Taylor series expansion

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Definition (**Sequences**)

A (infinite) sequence a_n is a list of infinite many numbers in a pre-dfined order.

- A sequence a_n can be treated as a function defined for integer numbers,

$$a_n = f(n) : \mathcal{N} \to \mathcal{R}$$

Definition (**Series**)

Given a sequence $\{a_n\}_{n=1}^{\infty}$, the series is $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

- A partial sum of the series is defined by

$$S_n = \sum_{k=1}^{n} a_k$$

Definition

A series $\sum_{n=1}^{\infty} a_n$ *is convergent if the limit* $\lim_{n\to\infty} S_n$ *exists. If the series does not converge it is called divergent*

Theorem

If a series $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

- $\lim_{n\to\infty} a_n = 0$ is not enough to guarantee that a series converges.
- Consider $\sum_{n=1}^{\infty} n$, $\sum_{n=0}^{\infty} ar^n$, $\sum_{n=1}^{\infty} 1/n$, $\sum_{n=1}^{\infty} 1/(n^2+n)$.

Ratio test

We will not cover proof of the results in this class. Let $\lim_{n\to\infty} |a_{n+1}/a_n| = L$.

- *If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.*
- *If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.*
- When $L = 1$, carefully apply the result.

Power series

Definition

A power series centered at $x = a$ has the form of $f(x) = \sum_{n=1}^{\infty} b_n(x-a)^n$, i.e. a series where the terms are $a_n = b_n(x-a)^n$.

- The domain of the function $f(x)$ consists of all real numbers x such that the power series is convergent, i.e.

$$D(f) = \left\{ x : \sum_{n=1}^{\infty} b_n(x-a)^n \ \ \textit{is convergent} \right\}$$

- Let's apply the ratio test to study domain of power series.

Radius of convergencs

With $a_n = b_n(x-a)^n$, compute

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{b_{n+1}(x-a)^{n+1}}{b_n(x-a)^n}\right| \\ &= |x-a|\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|.\end{aligned}$$

Define a number R by

$$R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|.$$

Thus, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-a|}{R},$$

and the power series is convergent if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-a|}{R} < 1, \quad i.e. \;\; |x-a| < R.$$

Definition (Radius of convergence)

The radius of convergence R of a power series $\sum_{n=1}^{\infty} b_n(x-a)^n$: A power series is **convergent** *for all x with $|x-a| < R$ and* **diverges** *for all x with $|x-a| > R$.*

- There is no general rule for $|x-a| = R$.
- If $R = 0$ the series converges only at $x = a$.
- If $R = \infty$ the series converges for all $x \in \mathcal{R}$.

ex 1.) $\sum_{n=0}^{\infty}(n!)x^n$.

ex 2.) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$.

ex 3.) $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$.

Derivatives and Integrals

Theorem

Supposet that a power series

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n,$$

has the radius of convergence R. Then, if differentiation and summation in the power series for f can be interchanged, for all x with $|x-a| < R$, we have

$$\begin{aligned} \frac{df(x)}{dx} &= \sum_{n=0}^{\infty} (b_n (x-a)^n,)' = \sum_{n=1}^{\infty} n b_n (x-a)^{n-1} \\ \frac{d^2 f(x)}{dx^2} &= \sum_{n=2}^{\infty} n(n-1) b_n (x-a)^{n-2} \\ &\vdots \end{aligned}$$

Derivatives and Integrals

Theorem

Supposet that a power series

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n,$$

has the radius of convergence R. *Then, if integration and summation in the power series for* f *can be interchanged, for all* x *with* $|x-a| < R$, *we have*

$$\int f(x)dx = \sum_{n=0}^{\infty} \int b_n (x-a)^n dx C + \sum_{n=0}^{\infty} \frac{b_n}{n+1} (x-a)^{n+1}$$

Taylor series expansion

- Using $f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$, what is $f(a)$?
- Using $\frac{df(x)}{dx} = \sum_{n=1}^{\infty} nb_n(x-a)^{n-1}$, what is $f'(a)$?
- $f''(a) = ?$
- $f^{(k)}(a) = ?$

Definition (Taylor series expansion around $x = a$)

The Taylor expansion around $x = a$ of a function f has the following form,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- When $a = 0$, we call it Maclaurin series of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n.$$

- Find Maclaurin series for
  1. $f(x) = e^x$
  2. $f(x) = a^x, (a > 0)$
  3. $f(x) = sin`x$
  4. $f(x) = cos`x$
- Find the value $\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$.

Theorem

The Taylor expansion to order N around $x = a$ of a function f has the following form,

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x).$$

If $f^{(n+1)}$ is continuous on an interval I that contains a, and x is in I, then there exists a number c between a and x such that

$$R_N(x) = \frac{1}{N!}\int_a^x (x-t)^N f^{(N+1)}(t)dt = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1},$$

- $R_N(x)$ is called the remainder of the Taylor series.
- For x with $|x-a| < R$, $\lim_{n\to\infty} R_n(x) = 0$, since the series is convergent.

Q & A

Thank You !