Complex Numbers

Complex numbers are an extension of the ordinary numbers used in everyday math. They have the unique property of representing and manipulating *two* variables as a *single* quantity. This fits very naturally with Fourier analysis, where the frequency domain is composed of two signals, the real and the imaginary parts. Complex numbers shorten the equations used in DSP, and enable techniques that are difficult or impossible with real numbers alone. For instance, the Fast Fourier Transform is based on complex numbers. Unfortunately, complex techniques are very mathematical, and it requires a great deal of study and practice to use them effectively. Many scientists and engineers regard complex techniques as the dividing line between DSP as a *tool*, and DSP as a *career*. In this chapter, we look at the mathematics of complex numbers, and elementary ways of using them in science and engineering. The following three chapters discuss important techniques based on complex numbers: the *complex Fourier transform*, the *Laplace transform*, and the *z-transform*. These complex transforms are the heart of theoretical DSP. Get ready, here comes the math!

The Complex Number System

To illustrate complex numbers, consider a child throwing a ball into the air. For example, assume that the ball is thrown straight up, with an initial velocity of 9.8 meters per second. One second after it leaves the child's hand, the ball has reached a height of 4.9 meters, and the acceleration of gravity (9.8 meters per second^2) has reduced its velocity to zero. The ball then accelerates toward the ground, being caught by the child two seconds after it was thrown. From basic physics equations, the height of the ball at any instant of time is given by:

$$h \;=\; \frac{-g\,t^2}{2} \,+\, v\,t$$

where h is the height above the ground (in meters), g is the acceleration of gravity (9.8 meters per second^2), v is the initial velocity (9.8 meters per second), and t is the time (in seconds).

Now, suppose we want to know *when* the ball passes a certain height. Plugging in the known values and solving for t:

$$t = 1 \pm \sqrt{1 - h/4.9}$$

For instance, the ball is at a height of 3 meters *twice*: $t = 0.38$ (going up) and $t = 1.62$ seconds (going down).

As long as we ask reasonable questions, these equations give reasonable answers. But what happens when we ask unreasonable questions? For example: At what time does the ball reach a height of 10 meters? This question has no answer in reality because the ball *never* reaches this height. Nevertheless, plugging the value of $h = 10$ into the above equation gives two answers: $t = 1 + \sqrt{-1.041}$ and $t = 1 - \sqrt{-1.041}$. Both these answers contain the square-root of a negative number, something that does not exist in the world as we know it. This unusual property of polynomial equations was first used by the Italian mathematician Girolamo Cardano (1501-1576). Two centuries later, the great German mathematician Carl Friedrich Gauss (1777-1855) coined the term **complex numbers**, and paved the way for the modern understanding of the field.

Every complex number is the sum of two components: a **real part** and an **imaginary part**. The real part is a **real number**, one of the ordinary numbers we all learned in childhood. The imaginary part is an **imaginary number**, that is, the *square-root of a negative number*. To keep things standardized, the imaginary part is usually reduced to an ordinary number multiplied by the square-root of negative one. As an example, the complex number: $t = 1 + \sqrt{-1.041}$, is first reduced to: $t = 1 + \sqrt{1.041}\sqrt{-1}$, and then to the final form: $t = 1 + 1.02\sqrt{-1}$. The real part of this complex number is 1, while the imaginary part is $1.02\sqrt{-1}$. This notation allows the abstract term, $\sqrt{-1}$, to be given a special symbol. Mathematicians have long used i to denote $\sqrt{-1}$. In comparison, electrical engineers use the symbol, j, because i is used to represent electrical current. Both symbols are common in DSP. In this book the electrical engineering convention, j, will be used.

For example, all the following are valid complex numbers: $1 + 2j$, $1 - 2j$, $-1 + 2j$, $3.14159 + 2.7183j$, $(4/3) + (19/2)j$, etc. All ordinary numbers, such as: 2, 6.34, and -1.414, can be viewed as a complex number with *zero* for the imaginary part, i.e., $2 + 0j$, $6.34 + 0j$, and $-1.414 + 0j$.

Just as real numbers are described as having positions along a number line, complex numbers are represented by locations in a two-dimensional display called the **complex plane**. As shown in Fig. 30-1, the horizontal axis of the

FIGURE 30-1
The complex plane. Every complex number has a unique location in the complex plane, as illustrated by the three examples shown here. The horizontal axis represents the real part, while the vertical axis represents the imaginary part.
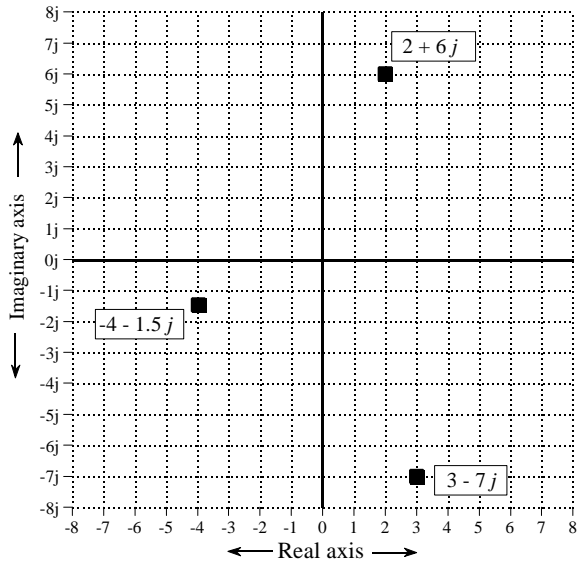
complex plane is the real part of the complex number, while the vertical axis is the imaginary part. Since real numbers are those complex numbers that have an imaginary part equal to zero, the *real number line* is the same as the *x-axis* of the complex plane.

