Derivatives

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March 15, 2021

Definition

The **derivative of the function** *f at a is the value of the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

- f is said to be **differentiable at** a if this limit exists.
- f is called **differentiable on the interval** (c, d) if it is differentiable at every point a in (c, d).
- Other notations

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Derivatives

- Instead of a and h, use x and Δ_x.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{\Delta_x \to 0} \frac{f(x+\Delta_x)-f(x)}{\Delta_x} \\ &= \lim_{\Delta_x \to 0} \frac{\Delta_y}{\Delta_x}, \\ &= \frac{dy}{dx} \end{aligned}$$

where $\Delta_y = f(x+\Delta_x) - f(x)$.

- Hence, we can write $f'(x) = \frac{df}{dx}$.

Examples

1. $f(x) = ax + b$
2. $f(x) = ax^2 + bx + c$
3. $f(x) = x^n,\ n \in \mathbb{N}^+$
4. $f(x) = x^a,\ a \in \mathbb{R}$
5. $f(x) = a^x$
6. $f(x) = sin(x), cos(x), tan(x)$
7. $f(x) = log(x)$

The differentiation Rules

- Constant rule : $c' = 0$
- Sum rule : $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- Product rule : $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
- Quotient rule : $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
- Chain rule : $f(g(x))' = f'(g(x))g'(x)$

Find the derivative of $f(x)$:

1. $f(ax)$
2. $f(x) = x^2 sin(x)$
3. $f(x) = e^{x^n}$
4. $f(x) = x^x$
5. $f(x)g(x)h(x)$
6. $f(g(h(x)))$
7. $V = \frac{4}{3}\pi r^3$, where $r = f(t)$. DV/dt?

Differentiable implies Continuous

Theorem

If a function f is differentiable at some a in its domain, then f is also continuous at a.

Proof)

- The reverse is not true!
  - $f(x) = |x|$
  - $f(x) = \sqrt{|x|}$

Higher Derivatives

- The nth derivative of f is denoted $f^{(n)}$:

$$\frac{d^n y}{dx^n} = f^{(n)}(x)$$

- Usually,

$$\frac{d^2 y}{dx^2} \neq (\frac{dy}{dx})^2$$

Implicit differentiation

- Consider the defining equation for the function $y = f(x)$;

$$F(x, y) = 0.$$

Ex)

$$x^2 + y^2 = 1$$
$$y + sin(y) = x$$

Derivative of inverse function

- $f^{-1}(x)$: solution of the equation $f^{-1}(f(x)) = x$ for all x, with given f. How to find
$$\frac{d}{dx} f^{-1}(x)?$$
- $f^{-1}(x) = arcsin(x)$, with $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Parameteric

- If $x = f(t)$ and $y = g(t)$, how to find dy/dx?

Proof) Let z=x(t+h)-x(t).

Ex) $x(t) = e^t$ and $y(t) = tan(t)$.

Tangent and Normal lines to graph

- The slope of the tangent line to the graph of f at the point $(a, f(a))$ is
$$m = f'(a)$$
- Tangent line at the point $(a, f(a))$:
$$y = f(a) + f'(a)(x - a)$$
- Normal line at the point $(a, f(a))$:
$$y = f(a) + f'(a)(x - a)$$

- The linearization of f at a $L(x)$:

$$L(x) = f'(a)(x-a) + f(a)$$

- The tangent line is a good approximation (linear approximation) to the values of the function as long as x is close to a.

Ex) Find the approximate value of $\sqrt{9.1}$ (Hint: use $f(x) = \sqrt{x+2}$).

- $dy = f'(x)dx$ and $\Delta_y = f(x + \Delta_x) - f(x)$.

Definition

1. *A function is called* **increasing** *if* $a < b$ *implies* $f(a) < f(b)$ *for all* a, b *in the domain of* f.
2. *A function is called* **decreasing** *if* $a < b$ *implies* $f(a) > f(b)$ *for all* a, b *in the domain of* f.
3. *A function is called* **non-decreasing** *if* $a < b$ *implies* $f(a) \leq f(b)$ *for all* a, b *in the domain of* f.
4. *A function is called* **non-increasing** *if* $a < b$ *implies* $f(a) \geq f(b)$ *for all* a, b *in the domain of* f.

Theorem

If a function is non-decreasing (non-increasing) on an interval $a < x < b$ *then* $f'(x) \geq 0 (f'(x) \leq 0)$ *for all* x *in the interval.*

Proof)

Increasing and decreasing function

Theorem

Suppose f is a differentiable function on an interval (a, b).

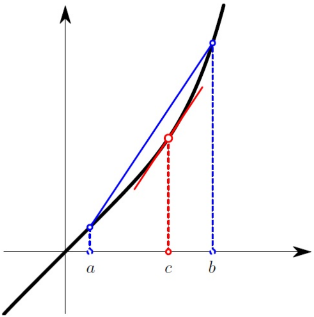
1. *If $f'(x) > 0$ for all $a < x < b$, then f is increasing.*
2. *If $f'(x) < 0$ for all $a < x < b$, then f is decreasing.*

Proof) Use the mean value theorem.

