \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & -6\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right) \bar{L}^3\left(\frac{H(\rho)}{\rho_w^3}\right) - 4\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}^2\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right) \bar{L}^2\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & -\left[\frac{\mathrm{L}(\rho)}{\rho_w^2}\right]^2 \bar{L}^3\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right) \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) + 15\frac{\mathrm{L}(\rho)}{\rho_w^2}\left[\mathrm{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\right]^2 \bar{L}^2\left(\frac{H(\rho)}{\rho_w^3}\right) \\ & +10\frac{\mathrm{L}(\rho)}{\rho_w^2}\bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\bar{L}^2\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right) - 15\left[\bar{L}\left(\frac{\mathrm{L}(\rho)}{\rho_w^2}\right)\right]^3 \bar{L}\left(\frac{H(\rho)}{\rho_w^3}\right), \end{aligned}$$

which the authors have demonstrated that when expanded in full, there are potentially about a million terms. In the special case where $M$ is an ellipsoid given by

$$ax^2 + y^2 + bu^2 + v^2 = 1 \qquad \big(a\geqslant 1,\ b\geqslant 1,\ (a,b)\neq(1,1)\big),$$

in a joint work with Joël Merker and The-Anh Ta, we managed to show that $I_{[w]}$ vanishes on a certain curve contained in $M$, and thus verifying the existence of umbilical locus:

**Theorem 5.7** (Foo-Merker-Ta, 2018 [36])**.** *For every real numbers $a \geqslant 1$, $b \geqslant 1$ with $(a, b) \neq (1, 1)$, the curve parametrised by $\theta \in \mathbb{R}$ valued in $\mathbb{C}^2$:*

$$\gamma:\ \theta \longmapsto (x(\theta) + iy(\theta),\ u(\theta) + iv(\theta)),$$

*with*

$$x(\theta) := \sqrt{\frac{a-1}{a(ab-1)}}\cos\theta, \qquad y(\theta) := \sqrt{\frac{b(a-1)}{ab-1}}\sin\theta$$

*and*

$$u(\theta) := \sqrt{\frac{b-1}{(ab-1)}}\sin\theta, \qquad v(\theta) := -\sqrt{\frac{a(b-1)}{ab-1}}\cos\theta.$$

*has image contained in the CR-umbilical locus $\{I_{[w]} = 0\}$ of the ellipsoid given by $ax^2 + y^2 + bu^2 + v^2 = 1$.*

But even so, the umbilical locus for ellipsoids are already very complicated.

Coming back from the digression, we reiterate our observations that the question of finding homogeneous models for Levi non-degenerate real hypersurfaces of dimension 5 and above remains largely open. Only recently, centuries after Cartan's monumental works, Loboda [60], and Dubrov-Merker-The [26], have solved the classification problem in dimension 5. With the tools coming from Lie theory, Cartan theory and theory of normal forms, we are planning to work on the following

**Problem 5.8.** To embark on a project to classify Levi non-degenerate real hypersurfaces, and to obtain all homogeneous models for dimension 7 and above.

The results of Cartan do not appear out of thin air. As Cartan had explained in [12], his result was inspired by the observation of B. Segre that a Levi non-degenerate CR manifold can be interpreted in terms of second order ODE, which has already been extensively studied. For any general hypersurface with the defining function:

$$\rho(z, w, \overline{z}, \overline{w}) = 0,$$

assuming that $\partial_w \rho \neq 0$ everywhere, the implicit function theorem solves the equation for $w$:

$$w = \Theta(z, \overline{z}, \overline{w}),$$

so that

$$\rho\big(z, \Theta(z, \overline{z}, \overline{w}), \overline{z}, \overline{w}\big) \equiv 0. \tag{5.9}$$

Then set up the map

$$\begin{aligned} \varphi :\ \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (z, \overline{z}, w) &\longmapsto \big(z, \Theta, \Theta_z\big). \end{aligned} \tag{5.10}$$

By equation (5.9), the Levi non-degeneracy condition is equivalent to the non-degeneracy of the Jacobian of $\varphi$, and hence this map is a local diffeomorphism. With this identification, one can then apply the result from the classification of second order ODE $w_{zz} = \Theta_{zz}$. The details of this construction can be found in [65].

5.2. **Uniformly Levi degenerate cases.** In $\mathbb{C}^2$, all uniformly Levi degenerate real hypersurfaces are Levi flat, and so the classification is obvious. However in $\mathbb{C}^3$, either the Levi rank is zero, in which the manifold is Levi flat. Otherwise, we will assume that it is uniformly of Levi rank 1.

If the Levi rank of the hypersurface $M^5$ is uniformly equal to 1, Freeman gave a necessary and sufficient condition for $M^5$ to be a product $M^3 \times \mathbb{C}$ of Levi non-degenerate manifold and the complex plane. If $T^{1,0}M^5$ is the $(1,0)$-bundle of the complexified tangent bundle $\mathbb{C}TM^5$, and $K^{1,0}M$ is its kernel bundle, then a necessary and sufficient condition would be that the following bilinear pairing is degenerate

$$\begin{aligned} K_p^{1,0}M \times T_p^{1,0}M &\longrightarrow \mathbb{C} \\ (X_p, Y_p) &\longmapsto [X, Y]\big|_p \ \mathsf{mod}\ K^{1,0}M \oplus T^{0,1}M, \end{aligned} \tag{5.11}$$

for any $(1,0)$ vector fields $X$, $Y$ that respectively extend the vectors $X_p$, $Y_p$. This bilinear map is well-defined, and is independent of the choice of the extensions. In this case, we call the manifold *Freeman degenerate*, and it is not interesting since its classification falls back on the results of Cartan. Otherwise we let $\mathfrak{C}_{2,1}$ denote the class of 5-dimensional CR manifolds of constant Levi rank 1, and which are Freeman non-degenerate.

A well-known homogeneous model is the *tube over the future light cone*, which has the following defining equation

$$\mathsf{Re}(z_1)^2 + \mathsf{Re}(z_2)^2 = \mathsf{Re}(z_3)^2, \qquad \mathsf{Re}(z_3) > 0.$$

The classification of $\mathfrak{C}_{2,1}$ was first undertaken by P. Ebenfelt in 2001 [29] using Cartan's equivalence method and the result was found to be erroneous [30]. It was corrected in the articles of Gaussier-Merker in 2003 [45, 46] where the authors showed that the homogeneous model is locally CR diffeomorphic to what is now known as the Gaussier-Merker model (here we let $(z, \zeta, w = u + iv) \in \mathbb{C}^3$ be holomorphic coordinates):

$$u = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}},$$

whose Lie algebra of infinitesimal CR automorphism was shown by authors, and later by Samuel Pocchiola [70], to be 10 dimensional.

There were several attempts to find the Cartan connection by Isaev-Zaitsev [56], Medori-Spiro [64] with claim to its existence without explicitness. In 2015, S. Pocchiola [69] showed that there are two primary invariants in this problem using Cartan's method of moving frames. In the joint work with Joël Merker [34], we completed the $\{e\}$-structure.

**Theorem 5.12** (S. Pocchiola 2015 [69], Foo-Merker 2019 [34])**.** *The equivalence problem of classifying* $5$*-dimensional generic real hypersurfaces* $M^5 \subset \mathbb{C}^3$ *of constant Levi rank* $1$*, and are 2 non-degenerate in the sense of Freeman (the "*$\mathfrak{C}_{2,1}$*" manifolds) has a solution given by 10 dimensional principal bundle* $P^{10} \longrightarrow M^5$ *whose cotangent bundle is generated by* $10$ *differential* $1$*-forms* $\{\rho, \kappa, \zeta, \pi^1, \pi^2, \overline{\kappa}, \overline{\zeta}, \overline{\pi}^1, \overline{\pi}^2, \Lambda\}$ *that satisfy the Cartan's structure equations:*

$$\begin{aligned}
d\rho &= \pi^1 \wedge \rho + \overline{\pi}^1 \wedge \rho + i\kappa \wedge \overline{\kappa}, \\
d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \overline{\kappa}, \\
d\zeta &= i\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi}^1 \wedge \zeta + W\kappa \wedge \zeta + R\rho \wedge \zeta + J\rho \wedge \overline{\kappa}, \\
d\pi^1 &= \Lambda \wedge \rho - i\overline{\pi}^2 \wedge \kappa + \zeta \wedge \overline{\zeta} + \widehat{\Omega}_1, \\
d\pi^2 &= \Lambda \wedge \kappa + \pi^2 \wedge \overline{\pi}^1 - \overline{\pi}^2 \wedge \zeta + \widehat{\Omega}_2 + h\rho \wedge \kappa, \\
d\Lambda &= \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi}^1 + i\pi^2 \wedge \overline{\pi}^2 + \Phi,
\end{aligned}$$

