Reference Answer of Midterm Exam

1 Fill in the blanks

1. Let A and B be events. If $P(A) = a$ and $P(B) = b$, then the smallest possible value of $P(A \cap B)$ is $\underline{\max(a+b-1,0)}$, the greatest possible value of $P(A \cap B)$ is $\underline{\min(a,b)}$.
2. Let A and B be mutually independent events, $P(A \cup B) = 0.82$, $P(A \cap B) = 0.28$, $P(A) \geq P(B)$, then $P(A) = \underline{0.7}$, $P(B) = \underline{0.4}$.
3. A team consists of 5 boys and 5 girls. Now a small working group of 4 people are to be selected at random from the team. The probability that there are more girls than boys in the working group is $\underline{\frac{11}{42}}$.
4. Let X be a random variable whose c.d.f. is F. The probability $P(1 \leq X \leq 2)$ expressed in F is $\underline{F(2) - F(1-)}$. If $F(0) = F(10) = 0.3$, then $P(X > 4) = \underline{0.7}$.
5. Let X_1, X_2 be independent discrete random variables that both have the uniform distribution on integers $1, 2, 3$. Let $Y = X_1 - X_2$, then $E(Y) = \underline{0}$, $\text{Var}(Y) = \underline{\frac{4}{3}}$.

2

Assume that the probability that any child born will be a girl is 1/2 and that all births are independent. A family has 3 children, determine the conditional probability that this family has at least one boy on the condition that this family has at least one girl.

Solution: The probability space is simple whose sample space is

$$S = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

where 0 indicates a girl and 1 indicates a boy.

The event "at least one girl"$=\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0)\}$.

The event "at least one boy and at least one girl"$=\{(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0)\}$.

Therefore $P(\text{at least one boy}|\text{at least one girl}) = \frac{6}{7}$.

3

A random variable X has a continuous distribution with the following p.d.f.

$$f(x) = \begin{cases} \frac{c}{1+x^2} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of c. (b) Let $Y = \ln(X)$. Determine the p.d.f. of Y.

Solution: (a)

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} \frac{c}{1+x^2}dx = c \cdot \frac{\pi}{2}.$$

Therefore

$$c = \frac{2}{\pi}.$$

$$f(x) = \begin{cases} \frac{2}{\pi(1+x^2)} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) The function $r(x) = \ln(x)$ maps $(0, \infty)$ to $(-\infty, \infty)$ one-to-one, therefore we can use Theorem 3.8.4.

Let g be the p.d.f. of Y. We have

$$f(x)|dx| = g(y)|dy|.$$

Since $x = e^y$, we further have

$$g(y) = f(x)|\frac{dx}{dy}| = \frac{2}{\pi(1+e^{2y})}|\frac{dx}{dy}| = \frac{2}{\pi(1+e^{2y})} \cdot e^y = \frac{2e^y}{\pi(1+e^{2y})}.$$

4

Suppose that random variables X and Y have the following joint p.d.f.

$$f(x,y) = \begin{cases} 3y & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Evaluate $P(X + Y < 1)$. (b) Find $E(X)$ and $E(Y)$. (c) Find $\mathrm{Cov}(X, Y)$

Solution: (a)

$$P(X+Y<1) = \int_0^{1/2} dx \int_x^{1-x} 3y dy = \int_0^{1/2} (\frac{3}{2} - 3x)dx = \frac{3}{8}.$$

(b)

$$E(X) = \int_0^1 dy \int_0^y 3xy dx = \int_0^1 dy \frac{3y^3}{2} = \frac{3}{8}.$$

$$E(Y) = \int_0^1 dy \int_0^y 3y^2 dx = \frac{3}{4}.$$

(c)

$$E(XY) = \int_0^1 dy \int_0^y 3xy^2 dx = \frac{3}{10}.$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{3}{10} - \frac{3}{8} \cdot \frac{3}{4} = \frac{3}{160}.$$

5

Let X and Y be random variables such that $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$, and $\rho(X,Y) = r$. Let $U = aX + bY$ and $V = aX - bY$. Evaluate $\rho(U, V)$.

