Probability and Statistics – Lancaster_BJTU Final Exam 2017 Paper A

1 Fill in the blanks (20 points)

1. Suppose $P(A) = 0.3$, $P(B) = 0.8$, and $P(A) = P(A|B)$. Then $P(B^c|A^c) =$ ________.

2. Of the following six families of distributions: binomial, Poisson, negative binomial, uniform, normal, and Gamma, there are two families that a random variable having distribution in them may take negative values with positive probability. These two families of distributions are the ________ distributions and the ________ distributions.

3. Suppose that there is a binomial distribution with parameter n and p, where n is large. 1) If np is moderate, then it can be approximated by a ________ distribution. 2) If np is large, then according to the ________ theorem, it can be approximated by a ________ distribution.

4. The lower and upper limits of correlation coefficients are ________ and ________, respectively.

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2, then an unbiased and consistent estimator of μ is ________, and an unbiased and consistent estimator of σ^2 is ________.

2

(8 points) Suppose that a box contains r red balls and w white balls. Suppose also that balls are drawn from the box one at a time, at random, without replacement.

(a) What is the probability that all red balls will be obtained before any white balls are obtained?

(b) What is the probability that the first two balls drawn will be of the same color?

3

(12 points) Two students, Adam and Brian, are going to take a test. They take the test independently of each other. The probability for Adam to pass the test is a, and the probability for Brian to pass the test is b. Let X be the number of people of these two students who pass the test. Prove that $\mathrm{Var}(X) = a - a^2 + b - b^2$.

4

(12 points) Suppose that random variables X and Y are **independent**. X has the uniform distribution on the interval $[0, 1]$, and Y has the exponential distribution with parameter β. Suppose also that $E(X) = E(Y)$.

(a) Determine the value of β.

(b) Find $P(X > Y)$.

5

(8 points) Suppose that X and Y are independent random variables that both have the Poisson distribution and are such that $\text{Var}(X) + \text{Var}(Y) = 5$. Evaluate $P(X + Y < 2)$.

6

(14 points) A random sample $X_1, \ldots, X_n$ is to be taken from a distribution with mean μ and standard deviation σ^2.

(a) Use the Chebyshev inequality to determine the smallest n such that the following relation will be satisfied:

$$P(|\overline{X}_n - \mu| < \frac{\sigma}{4}) > 0.99$$

(b) Suppose additionally that the random sample $X_1, \ldots, X_n$ has normal distribution. Determine the smallest n such that the above relation will be satisfied.

7

(14 points) Suppose that X and Y have the **bivariate normal distribution** with covariance matrix $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. The real number a is such that

$$aX + Y \text{ and } aX - Y \text{ are independent, and}$$

$$\text{Var}(aX + Y) = 1.$$

(a) Determine the value of a.

(b) Compute $\text{Var}(aX - Y)$.

8

(12 points) Suppose that $X_1, \ldots, X_n$ form a random sample from a distribution for which the p.d.f. is as follows:

$$f(x; \theta) = \begin{cases} \theta x^{\theta - 1} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $\theta > 1$ is the parameter to be estimated. Derive the M.L.E. of θ.

Table of the Standard Normal Distribution Function

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du$$

z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Probability and Statistics – Lancaster_BJTU Final Exam 2017 Paper B

1 Fill in the blanks (20 points)

1. In a simple probability space that has 50 outcomes, the probability of an event containing 5 outcomes is ________.

2. If X has the uniform distribution on interval $[1, 5]$, then $\text{Var}(X) =$ ________. If X has the binomial distribution with parameter $n = 5$ and $p = 0.4$, then $\text{Var}(X) =$ ________.

3. Of the following six families of distributions: binomial, Poisson, geometric, uniform, normal, and exponential, the family of distributions for which the mean and the variance are always equal are the ________ distributions; the family of distributions for which the variance is always the square of the mean are the ________ distributions.

4. Suppose that X and Y are two random variables and $\text{Var}(X) > 0$. If $Y = -7X$, then their correlation coefficient $\rho(X, Y) =$ ________; if X and Y are independent, then $\rho(X, Y) =$ ________.

5. A sampling distribution is the distribution of a ________. For a sequence of estimators to be practically usable, it must be ________, which means that it must converge ________ to the parameter being estimated.

2

(10 points) Suppose that A, B, and C are events such that A and B are independent, $P(A \cap B \cap C) = 0.04$, $P(C|A \cap B) = 0.25$, and $P(B) = 4P(A)$. Evaluate $P(A \cup B)$.

