MATH 423 Real Analysis
Homework 8

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Problem 17

(a) Denote $f_n(x) = f(x, t_n)$ where $t_n \in [a, b]$ and $t_n \to t_0, \forall n \in \mathbb{N}$.

Since $|f_n(x)| = |f(x, t_n)| \leq g(x)$, for all x, t_n and $lim_n f_n(x) = lim_{t \to t_0} f(x, t) = f(x, t_0)$, according to Dominated Convergence Theorem, we have $\int f(x, t_0) = lim_n \int f_n$.

According to definition of F, we have $F(t_0) = \int f(x, t_0)$ and $F(t_n) = \int f_n$. Therefore, $F(t_0) = lim_n F(t_n) = lim_{t \to t_0} F(t)$.

In particular, if $f(x, \cdot)$ is continuous for each x, which means $lim_{t \to t_0} f(x, t) = f(x, t_0)$ for all x and $t_0 \in [a, b]$, then it's easy to get $F(t_0) = lim_{t \to t_0} F(t)$ for all $t_0 \in [a, b]$. Thus F is continuous.

(b) Denote $h_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}$ where $t_n \in [a, b]$ and $t_n \to t_0, \forall n \in \mathbb{N}$.

Since $\frac{\partial f}{\partial t}$ exists, we know that $\frac{\partial f}{\partial t}(x, t_0) = lim\, h_n(x)$. In addition, $|h_n(x)| \leq sup_{t \in [a.b]} |\frac{\partial f}{\partial t}(x, t)| \leq g(x)$. Therefore, according to Dominated Convergence Theorem, $\int \frac{\partial f}{\partial t}(x, t_0) = lim \int h_n(x)$.

Besides, we know

$$F'(t_0) = lim \frac{F(t_n) - F(t_0)}{t_n - t_0} = lim \frac{\int f(x, t_n) - \int f(x, t_0)}{t_n - t_0} = lim \int \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0} = lim \int h_n(x)$$

Therefore, we have $F'(t_0) = \int \frac{\partial f}{\partial t}(x, t_0)$ for all $t_0 \in [a, b]$.

Problem 18

Let's assume the notation as in Theorem 2.28 and Exercise 2.23 in Folland.

$$
\begin{aligned}
&\{x \in [a,b] : f \text{ is discontinuous at } x\} \text{ has Lebesgue measure zero} \\
\Longleftrightarrow & f \text{ is continuous } a.e. \\
\Longleftrightarrow & H = h \ a.e. \text{ (according to Exercise 2.23(a) in Folland)} \\
\Longleftrightarrow & G = g \ a.e. \text{ (since } H = G \ a.e. \text{ and } h = g \ a.e. \text{ from Exercise 2.23(b) in Folland)} \\
\Longleftrightarrow & \int G \ dm = \int g \ dm \text{ (according to Proposition 2.23(b) in Folland)} \\
&\text{(since } \int G \ dm = \int lim \, G_{P_k} \ dm = lim \int G_{P_k} \ dm = \bar{I}_a^b(f) \text{ and similarly } \int G \ dm = \underline{I}_a^b(f)) \\
\Longleftrightarrow & \bar{I}_a^b(f) = \underline{I}_a^b(f) \\
\Longleftrightarrow & f \text{ is Riemann integrable}
\end{aligned}
$$

Note that $\int lim \, G_{P_k} \ dm = lim \int G_{P_k} \ dm$ is due to Dominated Convergence Theorem

Problem 19

$$
\begin{aligned}
&\int_0^\infty x^{a-1}(e^x - 1)^{-1} \ dx \\
=&\int_0^\infty x^{a-1} \frac{e^{-x}}{1 - e^{-x}} \ dx \\
=&\int_0^\infty x^{a-1} \sum_{n=1}^\infty (e^{-x})^n \ dx \\
=&\sum_{n=1}^\infty \int_0^\infty x^{a-1} e^{-nx} \ dx \text{ (according to Theorem 2.15 in Folland)} \\
=&\sum_{n=1}^\infty n^{-a} \int_0^\infty x^{a-1} e^{-x} \ dx \\
=&\zeta(a)\Gamma(a)
\end{aligned}
$$