Probability and Statistics
Model Answer of 2020 Midterm Exam
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1. (20pt) Fill in the blanks:

1. Two events A and B are independent, and $P(A) = 0.4$, $P(B) = 0.7$, then $P(A\overline{B}) =$ $\underline{\mathbf{0.12}}$.

2. A box contains 3 left shoes and 3 right shoes. If two shoes are randomly chosen from the box, then the probability that they are a pair (i.e. a left shoe and a right shoe) is $\underline{\frac{\mathbf{C_3^1 C_3^1}}{\mathbf{C_6^2}} = \frac{\mathbf{3}}{\mathbf{5}}}$.

3. Let the CDF of random variable X be $F(x) = \frac{1}{1+2^{-x}}$. Then $P(X < 1 | X > 0) = \underline{\frac{\mathbf{1}}{\mathbf{3}}}$.

4. Let the PDF of random variable X be $f(x)$. $Y = 2X$. Then the PDF of Y is $g(y) = \underline{\frac{\mathbf{1}}{\mathbf{2}}\mathbf{f}(\frac{\mathbf{y}}{\mathbf{2}})}$.

5. Suppose that $X \sim N(1, 5)$. $Y = 3X - 2$. Then $Y \sim \underline{\mathbf{N(1, 45)}}$.

6. Suppose that X has the uniform distribution on the interval $[2, 5]$. Then $P(3 < X \leq 4) = \underline{\frac{\mathbf{1}}{\mathbf{3}}}$.

7. A neighborhood, which has a gymnasium in it, has 1000 residents. Every morning each resident independently has 0.05 chance to go to the gymnasium. Then the number of gymnasium goers every morning approximately has the **Poisson** distribution with parameter $\lambda = \underline{\mathbf{50}}$.

8. Let the joint CDF of X and Y be $F(x, y) = \frac{1}{(1+e^{-x})(1+3^{-y})}$. Then the CDF of X is $\underline{\frac{\mathbf{1}}{\mathbf{1+e^{-x}}}}$ and the PDF of X is $\underline{\frac{\mathbf{e^{-x}}}{\mathbf{(1+e^{-x})^2}}}$.

2. (10pt) A company produces products at three different factories A, B, and C. Of the company's total volume, factory A produces 20%, factory B produces 50%, and factory C produces the rest. The product defective rates at the factories are 5% at factory A, 2% at factory B, and 10% at factory C. If you buy this product and it turns out to be defective, what is the probability that it was produced at factory A?

Solution: We have

$$P(\text{“made\_at\_A”}) = 0.2,\ P(\text{“made\_at\_B”}) = 0.5,\ P(\text{“made\_at\_C”}) = 0.3$$

$$P(\text{“defective”}|\text{“made\_at\_A”}) = 0.05,\ P(\text{“defective”}|\text{“made\_at\_B”}) = 0.02,$$

$$P(\text{“defective”}\,|\,\text{“made\_at\_C”}) = 0.1.$$

$$P(\text{“defective”}) = P(\text{“made\_at\_A”})P(\text{“defective”}\,|\,\text{“made\_at\_A”})$$

$$+P(\text{“made\_at\_B”})P(\text{“defective”}\,|\,\text{“made\_at\_B”})+P(\text{“made\_at\_C”})P(\text{“defective”}\,|\,\text{“made\_at\_C”})$$

$$= 0.2 * 0.05 + 0.5 * 0.02 + 0.3 * 0.1 = 0.05.$$

So

$$P(\text{“made\_at\_A”}\,|\,\text{“defective”}) = \frac{P(\text{“made\_at\_A”})P(\text{“defective”}\,|\,\text{“made\_at\_A”})}{P(\text{“defective”})}$$

$$= \frac{0.2 * 0.05}{0.05} = 0.2.$$

3. (10pt) An airline sells 200 tickets for a certain flight on an airplane that has only 198 seats because, on the average, 1% of purchasers of airline tickets do not appear for the departure of their flight. Determine the probability that everyone who appears for the departure of this flight will have a seat.

Solution: Let X denote the number of people who bought the ticket but do not appear. Then $X \sim B(200, 0.01)$, and approximately $X \sim P(2)$. The event that everyone who appears for the departure of this flight has a seat is $X \geq 2$. Thus the desired probability is

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-2}(\frac{2^0}{0!} + \frac{2^1}{1!}) = 0.594.$$

4. (12pt) In a population, the men's weight (in kilograms) has the distribution $N(60, 25)$ and the women's weight (in kilograms) has the distribution $N(50, 25)$. Randomly and independently pick a man and a woman from the population.
(1) What is the probability that the two persons' total weight exceeds 130 kilograms?
(2) What is the probability that the difference between the two persons' weights is smaller than 10 kilograms?

Solution: Let X be the selected man's weight and Y be the selected woman's weight, both in kilograms. Then $X \sim N(60, 25)$, $Y \sim N(50, 25)$, and X and Y are independent.