*where the torsions* $\widehat{\Omega}_1$*,* $\widehat{\Omega}_2$ *have the following explicit expressions:*

$$\begin{aligned}
\widehat{\Omega}_1 = &-\tfrac{1}{4}W\pi^2 \wedge \rho + \tfrac{1}{4}\overline{W}\overline{\pi}^2 \wedge \rho - \tfrac{1}{2}(R_\kappa - \overline{J_\zeta})\rho \wedge \kappa - \tfrac{1}{2}R_\zeta \rho \wedge \zeta \\
&+ \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\rho \wedge \overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\rho \wedge \overline{\zeta} + \left(\tfrac{1}{2}W_{\overline{\kappa}} - iR\right)\kappa \wedge \overline{\kappa} - \overline{W}\kappa \wedge \overline{\zeta} - W\zeta \wedge \overline{\kappa}, \\
\widehat{\Omega}_2 = &-R\pi^2 \wedge \rho - \tfrac{1}{4}W\pi^2 \wedge \kappa + \tfrac{1}{4}\overline{W}\overline{\pi}^2 \wedge \kappa - i(W_\rho - 2R_\kappa + \overline{J_\zeta})\rho \wedge \zeta \\
&- i(WJ - J_\kappa)\rho \wedge \overline{\kappa} - iJ\rho \wedge \overline{\zeta} - \tfrac{1}{2}R_\zeta \kappa \wedge \zeta + \tfrac{1}{2}(R_{\overline{\kappa}} - J_\zeta)\kappa \wedge \overline{\kappa} + \tfrac{1}{2}R_{\overline{\zeta}}\kappa \wedge \overline{\zeta} \\
&- R\zeta \wedge \overline{\kappa}.
\end{aligned}$$

*The function* $h$ *and the* $2$*-form* $\Phi$ *can be shown to depend on* $W$ *and* $J$*. There are two primary invariants* $W$*,* $J$*, and by Cartan's theory, they vanish identically if and only if the manifold is CR equivalent to the tube over the future light cone.*

However the following question remains open, and is of interest to parabolic geometers:

**Problem 5.13.** Find the Cartan connection in the equivalence problem.

In a paper published in Acta Mathematica, the authors Fels-Kaup [31] used Lie theory to classify all homogeneous models. Later in 2020, using Cartan's method, Merker-Nurowski [63] obtained the same result by applying the observation of Segre to interpret the tube over the future light cone in terms of systems of PDE. This manifold is in fact equivalent to

$$z = -\frac{(x+\overline{x})^2}{y-\overline{y}} + \overline{z},$$

which is the unique solution to

$$z_y = \frac{1}{4}(z_x)^2, \qquad z_{xxx} = 0,$$

but the existence of Cartan connection still remains unclear.

In 2019, Kolar-Kossovskiy [59] calculated the normal form for $\mathfrak{C}_{2,1}$-manifolds. However, the result was not complete because of the lack of explicit formula of the generators of the isotropy subgroup, and the lack of proof of the existence of the normal form. Independently in 2020, in a joint work with Joël Merker and The-Anh Ta [40], we closed these gaps and obtained the following:

**Theorem 5.14** (Foo-Merker-Ta [40], Kolar-Kossovskiy [59])**.** *There exists a biholomorphism $(z,\zeta,w) \to (z',\zeta',w')$ fixing the origin which maps $(M,0)$ to $(M',0')$ of normalised equation*

$$\begin{aligned}
u = {} & \frac{z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \frac{1}{2}z^2\overline{\zeta}}{1-\zeta\overline{\zeta}} \\
& + 2\mathsf{Re}\big(z^3\overline{\zeta}\, F_{3002}(v) + \zeta\overline{\zeta}\big(3z^2\overline{z}\overline{\zeta}F_{3002}(v)\big)\big) \\
& + 2\mathsf{Re}\big(z^5\overline{\zeta}F_{5001}(v) + z^4\overline{\zeta}^2F_{4002}(v) + z^3\overline{z}^2F_{3021}(v) \\
& \qquad\qquad + z^3\overline{z}\overline{\zeta}^2F_{3012}(v) + z^3\overline{\zeta}^3F_{3003}(v)\big) \\
& + z^3\overline{z}^3O_{z\overline{z}}(1) + \overline{z}^3\zeta O_{z\zeta\overline{z}}(3) + z^3\overline{\zeta}O_{z\overline{z}\overline{\zeta}}(3) + \zeta\overline{\zeta}O_{z\overline{z}}(3)O_{z\zeta\overline{z}\overline{\zeta}}(2).
\end{aligned} \tag{5.15}$$

*The map exists and is unique if it is assumed to be of the form*

$$z' = z + f_{\geqslant 2}(z,\zeta,w), \quad \zeta' = \zeta + g_{\geqslant 1}(z,\zeta,w), \quad w' = w + h_{\geqslant 3}(z,\zeta,w),$$

*where $f_w(0)=0$, $\mathsf{Im} h_{ww}(0) = 0$.*

We also showed that analogous to Chern-Moser, this is unique up to the five-dimensional isotropy subgroup generated by the following explicit formula in [40] never existed before:

$$\begin{aligned}
z' &= \lambda\frac{z + i\alpha z^2 + (i\alpha\zeta - i\overline{\alpha})\overline{w}}{1 + 2i\alpha - \alpha^2z^2 - (\alpha^2\zeta - \alpha\overline{\alpha} + ir)w}, \\
\zeta' &= \frac{\lambda}{\overline{\lambda}}\frac{\zeta + 2i\overline{\alpha}z - (\alpha\overline{\alpha} + ir)z^2 + (\overline{\alpha}^2 - ir\zeta - \alpha\overline{\alpha}\zeta)w}{1 + 2i\alpha - \alpha^2z^2 - (\alpha^2\zeta - \alpha\overline{\alpha} + ir)w}, \\
w' &= \lambda\overline{\lambda}\frac{w}{1 + 2i\alpha - \alpha^2z^2 - (\alpha^2\zeta - \alpha\overline{\alpha} + ir)w},
\end{aligned} \tag{5.16}$$

where $\alpha$, $\lambda$ are complex constants, and $r$ is a real constant. This year in 2021, in a joint work with Joël Merker, Paweł Nurowski, and The-Anh Ta [39], we pushed this result further to obtain all homogeneous models of Fels-Kaup.

Related to this problem is the classification of rigid $\mathfrak{C}_{2,1}$ manifolds up to rigid CR diffeomorphisms. Recall that a real hypersurface $M^5 \subset \mathbb{C}^3$ is rigid if locally it can be written in the form

$$u = F(z_1, z_2, \overline{z}_1, \overline{z}_2),$$

that is $F$ is independent of the variable $v$. A local biholomorphism $\varphi : \mathbb{C}^3 \to \mathbb{C}^3$ is *rigid* if it can be written in the form

$$\begin{aligned} \varphi :\ \mathbb{C}^3 &\longrightarrow \mathbb{C}^3 \\ (z_1, z_2, w) &\longmapsto \big(f(z_1, z_2), g(z_1, z_2), \alpha w + h(z_1, z_2)\big), \end{aligned} \tag{5.17}$$

where $f$, $g$, $h$ are holomorphic functions, and $\alpha \in \mathbb{R}^\times$. We will also assume that the rigid manifold $M^5$ is belongs to class $\mathfrak{C}_{2,1}$.