In mathematical equations, a complex number is represented by a single variable, even though it is composed of two parts. For example, the three complex variables in Fig. 30-1 could be written:

$$A = 2 + 6j$$
$$B = -4 - 1.5j$$
$$C = 3 - 7j$$

where A, B, & C are complex variables. This illustrates a strong *advantage* and a strong *disadvantage* of using complex numbers. The advantage is the inherent shorthand of representing two things by a single symbol. The disadvantage is having to remember which variables are complex and which variables are ordinary numbers.

The mathematical notation for separating a complex number into its real and imaginary parts uses the operators: *Re*( ) and *Im*( ). For example, using the above complex numbers:

$$Re\ A = 2 \qquad\qquad Im\ A = 6$$
$$Re\ B = -4 \qquad\qquad Im\ B = -1.5$$
$$Re\ C = 3 \qquad\qquad Im\ C = -7$$

Notice that the value returned by the mathematical operator, $Im(\ )$, does not include the j. For example, $Im(3 + 4j)$ is equal to 4, not $4j$.

Complex numbers follow the same algebra as ordinary numbers, treating the quantity, j, as a constant. For instance, addition, subtraction, multiplication and division are given by:

EQUATION 30-1
Addition of complex numbers.

$$(a + bj) + (c + dj) \;=\; (a + c) + j(b + d)$$

EQUATION 30-2
Subtraction of complex numbers.

$$(a + bj) - (c + dj) \;=\; (a - c) + j(b - d)$$

EQUATION 30-3
Multiplication of complex numbers.

$$(a + bj)(c + dj) \;=\; (ac - bd) + j(bc + ad)$$

EQUATION 30-4
Division of complex numbers.

$$\frac{(a + bj)}{(c + dj)} \;=\; \left(\frac{ac + bd}{c^2 + d^2} \right) + j \left(\frac{bc - ad}{c^2 + d^2} \right)$$

Two tricks are used when manipulating equations such as these. First, whenever a j^2 term is encountered, it is replaced by -1. This follows from the definition of j, that is: $j^2 = (\sqrt{-1})^2 = -1$. The second trick is a way to eliminate the j term from the denominator of a fraction. For instance, the left side of Eq. 30-4 has a denominator of $c + dj$. This is handled by multiplying the numerator and denominator by the term $c - jd$, cancelling all the imaginary terms from the denominator. In the jargon of the field, switching the sign of the imaginary part of a complex number is called taking the **complex conjugate**. This is denoted by a star at the upper right corner of the variable. For example, if $Z = a + bj$, then $Z^* = a - bj$. In other words, Eq. 30-4 is derived by multiplying both the numerator and denominator by the complex conjugate of the denominator.

The following properties hold even when the variables A, B, and C are complex. These relations can be proven by breaking each variable into its real and imaginary parts and working out the algebra.

EQUATION 30-5
Commutative property.

$$AB \;=\; BA$$

EQUATION 30-6
Associative property.

$$(A + B) + C \;=\; A + (B + C)$$

EQUATION 30-7
Distributive property.

$$A(B + C) \;=\; AB + AC$$

Polar Notation

Complex numbers can also be expressed in *polar notation*, besides the *rectangular notation* just described. For example, Fig. 30-2 shows three complex numbers in polar form, the same ones previously presented in Fig. 30-1. The **magnitude** is the length of the vector starting at the origin and ending at the complex point, while the **phase angle** is measured between this vector and the positive x-axis. Complex numbers can be converted between rectangular and polar notation by the following equations (paying attention to the polar notation *nuisances* discussed in Chapter 8):

EQUATION 30-8
Rectangular-to-polar conversion. The complex variable, A, can be changed from rectangular form: *Re* A & *Im* A, to polar form: *M* & θ.

$$M = \sqrt{(Re\ A)^2 + (Im\ A)^2}$$

$$\theta = \arctan\left[\frac{Im\ A}{Re\ A}\right]$$

EQUATION 30-9
Polar-to-rectangular conversion. This is changing the complex number from *M* & θ to *Re* A & *Im* A.
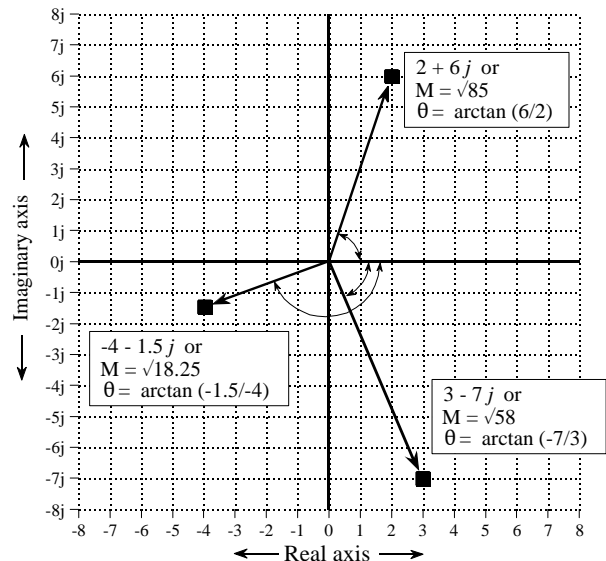
$$Re\ A = M\cos(\theta)$$

