



Lancaster University College
at Beijing Jiaotong University

2022/23 Examinations

Course code: [C108005W](#)

Course name: [Probability Theory and Mathematical Statistics \(B\)](#)

Midterm examination (November)

INSTRUCTIONS TO STUDENTS

- 1) Duration of the exam: [120 minutes](#)
- 2) This paper contains [3](#) pages. There are [8](#) questions.
- 3) You must answer all questions.
- 4) This is a closed book exam. No books or notes may be brought into the exam room.
- 5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
- 6) Some values that might be useful:

$$\Phi(0.94281) = 0.82711, \quad \Phi(1.33333) = 0.90879,$$

$$\Phi(1.88562) = 0.97033, \quad \Phi(2) = 0.97725,$$

$$\Phi(2.66667) = 0.99617$$

1. (20pt, 2pt for each blank)

- (1) Events A and B are disjoint and independent, $P(A) = 0.5$, then $P(A) + P(B) =$ _____.
- (2) Out of the students in a class, 50% are geniuses, 70% love chocolate. Let p be the conditional probability that a randomly selected student is a chocolate lover given that he/she is a genius. Then the theoretically lowest value of p is _____.
- (3) $2n$ students are assigned randomly to 2 classes each having n students. A couple of boyfriend and girlfriend are in the $2n$ students. The probability that this couple are in the same class is _____.
- (4) When it rains, there is a 40% chance that students play football on that day. On the other hand, when it does not rain there is a 90% chance that they play. The probability that it will rain tomorrow is 0.2. The probability that students play football tomorrow is _____.
- (5) Let $X \sim B(20, 0.8)$, then the most likely value of X is _____.
- (6) The random variable X has PDF $f(x) = Ce^{-3|x|}$. Then $C =$ _____ and $P(|X| < 1) =$ _____.
- (7) Let $X \sim U(0, 1)$, $Y \sim U(0, 1)$, and X and Y be independent. Then $P(X = Y) =$ _____ and $P\left((X - \frac{1}{2})^2 + (Y - \frac{1}{2})^2 < \frac{1}{4}\right) =$ _____.
- (8) The joint CDF of X and Y is $F(x, y) = \begin{cases} \Phi(x)(1 - e^{-y}), & y > 0 \\ 0, & \text{otherwise.} \end{cases}$. Then the joint PDF of X and Y is $f(x, y) =$ _____.

2. (10pt) Approximately 0.04% of human have liver cancer. A person with liver cancer has a 95% chance of a positive test, while a person without liver cancer has a 2% chance of a false positive result. What is the probability a person has liver cancer given that the person just had a positive test?

3. (10pt) Suppose that random variable X and X itself are independent. Show that there must exist a constant C such that $P(X = C) = 1$. (Hint: consider its CDF and use the properties of CDFs)

4. (10pt) A fair coin is tossed two times independently. Let

$$H_i = \begin{cases} 1, & \text{if head is obtained at the } i\text{th toss} \\ 0, & \text{otherwise.} \end{cases}$$

Let $X = H_1 + H_2$, $Y = |H_1 - H_2|$. Give the joint PMF of X and Y in table form.

5. (10pt) A book has 300 pages. For each page, whether it contains typing error has Bernoulli distribution $B(1, 0.02)$. The events for each page to contain typing error are independent. What is (approximately) the probability that this book contains at most 3 pages with typing error?

6. (12pt) Scores on a certain standardized test, IQ scores, are approximately normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. Here we are referring to the distribution of scores over a very large population, and we approximate that discrete cumulative distribution function by a normal continuous cumulative distribution function.

(a) An individual is selected at random. What is the probability that his score X satisfies $120 < X < 130$?

(b) Two individuals are selected at random, independently. What is the probability that their scores X and Y satisfy $\max(X, Y) > 120$?

7. (14pt) Suppose that the joint distribution of X and Y is uniform over the region in the xy -plane bounded by the four lines $x = -1$, $x = 1$, $y = x + 1$, and $y = x - 1$. Determine

(a) $P(XY > 0)$

(b) the joint PDF of X and Y

(c) the PDF of X

(d) the conditional PDF of Y given that $X = x$

(e) $P(Y > 0 | X = \frac{1}{2})$.

8. (14pt) Let X and Y be random variables for which the joint PDF is as follows:

$$f(x, y) = \begin{cases} 8xy & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the PDF of $Z = (X - Y)^2$.