**Theorem 5.18** (Foo-Merker-Ta [38])**.** *The equivalence problem under local rigid biholomorphisms of $\mathscr{C}^\omega$ rigid real hypersurfaces $\{u = F(z_1, z_2, \overline{z}_1, \overline{z}_2)\}$ in $\mathbb{C}^3$ whose Levi form has constant rank $1$ and which are everywhere $2$-nondegenerate reduces to classifying $\{e\}$-structures on the $7$-dimensional bundle $M^5 \times \mathbb{C}$ equipped with coordinates $(z_1, z_2, \overline{z}_1, \overline{z}_2, v, \mathsf{c}, \overline{\mathsf{c}})$ together with a coframe of $7$ differential $1$-forms:*

$$\{\rho,\ \kappa,\ \zeta,\ \overline{\kappa},\ \overline{\zeta},\ \alpha,\ \overline{\alpha}\},$$

*which satisfy invariant structure equations of the shape:*

$$\begin{aligned}
d\rho &= (\alpha + \overline{\alpha}) \wedge \rho + i\kappa \wedge \overline{\kappa}, \\
d\kappa &= \alpha \wedge \kappa + \zeta \wedge \overline{\kappa}, \\
d\zeta &= (\alpha - \overline{\alpha}) \wedge \zeta + \frac{1}{\mathsf{c}} \boldsymbol{I}_0\ \kappa \wedge \zeta + \frac{1}{\mathsf{c}\overline{\mathsf{c}}} \boldsymbol{V}_0\ \kappa \wedge \overline{\kappa}, \\
d\alpha &= \zeta \wedge \overline{\zeta} - \frac{1}{\mathsf{c}} \boldsymbol{I}_0\ \zeta \wedge \overline{\kappa} + \frac{1}{\mathsf{c}\overline{\mathsf{c}}} \boldsymbol{Q}_0\ \kappa \wedge \overline{\kappa} + \frac{1}{\overline{\mathsf{c}}} \overline{\boldsymbol{I}}_0\ \overline{\zeta} \wedge \kappa,
\end{aligned}$$

*conjugate equations for $d\overline{\kappa}$, $d\overline{\zeta}$, $d\overline{\alpha}$ being understood.*

Only two relative invariants are primary: $\boldsymbol{I}_0$ and $\boldsymbol{V}_0$, while the $\boldsymbol{Q}_0$ is a real-valued secondary relative invariant. By Cartan's theory, the identical vanishing of $\boldsymbol{I}_0$ and $\boldsymbol{V}_0$ implies that the manifold is rigidly CR equivalent to the tube over the future light cone. Our next step is to compute the normal forms:

**Theorem 5.19** (Chen-Foo-Merker-Ta [17])**.** *Every hypersurface $M^5 \in \mathfrak{C}_{2,1}$ is equivalent, through a local rigid biholomorphism, to a rigid real-analytic hypersurface $M' \subset \mathbb{C}^3$, which dropping primes for target coordinates, is a perturbation of the Gaussier-Merker model*

$$u = \frac{z_1 \overline{z}_1 + \frac{1}{2} z_1^2 \overline{z}_2 + \frac{1}{2} \overline{z}_1^2 z_2}{1 - z_2 \overline{z}_2} + \sum_{\substack{a,b,c,d \in \mathbb{N} \\ a+c \geqslant 3}} G_{a,b,c,d}\ z_1^a z_2^b \overline{z}_1^c \overline{z}_2^d,$$

*with a simplified $G$, which*

(1) *is normalised to $O_{z_1, \overline{z}_1}(3)$,*

(2) *satisfying prenormalisation conditions $G = O_{\overline{z}_1}(3) + O_{\overline{z}_2}(1)$:*

$$\begin{aligned} G_{a,b,0,0} &= 0 = G_{0,0,c,d}, \\ G_{a,b,1,0} &= 0 = G_{1,0,c,d}, \\ G_{a,b,2,0} &= 0 = G_{2,0,c,d}; \end{aligned}$$

(3) *satisfying in addition the sporadic normalisations:*

$$\begin{aligned} G_{3,0,0,1} &= 0 = G_{0,1,3,0}, \\ \mathsf{Im} G_{3,0,1,1} &= 0 = \mathsf{Im} G_{1,1,3,0}. \end{aligned}$$

*Furthermore, two such rigid real-analytic hypersurfaces $M$, $M'$ both brought into normal forms, are rigidly biholomorphically equivalent if and only if there exist two constants $\rho \in \mathbb{R}_+^*$, $\varphi \in \mathbb{R}$, such that*

$$G_{a,b,c,d} = G'_{a,b,c,d}\rho^{\frac{a+c-2}{2}} e^{i\varphi\cdot(a+2b-c-2d)}.$$

**Problem 5.20.** To find all homogeneous models of rigid $\mathfrak{C}_{2,1}$ manifolds under rigid biholomorphisms.

5.2.1. *Remark.* Much of the foundation of the 5-dimensional CR manifolds has been laid down in the works of Joël Merker [66–68] and thus in author's views, the classification problem will immediately be accessible for dimension 5. The next few years will be devoted to the classification of 7-dimensional CR manifolds, including those of higher codimensions.

## 6. Holomorphic Foliations in CR manifolds

As part of the problem of classification, one may ask the following

**Problem 6.1.** Determine if a CR manifold contains a holomorphic sub-manifold.

Here, we will only consider Cartan's method of exterior differential system (EDS), as has been done by Robert Bryant [7] in the case of Lorentzian CR manifolds. E.M. Chirka in one of his survey papers [23], remarked while mentioning Bryant's paper that "the existence and the structure of complex foliations for hypersurfaces with Levi forms of varying signs remain an urgent one". Since then till our current publication [35], to the author's knowledge, there seems to be no reference regarding the problem of Chirka. To address this problem, as well as in the interest of elucidating ideas, we have decided to consider real hypersurfaces of lowest dimension possible that are not Lorentzian, or in other words whose Levi signature is $(2, 2)$. We first give a brief overview of the problem before discussing the result of R. Bryant in the 5-dimensional case. Then we present our current result.

6.1. **General Overview.** Let $(z_1, \cdots, z_{n+1})$ with $z_i = x_i + iy_i$ be holomorphic coordinates, and let $J : T\mathbb{C}^{n+1} \to T\mathbb{C}^{n+1}$ be the standard complex structure

$$J(\partial_{x_i}) = \partial_{y_i}, \qquad J(\partial_{y_i}) = -\partial_{x_i}.$$

The real analytic CR manifold may be (locally) defined by a smooth real analytic real-valued function $M = \{\rho = 0\}$, with the CR structure naturally inherited from the ambient space

$$T^{1,0}M = \mathbb{C}TM \cap T^{1,0}\mathbb{C}^{n+1}.$$

There is a canonical real 1-form on $M$, $\theta := i\bar{\partial}\rho|_M$, such that

$$\mathsf{ker} i\bar{\partial}\rho|_M = T^{1,0}M \oplus T^{0,1}M.$$

Assuming that $M$ is a real analytic, Levi non-degenerate CR manifold, there is a $T^{1,0*}M$ co-frame $\{\alpha^1, \dots, \alpha^n\}$ which diagonalises $d\theta$ in the sense that

$$d\theta \equiv i\big(\alpha^1 \wedge \bar{\alpha}^1 + \cdots + \alpha^p \wedge \bar{\alpha}^p - \alpha^{p+1} \wedge \bar{\alpha}^{p+1} - \cdots - \alpha^n \wedge \bar{\alpha}^n\big) \qquad \mathsf{mod}\ \theta.$$

Therefore, the signature of the Levi form is $(n_+, n_-) = (p, n-p)$. Multiplying by $-1$ if necessary, it may be assumed that $n_+ := p$ is less than $n_- := n - p$. If $\varphi : \mathbb{D}^k \to M$ is a holomorphic immersion, the signature of the Levi form does not allow $k$ to be greater than $p$. For simplicity, let $p = k$, and let $(s_1, \dots, s_p) \in \mathbb{D}^p$ be holomorphic coordinates.

One of the steps in Cartan's equivalence method is prolongation. To illustrate this, consider the pushforward of the sections of the $T^{1,0}\mathbb{D}^p$ bundle. For each $i$, let $\{\mathscr{A}_i : 1 \leqslant i \leqslant n\}$ be a change of $T^{1,0}M$-frame that is dual to $\alpha^i$, obtained from the Gram-Schmidt process. Since $\varphi$ is a holomorphic immersion, there exist $p$ linearly independent vector fields on $\varphi(\mathbb{D}^p)$ which can be expressed as

$$\varphi_* \partial_{z_i} = \sum_{1 \leqslant j \leqslant n} f_{i,j}(s_1, \ldots, s_p, \bar{s}_1, \ldots, \bar{s}_p) \mathscr{A}_j$$

for certain real-analytic functions $f_{i,j}$, satisfying the vanishing condition

$$d\theta(\varphi_* \partial_{z_i} \wedge \varphi_* \partial_{\bar{z}_j}) \equiv 0. \tag{6.2}$$