$$Im\ A = M\sin(\theta)$$

This brings up a giant leap in the mathematics. (Yes, this means you should pay extra attention). A complex number written in rectangular notation

FIGURE 30-2
Complex numbers in polar form. Three example points in the complex plane are shown in polar coordinates. Figure 30-1 shows these same points in rectangular form.

is in the form: $a + bj$. The information is carried in the variables: a & b, but the proper complex number is the entire expression: $a + bj$. In polar form, the key information is contained in M & θ, but what is the full expression for the proper complex number?

The key to this is Eq. 30-9, the polar-to-rectangular conversion. If we start with the proper complex number, $a + bj$, and apply Eq. 30-9, we obtain:

EQUATION 30-10
Rectangular and polar complex numbers.
The left side is the rectangular form of a
complex number, while the expression on
the right is the polar representation. The
conversion between: M & θ and a & b, is
given by Eqs. 30-8 and 30-9.

$$a + jb \ = \ M(\cos\theta + j\sin\theta)$$

The expression on the left is the proper *rectangular* description of a complex number, while the expression on the right is the proper *polar* description.

Before continuing with the next step, let's review how we arrived at this point. First, we gave the rectangular form of a complex number a graphical representation, that is, a location in a two-dimensional plane. Second, we defined the terms M & θ to be consistent with our previous experience about the relationship between polar and rectangular coordinates (Eq. 30-8 and 30-9). Third, we followed the mathematical consequences of these actions, arriving at what the correct polar form of a complex number must be, i.e., $M(\cos\theta + j\sin\theta)$. Even though this logic is straightforward, the result is difficult to see with "intuition." Unfortunately, it gets worse.

One of the most important equations in complex mathematics is **Euler's relation**, named for the clever and very prolific Swiss mathematician, Leonhard Euler (1707-1783; Euler is pronounced: "Oiler"):

EQUATION 30-11
Euler's relation. This is a key equation
for using complex numbers in science
and engineering.

$$e^{jx} \ = \ \cos x + j\sin x$$

If you like such things, this relation can be proven by expanding the exponential term into a Taylor series:

$$e^{jx} \ = \ \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} \ = \ \left[\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \right] + j \left[\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right]$$

The two bracketed terms on the right of this expression are the Taylor series for $\cos(x)$ and $\sin(x)$. Don't spend too much time on this proof; we aren't going to use it for anything.

Rewriting Eq. 30-10 using Euler's relation results in the most common way of expressing a complex number in polar notation, a **complex exponential**:

EQUATION 30-12
Exponential form of complex numbers. The rectangular form, on the left, is equal to the exponential polar form, on the right.

$$a + jb = M e^{j\theta}$$

Complex numbers in this exponential form are the backbone of DSP mathematics. Start your understanding by memorizing Eqs. 30-8 through 30-12. A strong advantage of using this exponential polar form is that it is very simple to multiply and divide complex numbers:

EQUATION 30-13
Multiplication of complex numbers.

$$M_1 e^{j\theta_1} M_2 e^{j\theta_2} = M_1 M_2 e^{j(\theta_1 + \theta_2)}$$

EQUATION 30-14
Division of complex numbers.

$$\frac{M_1 e^{j\theta_1}}{M_2 e^{j\theta_2}} = \left[\frac{M_1}{M_2} \right] e^{j(\theta_1 - \theta_2)}$$

That is, complex numbers in polar form are multiplied by multiplying their magnitudes and adding their angles. The easiest way to perform addition and subtraction in polar form is to convert the numbers to rectangular form, perform the operation, and reconvert back into polar. Complex numbers are usually expressed in rectangular form in computer routines, but in polar form when writing and manipulating equations. Just as $Re(\ )$ and $Im(\ )$ are used to extract the rectangular components from a complex number, the operators $Mag(\ )$ and $Phase(\ )$ are used to extract the polar components. For example, if $A = 5e^{j\pi/7}$, then $Mag(A) = 5$ and $Phase(A) = \pi/7$.

Using Complex Numbers by Substitution

Let's summarize where we are at. Solutions to common algebraic equations often contain the square-root of a negative number. These are called *complex numbers*, and represent solutions that cannot exist in the world as we know it. Complex numbers are expressed in one of two forms: $a + bj$ (rectangular), or $Me^{j\theta}$ (polar), where j is a symbol representing $\sqrt{-1}$. Using either notation, a single complex number contains two separate pieces of information, either a & b, or M & θ. In spite of their elusive nature, complex numbers follow mathematical laws that are similar (or identical) to those governing ordinary numbers.