Theorem

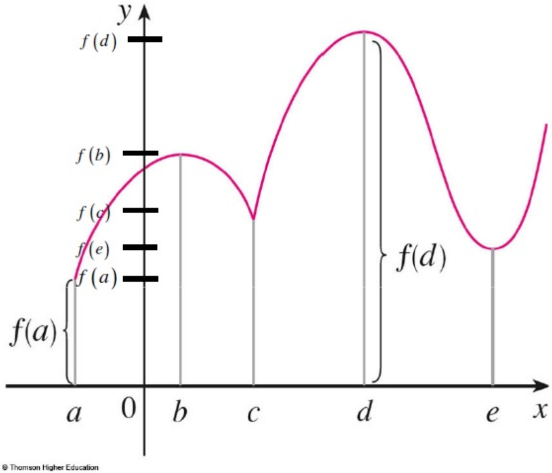
If f is a differentiable function on an interval $a < x < b$, then there is some number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



value.pdf

Maximum and minimum



Maximum and minimum

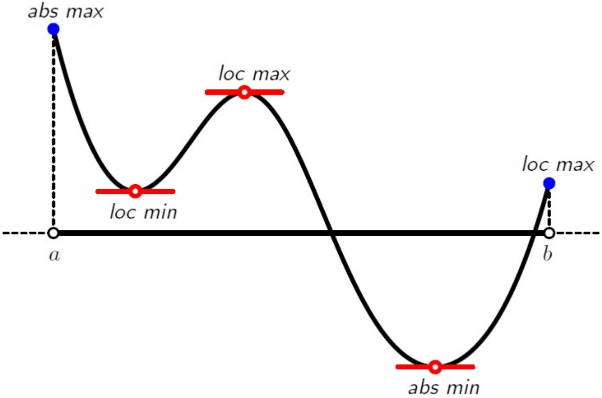
Definition

Any x value for which $f'(x) = 0$ is called a **stationary point** *for the function f.*

Definition

1. *A function has a* **global maximum** *at some a in its domain if $f(x) \leq f(a)$ for alll other x in the domain of f*
2. *A function has a* **global miniimum** *at some a in its domain if $f(x) \geq f(a)$ for alll other x in the domain of f*
3. *A function has a* **local maximum** *at some a in its domain if there exists a small $\delta > 0$ such tat $f(x) \leq f(a)$ for alll $x \in (a - \delta, a + \delta)$ which lie in the domain of f*
4. *A function has a* **locall minimum***at some a in its domain if there exists a small $\delta > 0$ such tat $f(x) \geq f(a)$ for alll $x \in (a - \delta, a + \delta)$ which lie in the domain of f*

Maximum and minimum



Maximum and minimum

Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an global maximum value $f(c)$ and global minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Theorem

If f has a local maximum or local minimum at c and $f'(c)$ exists then $f'(c) = 0$.

Theorem

Suppose f is a differentiable on $[a, b]$. Every local maximum or minimum of f is either one of the end points of the interval, or else it is a stationary point for f.

Ex) $y = x^3, y = 1/x^2$

Maximum and minimum

How to tell if a stationary point is a maximum, a minimum, or neither?

Theorem

If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, where $x \in (c - \delta, c + \delta), \delta > 0$ then f has a local maximum at $x = c$.

Theorem

If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, where $x \in (c - \delta, c + \delta), \delta > 0$ then f has a local minimum at $x = c$.

Convex and Concave

Definition

1. *A function f is* **convex** *on $[a, b]$ if for all x_1, x_2 satisfying $a \leq x_1 < x_2 \leq b$, and all $\lambda \in [0, 1]$, we have*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

2. *A function f is* **concave** *on $[a, b]$ if for all x_1, x_2 satisfying $a \leq x_1 < x_2 \leq b$, and all $\lambda \in [0, 1]$, we have*

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2).$$

- Geometrically, the line segment connecting two points $(x_1, f(x_1)$ and $(x_2, f(x_2))$ must sit above the graoh of the convex function$f$.
- f is concave if f is convex.

ex) $x, x^2, e^x, -log(x), xlogx$?

cf) Strictly convex and strictlry concave.

cf) Inflection point

First and second order characterizations of convex functions

Theorem

Suppose f is twice differentiable on (a, b). Then the followings are equivalent:

1. f *is convex.*
2. $f(x_2) \geq f(x_1) + f'(x_1)(x_2 - x_1)$, *for all* $x_1, x_2 \in (a, b)$.
3. $f''(x) \geq 0$ *for all* $x \in (a, b)$.

Corollary

If f is convex and differentiable then any point x that satisfies $f'(x) = 0$ is a global minimum.

(1) 종이에 그림을 인쇄하려고 한다. 종이의 정중앙에 그림이 인쇄되고 그림의 크기는 60 cm^2이다. 좌우 여백은 각각 5cm, 상하 여백은 각각 3cm이다. 이 때, 그림을 인쇄하기 위한 종이의 최소 면적을 구하시오.

(2) 뚜껑이 없는 사각 상자의 부피가 4000이라고 한다. 이러한 상자 중 최소 겉넓이를 갖는 상자의 규격을 구하시오. (규격: 가로*세로*높이)

(3) 타원 $4x^2+y^2=4$위의 점 중 점 $(1,0)$과 가장 멀리 떨어져 있는 점을 찾으시오.

(4) 한 극장에서 표 한 장의 가격이 8000원 일 때, 한 공연 당 1000장의 표가 팔린다고 한다. 표 한 장의 가격을 100원 내리면 20장이 더 팔린다고 할 때 수입을 최대로 올릴 수 있는 표의 가격을 찾으시오.

Q & A

Thank You !