Lancaster University College at Beijing Jiaotong University

2020/21 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Final examination (December)

INSTRUCTIONS TO STUDENTS

1) Duration of the exam: 120 minutes
2) This paper contains 3 pages. There are 8 questions.
3) You must answer all questions.
4) This is a closed book exam. No books or notes may be brought into the exam room.
5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
6) Some values that might be useful:

$$\Phi(0.2381) = 0.5941, \qquad \Phi(0.2405) = 0.5950,$$

$$\Phi(1) = 0.8413, \qquad \Phi(1.5921) = 0.9443,$$

$$\Phi(3.4503) = 0.999720, \qquad \Phi(3.4848) = 0.999754$$

1. (20pt)

(1) Albert and Brian are in a group of 10 members. They are seated in a random manner in a row of 10 seats. The probability that Albert and Brian are seated next to each other is __________________.

(2) Suppose $X \sim Geo(\frac{1}{3})$. Then $\boldsymbol{P(X}=4)=$__________________.

(3) Suppose random variables $\boldsymbol{X \sim B(2,p), Y \sim B(3,p)}$. If $\boldsymbol{P(X \geq 1) = 5/9}$, then $\boldsymbol{P(Y \geq 1)} =$____________________.

(4) The CDF of X is $F(x) = \frac{1}{2} + \frac{1}{\pi} arctan\,(x)$. Then the PDF of X is $f(x) =$____________________, and $\mathrm{P}(0 < X \leq 1) =$_____________________.

(5) Suppose that X has the uniform distribution on the interval $[2,5]$. Then $\mathrm{Var(X)} =$__________________.

(6) If $\mathrm{E(X)} = -1, \mathrm{Var(X)} = 2$, and $\mathrm{Y} = 3\mathrm{X} + 5$, then $\mathrm{E(Y)} =$ __________________ and $\mathrm{Var(Y)} =$ __________________.

(7) Suppose that random variables X and Y are independent and both have the standard normal distribution, then $\mathrm{Cov(2X + 3Y,\ X - Y)} =$___________________.

(8) Suppose $\boldsymbol{X_1, X_2, X_3}$ is a random sample from population $\boldsymbol{X \sim N(\mu, 1)}$, then $\boldsymbol{\frac{1}{2}X_1 + \frac{1}{3}X_2 + kX_3}$ is an unbiased estimator for $\boldsymbol{\mu}$ when $\boldsymbol{k} =$__________________.

2. (**10pt**) A professor provides his students with two types of exam papers: “easy” and “hard”. The students have a 0.80 chance of getting a “hard” paper. The probability that the first question on the exam paper is marked as difficult is 0.90 if the exam paper is “hard”, and is 0.15 if the exam paper is “easy”.

(1) (5**pt**) What is the probability that the first question on your exam is marked as difficult?

(2) (5**pt**) What is the probability that your exam is of the “hard” type given that the first question on the exam is marked as difficult?

3. (**12pt**) Suppose $X \sim N(0,1)$, find the PDF of $Y = X^2 + 1$.

4. (**10pt**) The joint PF of X and Y is shown in the following table.

	Y=1	Y=2
X=1	$\frac{1}{6}$	$\frac{1}{3}$
X=2	$\frac{1}{9}$	a
X=3	b	$\frac{1}{9}$

Suppose that X and Y are independent. Determine the values of a and b.

5. (**16pt**) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} cxy, & 0 < y < x < 1, \\ 0 \ , & \text{otherwise.} \end{cases}$$

Determine

(1) (**2pt**) the constant c;

(2) (**5pt**) $P(X + \mathrm{Y} \leq 1)$;

(3) (**5pt**) the marginal PDF $f_X(x)$;

(4) (**4pt**) the conditional PDF $f_{Y|X}(y|x)$.

6. (**12pt**) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} 6(x-y), & 0 < y < x < 1, \\ 0 \ , & \text{otherwise.} \end{cases}$$

Find $\mathrm{Cov}(X,Y)$.

7. (10pt) In a building there are 1000 lights, and each light independently has 0.7 probability to be turned on. Using the central limit theorem, approximately evaluate the probability that the total number of turned-on lights exceeds 750.

8. (**10pt**) Suppose that $X_1, X_2, \cdots, X_n$ is a random sample from population $X \sim P(\lambda)$. Derive the maximum likelihood estimator of the parameter λ.