This describes what complex numbers are and how they fit into the world of pure mathematics. Our next task is to describe ways they are useful in science

and engineering problems. How is it possible to use a mathematics that has no connection with our everyday experience? The answer: *If the tool we have is a hammer, make the problem look like a nail.* In other words, we *change* the physical problem into a complex number form, manipulate the complex numbers, and then *change* back into a physical answer.

There are two ways that physical problems can be represented using complex numbers: a simple method of **substitution**, and a more elegant method we will call **mathematical equivalence**. Mathematical equivalence will be discussed in the next chapter on the *complex Fourier transform*. The remainder of this chapter is devoted to substitution.

Substitution takes two real physical parameters and places one in the real part of the complex number and one in the imaginary part. This allows the two values to be manipulated as a single entity, i.e., a single complex number. After the desired mathematical operations, the complex number is separated into its real and imaginary parts, which again correspond to the physical parameters we are concerned with.

A simple example will show how this works. As you recall from elementary physics, *vectors* can represent such things as: force, velocity, acceleration, etc. For example, imagine a sailboat being pushed in one direction by the wind, and in another direction by the ocean current. The resulting force on the boat is the vector sum of the two individual force vectors. This example is shown in Fig. 30-3, where two vectors, A and B, are added through the parallelogram law, resulting in C.

We can represent this problem with complex numbers by placing the east/west coordinate into the real part of a complex number, and the north/south coordinate into the imaginary part. This allows us to treat each vector as a single complex number, even though it is composed of two parts. For instance, the force of the wind, vector A, might be in the direction of 2 parts to the east and 6 parts to the north, represented as the complex number: $2 + 6j$. Likewise, the force of the ocean current, vector B, might be in the direction of 4 parts to the east and 3 parts to the south, represented as the complex number: $4 - 3j$. These two vectors can be added via Eq. 30-1, resulting in the complex number representing vector C: $6 + 3j$. Converting this back into a physical meaning, the combined force on the sailboat is in the direction of 6 parts to the north and 3 parts to the east.

Could this problem be solved without complex numbers? Of course! The complex numbers merely provide a formalized way of keeping track of the *two* components that form a *single* vector. The idea to remember is that some physical problems can be converted into a complex form by simply adding a *j* to one of the components. Converting back to the physical problem is nothing more than dropping the *j*. This is the essence of the *substitution* method.

Here's the rub. How do we know that the rules and laws that apply to complex mathematics are the same rules and laws that apply to the original

North

FIGURE 30-3
Adding vectors with complex numbers. The vectors A & B represent forces measured with respect to north/south and east/west. The east/west dimension is replaced by the real part of the complex number, while the north/south dimension is replaced by the imaginary part. This substitution allows complex mathematics to be used with an entirely *real* problem.
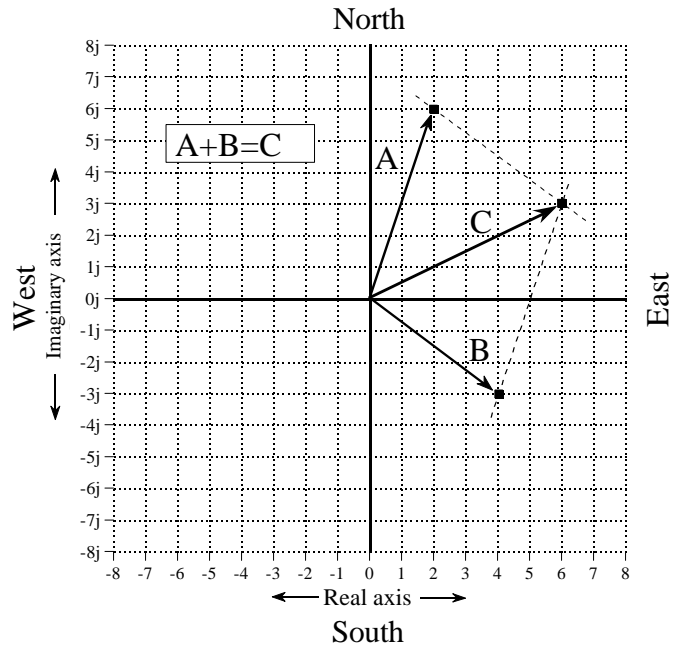
physical problem? For instance, we used Eq. 30-1 to add the force vectors in the sailboat problem. How do we know that the addition of complex numbers provides the same result as the addition of force vectors? In most cases, we know that complex mathematics can be used for a particular application because *someone else* said it does. Some brilliant and well respected mathematician or engineer worked out the details and published the results. The point to remember is that we cannot substitute just *any* problem into a complex form and expect the answer to make sense. We must stick to applications that have been shown to be applicable to complex analysis.

Let's look at an example where complex number substitution *does not* work. Imagine that you buy apples for \$5 a box, and oranges for \$10 a box. You represent this by the complex number: $5 + 10j$. During a particular week, you buy 6 boxes of apples and 2 boxes of oranges, which you represent by the complex number: $6 + 2j$. The total price you must pay for the goods is equal to number of items multiplied by the price of each item, that is, $(5 + 10j)(6 + 2j) = 10 + 70j$. In other words, the complex math indicates you must pay a total of \$10 for the apples and \$70 for the oranges. The problem is, the answer is completely wrong! The rules of complex mathematics *do not* follow the rules of this particular physical problem.

