Applied Algorithms, Summer 2017

Homework 3,

Due date: XX/7/17

**1**. (35 pts) Three selfish agents A,B,C, need to send messages from s to the bottom nodes (one message to each target, the target of agent X is node X, for X∈{A,B,C}). What is the social optimum cost and what are the Nash equilibrium routings in the fair cost- sharing model?

Your answer should distinguish between different ranges of x (≥0). What is the price of stability and the price of anarchy for each range?

Note: There are 5 different ranges.

3

2

1

1

x

6

**2.** (35 pts.)On unrelated machines, the processing time of a job depends on the machine on which it is assigned. The input is given as an n×m matrix that includes the processing times pij. For a given assignment, the cost of a job is the total processing time of the jobs assigned to its machine.

In this question you will prove that the *strong price of anarchy* for this model is at least m (the number of machines). The SPoA is the ratio between the makespan of the worst SE and the social optimum. Consider the following input (an example for m=5 is below): For each job Jj, j = 2, . . . ,m: pj,j = j, pj−1,j =1, and pi,j= ∞ for i ≠ j,j,-1. For job J1, p1,1=pm,1=1 and pi,1= ∞, for i ≠1,m.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | ∞ | ∞ | ∞ | 1 |
| 2 | 1 | 2 | ∞ | ∞ | ∞ |
| 3 | ∞ | 1 | 3 | ∞ | ∞ |
| 4 | ∞ | ∞ | 1 | 4 | ∞ |
| 5 | ∞ | ∞ | ∞ | 1 | 5 |

2.1. What is the makespan in the social optimum schedule (that achieves minimum makespan)?

2.2. Show that the schedule is which job j is assigned to machine j is a strong NE.

Hint: show by induction on j that job j is not part of a coalition.

2.3. Conclude that the SPoA in this game is at least m.

Remark: It is known that for any job scheduling game with m unrelated machines and n jobs, SPoA ≤ 2m−1.

**3**. (30 pts.) In class we saw that the Price of Anarchy for the following network with p=1 is 4/3.

Unfortunately, the ratio can be much worse when non-linear delay functions are allowed.

Assume that the delay on the lower edge is c(x)= xp.

3.1. What is the flow at Nash equilibrium?

What is the total cost?

3.2. What is the optimal flow?

What is the total cost of the optimal flow?

c(x)=1

c(x)=xp

3.3. What is the PoA as a function of p, and what is its value when p→∞ ?

Note: you will need simple calculus (derivative of a polynomial).

**Solution**

**1.** First note that Agent B will always route through V (this's the only possible path for him).

Therefore there are 4 possible routings:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  | A pays | B pays | C pays | Total cost | NE condition |
| 1 | A outer path  C outer path | 3 | x+6 | 2 | x+11 | x≥4 |
| 2 | A inner path  C outer path | x/2+1 | x/2+6 | 2 | x+9 | 3≤x≤4 |
| 3 | A outer path  C inner path | 3 | x/2+6 | x/2+1 | x+10 | never |
| 4 | A inner path  C inner path | x/3+1 | x/3+6 | x/3+1 | x+8 | x≤3 |

Agents A and C can chose between the inner or the outer path. Independent of x, the minimum total cost (Social optimum) is achieved when both A and C route via V. The total routing cost is x+1+1+6=x+8.

In order to calculate the condition for a routing to be a NE, we check when it is not beneficial for any of the agents to deviate.

For example, when both agents use the outer path (routing 1), A does not deviate if 3≤ x/2+1 and C does not deviate if 2≤ x/2+1. Any x≥4 achieves both conditions together.

The condition is calculated in a similar for the three other routings.

In order for routing 3 to be a NE it is required that both 2 ≥ x/2+1 (no deviation to routing 1) and 3 ≤ x/3+1 (no deviation to routing 4). That is x≤2 and x≥6. Both conditions cannot hold together.

Price of stability and price of anarchy:

For x>4, routing 1 is the only NE. PoS=PoA = (x+11)/(x+8).

For 3<x<4, routing 2 is the only NE, PoS=PoA= (x+9)/(x+8).

For x<3, routing 4 is the only NE. PoS=PoA=1.

For x=4, the best NE is routing 2 having cost x+9=13. The worst NE is routing 1 having cost x+11=15. Therefore, the PoA is 15/12=5/4 and the PoS is 13/12.

For x=3, the best NE is routing 4 with PoS =1. The worst NE is routing 2 with PoA=(x+9)/(x+8)=12/11.

**2.**

**1.**Themakespan is 1. Each job j is on a different machine i, for which pij=1.

**2**. We show by induction on j that job j is not part of a coalition.

Base case: Job 1 will not participate in any coalition because in any assignment it will be on a machine with load at least 1 – which is its current load, so there is no beneficial move for job 1.

Induction step: For any job j>1, the only beneficial move for j is to move to machine j-1. However, by the induction hypothesis, job j-1 is assigned on machine j-1 and will not leave it. This makes the move of j non-beneficial (together with job j-1, the load on machine j-1 will be at least j – the current cost of job j). Therefore, job j cannot be part of a coalition.

3. The resulting SPOA is m because we showed that some SE schedule has makespan m and the social optimum has cost 1.

**3**.

1. The flow at Nash equilibrium places the entire unit on the lower link, incurring a cost of 1.

2. Assume the cost split x on the lower and 1-x on the upper link. The total cost is *C(x)=x⋅xp + (1-x)=xp+1-x+1.* The minimal value of this function is achieved for *x=(p+1)-1/p*. (when *C’(x)=(p+1)xp-1=0)*. This solution has a total latency of 1*−p ·* (*p* + 1)*−*(*p*+1)*/p*, which tends to 0 as *p → ∞*.

3. The PoA is 1/ (1*−p ·* (*p* + 1)*−*(*p*+1)*/p)*, which tends to *∞* as *p → ∞*.