Probability and Statistics
Model Answer of 2020 Final Exam Paper B
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1. (20pt) Fill in the blanks:

1. $P(A) = 0.8$, $P(B) = 0.4$, $P(B|A) = 0.25$, then $P(A|B) = \underline{\mathbf{0.5}}$.
2. Let A, B, C be mutually independent events. If $P(A) = 0.8$, $P(AB) = 0.4$, $P(BC) = 0.3$, then $P(ABC) = \underline{\mathbf{0.24}}$.
3. Tom and Jerry are in group of 10 people. A team of 5 people are to be selected from the group. The probability that both Tom and Jerry are selected into the team is $\underline{\frac{C_8^3}{C_{10}^5} = \frac{2}{9}}$.
4. A fair coin is tossed 5 times. The probability of obtaining 4 or more heads is $\underline{\frac{3}{16}}$.
5. Suppose that $P(X = k) = \frac{c\lambda^k}{k!}, (k = 1, 2, \cdots, \lambda > 0)$ is a probability function, then $c = \underline{\frac{1}{e^\lambda - 1}}$.
6. Suppose that random vector $(X, Y) \sim N(0, 1; 0, 1; -0.5)$, $aX + Y$ and Y are independent, then $a = \underline{\mathbf{2}}$.
7. Suppose that $X \sim Exp(\lambda)$, $P(X \geq 1) = e^{-2}$, then $E(X) = \underline{\mathbf{0.5}}$, $E(X^2) = \underline{\mathbf{0.5}}$.
8. Given observed data (6,-7,4,-8,10) of a random sample from a normal distribution $N(\mu, \sigma^2)$. Then the maximum likelihood estimate of μ is $\underline{\mathbf{1}}$ and the maximum likelihood estimate of σ^2 is $\underline{\mathbf{52}}$.

2. (12pt) The CDF of X is given by

$$F(x) = \begin{cases} A + Be^{-2x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Find
(1) A and B;
(2) $P(-1 < X < 1)$;
(3) PDF $f(x)$ of X.

Solution: (1) By right continuity, $A + B = 0$. Since $F(\infty) = 1$, we have $A = 1$. So $B = -1$.

(2)

$$P(-1 < X < 1) = F(1-) - F(-1) = 1 - e^{-2}.$$

(3)

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

3. (10pt) The diameter of the dot produced by a printer is normally distributed with a mean diameter of 2 micro-inch and a standard deviation of 0.4 micro-inch.
(1) What is the probability that the diameter of a dot exceeds 2.6 micro-inch?
(2) What is the probability that a diameter is between 1.4 and 2.6 micro-inch?

Solution: $X \sim N(2, 0.4^2)$. We then have

$$P(X > 2.6) = P(\frac{X-2}{0.4} > 1.5) = 1 - \Phi(1.5) = 1 - 0.93319 = 0.06681.$$

$$P(1.4 < X < 2.6) = P(-1.5 < \frac{X-2}{0.4} < 1.5) = \Phi(1.5) - \Phi(-1.5)$$
$$= 2\Phi(1.5) - 1 = 2 * 0.93319 - 1 = 0.86638.$$

4. (10pt) Suppose that $X \sim \mathrm{P}(2)$, evaluate $E[(X+3)^2]$.

Solution:

$$E[(X+3)^2] = E(X^2 + 6X + 9] = E(X^2) + 6E(X) + 9$$
$$= E^2(X) + \mathrm{Var}(X) + 6E(X) + 9 = 2^2 + 2 + 6 * 2 + 9 = 27.$$

5. (14pt) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} e^{-x}, 0 < y < x, \\ 0, \textbf{otherwise}. \end{cases}$$

Find
(1) the marginal PDF $f_Y(y)$;
(2) the conditional PDF $f_{X|Y}(x|y)$;
(3) $P(X \leq 1 | Y \leq 1)$.