Complex Representation of Sinusoids

Complex numbers find a niche in electronics and signal processing because they are a compact way to represent and manipulate the most useful of all waveforms: sine and cosine waves. The conventional way to represent a sinusoid is: $M\cos(\omega t + \phi)$ or $A\cos(\omega t) + B\sin(\omega t)$, in polar and rectangular

notation, respectively. Notice that we are representing frequency by ω, the *natural frequency* in *radians per second*. If it makes you more comfortable, you can replace each ω with $2\pi f$ to make the expressions in hertz. However, most DSP mathematics is written using the shorter notation, and you should become familiar with it. Since it requires two parameters to represent a single sinusoid (i.e., *A* & *B*, or *M* & ϕ), the use of complex numbers to represent these important waveforms is a natural. Using substitution, the change from the conventional sinusoid representation to a complex number is straight-forward. In rectangular form:

$$A\cos(\omega t) + B\sin(\omega t) \quad \rightleftarrows \quad a + jb$$

(conventional representation)   (complex number)

where $A \rightleftharpoons a$, and $B \rightleftharpoons -b$. Put in words, the amplitude of the cosine wave becomes the real part of the complex number, while the *negative* of the sine wave's amplitude becomes the imaginary part. It is important to understand that this is *not* an equation, but merely a way of letting a complex number *represent* a sinusoid. This substitution also can be applied in polar form:

$$M\cos(\omega t + \phi) \quad \rightleftarrows \quad Me^{j\theta}$$

(conventional representation) (complex number)

where $M \rightleftharpoons M$, and $\theta \rightleftharpoons -\phi$. In words, the polar notation substitution leaves the magnitude the same, but changes the sign of the phase angle.

Why change the sign of the imaginary part & phase angle? This is to make the substitution appear in the same form as the *complex Fourier transform* described in the next chapter.   The *substitution* techniques of this chapter gain nothing from this sign change, but it is almost always done to keep things consistent with the more advanced methods.

Using complex numbers to represent sine and cosine waves is a common technique in electrical circuit analysis and DSP. This is because many (but not all) of the rules and laws governing complex numbers are the same as those governing sinusoids.   In other words, we can represent the sine and cosine waves with complex numbers, manipulate the numbers in various ways, and have the resulting answer match the way the sinusoids behave.

However, we must be careful to use only those mathematical operations that mimic the physical problem being represented (sinusoids in this case). For example, suppose we use the complex variables, *A* and *B*, to represent two sinusoids of the same frequency, but with different amplitudes and phase shifts. When the two complex numbers are *added*, a third complex number is produced. Likewise, a third sinusoid is created when the two sinusoids are

added. As you would hope, the third complex number represents the third sinusoid. The complex addition matches the physical system.

Now, imagine multiplying the complex numbers *A* and *B*, resulting in another complex number. Does this match what happens when the two sinusoids are multiplied? No! Multiplying two sinusoids does *not* produce another sinusoid. Complex multiplication fails to match the physical system, and therefore cannot be used.

Fortunately, the valid operations are clearly defined. Two conditions must be satisfied. First, all of the sinusoids must be at the *same frequency*. For example, if the complex numbers: $1 + 1j$ and $2 + 2j$ represent sinusoids at the same frequency, then the sum of the two sinusoids is represented by the complex number: $3 + 3j$. However, if $1 + 1j$ and $2 + 2j$ represent sinusoids with different frequencies, there is nothing that can be done with the complex representation. In this case, the sum of the complex numbers, $3 + 3j$, is meaningless.

In spite of this, frequency can be left as a variable when using complex numbers, but it must be the *same* frequency everywhere.   For instance, it is perfectly valid to add: $2\omega + 3\omega j$ and $3\omega + 1j$, to produce: $5\omega + (3\omega + 1)j$. These represent sinusoids where the amplitude and phase vary as frequency changes. While we do not know *what* the particular frequency is, we do know that it is the *same* everywhere, i.e., ω.

The second requirement is that the operations being represented must be *linear*, as discussed in Chapter 5. For instance, sinusoids can be combined by addition and subtraction, but not by multiplication or division. Likewise, systems may be amplifiers, attenuators, high or low-pass filters, etc., but not such actions as: squaring, clipping and thresholding. Remember, even convolution and Fourier analysis are only valid for linear systems.

Complex Representation of Systems

Figure 30-4 shows an example of using complex numbers to represent a sinusoid passing through a linear system. We will use continuous signals for this example, although discrete signals are handled the same way. Since the input signal is a sinusoid, and the system is linear, the output will also be a sinusoid, and at the same frequency as the input. As shown, the example input signal has a conventional representation of: $3\cos(\omega t + \pi/4)$, or the equivalent expression: $2.1213\cos(\omega t) - 2.1213\sin(\omega t)$.   When represented by a complex number this becomes: $3e^{-j\pi/4}$ or $2.1213 + j\,2.1213$. Likewise, the conventional representation of the output is: $1.5\cos(\omega t - \pi/8)$, or in the alternate form: $1.3858\cos(\omega t) + 0.5740\sin(\omega t)$. This is represented by the complex number: $1.5e^{j\pi/8}$ or $1.3858 - j\,0.5740$.

