Limits and Contiunous Functions

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Outline

- Limit
- Continuity

Limit

Definition ($\lim_{x \to a+0} f(x) = L$)

L is the right-limit of $f(x)$ as x aprroaches a, if

1. *$f(x)$ need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a.*
2. *for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all x in the domain of f, if $a < x < a + \delta$, then $|f(x) - L| < \epsilon$.*

Definition ($\lim_{x \to a-0} f(x) = L$)

L is the right-limit of $f(x)$ as x aprroaches a, if

1. *$f(x)$ need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a.*
2. *for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all x in the domain of f, if $a - \delta < x < a$, then $|f(x) - L| < \epsilon$.*

Limit

Definition ($\lim_{x \to a} f(x) = L$)

L is the limit of $f(x)$ as x aprroaches a, if

1. *$f(x)$ need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a.*
2. *for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all x in the domain of f, if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.*

Definition (Another definition of $\lim_{x \to a} f(x) = L$)

1. $\lim_{x \to a-0} f(x)$ *and* $\lim_{x \to a+0} f(x)$ *exist.*
2. $\lim_{x \to a-0} f(x) = \lim_{x \to a+0} f(x) = L$

Limit

- Properties of the Limit

  Let a and k be constants. Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ (with L and M finite).

  1. $\lim_{x \to a} x = a$
  2. $\lim_{x \to a} kf(x) = kL$
  3. $\lim_{x \to a} \{f(x) \pm g(x)\} = L \pm M$
  4. $\lim_{x \to a} \{f(x)g(x)\} = LM$
  5. $\lim_{x \to a} \{f(x)/g(x)\} = L/M, M \neq 0$
  6. $\lim_{x \to a} \{f(x)\}^n = L^n.$

Limit

- The value of the limit can become $\pm\infty$: $\lim_{x \to a} \frac{1}{x}$ when $a = 0$ or $\pm\infty$.
- $\frac{0}{0}$?
- $\frac{\infty}{\infty}$?
- $\infty - \infty$?
- $0 \cdot \infty$?

Limit

- mathmatical constant e:

$$e = \lim_{x \to 0}(1+x)^{\frac{1}{x}} = 2.71828\ldots.$$

- Natural log ln:

$$ln(x) = log_e(x)$$

- If $\lim_{x \to 0} f(x) = 0$, $\lim_{x \to 0}(1+f(x))^{\frac{1}{f(x)}} = e$

Theorem (The sandwich Theorem)

Suppose that for all x

$$f(x) \leq g(x) \leq h(x)$$

and that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$. *Then*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x).$$

Ex) $\lim_{x \to 0} \frac{\sin x}{x}$?

Limit

- Multiple limits:

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y) = \lim_{x \to x_0} (\lim_{y \to y_0} f(x, y)) \lim_{y \to y_0} \lim_{x \to x_0} f(x, y) = \lim_{y \to y_0} (\lim_{x \to x_0} f(x, y))$$

- The order of limits matters!
  - $f(x, y) = x^2 y - e^{-x-y}, (x, y) = (0, 1)$
  - $f(x, y) = (2x - y)/(x + 3y), (x, y) = (0, 0)$

Definition

A function f is **right − continuous at a** *if*

$$\lim_{x \to a+0} f(x) = f(a)$$

Definition

A function f is **left − continuous at a** *if*

$$\lim_{x \to a-0} f(x) = f(a)$$

Continuity

Definition

A function f is **continuous at a** *if*

$$\lim_{x \to a} f(x) = f(a)$$

Definition

A function is **continuous** *if it is continuous at every a in its domain.*

Continuity

- Propoerties for continuous function

  If function $f(x)$ and $g(x)$ are continuus at $x = a$, the followings are continuous at $x = a$.

  1. $kf(x)$, where k is constant
  2. $f(x) \pm g(x)$
  3. $f(x)g(x)$
  4. $\frac{f(x)}{g(x)}$, where $g(a) \neq 0$.

- Examples of continuous function
  - Polynomial($x^2 + 3$), Rational($1/x$), Irrational($\sqrt{x}$)
  - Exponential(e^x), log($\log x$),Trigonometric($sinx$, $cosx$)

Theorem (The Maximum-Minimum Theorem)

If $f(x)$ is continuous on $I = [a, b]$, a closed and bounded interval, then $f(x)$ contains both an absolute maximum and an absolute minimum on I.

Theorem (Intermediate value Theorem)

Suppose that $f(x)$ is continuous on $I = [a, b]$, a closed and bounded interval and that $f(a) \neq f(b)$. For all u s.t $f(a) < u < f(b)$, there is a $c \in (a, b)$ such that $f(c) = u$.

Q & A

Thank You !