Bottles and caps

Huafeng Xu

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It is rumored that some large language models can now solve the following math problem:

Suppose two empty bottles can exchange for one bottle of Fanta, and four bottle caps can exchange for one bottle of Fanta. If you start with $N = 5$ bottles of Fanta, how many total bottles of Fanta can you enjoy? (Fanta has not sponsored this article.)

(At the time of my writing, ChatGPT-4o still seems to struggle with the problem.)

The problem is straightforward to solve by recursion for general N. The bottles of drinks n_k, the number of empty bottles b_k, and the number of caps c_k after k rounds of exchanges satisfy the following recursion:

$$\begin{aligned} n_k &= \lfloor b_{k-1}/2 \rfloor + \lfloor c_{k-1}/4 \rfloor \\ b_k &= n_k + b_{k-1} - 2\lfloor b_{k-1}/2 \rfloor \\ c_k &= n_k + c_{k-1} - 4\lfloor c_{k-1}/4 \rfloor \end{aligned} \tag{1}$$

where $\lfloor x \rfloor$ denotes the largest integer no greater than x.

For $N = 5$, it is easy to see that after 5 exchanges one can have a total of $\sum_{k=0}^{5} n_k = 15$ bottles of drinks.

The recursion in Eq. 1 makes a closed form solution seem unlikely. But I found one, by starting with an approximation and making one small modification at the end that inexplicably makes the solution exact.

The residual "value" of a bottle of Fanta, *i.e.* the immediate exchange value of the empty bottle and the cap, is

$$r = 1/2 + 1/4 = 3/4 \tag{2}$$

where 1/2 comes from the empty bottle and 1/4 from the cap. This is the average number of bottles of drinks a consumed bottle of drink can exchange for.

The residual value can be exchanged for bottles of drinks that again carry their own residual values. Starting from N bottles of Fanta, after k rounds of exchanges, the residual value becomes Nr^k. The exchanges can continue until this residual value drops below 1 (when it can no longer be exchanged for one

N	True K	K from Eq. 3	True N_t	N_t from Eq. 4
1	0	0	1	1
2	1	2	3	4.6
3	3	3	7	8.2
4	4	4	11	12.2
5	5	5	15	16.4
6	6	6	19	20.8
7	7	6	23	24.3
8	7	7	27	28.8
9	8	7	31	32.4
10	8	8	35	37.0
20	11	10	75	76.6
30	12	11	115	116.2
40	13	12	155	156.2
50	14	13	195	196.4
60	15	14	235	236.8
70	15	14	275	276.3
80	16	15	315	316.8
90	16	15	355	356.4
100	16	16	395	397.0
1000	24	24	3995	3997.0
10000	32	32	39995	39997.0

Table 1: Total bottles of drinks and number of exchanges

additional bottle of drink). So the total rounds of exchanges K is approximately given by

$$\begin{aligned} & Nr^{K+1} < 1 \leq Nr^K \\ \Rightarrow \quad & K = \lfloor -\ln(N)/\ln(r) \rfloor \end{aligned} \tag{3}$$

The total number of bottles of drinks that one can get is thus

$$N_t = N \sum_{k=0}^{K} r^k = N \frac{1 - r^{K+1}}{1 - r} \tag{4}$$

Table 1 summarizes the results for some different values of N. For $N \geq 3$, K is approximated by Eq. 3 to within an error of 1, but N_t is overestimated by between 1 and 2.

Now let's take a leap of faith. In Eq. 3, we restrict K to be an integer. But we do not have to. Let's set instead

$$\begin{aligned} & Nr^K = 1 \\ \Rightarrow \quad & K = -\ln(N)/\ln(r) \end{aligned} \tag{5}$$

We assume that the last exchange never takes place because the residual value falls just below 1 and, without any reason whatsoever, that we still over

N	True K	K from Eq. 5	True N_t	N_t from Eq. 6
1	0	0	1	-1
2	1	2.4	3	3
3	3	3.8	7	7
4	4	4.8	11	11
5	5	5.6	15	15
6	6	6.2	19	19
7	7	6.8	23	23
8	7	7.2	27	27
9	8	7.6	31	31
10	8	8.0	35	35
20	11	10.4	75	75
30	12	11.8	115	115
40	13	12.8	155	155
50	14	13.6	195	195
60	15	14.2	235	235
70	15	14.8	275	275
80	16	15.2	315	315
90	16	15.6	355	355
100	16	16.0	395	395
1000	24	24.0	3995	3995
10000	32	32.0	39995	39995

Table 2: An improved approximation of total bottles of drinks and number of exchanges

estimate the total by 1, then the total bottles of drinks is

$$\begin{aligned} N_t &= N \sum_{k=0}^{K-1} r^k - 1 \\ &= N \frac{1-r^K}{1-r} - 1 \\ &= \frac{N-1}{1-r} - 1 \\ &= \frac{N}{1-r} - \frac{2-r}{1-r} \end{aligned} \tag{6}$$

where K is from Eq. 5.

Table 2 shows the numerical values of the new approximation. Except for the special case of $N = 1$, N_t from Eq. 6 is exact (when $r = 3/4$)!