This implies that over $\varphi(\mathbb{D}^p)$, the distribution of vector spaces spanned by $\varphi_* \partial_{z_i}$ lies in the isotropic cone of the Levi form. In other words, for each $v \in \varphi_* T^{1,0}\mathbb{D}^p$,

$$d\theta(v \wedge \bar{v}) \equiv 0.$$

The fact that the isotropic cone contains a distribution of $p$-dimensional vector space is a clue to the first prolongation process. From equation (6.2), a direct substitution results in

$$\sum_{1 \leqslant j \leqslant p} |f_{i,j}|^2 - \sum_{1 \leqslant j \leqslant n-p} |f_{i,p+j}|^2 = 0.$$

At this stage, define

$$U_{n_-,n_+} := \left\{ A \in M_{n_- \times n_+}(\mathbb{C}) : \; A^* A = \mathsf{Id}_{n_+ \times n_+} \right\}.$$

A theorem of Sommer shows that the positive and the negative part are related by a matrix in $U_{n_-,n_+}$,

$$\begin{pmatrix} f_{1,p+1} & \cdots & f_{p,p+1} \\ \vdots & \ddots & \vdots \\ f_{1,n} & \cdots & f_{p,n} \end{pmatrix} = \underbrace{\begin{pmatrix} f_{1,p+1} & \cdots & f_{p,p+1} \\ \vdots & \ddots & \vdots \\ f_{1,n} & \cdots & f_{p,n} \end{pmatrix} \begin{pmatrix} f_{1,1} & \cdots & f_{p,1} \\ \vdots & \ddots & \vdots \\ f_{1,p} & \cdots & f_{p,p} \end{pmatrix}^{-1}}_{:= \mathsf{U} \in U_{n_-,n_+}} \cdot \begin{pmatrix} f_{1,1} & \cdots & f_{p,1} \\ \vdots & \ddots & \vdots \\ f_{1,p} & \cdots & f_{p,p} \end{pmatrix}.$$

The fact that $\mathsf{U}^*\mathsf{U}$ is the identity matrix is due to $\boldsymbol{x}^* \cdot \boldsymbol{x} - (U\boldsymbol{x})^* \cdot U\boldsymbol{x} = 0$ for any $\boldsymbol{x} \in \mathbb{C}^p$ since the isotropic cone contains a $p$-dimensional vector space. If $n_- = n_+ := p$, then $U_{n_-,n_+} = U(p)$ is the usual set of unitary matrices of size $p$.

The first prolongation process treats $\mathsf{U}$ as any matrix in $U_{n_-,n_+}$ satisfying the *lifting condition*. More precisely, for any holomorphic immersion $\varphi : \mathbb{D}^p \to M$ into a CR real hypersurface with Levi signature $(p, n-p)$, there is a lift $\tilde{\varphi} : \mathbb{D}^p \to M \times U_{n_-,n_+}$ that sends every point $p \in \mathbb{D}^p$ to $(\varphi(p), \mathsf{U})$ so that the following diagram commutes

$$\begin{tikzcd} & M \times U_{n_-,n_+} \arrow[d, "\pi"] \\ \mathbb{D}^p \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & M. \end{tikzcd}$$

The map $\pi$ is just the projection onto the first component $M$.

Let $\mathsf{u_{i,j}}$ denote the coefficient of the $U_{n_-,n_+}$ matrix $\mathsf{U}$. Consider the Pfaffian system, which is a system of differential 1-forms:

$$\begin{aligned}
\omega^0 &:= \theta, \\
\omega^1 &:= \alpha^1, \\
&\vdots \\
\omega^p &:= \alpha^p, \\
\omega^{p+1} &:= \alpha^{p+1} - \sum_{1\leqslant k\leqslant p} \mathsf{u_{1,k}}\alpha^k, \\
&\vdots \\
\omega^n &:= \alpha^n - \sum_{1\leqslant k\leqslant p} \mathsf{u_{n-p,k}}\alpha^k.
\end{aligned} \tag{6.3}$$

These differential 1-forms constitute a $T^{1,0}M$ co-frame over $M$. Let $\mathscr{I}$ be the ideal generated by $\omega^0$, $\omega^k$ and $\bar{\omega}^k$ for $p+1 \leqslant k \leqslant n$. The ideal then describes the bundle $\varphi_* T^{1,0}\mathbb{D}^p$ over $\varphi(\mathbb{D}^p)$, and has to be expanded to a larger ideal $\mathscr{I}_+$ consisting of some differential 1-forms on $M \times U_{n_-,n_+}$ so that $d\mathscr{I}_+ \equiv 0 \mod \mathscr{I}_+$. In the CR-Lorentzian case [7], this allows certain values of $\mathsf{u_{i,j}}$ to be computed, giving the possible directions of the tangent vectors to $\varphi(\mathbb{D}^p)$.

6.2. **The result of R. Bryant.** Assume now that signature of the Levi form $d\theta$ is $(1,1)$ everywhere. The 2-form can simply be expressed in terms of the adapted coframe as:

$$d\theta \equiv i\big(\alpha^1 \wedge \overline{\alpha}^1 - \alpha^2 \wedge \overline{\alpha}^2\big) \qquad \mathrm{mod}\ \theta.$$

He then considered the problem of existence of holomorphic immersion of the unit disk $\mathbb{D}$ into $M$. Let $\varphi : \mathbb{D} \to M^5$ be such an immersion. Denoting by $t$ a holomorphic coordinate of $\mathbb{D}$, the holomorphic tangent vector to the image $\varphi(\mathbb{D})$ may be written as

$$\mathscr{L}_{\varphi(t)} = f_1(t,\bar{t})\mathscr{A}_1 + f_2(t,\bar{t})\mathscr{A}_2.$$

This vector field lies in the isotropic cone of the Levi form, and hence there exists a circle-valued function $\lambda : M^5 \to \mathbb{S}^1$ such that $f_2 = \lambda f_1$, inducing a lift $\tilde{\varphi} : \mathbb{D} \to M \times \mathbb{S}^1$ to the product space with fibre $\mathbb{S}^1$ so that $\pi \circ \tilde{\varphi} = \varphi$. Treating $\lambda$ as an unknown variable as part of the Cartan process, the following Pfaffian system is set up as before:

$$\begin{aligned}
\omega^0 &:= \theta, \\
\omega^1 &:= \alpha^1, \\
\omega^2 &:= \alpha^2 - \lambda\alpha^1.
\end{aligned} \tag{6.4}$$

It can be seen that $\varphi^*\theta \equiv 0 \equiv \varphi^*\omega^2$ with $\varphi^*(\alpha^1 \wedge \bar{\alpha}^1) \neq 0$. The 1-form

$$\tau := \bar{\lambda} d\lambda + L\bar{\alpha}^1 - \bar{L}\alpha^1,$$

which is purely imaginary, vanishes upon pullback by $\varphi$. Due to the property that $\bar{\tau} = -\tau$, for any holomorphic disk $\psi : \mathbb{D} \to M^9$ with $\psi^*\omega^0 \equiv \psi^*\omega^2 \equiv \psi^*\bar{\omega}^2 \equiv 0$ and $\psi^*(\alpha^1 \wedge \bar{\alpha}^1) \neq 0$, the pullback $\psi^*\tau$ vanishes. It is necessary to add $\tau$ to $\mathscr{I}$ and require that

$$d\tau \equiv 0 \qquad \mathrm{mod}\ \mathscr{I} + \langle\tau\rangle,$$

effectively solving for $\lambda$.

The discussion above can be related to the Chern-Moser theory, and a brief summary will be given here. Using the Appendix in Chern-Moser [22], the most general structure equation used is given in the following form:

$$\begin{aligned}
d\alpha^0 &= ig_{\gamma\bar{\beta}}\, \alpha^\gamma \wedge \alpha^{\bar{\beta}} + \alpha^0 \wedge \phi,\\
d\alpha^\gamma &= \alpha^\beta \wedge \phi_\beta^\gamma + \alpha^0 \wedge \phi^\gamma,\\
d\phi &= i\alpha_{\bar{\beta}} \wedge \phi^{\bar{\beta}} + i\phi_{\bar{\beta}} \wedge \alpha^{\bar{\beta}} + \alpha^0 \wedge \psi,\\
d\phi_\beta^\gamma &= \phi_\beta^\sigma \wedge \phi_\sigma^\gamma + i\alpha_\beta \wedge \phi^\gamma - i\phi_\beta \wedge \alpha^\gamma - i\delta_\beta^\gamma(\phi_\sigma \wedge \alpha^\sigma)\\
&\quad - \frac{1}{2}\delta_\beta^\gamma \psi \wedge \alpha^0 + \Phi_\beta^\gamma,\\
d\phi^\gamma &= \phi \wedge \phi^\gamma + \phi^\beta \wedge \phi_\beta^\gamma - \frac{1}{2}\psi \wedge \alpha^\gamma + \Phi^\gamma,\\
d\psi &= \phi \wedge \psi + 2i\phi^\beta \wedge \phi_\beta + \Psi.
\end{aligned} \tag{6.5}$$

Here the Einstein convention for summation is adopted. The 1-forms $\phi_{\bullet\bullet}$ satisfy

$$\phi_{\gamma\bar{\beta}} + \phi_{\bar{\beta}\gamma} - g_{\gamma\bar{\beta}}\phi = 0. \tag{6.6}$$

The 2-forms that are crucial to the study of CR geometry are

$$\Phi_{\alpha\bar{\rho}} = S_{\alpha\beta\bar{\rho}\bar{\sigma}}\, \alpha^\rho \wedge \alpha^{\bar{\sigma}} + \cdots$$

giving rise to the well-known S-tensor appearing in the expansion.