3

(8 points) Two people are playing a game by rolling dice. The rule of the game is as follows:

- One of the people is designated as player A and the other as player B.
- Player A rolls the die first. If a 5 or a 6 appears on the die, then player A wins and the game is over, otherwise the game proceeds.
- Now player B rolls the die. If the number that appears on his die is equal to or greater than the number that appeared on player A's die, then player B wins, otherwise player A wins.

Determine the probability for player A to win the game and decide whether either player has any advantage.

4

(14 points) Suppose that random variables X and Y have the following joint p.d.f.

$$f(x,y) = \begin{cases} 2(x+y) & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine
(a) the marginal p.d.f. of X;
(b) $P(X < 1/2)$;
(c) the conditional p.d.f. of Y given that $X = x$.

5

(12 points) Suppose that $X_1, \ldots, X_n$ form a random sample of size n from a continuous distribution with the following p.d.f.

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = \max(X_1, \ldots, X_n)$. Evaluate $E(Y)$.

6

(9 points) In a small town there are 500 people and there is a shop. Each hour, each person in the town has 0.01 probability to go to the shop and they go independently of one another. Approximately evaluate the probability that in a given hour, less than 3 people go to the shop.

7

(11 points) In a building there are 400 lights, and they are on or off independently of one another. At night, each light has 0.7 probability to be on. Using correction for continuity, approximately evaluate the probability that in the building there are between 270 (inclusive) and 290 (inclusive) that are on.

8

(16 points) Suppose that $X_1, \ldots, X_n$ form a random sample from a distribution for which the p.d.f. is as follows:

$$f(x;\theta) = \begin{cases} \theta x^{-(\theta+1)} & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $\theta > 1$ is the parameter to be estimated.
(a) Derive the method-of-moments estimator of θ.
(b) Derive the maximum likelihood estimator of θ.

Table of the Standard Normal Distribution Function

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2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Lancaster University College
at Beijing Jiaotong University

2019/20 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Final examination (January)

INSTRUCTIONS TO STUDENTS

1) Duration of the exam: 120 minutes
2) This paper contains 3 pages. There are 8 questions.
3) You must answer all questions.
4) This is a closed book exam. No books or notes may be brought into the exam room.
5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
6) Some values that might be useful:

$$\Phi(1) = 0.8413, \qquad \Phi(1.5921) = 0.9443,$$

$$t_{0.025}(15) = 2.1314.$$

1. (20pt)

(1) Two events A and B are independent, and $P(A) = 0.6$, $P(B) = 0.5$, then $P(A|A \cup \overline{B}) =$________________.

(2) If n is large and p is small, then the binomial distribution $B(n,p)$ can be approximated by ______________ distribution with parameter $\lambda =$___________.

(3) If random variables $X \sim B\left(18, \frac{1}{3}\right)$, $Y \sim P(3)$, and X and Y are independent, then $\mathrm{Var}(X - Y) =$_______________.

(4) Suppose that random variable X has $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. By Chebyshev's inequality, $P(|X - \mu| \geq 3\sigma) \leq$________________.

(5) Given observed data of a random sample: $(14, 20, 2, 16, 3)$, its sample mean is ________________, sample variance is ____________________.

(6) Random variables $X_1, X_2, X_3, X_4, X_5, X_6$ are independent and all have $N(0,1)$. If $\frac{c(X_1-X_2)}{\sqrt{(X_3-X_4)^2+(X_5-X_6)^2}} \sim t(m)$, then $c =$__________ and $m =$________.

(7) One class of 16 students had an English test. The sample mean and standard deviation of scores are 80 and 8, respectively. A 95% confidence interval of the mean score is (rounded to the nearest thousandth) ___________________________.

2. **(10pt)** Suppose that (X, Y) is uniformly distributed in the unit disk $\{(x,y)|x^2 + y^2 \leq 1\}$. Then we define two discrete random variables U and V as follows:

$$U = \begin{cases} 1, & X^2 + Y^2 < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$V = \begin{cases} 1, & X > 0 \text{ and } Y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(1) Determine the joint PF of U and V.

(2) Are U and V independent?

3. **(12pt)** The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{2x^2 y}, & x \geq 1, \frac{1}{x} \leq y \leq x, \\ 0\ , & \text{otherwise.} \end{cases}$$

(1) Find the PDF of Y.

(2) Evaluate $E(\frac{Y}{X})$.

4. **(10pt)** Suppose that the PDF of a random variable X is

$$f_X(x) = \begin{cases} 3x^2, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The conditional PDF of Y given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x}, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Find the joint PDF of X and Y.
(2) Find $P(X + Y < 1)$.