The system's characteristics can also be represented by a complex number. The magnitude of the complex number is the ratio between the magnitudes
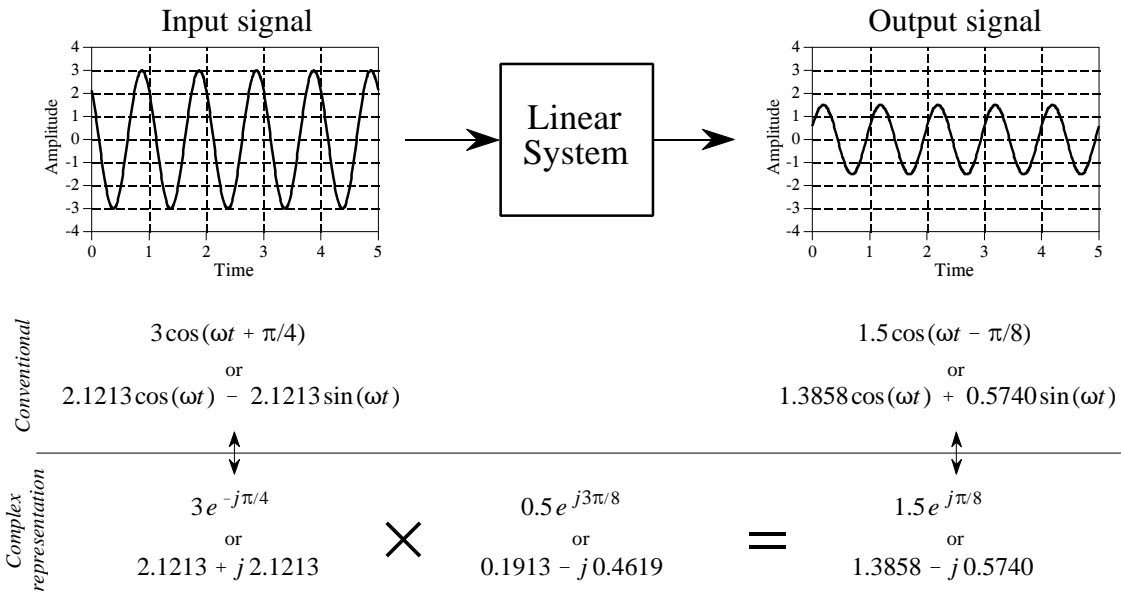
Input signal

Output signal

Linear System

Conventional

$3\cos(\omega t + \pi/4)$

or

$2.1213\cos(\omega t) - 2.1213\sin(\omega t)$

$1.5\cos(\omega t - \pi/8)$

or

$1.3858\cos(\omega t) + 0.5740\sin(\omega t)$

Complex representation

$3e^{-j\pi/4}$

or

$2.1213 + j\,2.1213$

$\times$

$0.5\,e^{\,j3\pi/8}$

or

$0.1913 - j\,0.4619$

$=$

$1.5\,e^{\,j\pi/8}$

or

$1.3858 - j\,0.5740$

FIGURE 30-4

Sinusoids represented by complex numbers. Complex numbers are popular in DSP and electronics because they are a convenient way to represent and manipulate *sinusoids*. As shown in this example, sinusoidal input and output signals can be represented as complex numbers, expressed in either polar or rectangular form. In addition, the *change* that a linear system makes to a sinusoid can also be expressed as a complex number.

of the input and output (i.e., M_{out}/M_{in}). Likewise, the angle of the complex number is the *negative* of the difference between the input and output angles (i.e., $-[\phi_{out} - \phi_{in}]$). In the example used here, the system is described by the complex number, $0.5\,e^{\,j3\pi/8}$. In other words, the amplitude of the sinusoid is reduced by 0.5, while the phase angle is changed by $-3\pi/8$.

The complex number representing the system can be converted into rectangular form as: $0.1913 - j\,0.4619$, but we must be careful in interpreting what this means. It does *not* mean that a sine wave passing through the system is changed in amplitude by 0.1913, nor that a cosine wave is changed by -0.4619. In general, a pure sine or cosine wave entering a linear system is converted into a *mixture* of sine and cosine waves.

Fortunately, the complex math automatically keeps track of these cross-terms. When a sinusoid passes through a linear system, the complex numbers representing the input signal and the system are *multiplied*, producing the complex number representing the output. If any two of the complex numbers are known, the third can be found. The calculations can be carried out in either polar or rectangular form, as shown in Fig. 30-4.

In previous chapters we described how the Fourier transform decomposes a signal into cosine and sine waves. The amplitudes of the cosine waves are called the *real part*, while the amplitudes of the sine waves are called the *imaginary part*. We stressed that these amplitudes are ordinary numbers, and

the terms *real* and *imaginary* are just names used to keep the two separate. Now that complex numbers have been introduced, it should be quite obvious were the names come from. For example, imagine a 1024 point signal being decomposed into 513 cosine waves and 513 sine waves. Using substitution, we can represent the spectrum by 513 complex numbers. However, don't be misled into thinking that this is the *complex Fourier transform*, the topic of Chapter 31. This is still the *real Fourier transform*; the spectrum has just been placed in a complex format by using substitution.

