Probability and Statistics
Model Answer of 2021 Midterm Exam Paper
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1. (20pt) Fill in the blanks:

(1) If events A and B are disjoint, C is the complement of A, then the relationship between B and C is $\underline{\mathbf{B \subseteq C}}$.

(2) Let the joint cumulative distribution function of random variables X and Y be $F(x, y)$. Then $P(1 < X \leq 2, Y \leq 3) = \underline{\mathbf{F(2,3) - F(1,3)}}$.

(3) Let random variables X, Y, and Z be independent. Which of the following sets of random variables are guaranteed independent? Your answer: **b**.

(a) $X^2 - Z$, Y, Z
(b) $\cos(X)$, e^{Y+Z}
(c) $Z^2 + Y^2$, $\sin(X + Z)$
(d) $X + 3Y$, $Y - 2Z$

(4) Of the following three functions: CDF, PF, and PDF, the function that is applicable to all random variables is **CDF** and the function that **NOT** necessarily takes value between 0 and 1 is **PDF**.

(5) The PDF of random variable X is $f(x) = \begin{cases} ax^{-2}, & \text{if } x > 1 \\ 0, & \text{otherwise.} \end{cases}$ Then $a = \underline{\mathbf{1}}$.

(6) A box contains 6 red balls and 4 white balls. Three balls are chosen without replacement, then the probability that there is exactly one red ball is $\underline{\mathbf{\frac{C_6^1 C_4^2}{C_{10}^3} = 0.3}}$.

(7) If $X \sim B(49, 0.2)$, then its variance is $\underline{\mathbf{7.84}}$.

(8) If $X \sim N(1, 4)$, then $2X + 3 \sim \underline{\mathbf{N(5, 16)}}$.

(9) Let $E(X) = 1$ and $\mathrm{Var}(X) = 2$. Then $E[X(X - 1)] = \underline{\mathbf{2}}$.

2. (10pt) There are five children in a family. In this family, each child, independent from each other, has 1/4 chance to have blue eyes. If it is known that at least one of the children has blue eyes, then what is the probability that all the five children have blue eyes?

Solution: Let X be the number of children in this family who have blue eyes. Then $X \sim B(5, \frac{1}{4})$.

$$
\begin{aligned}
P(X \geq 5 | X \geq 1) &= \frac{P(X \geq 5, X \geq 1)}{P(X \geq 1)} \\
&= \frac{P(X \geq 5)}{P(X \geq 1)} \\
&= \frac{(\frac{1}{4})^5}{1 - (\frac{3}{4})^5} \\
&= \frac{1}{781}
\end{aligned}
$$

3. (12pt) Random variable X has the following PDF

$$
f(x) = \begin{cases} 3x^2, & \textbf{if } 0 < x < 1 \\ 0, & \textbf{otherwise}. \end{cases}
$$

Let $Y = e^X$. Find the PDF $g(y)$ of Y.

Solution: The mapping $y = r(x) = e^x$ is one-to-one and strictly increasing from $(0, 1)$ to $(1, e)$. Its inverse is $x = s(y) = \ln y$. Thus for $1 < y < e$, we have

$$
\begin{aligned}
g(y) &= f[s(y)] |\frac{ds(y)}{dy}| \\
&= 3 \ln^2(y) |\frac{d \ln y}{dy}| \\
&= \frac{3 \ln^2(y)}{y}
\end{aligned}
$$

To sum up,

$$
g(y) = \begin{cases} \frac{3 \ln^2(y)}{y}, & \text{if } 1 < y < e \\ 0, & \text{otherwise}. \end{cases}
$$

4. (14pt) Random variables X and Y have the following joint PDF

$$
f(x, y) = \begin{cases} 6xy, & \textbf{if } 0 < x < 1 \textbf{ and } 0 < y < \sqrt{x} \\ 0, & \textbf{otherwise}. \end{cases}
$$

(1) Find $P(X \leq Y)$.

(2) Find marginal density $f_X(x)$.

(3) Find marginal density $f_Y(y)$.

(4) Check if X and Y are independent.