5. **(10pt)** Suppose that random variables X, Y, and Z are independent and all have the standard normal distribution. Evaluate $P(3X + 2Y < 6Z - 7)$.

6. **(12pt)** Let random variables X and Y be independent, and $\mathrm{Var}(X) = \mathrm{Var}(Y) = \sigma^2$. Let $U = 2X + Y$ and $V = 2X - Y$. Determine $\rho_{U,V}$.

7. **(12pt)** An election is held in a small town between two candidates A and B. An initial counting of the votes shows 1422 votes for A and 1405 votes for B. However, further counting reveals that 101 votes are illegal and have to be thrown out. Suppose that the illegal votes are independent from each other and each illegal vote is equally probable for A or for B. Use the central limit theorem with correction for continuity to approximately evaluate the probability that the removal of the illegal votes changes the result of the election. Note that there are only 3 results: A wins, B wins, or they tie.

8. **(14pt)** Suppose that $X_1, \cdots, X_n$ form a random sample from a continuous X which has the following PDF with parameter $\theta > 0$:

$$f(x|\theta) = \begin{cases} \frac{2}{\sqrt{\pi\theta}} \exp\left(-\frac{x^2}{\theta}\right), & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is known that $\int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \sqrt{\pi}$ and $\int_0^\infty t^{\frac{1}{2}} e^{-t} dt = \frac{\sqrt{\pi}}{2}$.

(1) Find $\mu_2 = E(X^2)$, and based on this, derive a moment estimator of θ.
(2) Derive the maximum likelihood estimator of θ.

Lancaster University College
at Beijing Jiaotong University

2019/20 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Resit examination (March)

INSTRUCTIONS TO STUDENTS

1) Duration of the exam: 120 minutes
2) This paper contains 3 pages. There are 8 questions.
3) You must answer all questions.
4) This is a closed book exam. No books or notes may be brought into the exam room.
5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
6) Some values that might be useful:

$$\Phi(0.5634) = 0.7134, \qquad \Phi(1) = 0.8413,$$

$$\Phi(1.3363) = 0.9093, \qquad t_{0.05}(4) = 2.1318.$$

1. (20pt)

(1) The joint PDF of X and Y is $f(x,y) = \begin{cases} ae^{-2x-4y}, & x > 0, y > 0, \\ 0\ , & \text{otherwise.} \end{cases}$ then $a =$____________.

(2) A family is chosen at random from all three-child families. If it is known that the family has a boy among the three children, then the conditional probability that the chosen family has one boy and two girls is ____________.

(3) If random variables $X \sim B(4, 0.3)$, $Y \sim B(11, 0.3)$, and X, Y are independent, then $X + Y \sim$_______________.

(4) If random variables X, Y have a joint continuous distribution, then $P(X = Y) =$ _______________.

(5) If random variable $X \sim U(1, 3)$, then $\mathrm{Var}(X) =$_______________.

(6) Suppose that X_1, X_2 have a bivariate normal distribution. Then X_1, X_2 are independent if and only if _________________$= 0$.

(7) Let $X_1, X_2, \cdots, X_n$ be a random sample from X with $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Then $E[(X_2 - X_1)^2] =$____________________, and if $c\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2$ is an unbiased estimator of σ^2 then $c =$____________________.

(8) Let X_1, X_2, X_3, X_4 be a random sample from distribution $N(\mu, \sigma^2)$. $\bar{X}$ is the sample mean. Suppose that $c\sum_{i=1}^{4}(X_i - \bar{X})^2 \sim \chi^2(m)$. Then $c =$_____________ and $m =$________________.

2. **(10pt)** In a world, 40% of products are highly successful, 35% are moderately successful, and 25% are poor products. In addition, 95% of highly successful products receive good reviews, 60% of moderately successful products receive good reviews, and 10% of poor products receive good reviews.

(1) What is the probability that a product attains a good review?

(2) If a new product attains a good review, what is the probability that it will be a highly successful product?

(3) If a product does not attain a good review, what is the probability that it will be a highly successful product?

3. **(10pt)** Two random variables X and Y are independent, $X \sim N(2,4)$, $Y \sim N(3,5)$. Determine $P(3X + 2Y \leq 22)$.

4. **(12pt)** The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} 12\ y^2, & 0 \leq x \leq 1, 0 \leq y \leq x, \\ 0\ , & \text{otherwise.} \end{cases}$$

(1) Find $\mathrm{Cov}(X,Y)$.

(2) Find $P(X + Y \leq 1)$.