Electrical Circuit Analysis

This method of substituting complex numbers for cosine & sine waves is called the **Phasor transform**. It is the main tool used to analyze networks composed of resistors, capacitors and inductors. [More formally, electrical engineers define the phasor transform as multiplying by the complex term: $e^{j\omega t}$ and taking the real part. This allows the procedure to be written as an equation, making it easier to deal with in mathematical work. "Substitution" achieves the same end result, but is less elegant].

The first step is to understand the relationship between the current and voltage for each of these devices. For the resistor, this is expressed in Ohm's law: $v = iR$, where i is the instantaneous current through the device, v is the instantaneous voltage across the device, and R is the resistance. In contrast, the capacitor and inductor are governed by the differential equations: $i = C\, dv/dt$, and $v = L\, di/dt$, where C is the capacitance and L is the inductance. In the most general method of circuit analysis, these nasty differential equations are combined as dictated by the circuit configuration, and then solved for the parameters of interest. While this will answer *everything* about the circuit, the math can become a real mess.

This can be greatly simplified by restricting the signals to be sinusoids. By representing these sinusoids with complex numbers, the difficult *differential* equations can be directly replaced with much simpler *algebraic* equations. Figure 30-5 illustrates how this works. We treat each of these three components (resistor, capacitor & inductor) as a *system*. The input to the system is the sinusoidal current through the device, while the output is the sinusoidal voltage across its two terminals. This means we can represent the input and output of the system by the two complex variables: I (for current) and V (for voltage), respectively. The relation between the input and output can also be expressed by a complex number. This complex number is called the **impedance**, and is given the symbol: Z. This means:

$$I \times Z = V$$

In words, the complex number representing the sinusoidal voltage is equal to the complex number representing the sinusoidal current multiplied by the impedance (another complex number). Given any two, the third can be
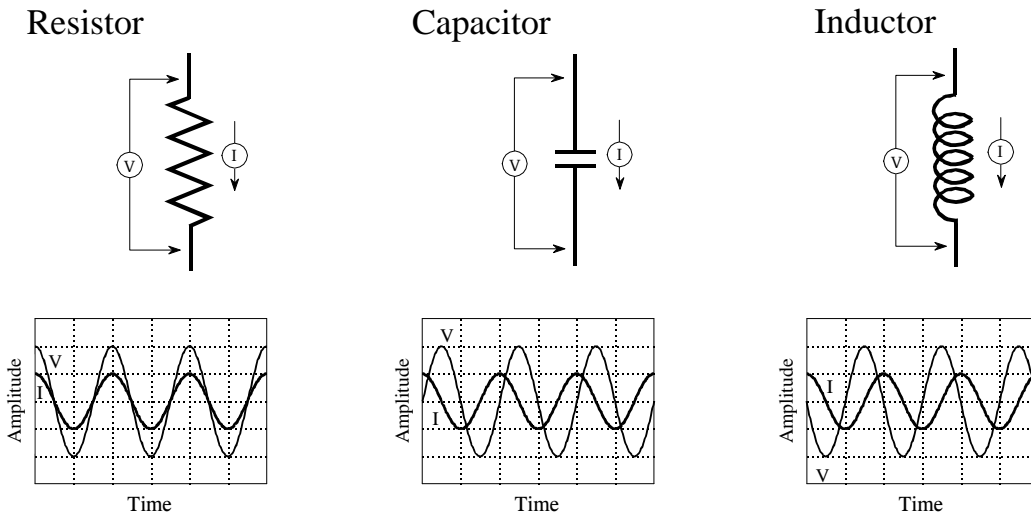
Resistor

Capacitor

Inductor

FIGURE 30-5

Definition of impedance. When sinusoidal voltages and currents are represented by complex numbers, the ratio between the two is called the *impedance*, and is denoted by the complex variable, Z. Resistors, capacitors and inductors have impedances of R, $-j/\omega C$, and $j\omega L$, respectively.

found. In polar form, the magnitude of the impedance is the ratio between the amplitudes of V and I. Likewise, the phase of the impedance is the phase difference between V and I.

This relation can be thought of as *Ohm's law for sinusoids*. Ohms's law ($v = iR$) describes how the *resistance* relates the instantaneous current and voltage in a resistor. When the signals are sinusoids represented by complex numbers, the relation becomes: $V = IZ$. That is, the *impedance* relates the current and voltage. Resistance is an ordinary number, since it deals with two ordinary numbers. Impedance is a complex number, since it relates two complex numbers. Impedance contains more information than resistance, because it dictates both the amplitudes *and* the phase angles.