Solution:

$$\mathrm{Var}(U) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X,Y) = a^2 + b^2 + 2abr.$$

$$\mathrm{Var}(V) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) - 2ab\mathrm{Cov}(X,Y) = a^2 + b^2 - 2abr.$$

$$\mathrm{Cov}(U,V) = a^2\mathrm{Var}(X) - b^2\mathrm{Var}(Y) = a^2 - b^2.$$

Therefore

$$\rho(U,V) = \frac{\mathrm{Cov}(U,V)}{\sqrt{\mathrm{Var}(U)\mathrm{Var}(V)}} = \frac{a^2 - b^2}{\sqrt{(a^2+b^2+2abr)(a^2+b^2-2abr)}}$$

$$= \frac{a^2 - b^2}{\sqrt{(a^2-b^2)^2 + 4a^2b^2(1-r^2)}}.$$

6

Suppose that the joint p.d.f of two random variables X and Y is as follows:

$$f(x,y) = \begin{cases} c\sin(x) & \text{for } 0 \le x \le \frac{\pi}{2}, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the conditional p.d.f. of Y for every given value of X.

Solution:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 c\sin(x)dy = \begin{cases} c\sin(x) & \text{for } 0 \le x \le \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}.$$

Therefore if $0 \le x \le \frac{\pi}{2}$, then

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} 1 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

7

Let X be a discrete random variable having the following p.f.

$$f(x) = \begin{cases} p(1-p)^x & \text{for } x = 0,1,2,\dots \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X \ge k)$ for each integer $k = 0,1,2,\dots$.
(b) Find $P(X \ge t+k|X \ge t)$ for each integer $t,k = 0,1,2,\dots$.

Solution: (a)

$$P(X \ge k) = \sum_{x=k}^{\infty} p(1-p)^x = \frac{p(1-p)^k}{1-(1-p)} = (1-p)^k.$$

(b)

$$P(X \ge t+k|X \ge t) = \frac{P(X \ge t+k, X \ge t)}{P(X \ge t)} = \frac{P(X \ge t+k)}{P(X \ge t)} = \frac{(1-p)^{t+k}}{(1-p)^t} = (1-p)^k.$$

8

Let X and Y be independent random variables that both have the uniform distribution on the interval [0,1]. Let $Z = \max(X^2, Y^2)$ and $W = \min(X^2, Y^2)$.
(a) Show that Z also has the uniform distribution on the interval [0,1].
(b) Find the p.d.f. of W.

Solution: (a) We have c.d.f.s of X^2 and Y^2:

$$F_{X^2}(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \sqrt{x} & \text{for } 0 < x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

and

$$F_{Y^2}(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \sqrt{x} & \text{for } 0 < x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

$$F_Z(x) = P(Z \le x) = P(X^2 \le x, Y^2 \le x) = P(X^2 \le x)P(Y^2 \le x) = F_{X^2}(x)F_{Y^2}(x)$$

$$= \begin{cases} 0 & \text{for } x \leq 0 \\ x & \text{for } 0 < x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

This is the c.d.f of the uniform distribution on the interval [0,1].

(b)

$$F_W(x) = P(W \leq x) = P(X^2 \leq x \text{ or } Y^2 \leq x) = 1 - P(X^2 > x, Y^2 > x)$$

$$= 1 - P(X^2 > x)P(Y^2 > x) = 1 - [1 - F_{X^2}(x)][1 - F_{Y^2}(x)] = F_{X^2}(x) + F_{Y^2}(x) - F_{X^2}(x)F_{Y^2}(x)$$

$$= \begin{cases} 0 & \text{for } x \leq 0 \\ 2\sqrt{x} - x & \text{for } 0 < x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, the p.d.f. of W is

$$f_W(x) = \begin{cases} \frac{1}{\sqrt{x}} - 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$