Probability and Statistics
Model Answer of 2020 Final Exam Paper A
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1. (20pt) Fill in the blanks:

(1) Albert and Brian are in a group of 10 members. They are seated in a random manner in a row of 10 seats. The probability that Albert and Brian are seated next to each other is $\underline{\mathbf{\frac{1}{5}}}$.

(2) Suppose $X \sim \text{Geo}(\frac{1}{3})$. Then $P(X = 4) = \underline{\mathbf{\frac{8}{81}}}$.

(3) Suppose random variables $X \sim B(2, p), Y \sim B(3, p)$. If $P(X \geq 1) = 5/9$, then $P(Y \geq 1) = \underline{\mathbf{\frac{19}{27}}}$.

(4) The CDF of X is $F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x)$. Then the PDF of X is $f(x) = \underline{\mathbf{\frac{1}{\pi(1+x^2)}}}$, and $P(0 < X \leq 1) = \underline{\mathbf{\frac{1}{4}}}$.

(5) Suppose that X has the uniform distribution on the interval $[2, 5]$. Then $\text{Var}(X) = \underline{\mathbf{\frac{3}{4}}}$.

(6) If $E(X) = -1$, $\text{Var}(X) = 2$, and $Y = 3X + 5$, then $E(Y) = \underline{\mathbf{2}}$ and $\text{Var}(Y) = \underline{\mathbf{18}}$.

(7) Suppose that random variables X and Y are independent and both have the standard normal distribution, then $\text{Cov}(2X + 3Y, X - Y) = \underline{\mathbf{-1}}$.

(8) Suppose X_1, X_2, X_3 is a random sample from population $X \sim N(\mu, 1)$, then $\frac{1}{2}X_1 + \frac{1}{3}X_2 + kX_3$ is an unbiased estimator for μ when $k = \underline{\mathbf{\frac{1}{6}}}$.

2. (10pt) A professor provides his students with two types of exam papers: "easy" and "hard". The students have a 0.80 chance of getting a "hard" paper. The probability that the first question on the exam paper is marked as difficult is 0.90 if the exam paper is "hard", and is 0.15 if the exam paper is "easy".
(1) What is the probability that the first question on your exam is marked as difficult?
(2) What is the probability that your exam is of the "hard" type given that the first question on the exam is marked as difficult?

Solution: We have

$$P(\text{“}Hardpaper\text{”}) = 0.8, \; P(\text{“}Easypaper\text{”}) = 0.2$$

$$P(\text{“}DiffQ1\text{”}|\text{“}Hardpaper\text{”}) = 0.9,\ P(\text{“}DiffQ1\text{”}|\text{“}Easypaper\text{”}) = 0.15.$$

(1)

$$\begin{aligned} P(\text{“}DiffQ1\text{”}) &= P(\text{“}Hardpaper\text{”})P(\text{“}DiffQ1\text{”}|\text{“}Hardpaper\text{”}) \\ &+P(\text{“}Easypaper\text{”})P(\text{“}DiffQ1\text{”}|\text{“}Easypaper\text{”}) \\ &= 0.8 \times 0.9 + 0.2 \times 0.15 = 0.75. \end{aligned}$$

(2)

$$P(\text{“}Hardpaper\text{”}|\text{“}DiffQ1\text{”}) = \frac{P(\text{“}Hardpaper\text{”})P(\text{“}DiffQ1\text{”}|\text{“}Hardpaper\text{”})}{P(\text{“}DiffQ1\text{”})}$$

$$= \frac{0.72}{0.75} = 0.96.$$

3. (12pt) Suppose $X \sim N(0,1)$, find the PDF of $Y = X^2 + 1$.

Solution: Let $G(y)$, $g(y)$ denote the CDF and PDF of Y, respectively.
1) If $y \le 1$, then $G(y) = 0$.
2) If $y > 1$, $G(y) = P(Y \le y) = P(X^2 + 1 \le y) = P(-\sqrt{y-1} \le X \le \sqrt{y-1})$

$$= \Phi(\sqrt{y-1}) - \Phi(-\sqrt{y-1}) = 2\Phi(\sqrt{y-1}) - 1.$$

Hence

$$g(y) = G'(y) = \begin{cases} 0, & \text{if } y \le 1, \\ 2\varphi(\sqrt{y-1})\frac{1}{2\sqrt{y-1}}, & \text{if } y > 1 \end{cases}$$

$$= \begin{cases} 0, & \text{if } y \le 1, \\ \frac{1}{\sqrt{2\pi(y-1)}}\exp(-\frac{y-1}{2}), & \text{if } y > 1. \end{cases}$$

4. (10pt) The joint PF of X and Y is shown in the following table. Suppose that X and Y are independent. Determine the values of a and b.

