Lancaster University College
at Beijing Jiaotong University

2019/20 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Resit examination (March)

INSTRUCTIONS TO STUDENTS

1) Duration of the exam: 120 minutes
2) This paper contains 3 pages. There are 8 questions.
3) You must answer all questions.
4) This is a closed book exam. No books or notes may be brought into the exam room.
5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
6) Some values that might be useful:

$$\Phi(0.5634) = 0.7134, \qquad \Phi(1) = 0.8413,$$

$$\Phi(1.3363) = 0.9093, \qquad t_{0.05}(4) = 2.1318.$$

1. (20pt)

(1) The joint PDF of X and Y is $f(x,y) = \begin{cases} ae^{-2x-4y}, & x > 0, y > 0, \\ 0\ , & \text{otherwise.} \end{cases}$ then $a =$____________.

(2) A family is chosen at random from all three-child families. If it is known that the family has a boy among the three children, then the conditional probability that the chosen family has one boy and two girls is ____________.

(3) If random variables $X \sim B(4, 0.3)$, $Y \sim B(11, 0.3)$, and X, Y are independent, then $X + Y \sim$_______________.

(4) If random variables X, Y have a joint continuous distribution, then $P(X = Y) =$ _______________.

(5) If random variable $X \sim U(1,3)$, then $\mathrm{Var}(X) =$_______________.

(6) Suppose that X_1, X_2 have a bivariate normal distribution. Then X_1, X_2 are independent if and only if _________________ $= 0$.

(7) Let $X_1, X_2, \cdots, X_n$ be a random sample from X with $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Then $E[(X_2 - X_1)^2] =$____________________, and if $c\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2$ is an unbiased estimator of σ^2 then $c =$____________________.

(8) Let X_1, X_2, X_3, X_4 be a random sample from distribution $N(\mu, \sigma^2)$. $\bar{X}$ is the sample mean. Suppose that $c\sum_{i=1}^{4}(X_i - \bar{X})^2 \sim \chi^2(m)$. Then $c =$_____________ and $m =$________________.

2. **(10pt)** In a world, 40% of products are highly successful, 35% are moderately successful, and 25% are poor products. In addition, 95% of highly successful products receive good reviews, 60% of moderately successful products receive good reviews, and 10% of poor products receive good reviews.

   (1) What is the probability that a product attains a good review?
   (2) If a new product attains a good review, what is the probability that it will be a highly successful product?
   (3) If a product does not attain a good review, what is the probability that it will be a highly successful product?

3. **(10pt)** Two random variables X and Y are independent, $X \sim N(2,4)$, $Y \sim N(3,5)$. Determine $P(3X + 2Y \leq 22)$.

4. **(12pt)** The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} 12\ y^2, & 0 \leq x \leq 1, 0 \leq y \leq x, \\ 0\ , & \text{otherwise.} \end{cases}$$

   (1) Find $\mathrm{Cov}(X,Y)$.
   (2) Find $P(X + Y \leq 1)$.

5. **(13pt)** Let X denote the total number of successes in 15 Bernoulli trials, with probability of success $p = 0.3$ on each trial.
   (1) Determine $P(X = 4)$.
   (2) Determine approximately the value of $P(X = 4)$ by using the central limit theorem with the correction for continuity.

6. **(12pt)** Suppose that X_1, X_2, X_3, X_4, X_5 are a random sample from distribution $N(\mu, \sigma^2)$, where μ and σ^2 are both unknown. We have the following observed data of the sample: $(x_1, x_2, x_3, x_4, x_5) = (29, 20, 11, 16, 14)$. Find a 90% confidence interval for μ.

7. **(10pt)** Let X_1, X_2 be a random sample from X with $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. We want $Y = a_1 X_1 + a_2 X_2$ to be an estimator of μ, where a_1, a_2 are real constants.
   (1) For Y to be unbiased, what requirement must a_1, a_2 fulfil?
   (2) Find $\mathrm{Var}(Y)$.
   (3) Determine a_1, a_2 if we want Y to be unbiased and the most efficient.

8. **(13pt)** Let $X_1, X_2, \cdots, X_n$ be a random sample from X whose PDF is

$$f(x) = \begin{cases} \theta x^{-(\theta+1)}, & x > 1, \\ 0 \quad , & \text{otherwise}, \end{cases}$$

where $\theta > 1$ is an unknown parameter.
   (1) Find the moment estimator of θ.
   (2) Find the maximum likelihood estimator of θ.