In the case of Lorentzian 5-dimensional real hypersurfaces, the matrix $g_{\alpha\bar{\beta}}$ may be chosen as

$$g_{\alpha\bar{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the first line in equation (6.5) reads

$$d\omega = i\big(\alpha^1 \wedge \alpha^{\bar{1}} - \alpha^2 \wedge \alpha^{\bar{2}}\big) + \alpha^0 \wedge \phi,$$

which is recognised as the the Levi form written using the adapted basis. The 1-form $\tau$ can alternatively be expressed in terms of $\phi_\bullet^\bullet$:

$$\tau = id\theta + e^{-i\theta}\phi_1^2 + (\phi_2^2 - \phi_1^1) - e^{i\theta}\phi_2^1.$$

The relations (6.6) imply that $\tau$ is purely imaginary, and hence it has to be added to the ideal $\mathscr{I}$. Taking the exterior differentiation of $\tau$, the components of the $S$-tensor appears in the 2-form $d\tau$:

$$d\tau \equiv -\big(S_{22\bar{1}\bar{1}}\lambda^2 + 4S_{21\bar{1}\bar{1}}\lambda + 6S_{11\bar{1}\bar{1}} + 4\overline{S_{21\bar{1}\bar{1}}}\bar{\lambda} + \overline{S_{22\bar{1}\bar{1}}}\bar{\lambda}^2\big)\, \alpha^1 \wedge \bar{\alpha}^1 \qquad \text{mod } \mathscr{I} + \langle\tau\rangle.$$

There are however differences in the case of $M^9 \subseteq \mathbb{C}^5$ with Levi signature $(2,2)$. These differences come from the fact that instead of dealing with 1-forms such as $\tau$, we are dealing with matrices of 1-forms.

6.3. **The $(2,2)$ case and current main result.** Reiterating Cartan's method, the Levi form may be written in terms of the adapted co-frame as

$$d\theta \equiv i\big(\alpha^1 \wedge \overline{\alpha}^1 + \alpha^2 \wedge \overline{\alpha}^2 - \alpha^3 \wedge \overline{\alpha}^3 - \alpha^4 \wedge \overline{\alpha}^4\big) \qquad \text{mod } \theta.$$

Assuming that a holomorphic immersion of the bi-disks $\varphi : \mathbb{D}^2 \to M^9$ exists, then there is a lift $\tilde{\varphi} : \mathbb{D}^2 \to M^9 \times U(2)$ so that for $(s,t) \in \mathbb{D}^2$:

$$\tilde{\varphi}(s,t,\bar{s},\bar{t}) = \left(\varphi(s,t), \begin{pmatrix} \mathsf{P} & \mathsf{Q} \\ \mathsf{R} & \mathsf{S} \end{pmatrix}\right) \qquad\qquad \begin{pmatrix} \mathsf{P} & \mathsf{Q} \\ \mathsf{R} & \mathsf{S} \end{pmatrix} \in U(2).$$

The following Pfaffian system is therefore set up:

$$\begin{aligned}
\omega^0 &:= \theta, \\
\omega^1 &:= \alpha^1, \\
\omega^2 &:= \alpha^2, \\
\omega^3 &:= \alpha^3 - \mathsf{P}\alpha^1 - \mathsf{Q}\alpha^2, \\
\omega^4 &:= \alpha^4 - \mathsf{R}\alpha^1 - \mathsf{S}\alpha^2.
\end{aligned} \tag{6.7}$$

Since $\varphi^*\omega^0 \equiv \varphi^*\omega^3 \equiv \varphi^*\omega^4 \equiv 0$ (along with their conjugates), it makes sense to let $\mathscr{I}$ be the ideal generated by $\omega^0$, $\omega^3$, $\omega^4$, $\bar{\omega}^3$ and $\bar{\omega}^4$, describing the holomorphic tangent bundle of the image $\varphi(\mathbb{D}^2)$. Their Poincaré derivatives also vanish upon pullbacks

$$\varphi^* d\omega^3 \equiv \varphi^* d\omega^4 \equiv 0 \qquad \mathsf{mod}\ \mathscr{I}.$$

There exist complex-valued functions $A,\ldots,J$ on $M \times U(2)$ such that

$$\begin{aligned}
\begin{pmatrix} \bar{\mathsf{P}} & \bar{\mathsf{R}} \\ \bar{\mathsf{Q}} & \bar{\mathsf{S}} \end{pmatrix} \begin{pmatrix} d\omega^3 \\ d\omega^4 \end{pmatrix} &\equiv - \begin{pmatrix} \bar{\mathsf{P}} & \bar{\mathsf{R}} \\ \bar{\mathsf{Q}} & \bar{\mathsf{S}} \end{pmatrix} \begin{pmatrix} d\mathsf{P} & d\mathsf{Q} \\ d\mathsf{R} & d\mathsf{S} \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} \\
&\quad - \begin{pmatrix} A\,\alpha^1 \wedge \overline{\alpha}^1 + B\,\alpha^1 \wedge \overline{\alpha}^2 + C\,\alpha^2 \wedge \overline{\alpha}^1 + D\,\alpha^2 \wedge \overline{\alpha}^2 + E\,\alpha^1 \wedge \alpha^2 \\ F\,\alpha^1 \wedge \overline{\alpha}^1 + G\,\alpha^1 \wedge \overline{\alpha}^2 + H\,\alpha^2 \wedge \overline{\alpha}^1 + I\,\alpha^2 \wedge \overline{\alpha}^2 + J\,\alpha^1 \wedge \alpha^2 \end{pmatrix} \mathsf{mod}\ \mathscr{I} \\
&\equiv 0\ \mathsf{mod}\ \mathscr{I}.
\end{aligned}$$

When the torsion

$$\begin{pmatrix} A\,\alpha^1 \wedge \overline{\alpha}^1 + B\,\alpha^1 \wedge \overline{\alpha}^2 + C\,\alpha^2 \wedge \overline{\alpha}^1 + D\,\alpha^2 \wedge \overline{\alpha}^2 + E\,\alpha^1 \wedge \alpha^2 \\ F\,\alpha^1 \wedge \overline{\alpha}^1 + G\,\alpha^1 \wedge \overline{\alpha}^2 + H\,\alpha^2 \wedge \overline{\alpha}^1 + I\,\alpha^2 \wedge \overline{\alpha}^2 + J\,\alpha^1 \wedge \alpha^2 \end{pmatrix}$$

is absorbed into the first term, the equation is simplified to

$$\begin{pmatrix} \bar{\mathsf{P}} & \bar{\mathsf{R}} \\ \bar{\mathsf{Q}} & \bar{\mathsf{S}} \end{pmatrix} \begin{pmatrix} d\omega^3 \\ d\omega^4 \end{pmatrix} = \begin{pmatrix} \mathsf{M}_{31} \wedge \alpha^1 + \mathsf{M}_{32} \wedge \alpha^2 \\ \mathsf{M}_{41} \wedge \alpha^1 + \mathsf{M}_{42} \wedge \alpha^2 \end{pmatrix} \equiv 0 \qquad \mathsf{mod}\ \mathscr{I},$$

which vanishes after being pulled back to the bi-disk. By Cartan's lemma, a suitable modification of the 1-forms $\varphi^*\mathsf{M}_{\mathsf{ij}}$ by linear combinations of $\varphi^*\alpha^1$ and $\varphi^*\alpha^2$ with coefficients $l$, $m$, $n$, $p$, $q$ and $r$ is needed such that the following matrix vanishes identically on $\mathbb{D}^2$:

$$\begin{pmatrix} \varphi^*\mathsf{M}_{31} - k\varphi^*\alpha^1 - l\varphi^*\alpha^2 & \varphi^*\mathsf{M}_{32} - l\varphi^*\alpha^1 - m\varphi^*\alpha^2 \\ \varphi^*\mathsf{M}_{41} - p\varphi^*\alpha^1 - q\varphi^*\alpha^2 & \varphi^*\mathsf{M}_{42} - q\varphi^*\alpha^1 - r\varphi^*\alpha^2 \end{pmatrix} \equiv 0. \tag{6.8}$$

