Lancaster University College
at Beijing Jiaotong University

2020/21 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Midterm examination (November)

INSTRUCTIONS TO STUDENTS

1) Duration of the exam: 120 minutes
2) This paper contains 3 pages. There are 8 questions.
3) You must answer all questions.
4) This is a closed book exam. No books or notes may be brought into the exam room.
5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
6) Some values that might be useful:

$$\Phi(0.2) = 0.57926, \qquad \Phi(1) = 0.8413,$$

$$\Phi(1.4142) = 0.92135, \qquad \Phi(2.8284) = 0.99766$$

1. (20pt)

(1) Two events A and B are independent, and $P(A) = 0.4$, $P(B) = 0.7$, then $P(A\overline{B}) =$__________________.

(2) A box contains 3 left shoes and 3 right shoes. If two shoes are randomly chosen from the box, then the probability that they are a pair (i.e. a left shoe and a right shoe) is __________________.

(3) Let the CDF of random variable X be $F(x) = \frac{1}{1+2^{-x}}$. Then $P(X < 1|X > 0) =$__________________.

(4) Let the PDF of random variable X be $f(x)$. $Y = 2X$. Then the PDF of Y is $g(y) =$____________________.

(5) Suppose that $X \sim N(1, 5)$. $Y = 3X - 2$. Then $Y \sim$____________________.

(6) Suppose that X has the uniform distribution on the interval $[2, 5]$. Then $P(3 < X \leq 4) =$__________________.

(7) A neighborhood, which has a gymnasium in it, has 1000 residents. Every morning each resident independently has 0.05 chance to go to the gymnasium. Then the number of gymnasium goers every morning approximately has the _______________ distribution with parameter $\lambda =$_____________.

(8) Let the joint CDF of X and Y be $F(x, y) = \frac{1}{(1+e^{-x})(1+3^{-y})}$. Then the CDF of X is __________________ and the PDF of X is ______________________.

2. **(10pt)** A company produces products at three different factories A, B, and C. Of the company's total volume, factory A produces 20%, factory B produces 50%, and factory C produces the rest. The product defective rates at the factories are 5% at factory A, 2% at factory B, and 10% at factory C. If you buy this product and it turns out to be defective, what is the probability that it was produced at factory A?

3. **(10pt)** An airline sells 200 tickets for a certain flight on an airplane that has only 198 seats because, on the average, 1% of purchasers of airline tickets do not appear for the departure of their flight. Determine the probability that everyone who appears for the departure of this flight will have a seat.

4. **(12pt)** In a population, the men's weight (in kilograms) has the distribution N(60, 25) and the women's weight (in kilograms) has the distribution N(50, 25). Randomly and independently pick a man and a woman from the population.
   (1) What is the probability that the two persons' total weight exceeds 130 kilograms?
   (2) What is the probability that the difference between the two persons' weights is smaller than 10 kilograms?

5. **(14pt)** The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} 1, & 0 < x < 1, 0 < y < 2x, \\ 0, & \text{otherwise.} \end{cases}$$

   (1) Find the marginal density $f_X(x)$ and the conditional PDF $f_{Y|X}(y|x)$.
   (2) Evaluate $P(Y < \frac{1}{2} | X < \frac{1}{2})$.

6. **(10pt)** Suppose that a fair coin is tossed repeatedly until two consecutive heads or two consecutive tails appear. Let X be the number of tosses required. Let $Y = 1$ if it is two consecutive heads and $Y = 0$ if it is two consecutive tails. Determine the joint PF of X and Y.

7. **(12pt)** Let random variables X and W be independent and distributed uniformly on the interval $[0,1]$. Let $Y = -W$.
   (1) What is the distribution of Y?
   (2) Are X and Y independent?
   (3) Find the PDF of $Z = X + Y = X - W$.

8. **(12pt)** Let random variables X and Y be independent and $X \sim \text{Exp}(\lambda_1)$ and $Y \sim \text{Exp}(\lambda_2)$.
   (1) Give the CDF of X and the CDF of Y.
   (2) Let $Z = \min(X, Y)$. Find the CDF and PDF of Z.