(1) $X + Y \sim N(110, 50)$, and thus

$$P(X + Y > 130) = P(\frac{X + Y - 110}{\sqrt{50}} > 2.8284) = 1 - \Phi(2.8284) = 1 - 0.99766 = 0.00234.$$

(2) $X - Y \sim N(10, 50)$, and thus

$$P(|X - Y| < 10) = P(-10 < X - Y < 10) = P(\frac{-10 - 10}{\sqrt{50}} < \frac{X - Y - 10}{\sqrt{50}} < \frac{10 - 10}{\sqrt{50}})$$

$$= \Phi(0) - \Phi(-2.8284) = 0.5 - 1 + \Phi(2.8284) = 0.49766.$$

5. (14pt) The joint PDF of X and Y is given by

$$f(x,y)=\begin{cases} 1, 0<x<1, 0<y<2x, \\ 0, \textbf{otherwise}. \end{cases}$$

Find
(1) the marginal PDF $f_X(x)$;
(2) the conditional PDF $f_{Y|X}(y|x)$;
(3) $P(Y \le \frac{1}{2}|X \le \frac{1}{2})$.

Solution: (1) If $x \le 0$ or $x \ge 1$, then $f_X(x)=0$.
If $0<x<1$, then

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_0^{2x} dy=2x.$$

Thus,

$$f_X(x)=\begin{cases} 2x, & \text{if } 0<x<1, \\ 0, & \text{otherwise}. \end{cases}$$

(2) If $0<x<1$, then

$$f_{Y|X}(y|x)=\begin{cases} \frac{1}{2x}, & \text{if } 0<x<1, 0<y<2x, \\ 0, & \text{otherwise}. \end{cases}$$

(3)

$$P(X \le \frac{1}{2})=\int_{-\infty}^{\frac{1}{2}} f_X(x)dy=\int_0^{\frac{1}{2}} 2xdx=\frac{1}{4}.$$

$$P(X \le \frac{1}{2}, Y \le \frac{1}{2})=\int_0^{\frac{1}{4}}\int_0^{2x} dydx+\int_{\frac{1}{4}}^{\frac{1}{2}}\int_0^{\frac{1}{2}} dydx=\frac{3}{16}.$$

$$P(Y \le \frac{1}{2}|X \le \frac{1}{2})=\frac{\frac{3}{16}}{\frac{1}{4}}=\frac{3}{4}.$$

6. (10pt) Suppose that a fair coin is tossed repeatedly until two consecutive heads or two consecutive tails appear. Let X be the number of tosses required. Let $Y=1$ if it is two consecutive heads and $Y=0$ if it is two consecutive tails. Determine the joint PF of X and Y.

Solution: The event $X=1$ is impossible.
For $n \ge 2$, the event $X=n$ and $Y=1$ is the sequence $TH...THH$ when n is odd, or $HT...HTHH$ when n is even. In either case, the probability is $\frac{1}{2^n}$. Hence $P(X=n, Y=1)=\frac{1}{2^n}$ for $n \ge 2$.
Similarly, we have $P(X=n, Y=0)=\frac{1}{2^n}$ for $n \ge 2$.

7. (12pt) Let random variables X and W be independent and distributed uniformly on the interval $[0,1]$. Let $Y=-W$.
(1) What is the distribution of Y?
(2) Are X and Y independent?
(3) Find the PDF of $Z=X+Y=X-W$.

Solution: (1) $Y \sim U[-1, 0]$.
(2) Yes.
(3) If $z \leq -1$ or $z \geq 1$, then obviously $f_Z(z) = 0$.
If $-1 < z < 0$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_W(x-z)dx = \int_0^{z+1} dx = z+1.$$

If $0 \leq z < 1$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_W(x-z)dx = \int_z^1 dx = 1-z.$$

To sum up,

$$f_Z(z) = \begin{cases} z+1, & \text{if } -1 < z < 0, \\ 1-z, & \text{if } 0 \leq z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

8. (12pt) Let random variables X and Y be independent and $X \sim \text{Exp}(\lambda_1)$ and $Y \sim \text{Exp}(\lambda_2)$.
(1) Give the CDF of X and the CDF of Y.
(2) Let $Z = \min(X, Y)$. Find the CDF and PDF of Z.

Solution: (1) Let $F_X()$ and $F_Y()$ be the CDF of X and Y, respectively. Then

$$F_X(x) = \begin{cases} 1 - e^{-\lambda_1 x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$F_Y(x) = \begin{cases} 1 - e^{-\lambda_2 x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(2)

$$F_Z(x) = P(Z \leq x) = P(\min(X, Y) \leq x) = 1 - P(\min(X, Y) > x)$$

$$= 1 - P(X > x, Y > x) = 1 - P(X > x)P(Y > x) = 1 - [1 - F_X(x)][1 - F_Y(x)].$$

If $x > 0$, then

$$F_Z(x) = 1 - [1 - (1 - e^{-\lambda_1 x})][1 - (1 - e^{-\lambda_2 x})] = 1 - e^{-(\lambda_1 + \lambda_2)x}.$$

Otherwise,

$$F_Z(x) = 1 - (1-0)(1-0) = 0.$$

To sum up,

$$F_Z(x) = \begin{cases} 1 - e^{-(\lambda_1 + \lambda_2)x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_Z(x) = \begin{cases} (\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$