From the differential equations that govern their operation, it can be shown that the impedance of the resistor, capacitor, and inductor are: R, $-j/\omega C$, and $j\omega L$, respectively. As an example, imagine that the current in each of these components is a unity amplitude cosine wave, as shown in Fig. 30-5. Using substitution, this is represented by the complex number: $1 + 0j$. The voltage across the resistor will be: $V = IZ = (1+0j)R = R+0j$. In other words, a cosine wave of amplitude R. The voltage across the capacitor is found to be: $V = IZ = (1+0j)(-j/\omega C)$. This reduces to: $0 - j/\omega C$, a sine wave of amplitude, $1/\omega C$. Likewise, the voltage across the inductor can be calculated: $V = IZ = (1+0j)(j\omega L)$. This reduces to: $0 + j\omega L$, a negative sine wave of amplitude, ωL.

The beauty of this method is that *RLC* circuits can be analyzed without having to resort to differential equations. The *impedance* of the resistors, capacitors,

Vin

Z1

Vout

Z2

Z3

FIGURE 30-6
RLC notch filter. This circuit removes a narrow band of frequencies from a signal. The use of complex substitution greatly simplifies the analysis of this and similar circuits.
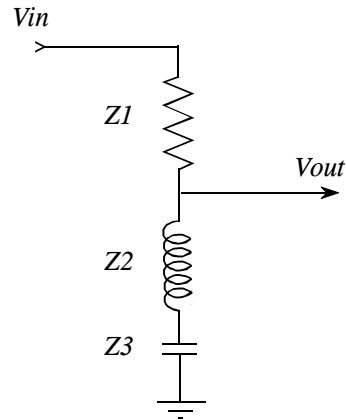
and inductors is treated the same as *resistance* in a DC circuit. This includes all of the basic combinations, such as: resistors in series, resistors in parallel, voltage dividers, etc.

As an example, Fig. 30-6 shows an *RLC* circuit called a **notch filter**, used to remove a narrow band of frequencies. For instance, it could eliminate 60 hertz interference in an audio or instrumentation signal. If this circuit were composed of three resistors (instead of the resistor, capacitor and inductor), the relationship between the input and output signals would be given by the voltage divider formula: $v_{out}/v_{in} = (R2+R3)/(R1+R2+R3)$. Since the circuit contains capacitors and inductors, the equation is rewritten with impedances:

$$\frac{Vout}{Vin} = \frac{Z2 + Z3}{Z1 + Z2 + Z3}$$

where: *Vout*, *Vin*, *Z1*, *Z2*, and *Z3* are all complex variables. Plugging in the impedance of each component:

$$\frac{Vout}{Vin} = \frac{j\omega L - \dfrac{j}{\omega C}}{R + j\omega L - \dfrac{j}{\omega C}}$$

Next, we crank through the algebra to separate everything containing a *j*, from everything that does not contain a *j*. In other words, we separate the equation into its real and imaginary parts. This algebra can be tiresome and long, but the alternative is to write and solve *differential equations*, an
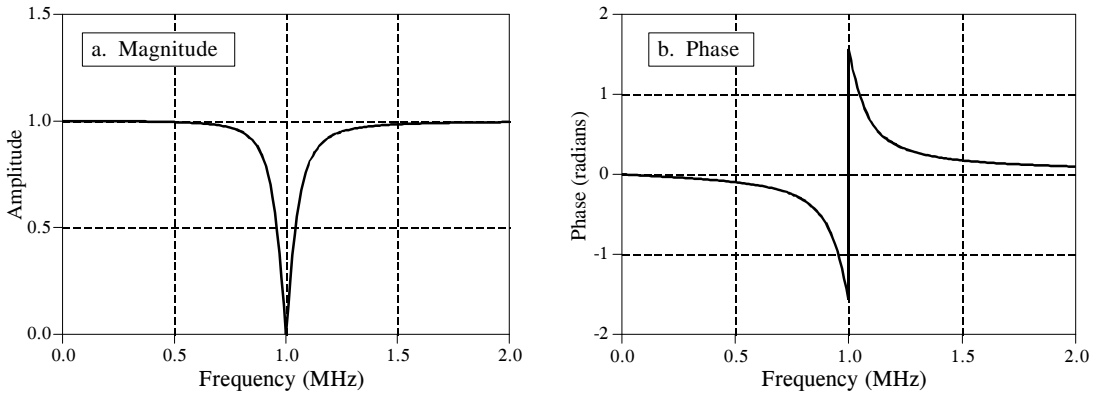
FIGURE 30-7
Notch filter frequency response. These curves are for the component values: $R = 50\Omega$, $C = 470\rho F$, and $L = 54\ \mu H$.

even nastier task. When separated into the real and imaginary parts, the complex representation of the notch filter becomes:

$$\frac{Vout}{Vin} = \frac{k^2}{R^2 + k^2} + j\frac{Rk}{R^2 + k^2}$$

where: $k = \omega L - 1/\omega C$

Lastly, the relation is converted to polar notation, and graphed in Fig. 30-7:

$$Mag = \frac{\omega L - 1/\omega C}{\left(R^2 + [\omega L - 1/\omega C]^2\right)^{1/2}} \qquad Phase = \arctan\left[\frac{R}{\omega L - 1/\omega C}\right]$$

The key point to remember from these examples is how *substitution* allows complex numbers to represent real world problems. In the next chapter we will look at a more advanced way to use complex numbers in science and engineering, *mathematical equivalence*.