	$Y=1$	$Y=2$
$X=1$	$\frac{1}{6}$	$\frac{1}{3}$
$X=2$	$\frac{1}{9}$	a
$X=3$	b	$\frac{1}{9}$

Solution: Since $P(X=1, Y=2) = P(X=1)P(Y=2)$, we have $\frac{1}{3} = (\frac{1}{6} + \frac{1}{3})(\frac{1}{3} + a + \frac{1}{9})$. Hence $a = \frac{2}{9}$.

Since $P(X=1, Y=1) = P(X=1)P(Y=1)$, we have $\frac{1}{6} = (\frac{1}{6} + \frac{1}{9} + b)(\frac{1}{6} + \frac{1}{3})$. Hence $b = \frac{1}{18}$.

5. (16pt) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} cxy, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine
(1) the constant c;

(2) $P(X+Y \le 1)$;
(3) the marginal PDF $f_X(x)$;
(4) the conditional PDF $f_{Y|X}(y|x)$.

Solution: (1)

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = \int_0^1\int_0^x cxydydx = \int_0^1 cx\frac{x^2}{2}dx = \frac{c}{8}.$$

So $c = 8$.
(2) $P(X+Y \le 1) =$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{1-x} f(x,y)dydx = \int_0^{\frac{1}{2}}\int_0^x 8xydydx + \int_{\frac{1}{2}}^1\int_0^{1-x} 8xydydx$$

$$= \int_0^{\frac{1}{2}} [4xy^2]_{y=0}^{y=x}dx + \int_{\frac{1}{2}}^1 [4xy^2]_{y=0}^{y=1-x}dx$$

$$= \int_0^{\frac{1}{2}} 4x^3dx + \int_{\frac{1}{2}}^1 4x(1-x)^2dx$$

$$= \int_0^{\frac{1}{2}} 4x^3dx + \int_0^{\frac{1}{2}} 4(1-x)x^2dx$$

$$= \int_0^{\frac{1}{2}} 4x^2dx = \frac{1}{6}.$$

(3) If $x \le 0$ or $x \ge 1$, then $f_X(x) = 0$.
If $0 < x < 1$, then

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 8xydy = 4x^3.$$

Thus,

$$f_X(x) = \begin{cases} 4x^3, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(4) If $0 < x < 1$, then

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{x^2}, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

6. (12pt) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} 6(x-y) & \textbf{if } 0 < y < x < 1, \\ 0, & \textbf{otherwise.} \end{cases}$$

Find $\mathrm{Cov}(X,Y)$.

Solution:

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dydx = \int_0^1\int_0^x x\cdot 6(x-y)dydx$$

$$= \int_0^1 [6x^2y - 3xy^2]_{y=0}^{y=x} dx = \int_0^1 3x^3 dx = \frac{3}{4}.$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dydx = \int_0^1\int_0^x y\cdot 6(x-y)dydx$$

$$= \int_0^1 [3xy^2 - 2y^3]_{y=0}^{y=x} dx = \int_0^1 x^3 dx = \frac{1}{4}.$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dydx = \int_0^1\int_0^x xy\cdot 6(x-y)dydx$$

$$= \int_0^1 [3x^2y^2 - 2xy^3]_{y=0}^{y=x} dx = \int_0^1 x^4 dx = \frac{1}{5}.$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{5} - \frac{3}{4}\cdot\frac{1}{4} = \frac{1}{80}.$$

7. (10pt) In a building there are 1000 lights, and each light independently has 0.7 probability to be turned on. Using the central limit theorem, approximately evaluate the probability that the total number of turned-on lights exceeds 750.

Solution: Let X be the number of lights that are on in the building. X has the binomial distribution with parameter $n = 1000$ and $p = 0.7$, and according to the central limit theorem, it approximately has the normal distribution with mean $\mu = np = 700$ and $\sigma^2 = np(1-p) = 210$. Using correction for continuity, the desired probability is

$$P(X > 750.5) = P(\frac{X-700}{\sqrt{210}} > \frac{750.5-700}{\sqrt{210}})$$

$$\approx 1 - \Phi(3.4848) = 1 - 0.999754 = 0.000246.$$

8. (10pt) Suppose $X_1, X_2, \cdots, X_n$ is a random sample from population $X \sim P(\lambda)$. Derive the maximum likelihood estimator of the parameter λ.

Solution: For observed values $x_1, \ldots, x_n$, the likelihood function is

$$L(\lambda) = \prod_{i=1}^n \exp(-\lambda)\frac{\lambda^{x_i}}{x_i!} = \exp(-n\lambda)\frac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}.$$

$$\log L(\lambda) = -n\lambda + \sum_{i=1}^n x_i \cdot \log(\lambda) - \log(\prod_{i=1}^n x_i).$$

The maximizer of $\log L(\lambda)$ is the solution of the following equation:

$$\frac{d\log L}{d\lambda} = -n + \frac{\sum_{i=1}^n x_i}{\lambda},$$

$$\lambda = \frac{\sum_{i=1}^n x_i}{n}.$$

Thus the maximum likelihood estimator of λ is $\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^n X_i}{n}$.