Solution: (1)

$$\begin{aligned}
P(X \leq Y) &= \int_0^1 \int_x^{\sqrt{x}} 6xydydx \\
&= \int_0^1 (3xy^2)|_{y=x}^{y=\sqrt{x}} dx \\
&= \int_0^1 3x(x - x^2)dx \\
&= [x^3 - \frac{3}{4}x^4]|_0^1 \\
&= \frac{1}{4}
\end{aligned}$$

(2) If $x \leq 0$ or $x \geq 1$, then $f_X(x) = 0$. If $0 < x < 1$, then

$$\begin{aligned}
f_X(x) &= \int_0^{\sqrt{x}} 6xydy \\
&= [3xy^2]|_{y=0}^{y=\sqrt{x}} \\
&= 3x^2
\end{aligned}$$

To sum up,

$$f_X(x) = \begin{cases} 3x^2, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $y \leq 0$ or $y \geq 1$, then $f_Y(y) = 0$. If $0 < y < 1$, then

$$\begin{aligned}
f_Y(y) &= \int_{y^2}^1 6xydx \\
&= [3x^2y]|_{x=y^2}^{x=1} \\
&= 3(y - y^5)
\end{aligned}$$

To sum up,

$$f_Y(y) = \begin{cases} 3(y - y^5), & \text{if } 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(4) Obviously not independent.

5. (12pt) Let $X \sim U[-3, 5]$, and let

$$Y = \begin{cases} 2, & \textbf{if } X > 0 \\ -2, & \textbf{otherwise.} \end{cases}$$

(1) Is Y continuous or discrete? If Y is continuous, give its PDF; and if it is discrete, give its PF.

(2) Find $E(Y)$ and $\mathrm{Var}(Y)$.

Solution: (1) Y is discrete.

$$\begin{aligned} P(Y=2) &= P(X>0) \\ &= \frac{5}{8} \end{aligned}$$

$$P(Y=-2) = \frac{3}{8}$$

(2)

$$\begin{aligned} E(Y) &= 2\times\frac{5}{8} - 2\times\frac{3}{8} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E(Y^2) &= 4\times\frac{5}{8} + 4\times\frac{3}{8} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(Y) &= E(Y^2) - E^2(Y) \\ &= 4 - \frac{1}{4} = 3.75. \end{aligned}$$

6. (10pt) Tom and Jerry agree to meet up in a station. They arrive at the station uniformly between 9:00 AM and 10:00 AM. Suppose that after arriving, each waits 20 minutes for the other person before leaving. What is the probability that they will meet?

Solution: Let X, Y be the times of arrival after 9:00 AM, in hours, of Tom and Jerry, respectively. Then $X \sim U[0,1]$, $Y \sim U[0,1]$, and X and Y are independent. The event that they will meet is $|X-Y| \le \frac{1}{3}$.

The area of this region is $1 - \frac{1}{2}\times\frac{2}{3}\times\frac{2}{3} - \frac{1}{2}\times\frac{2}{3}\times\frac{2}{3} = \frac{5}{9}$.

The area of the support is 1.

Thus the desired probability is $\frac{5}{9}$.

7. (10pt) The length in cm (X) of some bolts is normally distributed with a mean of 10.05 and a standard deviation of 0.06. A bolt is qualified if its length is between 10.05 ± 0.12 cm. Find the probability that a bolt is defective.

Solution: The event that the bolt is defective is $|X - 10.05| > 0.12$.

$$\begin{aligned} P(|X-10.05|>0.12) &= P(\frac{|X-10.05|}{0.06} > 2) \\ &= 1 - [\Phi(2) - \Phi(-2)] \\ &= 2[1-\Phi(2)] \\ &= 2*(1-0.977249868) = 0.045500264. \end{aligned}$$

8. (12pt) Random variables X and Y have the following joint PDF

$$f(x,y) = \begin{cases} 1, & \textbf{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \textbf{otherwise.} \end{cases}$$

Find the PDF of $Z = X + Y$.

Solution: Let the PDF of Z be $f_Z(z)$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)dx.$$

If $z < 0$, then $f_Z(z) = 0$.
If $0 \leq z < 1$, then

$$f_Z(z) = \int_0^z 1dx = z.$$

If $1 \leq z < 2$, then

$$f_Z(z) = \int_{z-1}^1 1dx = 2 - z.$$

If $z \geq 2$, then

$$f_Z(z) = 0.$$

To sum up,

$$f_Z(z) = \begin{cases} z, & \text{if } 0 \leq z < 1 \\ 2-z & \text{if } 1 \leq z < 2 \\ 0, & \text{otherwise.} \end{cases}$$