Introduction to Probability

Chapter 1 of Probability and Statistics

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Course Syllabus

- **Textbook:** Probability and Statistics (4th Edition), Morris H. DeGroot and Mark J. Schervish, Pearson
- **Chapters covered:** 1–8
- **Final grades:** Final examination (70%), Midterm examination (10%), Assignments and attendance (20%)

Chapter Contents

Mathematical Theory of Probability

Experiments and Events

An *experiment* is a process in which the possible outcomes can be identified ahead of time. An *event* is a well-defined set of possible outcomes of the experiment.

Examples

- A six-sided die is rolled. There are 6 outcomes of this experiment: 1,2,3,4,5,6. 

  One event A is that an even number is obtained, and it can be represented as $A = \{2, 4, 6\}$. The event B that a number greater than 2 is obtained is represented as $B = \{3, 4, 5, 6\}$.

- The rainfall (in millimeters) of a particular region in a future year is random. Its outcomes are numbers in the interval $[0, +\infty)$. The event A that the particular rainfall is greater than 1000 is represented as $A = (1000, +\infty)$.

The Sample Space

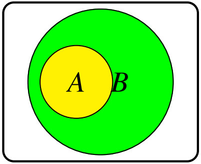
The collection of all possible outcomes of an experiment is called the *sample space* of the experiment. It will be denoted by S throughout.

- Each outcome can be thought of as a *point*, or an *element*, in the sample space.
- Events are subsets of the sample space. If in one finished experiment, the outcome belongs to event E, then we say E has occurred.

Examples

- The sample space of the rolling-die experiment is $\{1, 2, 3, 4, 5, 6\}$.
- The sample space of the rainfall experiment is $[0, \infty)$.
- Toss a coin three times. Denote heads by 1 and tails by 0. Then the sample space is $\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$.

Set Containment



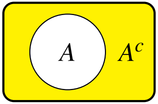
- **Containment**: For any sets A, B, we say A is contained in B, or B contains A, denoted by $A \subset B$ and $B \supset A$, if every element of A is also element of B. For events, $A \subset B$ means that if A occurs then so does B.
- Properties of Containment: If $A \subset B$ and $B \subset A$, then $A = B$. If $A \subset B$ and $B \subset C$, then $A \subset C$.
- To say A is contained in B is equivalent to say A is a subset of B.

Empty, finite, countable sets

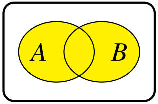
- **Empty set**: The set that has no element is called the empty set. It is denoted by $\emptyset$. The empty set is contained in any set. Empty event cannot occur.
- **Finite set**: A set that has finitely many elements is called a finite set.
- **Countable set**: A set A is said to be countable, if A is either finite or there is a one-to-one correspondence between the elements of A and the set of natural numbers $\{1, 2, 3, \dots\}$.

In ascending order of set bigness: Empty set, Nonempty finite sets, Countably infinite sets, Uncountable sets.

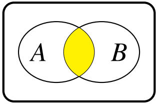
Set Operations – 1

- **Complement**: 

The *complement* of a set A, denoted by A^c, is the set of the elements of the universal set (which in our case is the sample space) S that do not belong to A.

- **Union**: 

The *union* of A and B, denoted by $A \cup B$, is the set of all elements that belong to A or to B.

- **Intersection**: 

The *intersection* of A and B, denoted by $A \cap B$, is the set of all elements that belong both to A and to B.

Set Operations – 2

- **Union of a collection of sets**: Let $\mathcal{A}$ be a collection of sets. Then the union of this collection, denoted by $\bigcup \mathcal{A}$, is defined as

$$\bigcup \mathcal{A} = \{x | \text{There is some } A \in \mathcal{A} \text{ such that } x \in A\}.$$

If $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$, then $\bigcup \mathcal{A}$ is also denoted by $\bigcup_{i=1}^{n} A_i$ or $A_1 \cup A_2 \cup \dots \cup A_n$. If $\mathcal{A} = \{A_1, A_2, \dots\}$, then $\bigcup \mathcal{A}$ is also denoted by $\bigcup_{i=1}^{\infty} A_i$.
Obviously, if $\mathcal{A} = \{A, B\}$, then $\bigcup \mathcal{A}$ is exactly the same as $A \cup B$ as defined in the previous slide. So these two definitions agree with each other.