The main point in this paper is that the Maurer-Cartan form

$$\begin{pmatrix} \bar{\mathsf{P}} & \bar{\mathsf{R}} \\ \bar{\mathsf{Q}} & \bar{\mathsf{S}} \end{pmatrix} \begin{pmatrix} d\mathsf{P} & d\mathsf{Q} \\ d\mathsf{R} & d\mathsf{S} \end{pmatrix}$$

is skew-hermitian, and so the matrix in equation (6.8) must also be skew-hermitian, leading to a set of equations involving additional variables $k$, $l$, $m$, $p$, $q$, $r$ that need to be solved. These equations have solutions if the following functions

$$\mathsf{T}_1 := \bar{B} + E - \bar{F}, \qquad \mathsf{T}_2 := \bar{D} + J - \bar{H}$$

vanish on $\tilde{\varphi}(\mathbb{D}^2)$. The following theorem summarises the new result obtained in a joint work with Professor Merker:

**Theorem 6.9** (Foo-Merker [35], 2019)**.** *Let $M^9 \subset \mathbb{C}^5$ be a CR generic smooth real hypersurface passing through the origin, and whose Levi form has signature of $(2,2)$ at each point in $M^9$. Suppose $\varphi : \mathbb{D}^2 \to M^9$ is a holomorphic immersion of bi-disk into $M^9$ with $\varphi(0) = 0$, then its unique lift $\tilde{\varphi} : \mathbb{D}^2 \to M^9 \times U(2)$ has the image lying in the zero set of two complex valued functions $\mathsf{T}_1$ and $\mathsf{T}_2$ on $M^9 \times U(2)$.*

Since we have treated the case where the Levi signature is $(2,2)$, we believe that the same technique may be applied to other more general cases. Thus we have the following

**Problem 6.10.** Find a good formulation for which this result may be generalised to higher dimensions with all other Levi signatures.

## 7. Analytic methods in CR geometry, the asymptotic expansion of the Szegö kernel

In several complex variables, the Bergman kernel function plays important roles such as in the existence of Kähler metrics, Berezin-Toeplitz quantisations, equidistribution of the zeroes of holomorphic sections, mathematical physics etc. to name a few (cf. [62] for a comprehensive survey of the topic). Its wide and profound applications in various areas of mathematics have generated great interests in computing its asymptotic expansion. For instance, the existence of its full asymptotic expansion was treated by Berman, Berndtsson, and Sjöstrand [2]. Dai, Liu and Ma [24, 25] obtained the full off-diagonal asymptotic expansion and Agmon estimates of the Bergman kernel by using the heat kernel method. Tian [83] used the peak section method to the asymptotic behaviour of the Bergman kernel. One method that is related to CR geometry is due to Catlin [16] and Zelditch [86], who computed its expansion by using the theorem of Boutet de Monvel-Sjöstrand on Szegö kernel.

Let $X$ be a compact $2n+1$ dimensional CR manifold. Its complexified tangent bundle $\mathbb{C}TX = TX \otimes_{\mathbb{R}} \mathbb{C}$ contains the sub-bundle $T^{1,0}X \oplus T^{0,1}X$. Let $\omega_0$ be a 1-form on $X$ such that $\ker \omega_0 = T^{1,0}X \oplus T^{0,1}X$, and $T$ be the vector field on $X$ satisfying $\omega_0(T) = 1$. The Levi form $\mathscr{L}_0$ the 2-form given by

$$\mathscr{L}_0(u,v) := \tfrac{1}{2i}\langle d\omega_0,\ u \wedge \overline{v}\rangle$$

for all $u$, $v$ in $T^{1,0}_x X$ at each $x \in X$. Assuming that $X$ is strictly pseudoconvex. There exist smooth $T^{1,0}X$ vector fields $\mathscr{L}_1, \cdots, \mathscr{L}_n$ that diagonalise the Levi form

$$\mathscr{L}_0(\mathscr{L}_i, \mathscr{L}_j) = \delta_{i,j}.$$

Let $\omega^1, \cdots, \omega^n$ be the $(1,0)$-forms so that $\omega^i(\mathscr{L}_j) = \delta_{i,j}$, and $\omega^i(\overline{\mathscr{L}}_j) = 0$ for all $i$ and $j$. The $T^{*0,1}X$ forms are given by their complex conjugates.

For any smooth function $f : X \to \mathbb{C}$, the *Cauchy-Riemann* tangential operator $\overline{\partial}_b$ is the projection of $df$ onto the space of smooth sections of the $T^{*0,1}X$ bundle $\Omega^{0,1}(X)$.

Let $L^2(X)$ be the space of square integrable functions on $X$ equipped with the inner product

$$(f,g) := \int_X f\overline{g}\, d\mu,$$

where $d\mu$ is the Lebesgue measure on $X$. Similarly one can define $L^2_{(0,1)}(X)$ to be the space of 1-forms

$$\alpha := \alpha_1\, \overline{\omega}^1 + \cdots + \alpha_n\, \overline{\omega}^n,$$

such that each of the functions $\alpha_i$ is square integrable on $X$. If $\beta$ is another $(0,1)$-form

$$\beta := \beta_1\, \overline{\omega}^1 + \cdots + \beta_n\, \overline{\omega}^n,$$

then the inner product can then be extended to this space by setting

$$(\alpha,\beta) := \int_X \sum_{i=1}^n \alpha_i \overline{\beta}_i\, d\mu.$$

This whole formalism can be extended to space of differential forms of $(p,q)$ type, but we will stick our discussion only to the $(0,1)$ forms. Let

$$\mathsf{Dom}(\overline{\partial}_b) := \big\{ f \in L^2(X) :\ \overline{\partial}_b f \in L^2_{(0,1)}(X) \big\},$$

which is dense in $L^2(X)$. Thus $\overline{\partial}_b$ is a densely defined operator

$$\overline{\partial}_b :\ L^2(X) \dashrightarrow L^2_{(0,1)}(X).$$

Since the operator $\overline{\partial}_b$ is closed, its kernel $\mathsf{ker}\,\overline{\partial}_b$ is a closed subspace of $L^2(X)$, and let $\Pi$ be the projection map

$$\Pi :\ L^2(X) \longrightarrow \mathsf{ker}\,\overline{\partial}_b.$$

This linear operator can be expressed in terms of the kernel function $S(x,y) :\ X \times X \to \mathbb{C}$ such that:

$$\Pi(f) = \int_X S(x,y) f(y)\, d\mu$$

for each $f \in L^2(X)$. To study the projection map, we have the following classical

**Theorem 7.1** (Boutet de Monvel-Sjöstrand [6]). *Suppose that $\overline{\partial}_b$ has closed range. Let $D \subset X$ be an open coordinate patch with local coordinates $(x_1, \cdots, x_{2n+1})$. Then*

$$S(x,y) = \int_0^\infty e^{it\phi(x,y)} a(x,y,t)\, dt \quad \mathsf{mod}\ \mathscr{C}^\infty(D \times D),$$

*where*

(1) $\phi(x,y) \in \mathscr{C}^\infty(D \times D)$ *and* $\mathsf{Im}\ \phi \geqslant 0$,
(2) $\phi(x,y) = 0$ *if and only if* $x = y$,
(3) $d_x\phi(x,x) = -d_y\phi(x,x) = -\omega_0(x)$ *for all* $x \in D$,
(4) $a(x,y,t) \sim \sum_{j=0}^\infty a_j(x,y) t^{n-j}$ *in* $S^n_{1,0}(D \times D \times \mathbb{R})$,
(5) $a_j(x,y) \in \mathscr{C}^\infty(D \times D)$ *for* $j = 0, 1, 2, \ldots$,
(6) $a_0(x,x) \neq 0$ *for any* $x \in D$.

