Simple note for Lie algebra for SLAM

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Abstract

This note is a simple review note for me.
Keywords:

1 Preliminaries

Definition 1.1 (Lie group)**.** Lie group G is a smooth manifold with smooth group structure. Smooth group structure means that

- smooth group multiplication (composition) $*: G \times G \to G$ defined by $(g, h) \mapsto gh$,
- smooth inverse map $(\cdot)^{-1}: G \to G$ defined by $g \mapsto g^{-1}$

Definition 1.2 (Left group action)**.** For any element $g \in G$, there is a unique diffeomorphism $L_g : G \to G$ which is defined by, for any $h \in G$,

$$L_g(h) = gh. \tag{1}$$

Here are building blocks of Lie groups.

Theorem 1.1. If G_1 and G_2 are Lie groups, then the product space $G_1 \times G_2$ is a Lie group.

Theorem 1.2 (Catan)**.** If $H \leq G$ is both closed subspace and subgroup (closed subgroup), then H is a Lie group.

Theorem 1.3. If $N \leq G$ is a closed normal subgroup, then G/N is a Lie group.

Theorem 1.4. The universal cover of a connected Lie group is a Lie group.

2 Examples

Example 2.1. Suppose that $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

$$\mathrm{GL}(2, \mathbb{F}) = \{A \ : \ \det A \neq 0\} \geq \mathrm{O}(2) \text{ or } \mathrm{U}(2) \geq \mathrm{SO}(2, \mathbb{F}). \tag{2}$$

Example 2.2.

$$\left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \ : \ a > 0, b \in \mathbb{R} \right\}. \tag{3}$$

3 Lie algebra

Abstract

The Lie algebra is a linearization of a Lie group.

Definition 3.1 (Lie algebra)**.** The tangent space $T_eG = (\mathfrak{g}, +, [\cdot,\cdot], \cdot)$ at identity with Lie bracket is the Lie algebra.

Definition 3.2 (Lie bracket)**.** Let $(\mathfrak{g}, +, \cdot)$ be a finite dimensional vector space. A bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

1. (**anti-symmetry**) $[x, x] = 0$ for any $x \in \mathfrak{g}$,

2. **(Jacobi identity)** For any $x, y, z \in \mathfrak{g}$,

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0. \tag{4}$$

Theorem 3.1. Assume that V is a vector space. For any bilinear operation $* : V \times V \to V$,

$$[x,y] := x * y - y * x \tag{5}$$

is to be a Lie bracket.

Example 3.1. For any vector space V, $\mathrm{End}(V)$ has operations

1. (**addition**)
2. (**scalar multiplication**)
3. (**composition***) $\quad\to\quad$ $[f,g] := f \circ g - g \circ f$ is a Lie bracket.

If we define the Lie bracket by the way above, then $\mathrm{End}(V)$ is called the general Lie algebra and denoted $\mathfrak{gl}(V)$.

Definition 3.3 (Liebniz rule)**.** Denote $D_x(y) := [x,y]$. Then, $D_x : \mathfrak{g} \to \mathfrak{g}$ is an adjoint map. Now let us rewrite the Jacobi identity as

$$\begin{aligned}
& D_x([y,z]) + D_y([z,x]) + D_z([x,y]) = 0 && (6)\\
\iff\ & D_x([y,z]) = -[y,[z,x]] - [z,[x,y]] && (7)\\
& \qquad\qquad = [y,[x,z]] + [[x,y],z] && (8)\\
& \qquad\qquad = [y, D_x(z)] + [D_x(y), z] = [D_x(y), z] + [y, D_x(z)]. && (9)
\end{aligned}$$

By the steps above, we call $D_x(\cdot) = [x,\cdot]$ the defferentiation on Lie algebra.

3.1 Operations on Lie algebra

Consider a simple example $\mathrm{GL}_n(\mathbb{R})$ and its corresponding Lie algebra is $\mathfrak{gl}(\mathbb{R})$. The definition of $\mathrm{GL}_n(\mathbb{R})$ is

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\} = f^{-1}(\mathbb{R} - \{0\}), \tag{10}$$

where $f : \mathbb{R}^{n\times n} \to \mathbb{R}^d$ is a polynomial on matrices. Hence, we guess the Lie group is not homeomorphic to Euclidean space. In this section, we will see some recipes of linearization (Lie algebra) of Lie group.

3.2 Exponential map and Logarithm map

Definition 3.4 (Exponential map)**.** Let G be a Lie group and $\mathfrak{g}$ be its tangent space at e. The exponential map $\exp : \mathfrak{g} \to G$ is defined by

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \tag{11}$$

Theorem 3.2 (Commutative diagram)**.** Let $\phi : G \to H$ be a group homomorphism between two Lie groups and $\phi_* : \mathfrak{g} \to \mathfrak{h}$ be its derivative at identity. Then,

$$\exp \circ \phi_* = \phi \circ \exp : \mathfrak{g} \to H. \tag{12}$$

Definition 3.5 (Logarithm map)**.** $\mathrm{Log} : G \to \mathfrak{g}$ is the inverse map of the exponential map.

3.3 Operations on Lie algebra

Note that $\mathfrak{g} = T_eG$ and $(\mathfrak{g}, +, [\cdot,\cdot], \cdot_{\mathbb{R}})$... XXXX