Bottles and caps

Huafeng Xu

November 24, 2024

It is rumored that some large language models can now solve the following math problem:

1 Problem

Suppose two empty bottles can exchange for one bottle of Fanta, and four bottle caps can exchange for one bottle of Fanta. If you start with $N = 5$ bottles of Fanta, how many total bottles of Fanta can you enjoy? (Fanta has not sponsored this article.)

(At the time of my writing, ChatGPT-4o still seems to struggle with the problem.)

2 Recursion

The problem is straightforward to solve by recursion for general N. The bottles of drinks n_t, the number of empty bottles b_t, and the number of caps c_t after t rounds of exchanges satisfy the following recursion:

$$\begin{aligned} n_t &= \lfloor b_{t-1}/2 \rfloor + \lfloor c_{t-1}/4 \rfloor \\ b_t &= n_t + b_{t-1} - 2\lfloor b_{t-1}/2 \rfloor \\ c_t &= n_t + c_{t-1} - 4\lfloor c_{t-1}/4 \rfloor \end{aligned} \tag{1}$$

where $\lfloor x \rfloor$ denotes the largest integer no greater than x.

For $N = 5$, it is easy to see that after 5 exchanges one can have a total of $\sum_{t=0}^{5} n_t = 15$ bottles of drinks.

The recursion in Eq. 1 makes a closed form solution seem unlikely. But I found one, by starting with an approximation and making one small modification at the end that inexplicably makes the solution exact. In Section 4, I give the exact solution for the general problem of arbitrary exchange rates of bottles and caps.

3 Approximation

The residual ”value” of a bottle of Fanta, *i.e.* the immediate exchange value of the empty bottle and the cap, is

$$r = 1/2 + 1/4 = 3/4 \tag{2}$$

where 1/2 comes from the empty bottle and 1/4 from the cap. This is the average number of bottles of drinks a consumed bottle of drink can exchange for.

The residual value can be exchanged for bottles of drinks that again carry their own residual values. Starting from N bottles of Fanta, after t rounds of exchanges, the residual value becomes Nr^t. The exchanges can continue until this residual value drops below 1 (when it can no longer be exchanged for one additional bottle of drink). So the total rounds of exchanges T is approximately given by

$$\begin{aligned} & Nr^{T+1} < 1 \leq Nr^T \\ \Rightarrow \quad & T = \lfloor -\ln(N)/\ln(r) \rfloor \end{aligned} \tag{3}$$

The total number of bottles of drinks that one can get is thus

$$N_T = N \sum_{t=0}^{T} r^t = N \frac{1 - r^{T+1}}{1 - r} \tag{4}$$

Table 1 summarizes the results for some different values of N. For $N \geq 3$, T is approximated by Eq. 3 to within an error of 1, but N_T is overestimated by between 1 and 2.

Now let’s take a leap of faith. In Eq. 3, we restrict T to be an integer. But we do not have to. Let’s set instead

$$\begin{aligned} & Nr^T = 1 \\ \Rightarrow \quad & T = -\ln(N)/\ln(r) \end{aligned} \tag{5}$$

We assume that the last exchange never takes place because the residual value falls just below 1 and, without any reason whatsoever, that we still over estimate the total by 1, then the total bottles of drinks is

$$\begin{aligned} N_T &= N \sum_{t=0}^{T-1} r^t - 1 \\ &= N \frac{1 - r^T}{1 - r} - 1 \\ &= \frac{N - 1}{1 - r} - 1 \\ &= \frac{N}{1 - r} - \frac{2 - r}{1 - r} \end{aligned} \tag{6}$$

N	True T	T from Eq. 3	True N_T	N_T from Eq. 4
1	0	0	1	1
2	1	2	3	4.6
3	3	3	7	8.2
4	4	4	11	12.2
5	5	5	15	16.4
6	6	6	19	20.8
7	7	6	23	24.3
8	7	7	27	28.8
9	8	7	31	32.4
10	8	8	35	37.0
20	11	10	75	76.6
30	12	11	115	116.2
40	13	12	155	156.2
50	14	13	195	196.4
60	15	14	235	236.8
70	15	14	275	276.3
80	16	15	315	316.8
90	16	15	355	356.4
100	16	16	395	397.0
1000	24	24	3995	3997.0
10000	32	32	39995	39997.0

Table 1: Total bottles of drinks and number of exchanges

N	True T	T from Eq. 5	True N_T	N_T from Eq. 6
1	0	0	1	-1
2	1	2.4	3	3
3	3	3.8	7	7
4	4	4.8	11	11
5	5	5.6	15	15
6	6	6.2	19	19
7	7	6.8	23	23
8	7	7.2	27	27
9	8	7.6	31	31
10	8	8.0	35	35
20	11	10.4	75	75
30	12	11.8	115	115
40	13	12.8	155	155
50	14	13.6	195	195
60	15	14.2	235	235
70	15	14.8	275	275
80	16	15.2	315	315
90	16	15.6	355	355
100	16	16.0	395	395
1000	24	24.0	3995	3995
10000	32	32.0	39995	39995

Table 2: An improved approximation of total bottles of drinks and number of exchanges

where T is from Eq. 5.