Here we explain some of the notations in the theorem. For $m \in \mathbb{R}$, let $\mathscr{S}^m_{1,0}$ be the space of all $a(x,y,t) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}_+)$ such that for all compact sets $K \Subset D \times D$, and all $\alpha,\ \beta \in \mathbb{N}_0^{2n+1}$, $\gamma \in \mathbb{N}_0$, there exists a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$\left|\partial_x^\alpha \partial_y^\beta \partial_t^\gamma a(x,y,t)\right| \leqslant C_{\alpha,\beta,\gamma} \cdot \big(1 + |t|\big)^{m-|\gamma|}$$

for all $(x,y,t) \in K \times \mathbb{R}_t$, and $t \geqslant 1$.

Put

$$\mathscr{S}^{-\infty}(D \times D \times \mathbb{R}_+) := \bigcap_{m \in \mathbb{N}} \mathscr{S}^m_{1,0}(D \times D \times \mathbb{R}_+),$$

and let $a_j \in \mathscr{S}^{m_j}(D \times D \times \mathbb{R}_+)$ for $j \in \mathbb{N}$ with $m_j \to -\infty$ as $j \to \infty$. Then there exists $a \in \mathscr{S}^{m_0}_{1,0}(D \times D \times \mathbb{R}_+)$ unique modulo $\mathscr{S}^{-\infty}$ such that

$$a - \sum_{j=0}^{k-1} a_j \in \mathscr{S}^{m_k}_{1,0}(D \times D \times \mathbb{R}_+)$$

for $k \geqslant 1$. In this case we write

$$a \sim \sum_{j=0}^\infty a_j \quad \text{in} \quad \mathscr{S}^{m_0}_{1,0}(D \times D \times \mathbb{R}_+).$$

The first few terms of the asymptotic expansion of $a(x,y,t)$ along the diagonal has geometric meanings. Boutet de Monvel-Sjöstrand showed that

$$a_0(x,x) = \tfrac{1}{2\pi i}\,\mathrm{det}\mathscr{L}_0.$$

Since the Levi form is diagonalised, usually one has

$$a_0(x,x) = \frac{1}{2\pi i}.$$

In 2021, Hsiao-Shen [51] computed the first term $a_1$ in terms of the Tanaka-Webster scalar curvature. We recall some of the notions.

7.1. **The Tanaka-Webster Connection.** Let $H(X) := \mathrm{Re}(T^{1,0}X \oplus T^{0,1}X)$ be the contact structure of $X$, and let $J$ be the contact structure on $H(X)$. Let $\theta_0 := -\omega_0$. There exists an affine connection known as the *Tanaka-Webster connection* (cf. [85]):

$$\nabla :\ C^{\infty}(X, TX) \longrightarrow C^{\infty}(X, T^*X \otimes TX),$$

satisfying the infinitesimal description of the equivalence problem condition:

(1) for any $f \in \mathscr{C}^{\infty}(X, HX)$ and $U \in \mathscr{C}^{\infty}(X, TX)$, one has $\nabla_U f \in \mathscr{C}^{\infty}(X, HX)$,
(2) $\nabla T = \nabla J = \nabla d\theta_0 = 0$,
(3) The torsion of $\nabla$ satisfies $\tau(U,V) = d\theta_0(U,V)T$, and $\tau(T, JU) = -J\tau(T,U)$ for all $U,\ V \in \mathscr{C}^{\infty}(X, HX)$.

Then there exist 1-forms $\theta^{\alpha}$ such that

$$\nabla \mathscr{L}_{\alpha} = \theta_{\alpha}^{\beta} \otimes \mathscr{L}_{\beta}.$$

The Tanaka-Webster 2-form is given by the Cartan's structure equation:

$$\Theta_{\alpha}^{\beta} = d\theta_{\alpha}^{\beta} - \theta_{\alpha}^{\gamma} \wedge \theta_{\gamma}^{\beta}.$$

After expansion, we obtain

$$\Theta_{\alpha}^{\beta} = R_{\alpha j\overline{k}}^{\beta}\, \omega^j \wedge \overline{\omega}^k + A_{\alpha jk}^{\beta}\, \omega^j \wedge \omega^k + B_{\alpha jk}^{\beta}\, \overline{\omega}^j \wedge \overline{\omega}^k + C_0 \wedge \omega_0,$$

for some 1-form $C_0$. The $R_{\alpha j\overline{k}}^{\beta}$ is called the pseudo-hermitian curvature tensor, whose trace

$$R_{\alpha\overline{k}} := \sum_{j=1}^{n} R_{\alpha j\overline{k}}^{j}$$

is called the pseudo-hermitian Ricci curvature. Write

$$d\theta_0 = i g_{\alpha\overline{\beta}}\, \omega^{\alpha} \wedge \overline{\omega}^{\beta},$$

and $g^{\overline{c}d}$ be the inverse of $g_{a\overline{b}}$. The Tanaka-Webster scalar curvature $R_{scal}$ with respect to the pseudo-hermitian structure $\theta_0$ is then given by

$$R_{scal} := g^{\overline{k}\alpha} R_{\alpha\overline{k}}.$$

7.2. **The result of Hsiao-Shen, and future research plans.** The technique that Hsiao-Shen [51] used consists of using Malgrange's preparation theorem to write the phase function $\phi$ in the following way:

$$\phi(x,y)=f(x,y)(x_{2n+1}+g(x',y)),$$

where $x':=(x_1,\cdots,x_{2n})$. Let

$$\hat{\phi}(x,y):=x_{2n+1}+g(x',y).$$

Then $\phi(x,y)$ and $\hat{\phi}$ are equivalent in the sense of Melin-Sjöstrand. Moreover, it satisfies

$$\frac{\partial^2}{\partial x_{2n+1}^2}\hat{\phi}\equiv 0.$$

With this preparation, we have the following

**Theorem 7.2** (Hsiao-Shen, 2021, [51])**.** *There exists $A(x,y,t)\in S^n_{cl}(D\times D\times\mathbb{R}_+)$ satisfying:*

(1) $A_0(x,x)=\frac{1}{2\pi^{n+1}}$ *for all $x,\ y$ in $D$,*
(2) $\partial_{x_{2n+1}}A_0=0$ *in $D$,*
(3) $A_1(x,x)=\frac{1}{4\pi^{n+1}}R_{scal}(x)$,

*such that*

$$\Pi(x,y)=\int_0^\infty e^{i\hat{\phi}(x,y)}A(x,y,t)\,dt\quad \operatorname{mod}\mathscr{C}^\infty(D\times D).$$

We also remark that similar result was also obtained by Hsiao-Huang-Shao for the case where the CR manifold is equipped with $\mathbb{S}^1$ action.

From the discussion above, it is conjectured that the higher order terms of the Szegö kernel might be expressed in terms of higher order invariants, similar to the case of the normal form. Since the sixth order Cartan invariant determines whether a manifold is CR diffeomorphic to $v=z\overline{z}$, in a joint work with Chin-Yu Hsiao, we will address the following natural question:

**Problem 7.3.** Under what circumstances do we see the Cartan invariant appearing in the asymptotic expansion of the Szegö kernel?

7.3. **The weakly pseudoconvex case.** Recently, Hsiao-Savale [54] computed the asymptotic expansion of the Szegö kernel for 3-dimensional weakly pseudoconvex CR manifolds $X$, with $HX$-bundle being of finite type. This means that the smooth sections of the $HX$ bundle $\mathscr{C}^\infty(HX)$ generate $\mathscr{C}^\infty(TX)$ under finite number of iterations of Lie brackets. For $m\in\mathbb{R}$, $\rho,\ \delta\in(0,1]$, and $U\subset\mathbb{R}^2$, let $S^m_{\rho,\delta}(U\times\mathbb{R}_t)$ denote the Hörmander's symbol class, and let $S^m_{\delta,cl}(U\times\mathbb{R}_t)\subset S^m_{1,\delta}(U\times\mathbb{R}_t)$ be the space of $a(x,t)$ for which there exist for $j=0,1,2,\cdots$ functions $a_j\in S(\mathbb{R}^2)$ such that for all $n\in\mathbb{N}$:

$$a(x,t)-t^m\left(\sum_{j=0}^N t^{-\delta_j}a_j(t^\delta x)\right)\in S^{m-\delta N}_\delta(U\times\mathbb{R}_+)$$

**Theorem 7.4** (Hsiao-Savale, [54])**.** *Let $X$ be a compact weakly pseudoconvex 3-dimensional CR manifolds of finite type for which the tangential CR operator $\overline{\partial}_b$ is closed. At any point of type $r=r(x)$, there exists a set of coordinates $(x_1,x_2,x_3)$ centred at a point $x'$, and a classical symbol $a\in S^{\frac{2}{r}}_{\frac{1}{r},cl}(\mathbb{R}^2_{x_1,x_2}\times\mathbb{R}_t)$ with $a_0>0$ such that the* *poinwise Szegö kernel at $x'$ satisfies*

$$S(x,x') = \int_0^\infty e^{itx_3} a(x;t)\, dt + C^\infty(x).$$

**Problem 7.5.** To obtain a similar result for 5-dimensional $\mathfrak{C}_{2,1}$-manifolds, whose Levi form is uniformly degenerate of rank 1, and are Freeman non-degenerate.