- **Intersection of a collection of sets**: Let $\mathcal{A}$ be a collection of sets. Then the intersection of this collection, denoted by $\bigcap \mathcal{A}$, is defined as

$$\bigcap \mathcal{A} = \{x | \text{For all } A \in \mathcal{A}, \text{ we have } x \in A\}.$$

If $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$, then $\bigcap \mathcal{A}$ is also denoted by $\bigcap_{i=1}^{n} A_i$ or $A_1 \cap A_2 \cap \dots \cap A_n$. If $\mathcal{A} = \{A_1, A_2, \dots\}$, then $\bigcap \mathcal{A}$ is also denoted by $\bigcap_{i=1}^{\infty} A_i$.
Obviously, if $\mathcal{A} = \{A, B\}$, then $\bigcap \mathcal{A}$ is exactly the same as $A \cap B$ as defined in the previous slide. So these two definitions agree with each other.

Properties of Set Operations – 1

- Complement:

$$(A^c)^c = A, \quad \emptyset^c = S, \quad S^c = \emptyset.$$

- Union:

$$\begin{aligned} &A \cup B = B \cup A, \quad A \cup A = A, \quad A \cup A^c = S, \\ &A \cup \emptyset = A, \quad A \cup S = S. \end{aligned}$$

$$A \subset B \Leftrightarrow A \cup B = B.$$

- Intersection:

$$\begin{aligned} &A \cap B = B \cap A, \quad A \cap A = A, \quad A \cap A^c = \emptyset, \\ &A \cap \emptyset = \emptyset, \quad A \cap S = A. \end{aligned}$$

$$A \subset B \Leftrightarrow A \cap B = A.$$

Properties of Set Operations – 2

- Associative Properties:

$$\begin{aligned} A \cup B \cup C = \quad & (A \cup B) \cup C & = A \cup (B \cup C) \\ A \cap B \cap C = \quad & (A \cap B) \cap C & = A \cap (B \cap C). \end{aligned}$$

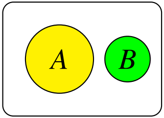
- Distributive Properties:

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C). \end{aligned}$$

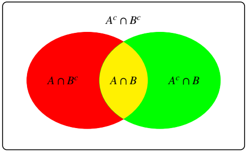
- De Morgan's Laws:

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c. \end{aligned}$$

Disjoint (Mutually Exclusive) Sets

- Two sets A and B are said to be disjoint (or mutually exclusive), if $A \cap B = \emptyset$. 

- The sets $A_1, \ldots, A_n$, or the sets $A_1, A_2, \ldots$, are said to be disjoint (or mutually exclusive), if for every $i \neq j$, we have $A_i \cap A_j = \emptyset$.
- As an example, given two sets A and B, then the sample space can be partitioned into four disjoint sets shown below:



Conditions on Events

Ideally, one would expect all subsets of the sample space to be events, which are each assigned a probability. However, it has been known that this may be unachievable when the sample space is uncountable. In order to be able to do all of the probability calculations that we might find interesting, there are three simple conditions that must be met by the collection of sets that we call events.

1. The sample space S must be an event.
2. If A is an event, then A^c is also an event.
3. If $A_1, A_2, \ldots$ are events, then $\bigcup_{i=1}^{\infty} A_i$ is an event.

Consequences of the Conditions on Events

1. The sample space S must be an event.
2. If A is an event, then A^c is also an event.
3. If $A_1, A_2, \dots$ are events, then $\bigcup_{i=1}^{\infty} A_i$ is an event.

The following are some consequences of the above conditions:

- The empty set must be an event.
- If $A_1, \dots, A_n$ are events, then $\bigcup_{i=1}^{n} A_i$ is an event.
- If $A_1, A_2, \dots$ are events, then $\bigcap_{i=1}^{\infty} A_i$ is an event.
- If $A_1, \dots, A_n$ are events, then $\bigcap_{i=1}^{n} A_i$ is an event.