5. **(13pt)** Let X denote the total number of successes in 15 Bernoulli trials, with probability of success $p = 0.3$ on each trial.
   (1) Determine $P(X = 4)$.
   (2) Determine approximately the value of $P(X = 4)$ by using the central limit theorem with the correction for continuity.

6. **(12pt)** Suppose that X_1, X_2, X_3, X_4, X_5 are a random sample from distribution $N(\mu, \sigma^2)$, where μ and σ^2 are both unknown. We have the following observed data of the sample: $(x_1, x_2, x_3, x_4, x_5) = (29, 20, 11, 16, 14)$. Find a 90% confidence interval for μ.

7. **(10pt)** Let X_1, X_2 be a random sample from X with $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$. We want $Y = a_1 X_1 + a_2 X_2$ to be an estimator of μ, where a_1, a_2 are real constants.
   (1) For Y to be unbiased, what requirement must a_1, a_2 fulfil?
   (2) Find $\text{Var}(Y)$.
   (3) Determine a_1, a_2 if we want Y to be unbiased and the most efficient.

8. **(13pt)** Let $X_1, X_2, \cdots, X_n$ be a random sample from X whose PDF is

$$f(x) = \begin{cases} \theta x^{-(\theta+1)}, & x > 1, \\ 0 \quad , & \text{otherwise}, \end{cases}$$

where $\theta > 1$ is an unknown parameter.
   (1) Find the moment estimator of θ.
   (2) Find the maximum likelihood estimator of θ.

Probability and Statistics Mock Exam 2020

1. (20pt) Fill in the blanks:

(1) A and B are two events with $P(A) = 0.4$, $P(A \cup B) = 0.7$, then $P(B|\overline{A}) =$ ____________.

(2) Three people are assigned randomly and independently into 4 rooms numbered A–D. Then the expected number of people in room A is ____________ and the probability that room A has exactly 2 people is ____________.

(3) The PDF of X which has the standard normal distribution is $\varphi(x) =$ ________________.

(4) Suppose that random variables X and Y are independent, $X \sim Exp(1)$ and $Y \sim U[0, 1]$. Then the joint PDF of (X, Y) is $f(x, y) =$ ____________________, and $\text{Var}(X - Y) =$ ____________.

(5) Of the following families of distributions: binomial, Poisson, geometric, uniform, normal, and exponential, the ____________ and ____________ distributions are completely additive. A distribution family is additive if X and Y are independent and belong to this distribution family, then $X + Y$ also belongs to this distribution family.

(6) For an estimator (actually a sequence of estimators) to be practically usable, if must be ____________. For any distribution, the unbiased and most efficient estimator of population expectation is ________________.

2. (10pt) Suppose that random variables X and Y are independent, $X \sim N(0, 1)$ and $Y \sim U[0, 1]$. Let

$$Z = \begin{cases} X, & \text{if } Y < 0.4, \\ 2X - 1, & \text{otherwise.} \end{cases}$$

Find the CDF and PDF of Z. (You can simply write the CDF and PDF of the standard normal distribution by $\Phi(\cdot)$ and $\varphi(\cdot)$, respectively.)

3. (12pt) The PDF of X is given by

$$f(x) = \begin{cases} 1 - |x|, & \text{if } -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = \cos(\frac{\pi X}{2})$. Find the PDF of Y.

4. (10pt) Suppose that random point (X, Y) is uniformly distributed in the disk $x^2 + y^2 < 9$.
(1) Find the conditional PDF $f_{Y|X}(y|x)$.
(2) Determine $P(Y > 0|X = 2)$.

5. (14pt) The joint PDF of X and Y is given by

$$f(x, y) = \begin{cases} x + y, & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find
(1) $E(X)$ and $E(Y)$;
(2) $\text{Var}(X)$ and $\text{Var}(Y)$;
(3) $\text{Cov}(X, Y)$.

6. (10pt) Suppose that random variables X, Y and Z have $E(X) = E(Y) = 1$, $E(Z) = -1$, $\text{Var}(X) = \text{Var}(Y) = \text{Var}(Z) = 1$, $\rho_{X,Y} = 0$, $\rho_{X,Z} = \frac{1}{2}$, $\rho_{Y,Z} = -\frac{1}{2}$. Let $W = X - Y + Z$. Find $E(W)$ and $\text{Var}(W)$.

7. (12pt) An insurance company runs a certain disease insurance policy, which has 10,000 policy holders. Each year, each policy holder pays the company a premium of \$170, and if he gets the disease in that year, he receives from the company a compensation of $\$20,000$. The probability for a person to get the disease in a year is 0.006. Using the central limit theorem, for this company approximately evaluate
(1) the probability that the annual profit of this policy is at least $\$200,000$;
(2) the probability that the annual profit of this policy is positive.