7.4. **CR Morse Inequality and Comparison Theorem.** In 2010, Hsiao-Marinescu considered the CR Morse inequality, analogous to Demailly's holomorphic morse inequality. Their approach was similar to Berman-Berndtsson's who approximated the Chern class of a holomorphic line bundle with a sequence of Bergman kernels.

Let $\Omega^{0,q}(X)$ be the space of smooth $(0,q)$-forms given by

$$f = \sum f_{i_1,\ldots,i_q}\, \overline{\omega}^{i_1} \wedge \cdots \wedge \overline{\omega}^{i_2},$$

where the $i_j$'s can be taken to be strictly increasing. The tangential Cauchy-Riemann operator $\overline{\partial}_b$ can be naturally be extended to $(0,q)$ forms:

$$\overline{\partial}_b :\ \Omega^{0,q}(X) \longrightarrow \Omega^{0,q+1}(X).$$

A line bundle $L \to X$ is CR if its transition functions are CR, and we denote the metric on $L$ by $\phi$.

Let $\Omega^{0,q}(X,L)$ be the space of smooth $(0,q)$-forms with values in $L$, and for $k \geqslant 1$, let

$$\overline{\partial}_{b,k} :\ \Omega^{0,q}(X,L^k) \longrightarrow \Omega^{0,q+1}(X,L^k)$$

be the corresponding tangential CR operator. The CR cohomology is described by

$$H_b^\bullet(X,L^k) = \frac{\mathsf{ker}\ \overline{\partial}_{b,k}}{\mathsf{Im}\ \overline{\partial}_{b,k}}.$$

We may extend $\overline{\partial}_b$ to $L^2_{0,q}(X)$ the space of $(0,q)$-forms that are square-integrable, and denote its $L^2$-adjoint by $\overline{\partial}_b^*$. Let $Y(q)$ be the condition that the Levi form has at least either $\mathsf{max}(q+1; n-q)$ eigenvalues of the same sign or $\mathsf{min}(q+1; n-q)$ pairs of eigenvalues with opposite signs. This condition is necessary because this would imply that the closed, self-adjoint Kohn laplacian

$$\square_{b,k}^{(q)} := \overline{\partial}_{b,k}^*\overline{\partial}_{b,k} + \overline{\partial}_{b,k}\overline{\partial}_{b,k}^* :\ L^2_{0,q}(X,L^k) \dashrightarrow L^2_{0,q}(X,L^k)$$

defined on $\mathsf{Dom}(\square_{b,k}^{(q)})$, is hypoelliptic and has compact resolvant. If $\mathscr{H}_b^{(0,q)}(X) := \mathsf{ker}\square_{b,k}^{(q)}$ is the space of harmonic forms, then the Hodge decomposition holds

$$H_b^q(X,L^k) \cong \mathscr{H}_b^{(0,q)}(X) := \mathsf{ker}\square_{b,k}^{(q)}.$$

Thus for each fixed $\lambda > 0$, the space $H^q_{b,\leqslant\lambda}(X,L^k)$ of harmonic forms of the Kohn laplacian whose eigenvalues are less than or equal to $\lambda$ will be finite dimensional, and let $\Pi^{(q)}_{k,\leqslant\lambda} :\ H^q_{b,\leqslant\lambda}(X,L^k) \to \mathsf{ker}\ \overline{\partial}_{b,k}$ be the Szegö projection.

The authors defined the analogue of the Chern class for the CR line bundle at $x \in X$:

$$M_x^\phi(U,\overline{V}) := \tfrac{1}{2}\langle U \wedge \overline{V},\ d\big(\overline{\partial}_b\phi - \partial_b\phi\big)x\rangle$$

for all $U,\ V \in T_p^{1,0}X$. Let $\Pi_k^{(q)}$ be the Szegö projection $L^2_{0,q}(X, L^k) \to \operatorname{ker} \overline{\partial}_{b,k}$, and set

$$\mathbb{R}_{\phi(x),q} := \Big\{ s \in \mathbb{R} :\ M_x^\phi + s\mathscr{L}_x \text{ has exactly } q \text{ negative eigenvalues and} \\ n-1-q \text{ positive eigenvalues.} \Big\}. \tag{7.6}$$

Hsiao-Marinescu [53] then showed the CR version of the Tian-Catlin-Zelditch's result on the quantisation of the Monge-Ampère masses (cf. next subsection):

**Theorem 7.7** (Hsiao-Marinescu, [53]). *Assume that condition $Y(q)$ holds at each point of $X$. Then for any sequence $\nu_k > 0$ with $\nu_k \to 0$ as $k \to \infty$, there exists a constant $C_0 > 0$ such that*

$$k^{-n}\Pi^{(q)}_{k,k\nu_k} < C_0,$$

*for all $x \in X$. Moreover, there is a sequence $\mu_k > 0$ with $\mu_k \to 0$ as $k \to \infty$, such that for any sequence $\nu_k > 0$ with $\operatorname{lim}\frac{\mu_k}{\nu_k} = 0$, we have for all $x \in X$:*

$$\operatorname{lim}_{k\to\infty} k^{-n}\Pi^{(q)}_{k,k\nu_k} = \frac{1}{2(2\pi)^n}\int_{\mathbb{R}_{\phi,q}} \Big|\operatorname{det}\big(M_x^\phi + s\mathscr{L}_x\big)\Big|\, ds.$$

Using this theorem, they obtained the CR Morse inequality:

**Theorem 7.8** (Hsiao-Marinescu, [53]). *Assume that $Y(q)$ holds at each point of $x \in X$. Then for $k \to \infty$,*

$$\operatorname{dim} H_b^q(X, L^k) \leqslant \frac{k^n}{2(2\pi)^n}\int_X \int_{\mathbb{R}_{\phi,q}} \Big|\operatorname{det}\big(M_x^\phi + s\mathscr{L}_x\big)\Big|\, ds\, dv_X(x).$$

**Problem 7.9.** Extend the result of Hsiao-Marinescu's CR Morse inequality to vector bundles.

7.4.1. *The Comparison Theorem.* Let $X$ be a complex manifold, and $\phi$, $\psi$ be two plurisubharmonic functions on $X$. A classical comparison theorem of Bedford-Taylor states that

$$\int_{\{\psi<\phi\}} \big(dd^c\phi\big)^n \leqslant \int_{\{\phi<\psi\}} \big(dd^c\psi\big)^n.$$

Let $B_\phi$ and $B_\psi$ be the Bergman kernels with respect to $\phi$ and $\psi$ respectively. A theorem of Tian-Catlin-Zelditch says that

$$\operatorname{lim}_{k\to\infty} k^{-n} B_{k\phi}\, d\mu = c_n\big(dd^c\phi\big)^n,$$

which can be interpreted as approximation or quantisation of the Monge-Ampère measure of $\phi$. Based on this observation, Berndtsson [3] proved the analogue of the Bedford-Taylor theorem for Bergman kernels:

**Theorem 7.10.** *Let $L$ be a holomorphic line bundle over complex manifold $X$, and let $\phi$, $\psi$ be two possible singular metrics on $L$. Suppose that $dd^c\phi \geqslant -\omega$, $dd^c\psi \geqslant -\omega$ for some smooth hermitian $(1,1)$-form $\omega$. Assume also that for some constant $C$, $\phi \leqslant \psi + C$, and $\mu$ is given by strictly positive volume form. Then*

$$\int_{\{\phi<\psi\}} B_\phi\, d\mu \leqslant \int_{\{\psi<\phi\}} B_\psi\, d\mu.$$

*Moreover, if $\emptyset \neq \{\psi < \phi\} \neq X$, then the inequality holds unless both sides are zero or infinite.*

As Berndtsson has remarked, the proof of the statement is purely formal from the point of view of functional analysis. With the theorem of Hsiao-Marinescu 7.7, we have the following

**Problem 7.11.** Extend the comparison theorem of Bedford-Taylor for Monge-Ampère measures, and of Berndtsson for Bergman kernels, to Szegö kernels in CR situations.

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