Definition of Probability

In a probability space, each event A is assigned a number $P(A)$ that indicates the probability that A will occur. The assignment of probability must satisfy some axioms so that it has the properties that we intuitively expect it to have.

1. For every event A, $P(A) \geq 0$.
2. $P(S) = 1$.
3. For every infinite sequence of disjoint events $A_1, A_2, \ldots$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

A probability measure on a sample space S is a specification of numbers $P(A)$ for all events A that satisfy axioms 1–3.

Properties of Probability – 1

1. $P(\emptyset) = 0$.

Proof.

Consider the infinite sequence of events $\emptyset, \emptyset, \ldots$. Then this is a sequence of disjoint events, since $\emptyset \cap \emptyset = \emptyset$. Furthermore, $\bigcup_{i=1}^{\infty} \emptyset = \emptyset$. Therefore, it follows from Axiom 3 that

$$P(\emptyset) = P(\bigcup_{i=1}^{\infty} \emptyset) = \sum_{i=1}^{\infty} P(\emptyset).$$

The only real number for $P(\emptyset)$ with this property is 0. $\square$

Properties of Probability – 2

2. For every finite sequence of n disjoint events $A_1, \dots, A_n$,

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i).$$

Proof.

Extend the finite sequence $A_1, \dots, A_n$ to infinite by letting $A_{n+1} = \dots = \emptyset$.

Then the events in this infinite sequence are disjoint and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} A_i$.

Therefore by Axiom 3,

$$P(\bigcup_{i=1}^{n} A_i) = P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i) + 0 = \sum_{i=1}^{n} P(A_i).$$

□

Properties of Probability – 3,4

3. For every event A, $P(A^c) = 1 - P(A)$.

Proof.

Since A and A^c are disjoint events and $A \cup A^c = S$, it follows from Property 2 that $P(S) = P(A) + P(A^c)$. Since $P(S) = 1$ by Axiom 2, then $P(A^c) = 1 - P(A)$. $\square$

4. If $A \subset B$, then $P(A) \leq P(B)$.

Proof.

The events A and $B \cap A^c$ are disjoint and $A \cup (B \cap A^c) = B$. It follows from Property 2 that $P(B) = P(A) + P(B \cap A^c)$. Since $P(B \cap A^c) \geq 0$, then $P(B) \geq P(A)$. $\square$

Properties of Probability – 5–7

5. For every event A, $0 \le P(A) \le 1$.

Proof.

It is known from Axiom 1 that $P(A) \ge 0$. Since $A \subset S$, then by Property 4, $P(A) \le P(S) = 1$, by Axiom 2. □

6. For every events A and B, $P(A \cap B^c) = P(A) - P(A \cap B)$.

Proof.

The events $A \cap B^c$ and $A \cap B$ are disjoint and $A = (A \cap B^c) \cup (A \cap B)$. It follows from Property 2 that $P(A) = P(A \cap B^c) + P(A \cap B)$. □

7. For every events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof.

The events B and $A \cap B^c$ are disjoint and $A \cup B = B \cup (A \cap B^c)$. It follows from Property 2 that $P(A \cup B) = P(B) + P(A \cap B^c) = P(B) + P(A) - P(A \cap B)$, by Property 6. □

Probability Spaces

- A probability space is a triple $(S, \mathcal{E}, P)$, where S is the sample space, $\mathcal{E}$ is the class of events, and P is the probability measure, that satisfy the previously stated axioms.
- For the same sample space S, there may exist many legal classes of events $\mathcal{E}$. And for the same pair $(S, \mathcal{E})$, there may exist many legal probability measures P. In a practical point of view, we should build probability spaces that are not only mathematically legal but also well reflect the behavior of the random experiment.
- An impossible event is the empty event. An event having probability 0 is not necessarily empty event and hence not necessarily impossible.