Solution: (1) If $y \leq 0$, then $f_Y(y) = 0$.
If $y > 0$, then

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^{\infty} e^{-x}dx = e^{-y}.$$

Thus,

$$f_Y(y) = \begin{cases} e^{-y}, & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $y > 0$, then

$$f_{X|Y}(x|y) = \begin{cases} e^{y-x}, & \text{if } 0 < y < x, \\ 0, & \text{otherwise.} \end{cases}$$

(3)

$$P(Y \leq 1) = \int_{-\infty}^{1} f_Y(y)dy = \int_0^1 e^{-y}dy = 1 - e^{-1}.$$

$$P(X \le 1, Y \le 1) = \int_0^1 \int_0^x e^{-x} dy dx = \int_0^1 x e^{-x} dx = [(x+1)e^{-x}]_1^0 = 1 - 2e^{-1}.$$

$$P(X \le 1 | Y \le 1) = \frac{1 - 2e^{-1}}{1 - e^{-1}}.$$

6. (12pt) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(x+y), 0 < y < x < 2, \\ 0, \textbf{otherwise}. \end{cases}$$

(1) Evaluate $E(X)$ and $E(Y)$.
(2) Evaluate $\mathrm{Cov}(X,Y)$.

Solution: (1)

$$E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y) dy dx = \int_0^2 \int_0^x x \cdot \frac{1}{4}(x+y) dy dx$$

$$= \frac{1}{4} \int_0^2 [x^2 y + \frac{1}{2} x y^2]_{y=0}^{y=x} dx = \frac{1}{4} \int_0^2 \frac{3}{2} x^3 dx = \frac{3}{2}.$$

$$E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y) dy dx = \int_0^2 \int_0^x y \cdot \frac{1}{4}(x+y) dy dx$$

$$= \frac{1}{4} \int_0^2 [\frac{1}{2} x y^2 + \frac{1}{3} y^3]_{y=0}^{y=x} dx = \frac{1}{4} \int_0^2 \frac{5}{6} x^3 dx = \frac{5}{6}.$$

(2)

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) dy dx = \int_0^2 \int_0^x xy \cdot \frac{1}{4}(x+y) dy dx$$

$$= \frac{1}{4} \int_0^2 [\frac{1}{2} x^2 y^2 + \frac{1}{3} x y^3]_{y=0}^{y=x} dx = \frac{1}{4} \int_0^2 \frac{5}{6} x^4 dx = \frac{4}{3}.$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{3} - \frac{3}{2} \cdot \frac{5}{6} = \frac{1}{12}.$$

7. (12pt) A company has 260 telephone extensions. In any time, each extension, independently from each other, has 4% probability to request an external communication channel. Using the central limit theorem, approximately estimate how many external communication channels should be equipped, so that the probability of channel request being satisfied exceeds 95%.

Solution: Let X be the number of channel requests. X has the binomial distribution with parameter $n = 260$ and $p = 0.04$, and by the central limit theorem,, it approximately has the normal distribution with mean $\mu = np = 10.4$ and variance $\sigma^2 = np(1-p) = 9.984$.

Let m be the number of equipped channels. Then it is desired that $P(X \le m) \ge 0.95$. Using correction for continuity, $P(X \le m) = P(X < m + 0.5)$

$$0.95 < P(X < m + 0.5) = P(\frac{X - 10.4}{\sqrt{9.984}} < \frac{m + 0.5 - 10.4}{\sqrt{9.984}}) \approx \Phi(\frac{m - 9.9}{\sqrt{9.984}})$$

$$\frac{m - 9.9}{\sqrt{9.984}} \ge 1.645$$

$$m \ge 1.645 * \sqrt{9.984} + 9.9 = 15.09778353$$

So the smallest desired number of external communication channels to be equipped is 16.

8. (10pt) Suppose $X_1, X_2, \cdots, X_n$ is a random sample from population X, where the PDF of X is

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, x \ge 0, \\ 0, \textbf{otherwise}. \end{cases}$$

where $\theta > 0$ is unknown parameter. Find the maximum likelihood estimator of the parameter θ.

Solution: For observed values $x_1, \ldots, x_n$, the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \theta^{-n} \exp(-\frac{\sum_{i=1}^{n} x_i}{\theta}).$$

$$\log L(\theta) = -n \log \theta - \frac{\sum_{i=1}^{n} x_i}{\theta}.$$

The maximizer of $\log L(\theta)$ is the solution of the following equation:

$$\frac{d \log L}{d\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2},$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{n}.$$

Thus the maximum likelihood estimator of θ is $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n}$.