8. (12pt) Let $X_1, \cdots, X_n$ be a random sample from $B(100, p)$.
(1) Derive the maximum likelihood estimator of p.
(2) Find the bias of the estimator you just derived. Is it unbiased?

Lancaster University College at Beijing Jiaotong University

2020/21 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Final examination (December)

INSTRUCTIONS TO STUDENTS

1) Duration of the exam: 120 minutes
2) This paper contains 3 pages. There are 8 questions.
3) You must answer all questions.
4) This is a closed book exam. No books or notes may be brought into the exam room.
5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
6) Some values that might be useful:

$$\Phi(0.2381) = 0.5941, \qquad \Phi(0.2405) = 0.5950,$$

$$\Phi(1) = 0.8413, \qquad \Phi(1.5921) = 0.9443,$$

$$\Phi(3.4503) = 0.999720, \qquad \Phi(3.4848) = 0.999754$$

1. (20pt)

(1) Albert and Brian are in a group of 10 members. They are seated in a random manner in a row of 10 seats. The probability that Albert and Brian are seated next to each other is ________________.

(2) Suppose $X \sim Geo(\frac{1}{3})$. Then $\boldsymbol{P(X} = 4) =$________________.

(3) Suppose random variables $\boldsymbol{X \sim B(2, p), Y \sim B(3, p)}$. If $\boldsymbol{P(X \geq 1) = 5/9}$, then $\boldsymbol{P(Y \geq 1)} =$__________________.

(4) The CDF of X is $F(x) = \frac{1}{2} + \frac{1}{\pi} arctan\,(x)$. Then the PDF of X is $f(x) =$__________________, and $\mathrm{P}(0 < X \leq 1) =$____________________.

(5) Suppose that X has the uniform distribution on the interval $[2, 5]$. Then $\mathrm{Var(X)} =$________________.

(6) If $\mathrm{E(X) = -1, Var(X) = 2,}$ and $\mathrm{Y = 3X + 5,}$ then $\mathrm{E(Y)} =$ ________________ and $\mathrm{Var(Y)} =$ ________________.

(7) Suppose that random variables X and Y are independent and both have the standard normal distribution, then $\mathrm{Cov(2X + 3Y,\ X - Y)} =$_________________.

(8) Suppose $\boldsymbol{X_1, X_2, X_3}$ is a random sample from population $\boldsymbol{X \sim N(\mu, 1)}$, then $\boldsymbol{\frac{1}{2}X_1 + \frac{1}{3}X_2 + kX_3}$ is an unbiased estimator for $\boldsymbol{\mu}$ when $\boldsymbol{k} =$________________.

2. (**10pt**) A professor provides his students with two types of exam papers: “easy” and “hard”. The students have a 0.80 chance of getting a “hard” paper. The probability that the first question on the exam paper is marked as difficult is 0.90 if the exam paper is “hard”, and is 0.15 if the exam paper is “easy”.

(1) (5**pt**) What is the probability that the first question on your exam is marked as difficult?

(2) (5**pt**) What is the probability that your exam is of the “hard” type given that the first question on the exam is marked as difficult?

3. (**12pt**) Suppose $X \sim N(0,1)$, find the PDF of $Y = X^2 + 1$.

4. (**10pt**) The joint PF of *X* and *Y* is shown in the following table.

	Y=1	*Y*=2
X=1	$\frac{1}{6}$	$\frac{1}{3}$
X=2	$\frac{1}{9}$	a
X=3	b	$\frac{1}{9}$

Suppose that *X* and *Y* are independent. Determine the values of *a* and *b*.

5. (**16pt**) The joint PDF of X and Y is given by

$$f(x,y)=\begin{cases} cxy, & 0<y<x<1, \\ 0\ , & \text{otherwise.} \end{cases}$$

Determine

(1) (**2pt**) the constant c;

(2) (**5pt**) $P(X+\mathrm{Y}\le 1)$;

(3) (**5pt**) the marginal PDF $f_X(x)$;

(4) (**4pt**) the conditional PDF $f_{Y|X}(y|x)$.

6. (**12pt**) The joint PDF of X and Y is given by

$$f(x,y)=\begin{cases} 6(x-y), & 0<y<x<1, \\ 0\ , & \text{otherwise.} \end{cases}$$

Find $\mathrm{Cov}(X,Y)$.

7. (10pt) In a building there are 1000 lights, and each light independently has 0.7 probability to be turned on. Using the central limit theorem, approximately evaluate the probability that the total number of turned-on lights exceeds 750.