Probabilities in Simple Probability Spaces

Simple Probability Space - Definition

- **Finite Probability Space**: A probability space whose sample space is finite.
- **Simple Probability Space**: A probability space where
  - The sample space is finite;
  - All subsets of the sample space are events;
  - If the sample space $S = \{s_1, s_2, \ldots, s_n\}$, then

$$P(\{s_1\}) = P(\{s_2\}) = \ldots = P(\{s_n\}).$$

Based on the axioms, if the probability space is simple, and $S = \{s_1, s_2, \ldots, s_n\}$, then it must be that

$$P(\{s_1\}) = P(\{s_2\}) = \ldots = P(\{s_n\}) = \frac{1}{n}.$$

If an event A in this simple probability space contains exactly m outcomes, then

$$P(A) = \frac{m}{n}.$$

Simple Probability Space - Examples

- **Tossing Three Coins**. Three coins are tossed. We shall determine the probability of obtaining exactly two heads.
  Solution: The probability space is simple. $S = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$.
  Denote the event (exactly two heads) by A, then $A = \{(0,1,1),(1,0,1),(1,1,0)\}$, and

$$P(A) = \frac{\#A}{\#S} = \frac{3}{8}.$$

- **Rolling Two Dice**. Two balanced dice are rolled. We shall determine the probability of each of the possible values of the sum of the two numbers that may appear.
  Solution: The probability space is simple.

$$S = \left\{ \begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}.$$

  Let P_i denote the probability that the sum of the two numbers is i for $i = 2,3,\dots,12$. The only outcome in S for which the sum is 2 is the outcome $(1,1)$. Therefore, $P_2 = 1/36$. The sum will be 3 for either of the two outcomes $(1,2)$ and $(2,1)$. Therefore $P_3 = 2/36 = 1/18$. By continuing this manner, we obtain the following probability for each of the possible values of the sum: $P_2 = P_{12} = \frac{1}{36}$, $P_3 = P_{11} = \frac{2}{36}$, $P_4 = P_{10} = \frac{3}{36}$, $P_5 = P_9 = \frac{4}{36}$, $P_6 = P_8 = \frac{5}{36}$, $P_7 = \frac{6}{36}$.

Multiplication Rule for Two-Part Experiments

Consider an experiment that has the following characteristics:

- The experiment is performed in two parts.
- The first part of the experiment has m possible outcomes $x_1, \ldots, x_m$, and, regardless of which one of these outcomes x_i occurs, the second part of the experiment has n possible outcomes $y_1, \ldots, y_n$.

Each outcome in the sample space S of such an experiment will therefore be a pair having the form (x_i, y_j), and S will be composed of the following pairs:

$$\begin{array}{cccc} (x_1, y_1) & (x_1, y_2) & \cdots & (x_1, y_n) \\ (x_2, y_1) & (x_2, y_2) & \cdots & (x_2, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ (x_m, y_1) & (x_m, y_2) & \cdots & (x_m, y_n). \end{array}$$

Observation: The sample space S has exactly mn outcomes.

Multiplication Rule

Suppose that an experiment has k parts, that the ith part of the experiment can have n_i possible outcomes ($i = 1, \ldots, k$), and that all of the outcomes in each part can occur regardless of which specific outcomes have occurred in the other parts. Then the sample space S of the experiment has exactly $n_1 n_2 \cdots n_k$ outcomes.

Example

Tossing Several Coins. Suppose that we toss six coins. Each outcome in S will consist of a sequence of six heads and tails, such as “HTTHHH". Since there are two possible outcomes for each of the six coins, $\#S = 2^6 = 64$.

Multiplication Rule – Further Examples

Sampling with Replacement. This is a big class of experiments where multiplication rule is applied. As an example, consider a box that contains n balls numbered $1, \ldots, n$. First, one ball is selected at random from the box. This ball is then *put back* in the box and another ball is selected (it is possible that the same ball will be selected again). As many balls as desired can be selected in this way. This process is called *sampling with replacement*. It is assumed that each of the n balls is equally likely to be selected at each stage and that all selections are made independently of each other.

Suppose that a total of k selections are to be made. Then the sample space S of this experiment will contain all vectors of the form $(x_1, \ldots, x_k)$, where x_i is the outcome of the ith selection ($i = 1, \ldots, k$). Since there are n possible outcomes for each of the k selections, the total number of vectors in S is n^k. Furthermore, from our assumptions it follows that the probability space is simple. Hence, the probability assigned to each vector in S is $1/n^k$.