Table 2 shows the numerical values of the new approximation. Except for the special case of $N = 1$, N_T from Eq. 6 is exact (when $r = 3/4$)!

What's going on? Clearly Eq. 6 cannot be true for different exchange ratios of caps and empty bottles, as it will not be guaranteed to yield integer answers.

4 Exact solution

Let B bottles exchange for one bottle of drink and C caps exchange for one bottle of drink. The total number of drinks enjoyed after t exchanges satisfy the following recursion:

$$N_t = \sum_{\tau=0}^{t} n_\tau = N + \left\lfloor \frac{N_{t-1}}{B} \right\rfloor + \left\lfloor \frac{N_{t-1}}{C} \right\rfloor \tag{7}$$

This is because of all the drinks consumed, $\lfloor N_{t-1}/B \rfloor$ come from exchanging the empty bottles and $\lfloor N_{t-1}/C \rfloor$ come from exchanging the caps.

The number of drinks in round t is given by

$$n_t = N_t - N_{t-1} = N - \left(N_{t-1} - \left\lfloor \frac{N_{t-1}}{B} \right\rfloor - \left\lfloor \frac{N_{t-1}}{C} \right\rfloor \right) \tag{8}$$

We will show below that as the value of N_{t-1} increases, n_t will encounter 0. This fixed point satisfies the balance equation

$$N_T = N + \left\lfloor \frac{N_T}{B} \right\rfloor + \left\lfloor \frac{N_T}{C} \right\rfloor \tag{9}$$

where N_T is the total number of drinks consumed when no more exchanges can be made.

To solve Eq. 9, let $L = \mathrm{lcm}(B, C)$ be the least common multiple of B and C. We can write $N_T = LK + j$ where $j = 0, 1, \ldots, L-1$. Substituting this in Eq. 9, we have

$$\begin{aligned} LK + j &= N + \frac{L}{B}K + \left\lfloor \frac{j}{B} \right\rfloor + \frac{L}{C}K + \left\lfloor \frac{j}{C} \right\rfloor \\ \Rightarrow \left(L - \frac{L}{B} - \frac{L}{C} \right) K &= N + \left\lfloor \frac{j}{B} \right\rfloor + \left\lfloor \frac{j}{C} \right\rfloor - j \end{aligned} \tag{10}$$

We thus need to find j's such that

$$\left(L - \frac{L}{B} - \frac{L}{C} \right) \mid (N + \lfloor j/B \rfloor + \lfloor j/C \rfloor - j) \tag{11}$$

Because as j goes from 0 to L, $\lfloor j/B \rfloor + \lfloor j/C \rfloor - j$ decreases from 0 to $L/B + L/C - L - 1$, and the decrement with each unit increase of j is no greater than 1, thus there must be js where Eq. 11 hold.

Any j satisfying Eq. 11 in fact leads to a solution of Eq. 9. Indeed, our problem has multiple solutions if we allow borrowing empty bottles and empty caps and paying them back after consuming the exchanged drinks. Disallowing this borrowing, we are looking for the value j that satisfies Eq. 11 and leads to the smallest N_t in

$$N_T = \min_{j \text{ s.t. Eq. 11}} \frac{L}{L - \frac{L}{B} - \frac{L}{C}} \left(N + \left\lfloor \frac{j}{B} \right\rfloor + \left\lfloor \frac{j}{C} \right\rfloor - j \right) + j \tag{12}$$

Why is the smallest N_T that satisfies Eq. 9 the correct solution? Consider the following

$$N_t - \left\lfloor \frac{N_t}{B} \right\rfloor - \left\lfloor \frac{N_t}{C} \right\rfloor \tag{13}$$

This expression increases by at most 1 as N_t increases by 1, and it increases by $L - L/B - L/C$ when N_t increases from $L * k$ to $L * (k+1)$). This guarantees a fixed point N_T in Eq. 9. For any $N_t < N_T$ where N_T is the smallest value that satisfies Eq. 9,

$$N_t < N + \left\lfloor \frac{N_t}{B} \right\rfloor + \left\lfloor \frac{N_t}{C} \right\rfloor \tag{14}$$

Thus

$$n_{t+1} = N_{t+1} - N_t = N + \left\lfloor \frac{N_t}{B} \right\rfloor + \left\lfloor \frac{N_t}{C} \right\rfloor - N_t >= 1 \tag{15}$$

So there are more drinks to be had until the fixed point in Eq. 9 is reached.