8. (**10pt**) Suppose that $X_1,X_2,\cdots,X_n$ is a random sample from population $X\sim P(\lambda)$. Derive the maximum likelihood estimator of the parameter λ.

1. (20pt)

(1) $\boldsymbol{P(A) = 0.8, P(B) = 0.4, P(B|A) = 0.25, P(A|B) =}$ ________________.

(2) Let A, B, C be mutually independent events. If $P(A) = 0.8$, $P(AB) = 0.4$, $P(BC) = 0.3$, then $P(ABC) =$ ________________.

(3) Tom and Jerry are in group of 10 people. A team of 5 people are to be selected from the group. The probability that both Tom and Jerry are selected into the team is ________________.

(4) A fair coin is tossed 5 times. The probability of obtaining 4 or more heads is ________________.

(5) Suppose that $\boldsymbol{P(X = k) = \frac{c\lambda^k}{k!}, (k = 1, 2, \cdots, \lambda > 0)}$ is a probability function, then $\boldsymbol{c} =$________________.

(6) Suppose that random vector $\boldsymbol{(X, Y) \sim N(0, 1; 0, 1; -0.5)}$, $\boldsymbol{aX + Y}$ and $\boldsymbol{Y}$ are independent, then $\boldsymbol{a} =$______________.

(7) Suppose that $\boldsymbol{X \sim Exp(\lambda)}$, $\boldsymbol{P(X \geq 1) = e^{-2}}$, then $\boldsymbol{E(X)} =$____________, $\boldsymbol{E(X^2)} =$________________.

(8) Given observed data $(6, -7, 4, -8, 10)$ of a random sample from a normal distribution $\mathrm{N}(\mu, \sigma^2)$. Then the maximum likelihood estimate of μ is ______________ and the maximum likelihood estimate of σ^2 is ________________.

2. (**12pt**) The CDF of X is given by

$$F(x) = \begin{cases} A + Be^{-2x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Find

(1) (**5pt**) A and B;

(2) (**3pt**) $P(-1 < X < 1)$;

(3) (**4pt**) PDF $f(x)$ of X.

3. (**10pt**) The diameter of the dot produced by a printer is normally distributed with a mean diameter of 2 micro-inch and a standard deviation of 0.4 micro-inch.

(1) (**5pt**) What is the probability that the diameter of a dot exceeds 2.6 micro-inch?

(2) (**5pt**) What is the probability that a diameter is between 1.4 and 2.6 micro-inch?

4. (**10pt**) Suppose that $X \sim \mathrm{P}(2)$, evaluate $\mathrm{E}[(X + 3)^2]$.

5. (**14pt**) The joint PDF of X and Y is given by

$$f(x,y)=\begin{cases} e^{-x}, & 0<y<x, \\ 0, & \text{otherwise.} \end{cases}$$

Find

(1) **(4pt)** the marginal PDF $f_Y(y)$;

(2) **(4pt)** the conditional PDF $f_{X|Y}(x|y)$;

(3) (**6pt**) $P(X \le 1 | Y \le 1)$.

6. (**12pt**) The joint PDF of X and Y is given by

$$f(x,y)=\begin{cases} \frac{1}{4}(x+y), & 0<y<x<2, \\ 0, & \text{otherwise.} \end{cases}$$

(1) (**6pt**) Evaluate $E(X)$ and $E(Y)$.
(2) **(6pt)** Evaluate $Cov(X,Y)$.

7. (**12pt**) A company has 260 telephone extensions. In any time, each extension, independently from each other, has 4% probability to request an external communication channel. Using the central limit theorem, approximately estimate how many external communication channels should be equipped, so that the probability of channel request being satisfied exceeds 95%.

8. (**10pt**) Suppose $X_1, X_2, \cdots, X_n$ is a random sample from population X, where the PDF of X is

$$f(x)=\begin{cases} \frac{1}{\theta}e^{-\frac{x}{\theta}}, & x \ge 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is unknown parameter. Derive the maximum likelihood estimator of the parameter θ.

Some values that might be useful:

$$\Phi(0.6)=0.72575, \qquad \Phi(0.9487)=0.8286$$

$$\Phi(1)=0.8413, \qquad \Phi(1.5)=0.93319,$$

$$\Phi(1.645)=0.95$$

Complex Functions and Integral Transforms Model Answer of 2021 Final Exam Paper B BJTU Lancaster University College

Yiping Cheng

November 29, 2021

Note: x, y, u, v denote real numbers or real functions.

