# Probability and Statistics Mock Exam 2020

**1. (20pt)** Fill in the blanks:

(1) $A$ and $B$ are two events with $P(A) = 0.4$, $P(A \cup B) = 0.7$, then $P(B|\overline{A}) =$ ____________.

(2) Three people are assigned randomly and independently into 4 rooms numbered A–D. Then the expected number of people in room A is ____________ and the probability that room A has exactly 2 people is ____________.

(3) The PDF of $X$ which has the standard normal distribution is $\varphi(x) =$ ________________.

(4) Suppose that random variables $X$ and $Y$ are independent, $X \sim Exp(1)$ and $Y \sim U[0,1]$. Then the joint PDF of $(X, Y)$ is $f(x, y) =$ ____________________, and $\text{Var}(X - Y) =$ ____________.

(5) Of the following families of distributions: binomial, Poisson, geometric, uniform, normal, and exponential, the ____________ and ____________ distributions are completely additive. A distribution family is additive if $X$ and $Y$ are independent and belong to this distribution family, then $X + Y$ also belongs to this distribution family.

(6) For an estimator (actually a sequence of estimators) to be practically usable, if must be ____________. For any distribution, the unbiased and most efficient estimator of population expectation is ________________.

**2. (10pt)** Suppose that random variables $X$ and $Y$ are independent, $X \sim N(0,1)$ and $Y \sim U[0,1]$. Let

$$Z = \begin{cases} X, & \text{if } Y < 0.4, \\ 2X - 1, & \text{otherwise.} \end{cases}$$

Find the CDF and PDF of $Z$. (You can simply write the CDF and PDF of the standard normal distribution by $\Phi(\cdot)$ and $\varphi(\cdot)$, respectively.)

**3. (12pt)** The PDF of $X$ is given by

$$f(x) = \begin{cases} 1 - |x|, & \text{if } -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = \cos(\frac{\pi X}{2})$. Find the PDF of $Y$.

**4. (10pt)** Suppose that random point $(X, Y)$ is uniformly distributed in the disk $x^2 + y^2 < 9$.
(1) Find the conditional PDF $f_{Y|X}(y|x)$.
(2) Determine $P(Y > 0|X = 2)$.

**5. (14pt)** The joint PDF of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} x + y, & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find
(1) $E(X)$ and $E(Y)$;
(2) $\text{Var}(X)$ and $\text{Var}(Y)$;
(3) $\text{Cov}(X, Y)$.

**6. (10pt)** Suppose that random variables $X$, $Y$ and $Z$ have $E(X) = E(Y) = 1$, $E(Z) = -1$, $\text{Var}(X) = \text{Var}(Y) = \text{Var}(Z) = 1$, $\rho_{X,Y} = 0$, $\rho_{X,Z} = \frac{1}{2}$, $\rho_{Y,Z} = -\frac{1}{2}$. Let $W = X - Y + Z$. Find $E(W)$ and $\text{Var}(W)$.

**7. (12pt)** An insurance company runs a certain disease insurance policy, which has 10,000 policy holders. Each year, each policy holder pays the company a premium of \$170, and if he gets the disease in that year, he receives from the company a compensation of $\$20,000$. The probability for a person to get the disease in a year is 0.006. Using the central limit theorem, for this company approximately evaluate
(1) the probability that the annual profit of this policy is at least $\$200,000$;
(2) the probability that the annual profit of this policy is positive.

**8. (12pt)** Let $X_1, \cdots, X_n$ be a random sample from $B(100, p)$.
(1) Derive the maximum likelihood estimator of $p$.
(2) Find the bias of the estimator you just derived. Is it unbiased?