A geometric solution of the optimal launch angle of a projectile

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Suppose a projectile is launched from a height, h, at the speed v_0 and the launch angle θ (Fig. 1). What is the optimal angle θ to maximize the horizontal distance $D = v_x t = v_0 t \cos\theta$ that the projectile travels before hitting the ground, assuming that there is no air friction?

The change in velocity is

$$\mathbf{v}_t - \mathbf{v}_0 = \mathbf{g}t \tag{1}$$

where $\mathbf{g}$ is the gravitational acceleration.

By energy conservation, the final velocity $\mathbf{v}_t$ satisfies

$$v_t^2 = v_0^2 + 2gh \tag{2}$$

The horizontal distance travelled is

$$D = v_x t = \frac{v_0 \cos\theta(gt)}{g} \tag{3}$$

The numerator in the above expression of D is twice the area of the triangle formed by the vectors $\mathbf{v}_0$, $\mathbf{g}t$, and $\mathbf{v}_t$ (Fig. 1), where the lengths of the first and last sides, v_0 and $v_t = \sqrt{v_0^2 + 2gh}$, are constant. Thus the area of the triangle, hence D, is maximized when $\mathbf{v}_0$ and $\mathbf{v}_t$ are perpendicular to each other. When this happens, the angle between $\mathbf{v}_t$ and $\mathbf{g}$ is also θ, which implies

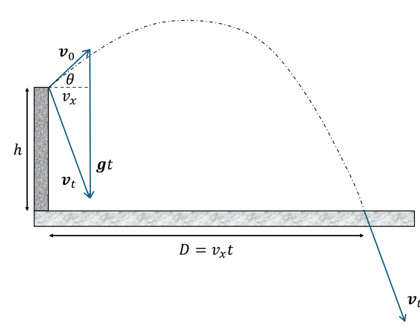

Figure 1: The geometry of the launch angle of projectiles.

$$\tan\theta^* = \frac{v_0}{v_t} = \frac{v_0}{\sqrt{v_0^2 + 2gh}} \tag{4}$$

The maximum distance travelled is thus

$$D^* = v_0 v_t / g = \frac{v_0\sqrt{v_0^2 + 2gh}}{g} \tag{5}$$

For completeness, the distance D is given in terms of the angle θ by

$$D = \frac{v_0^2}{g}\left(\sqrt{\frac{2gh}{v_0^2} + \sin^2\theta} + \sin\theta\right)\cos\theta \tag{6}$$