1. (12pt) Compute the following:
(a) $(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})^{85}$
(b) all values of $\sqrt[3]{i}$
(c) principal value of $(-i)^{\frac{1}{\pi}}$
(d) $\mathrm{Re}[\exp(2 - i\frac{\pi}{3})]$

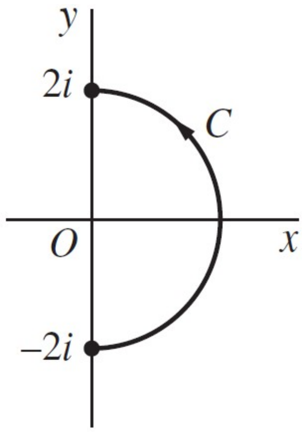
Solution: (a) $(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})^{85} = (-i)^{42}(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) = (-1)^{21}(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) = -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} + \frac{\mathbf{i}}{\sqrt{\mathbf{2}}}$.
(b) $\sqrt[3]{-1} = \sqrt[3]{e^{i\pi}} = e^{i\frac{\pi}{3}}, e^{i\frac{3\pi}{3}}, e^{i\frac{5\pi}{3}} = \frac{\mathbf{1}}{\mathbf{2}} + \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\mathbf{i}, -\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}} - \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\mathbf{i}$.
(c) P.V. $(-i)^{\frac{1}{\pi}} = \exp(\frac{1}{\pi}\mathrm{Log}(-i)) = \exp(\frac{1}{\pi}(-\frac{\pi}{2}i)) = \cos(\frac{\mathbf{1}}{\mathbf{2}}) - \mathbf{i}\sin(\frac{\mathbf{1}}{\mathbf{2}})$.
(d) $\mathrm{Re}[\exp(2 - i\frac{\pi}{3})] = \mathbf{e^2}$.

2. (10pt) Let $f(z) = x^2 + iy^2$ where $z = x + iy$ with x, y being real numbers. Find all values of z where $f'(z)$ exists using the Cauchy-Riemann equations, and give the value of $f'(z)$ where it exists.

Solution: $u_x = 2x$, $v_x = 0$, $u_y = 0$, $v_y = 2y$. For the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ to hold, we have $\mathbf{x = y}$. If $x = y$, then

$$f'(z) = u_x + iv_x = \mathbf{2x}.$$

3. (10pt) Evaluate the integral $\int_C \mathrm{Re}(z)dz$, where C is the contour shown below.



Solution: Let the parameterization of the circle be $z(\theta) = 2e^{i\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
So

$$\int_C \mathrm{Re}(z)dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta i e^{i\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{i\theta} + e^{-i\theta}) i e^{i\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i(e^{2i\theta} + 1) d\theta$$

$$= \pi \mathbf{i}.$$

4. (10pt) Evaluate the integral $\int_C \cos(\frac{z}{2})dz$ for the following two cases:
(a) C is the semi-circle from $-2i$ to $2i$ shown in the figure above.
(b) C is the line segment from $-2i$ to $2i$.

Solution: $f(z) = \cos(\frac{z}{2})$ has an anti-derivative $F(z) = 2\sin(\frac{z}{2})$ along both contours.
So

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = 2\sin(\frac{z}{2})|_{-2i}^{2i} = 4\sin i$$

$$= \mathbf{2(e - e^{-1})i}.$$

5. (12pt) Let $f(z) = \frac{z}{(z+1)(z+2)}$. Find the following:
(a) Taylor series representation of $f(z)$ in $|z| < 1$
(b) Laurent series representation of $f(z)$ in $1 < |z| < 2$.

Solution: (a) $f(z) = \frac{z}{(z+1)(z+2)} = \frac{-1}{z+1} + \frac{2}{z+2} = \frac{-1}{1+z} + \frac{1}{1+\frac{1}{2}z}$

$$= \sum_{\mathbf{n=0}}^{\infty} [\mathbf{-(-1)^n + (-\frac{1}{2})^n}]\mathbf{z^n}.$$

(b) $f(z) = \frac{-1}{z+1} + \frac{2}{z+2} = \frac{1}{1+\frac{1}{2}z} - z^{-1}\frac{1}{1+z^{-1}}$

$$= \sum_{\mathbf{n=0}}^{\infty} (\mathbf{-\frac{1}{2}})^{\mathbf{n}}\mathbf{z^n} + \sum_{\mathbf{n=1}}^{\infty} (\mathbf{-1})^{\mathbf{n}}\mathbf{z^{-n}}.$$