Permutations

- **Definition**: A k-permutation of n ($k \leq n$) is a mapping $\sigma : \{1, \ldots, k\} \mapsto \{1, \ldots, n\}$ with the property $\sigma(i) \neq \sigma(j)$ when $i \neq j$.
- **Counting Permutations**: Let P_n^k be the number of k-permutations of n, then

$$P_n^k = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

When $k = n$,

$$P_n^n = n(n-1)\cdots 1 = n!.$$

It is convenient to define $0!$ by the relation

$$0! = 1.$$

Permutations – Examples

- **Sampling without Replacement**. As an example, consider a box that contains n balls numbered $1, \ldots, n$. First, one ball is selected at random from the box. This ball is *NOT put back* in the box and another ball is selected. Continuing this manner, until k balls are selected. This process is called *sampling without replacement*. Since each outcome corresponds to a k-permutation of n, the number of outcomes in the sample space S is P_n^k.
- **Choosing Officers**. A club of 25 members needs a president and a secretary to be chosen from the membership. Since the positions can be filled by first choosing one of the 25 members to be president and then choosing one of the remaining 24 members to be secretary, the possible number of choices is $P_{25}^2 = 25 \times 24 = 600$.
- **Arranging Books**. Six different books are to be arranged on a shelf. The number of possible permutations of the books is $6! = 720$.

Combinations

- **Definition**: A k-combination of n ($k \leq n$) is a subset of $\{1, \ldots, n\}$ whose size is k.
- **Counting Combinations**: Let C_n^k be the number of k-combinations of n, then

$$C_n^k = \frac{P_n^k}{k!} = \frac{n!}{k!(n-k)!}.$$

Examples

- **Selecting an Unstructured Committee**. A committee of 8 people is to be selected from a group of 20 people. The number of different groups of people that might be on the committee is $C_{20}^8 = \frac{20!}{8!12!} = 125,970$.
- **Selecting a Structured Committee**. In previous example, the 8 people in the committee each get a different job to perform on the committee. The number of ways to choose 8 people out of 20 and assign them to the 8 different jobs is $P_{20}^8 = C_{20}^8 \times 8! = 125,970 \times 8! = 5,078,110,400$.

Combinations – Further Examples

Tossing a Coin. A fair coin is to be tossed 10 times, and it is desired to determine (a) the probability p of obtaining exactly three heads and (b) the probability p' of obtaining three or fewer heads.
(a) Solution:

- The total possible number of different sequences of 10 heads and tails is 2^{10}, and it may be assumed that each of these sequences is equally probable.
- The number of these sequences that contain exactly three heads will be equal to the number of different arrangements that can be formed with three heads and seven tails: C_{10}^{3}.
- Therefore, $p = \frac{C_{10}^{3}}{2^{10}} = 0.1172$.

(b) Solution: $p' = \frac{C_{10}^{0}+C_{10}^{1}+C_{10}^{2}+C_{10}^{3}}{2^{10}} = 0.1719$.

Combinations – Further Examples (Continued)

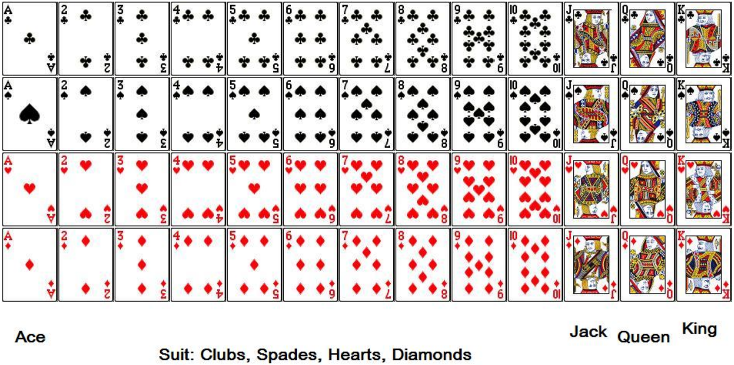
Forming a Team. A class contains 15 boys and 30 girls, and 10 students are to be selected at random for a special assignment. We shall determine the probability p that exactly three boys will be selected.
Solution:

- The number of different combinations of the 45 students that might be obtained in the sample of 10 students is C_{45}^{10}.
- These possible combinations are equally probable.
- A combination of three boys and seven girls can be formed in two steps: (1) select 3 boys, C_{15}^{3} ways; (2) 7 girls, C_{30}^{7} ways. Apply the multiplication rule, the number of combinations containing exactly three boys is $C_{15}^{3} \times C_{30}^{7}$.

Therefore, the desired probability is $p = \frac{C_{15}^{3} C_{30}^{7}}{C_{45}^{10}} = 0.2904$.

Combinations – Further Examples (Continued 2)

Playing Cards. A deck of 52 cards containing four aces is shuffled thoroughly and the cards are then distributed among four players so that each player receives 13 cards. We shall determine the probability p that each player will receive one ace.



Combinations – Further Examples (Continued 3)

Solution 1: Let each outcome be a complete ordering of the 52 cards, so there are $52!$ outcomes. How many of them have one ace in each player? There are 13^4 ways to choose the four positions for the four aces, $4!$ ways to arrange the four aces in these four positions, there are $48!$ ways to arrange the remaining 48 cards in the 48 remaining positions. We then calculate

$$p = \frac{13^4 \times 4! \times 48!}{52!} = 0.1055.$$

Solution 2: Consider the 4 positions occupied by the 4 aces, disregarding what other cards are on other positions.

$$p = \frac{13^4}{C_{52}^4} = 0.1055.$$

Solution 3: Consider the 4 groups of cards distributed to the 4 players, disregarding the order of cards distributed to the same player.

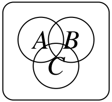
$$p = \frac{4! \times C_{48}^{12} \times C_{36}^{12} \times C_{24}^{12}}{C_{52}^{13} \times C_{39}^{13} \times C_{26}^{13}} = 0.1055.$$

Some Probability Formulae

Probability of the Union of Three Events

Recall properties 2 and 7 of probability,

- If events $A_1, \dots, A_n$ are disjoint, $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$.
- For every events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Further, we have for every events A, B, C, 

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$-P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Proof.

$P(A \cup B \cup C) = P(A) + P(B \cup C) - P(A \cap (B \cup C)) =$
$P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$
$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C).$ □

Probability of the Union of a Finite Number of Events

For every n events $A_1, \dots, A_n$,

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$- \sum_{i<j<k<l} P(A_i \cap A_j \cap A_k \cap A_l) + \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

Example – The Matching Problem

A person types n letters, types the corresponding addresses on n envelopes, and then places the n letters in the n envelopes in a random manner. It could be desired to determine the probability that at least one letter will be placed in the correct envelope.
Solution: Let A_i be the event that letter i is placed in the correct envelope $(i = 1, \ldots, n)$, we shall determine the value $p_n = P(\bigcup_{i=1}^{n} A_i)$.

$$\sum_{i=1}^{n} P(A_i) = n \times \frac{(n-1)!}{n!} = 1$$

$$\sum_{i<j} P(A_i \cap A_j) = C_n^2 \frac{(n-2)!}{n!} = \frac{1}{2!}$$

$$\sum_{i<j<k} P(A_i \cap A_j \cap A_k) = C_n^3 \frac{(n-3)!}{n!} = \frac{1}{3!}$$

$$\vdots$$

$$p_n = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n-1} \frac{1}{n!}$$

$$1 - p_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^{n} \frac{1}{n!}$$

$$\lim_{n\to\infty} 1 - p_n = e^{-1}, \text{ as } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\lim_{n\to\infty} p_n = 1 - e^{-1} = 0.63212....$$

Homework for Chapter 1

- P15: 6
- P21: 3, 4, 5, 6, 7, 8
- P25: 3, 6
- P32: 2, 5, 6, 7, 8, 9
- P41: 4, 10, 13, 17
- P50: 2, 3