

Probability and Statistics
Model Answer of 2022 Midterm Exam
BJTU Lancaster University College

Yiping Cheng

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1. (20pt) Fill in the blanks:

- (1) Events A and B are disjoint and independent, $P(A) = 0.5$, then $P(A) + P(B) = \underline{\hspace{1cm}}$.
- (2) Out of the students in a class, 50% are geniuses, 70% love chocolate. Let p be the conditional probability that a randomly selected student is a chocolate lover given that he/she is a genius. Then the theoretically lowest value of p is $\underline{\hspace{1cm}}$
- (3) $2n$ students are assigned randomly to 2 classes each having n students. A couple of boyfriend and girlfriend are in the $2n$ students. The probability that this couple are in the same class is $\underline{\hspace{1cm}}$. (or $\underline{\hspace{1cm}}$)
- (4) When it rains, there is a 40% chance that students play football on that day. On the other hand, when it does not rain there is a 90% chance that they play. The probability that it will rain tomorrow is 0.2. The probability that students play football tomorrow is $\underline{\hspace{1cm}}$
- (5) Let $X \sim B(20, 0.8)$, then the most likely value of X is $\underline{\hspace{1cm}}$
- (6) The random variable X has PDF $f(x) = Ce^{-3|x|}$. Then $C = \underline{\hspace{1cm}}$ and $P(|X| < 1) = \underline{\hspace{1cm}}$
- (7) Let $X \sim U(0, 1)$, $Y \sim U(0, 1)$, and X and Y be independent. Then $P(X = Y) = \underline{0}$ and $P((X - 1/2)^2 + (Y - 1/2)^2 < 1/4) = \underline{\hspace{1cm}}$.
- (8) The joint CDF of X and Y is $F(x, y) = \begin{cases} \Phi(x)(1 - e^{-y}), & \text{for } y > 0 \\ 0, & \text{otherwise.} \end{cases}$ Then the joint PDF of X and Y is $f(x, y) = \underline{\hspace{1cm}}$

2. (10pt) Approximately 0.04% of human have liver cancer. A person with liver cancer has a 95% chance of a positive test, while a person without liver cancer has a 2% chance of a false positive result. What is the probability a person has liver cancer given that the person just had a positive test?

3. (10pt) Suppose that random variable X and X itself are independent. Show that there must exist a constant C such that $P(X = C) = 1$. (Hint: consider its CDF and use the properties of CDFs)

4. (10pt) A fair coin is tossed two times independently. Let

$$H_i = \begin{cases} 1, & \text{if head is obtained at the } i\text{th toss,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $X = H_1 + H_2$, $Y = |H_1 - H_2|$. Give the joint PMF of X and Y in table form.

5. (10pt) A book has 300 pages. For each page, whether it contains typing error has Bernoulli distribution $B(1, 0.02)$. The events for each page to contain typing error are independent. What is (approximately) the probability that this book contains at most 3 pages with typing error?

6. (12pt) Scores on a certain standardized test, IQ scores, are approximately normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. Here we are referring to the distribution of scores over a very large population, and we approximate that discrete cumulative distribution function by a normal continuous cumulative distribution function.

(a) An individual is selected at random. What is the probability that his score X satisfies $120 < X < 130$?

(b) Two individuals are selected at random, independently. What is the probability that their scores X and Y satisfy $\max(X, Y) > 120$?

7. (14pt) Suppose that the joint distribution of X and Y is uniform over the region in the xy -plane bounded by the four lines $x = -1$, $x = 1$, $y = x + 1$, and $y = x - 1$. Determine

(a) $P(XY > 0)$

(b) the joint PDF of X and Y

(c) the PDF of X

(d) the conditional PDF of Y given that $X = x$

(e) $P(Y > 0 | X = 1/2)$.

8. (14pt) Let X and Y be random variables for which the joint PDF is as follows:

$$f(x, y) = \begin{cases} 8xy & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the PDF of $Z = (X - Y)^2$.