6. (10pt) Determine the order and residue of the pole $z = 0$ of $f(z)$:
(a) $f(z) = \frac{e^z - 1}{z^3}$
(b) $f(z) = \frac{1}{z \sin z}$

Solution: (a) $f(z) = \frac{1}{z^2}\frac{e^z-1}{z} = \frac{1}{z^2}\frac{z+\frac{z^2}{2}+\cdots}{z} = \frac{1}{z^2}(1 + \frac{z}{2} + \cdots)$. **2-order pole**. $\mathrm{Res}(\mathbf{f}, \mathbf{0}) = \frac{\mathbf{1}}{\mathbf{2}}$.
(b) $f(z) = \frac{1}{z \sin z} = \frac{1}{z^2}\frac{z}{\sin z}$. **2-order pole**. $\mathrm{Res}(\mathbf{f}, \mathbf{0}) = \mathbf{0}$.

7. (14pt) Use the residue theorem to evaluate the following integrals. All the contours are in the counterclockwise direction.
(a) $\int_{|z|=2} \frac{e^{2z}}{(z-1)^2} dz$
(b) $\int_{|z|=2} \frac{z}{z^2-1} dz$.

Solution: (a) $\frac{e^{2z}}{(z-1)^2}$ has one isolated singular point in $|z| < 2$: 1, which is a 2nd-order pole. $\mathrm{Res}(\frac{e^{2z}}{(z-1)^2}, 1) = [e^{2z}]'_{z=1} = 2e^2$. Therefore,

$$\int_{|z|=2} \frac{e^{2z}}{(z-1)^2} dz = 2\pi i \mathrm{Res}(\frac{e^{2z}}{(z-1)^2}, 1) = \mathbf{4\pi e^2 i}.$$

(b) $\frac{z}{z^2-1}$ has two isolated singular points in $|z| < 3$: -1, 1, which are both simple poles. $\mathrm{Res}(\frac{z}{z^2-1}, -1) = [\frac{z}{2z}]_{z=-1} = \frac{1}{2}$. Likewise, $\mathrm{Res}(\frac{z}{z^2-1}, 1) = \frac{1}{2}$. Therefore,

$$\int_{|z|=3} \frac{z}{z^2-1} dz = 2\pi i[\mathrm{Res}(\frac{z}{z^2-1}, -1) + \mathrm{Res}(\frac{z}{z^2-1}, 1)] = \mathbf{2\pi i}.$$

8. (12pt) Use the residue theorem to evaluate the real integral $\int_0^{2\pi} \frac{1}{2-\sin\theta} d\theta$.

Solution: Let C be the contour $z = e^{i\theta}, \quad \theta \in [0, 2\pi]$. Then $dz = ie^{i\theta}d\theta = izd\theta$, and $d\theta = \frac{dz}{iz}$. $\cos\theta = \frac{z+z^{-1}}{2}$.

$$\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta} = \int_C \frac{1}{2 - \frac{z-z^{-1}}{2i}} \frac{dz}{iz}$$

$$= \int_C \frac{-2}{z^2 - 4iz - 1} dz = \int_C \frac{-2}{(z - (2+\sqrt{3})i)(z - (2-\sqrt{3})i)} dz$$

The integrand function has a pole inside the contour, which is $(2 - \sqrt{3})i$, the residue there is

$$[\frac{-2}{2z - 4i}]_{z=(2-\sqrt{3})i} = \frac{1}{\sqrt{3}i}.$$

So by Cauchy's residue theorem, the desired integral is

$$2\pi i \cdot \frac{1}{\sqrt{3}i} = \frac{\mathbf{2\pi}}{\sqrt{\mathbf{3}}}.$$

9. (10pt) Find the Fourier transform of the following signal

$$f(t) = \begin{cases} -1, & \textbf{if } -1 < t < 0, \\ 1, & \textbf{if } 0 \le t < 1, \\ 0, & \textbf{otherwise.} \end{cases}$$

Solution: Let $F(\omega)$ denote the Fourier transform of $f(t)$. Then

$$F(\omega) = \int_{-1}^{0} -e^{-i\omega t} dt + \int_{0}^{1} e^{-i\omega t} dt = -\int_{0}^{1} e^{i\omega u} du + \int_{0}^{1} e^{-i\omega t} dt$$

$$= -\int_{0}^{1} e^{i\omega t} dt + \int_{0}^{1} e^{-i\omega t} dt = \int_{0}^{1} -2i\sin(\omega t) dt = [\frac{2i\cos(\omega t)}{\omega}]|_0^1$$

$$= \mathbf{2i} \cdot \frac{\cos\omega - \mathbf{1}}{\omega}.$$