Probability and Statistics – Lancaster_BJTU Final Exam 2017 Paper A

1 Fill in the blanks (20 points)

1. Suppose $P(A) = 0.3$, $P(B) = 0.8$, and $P(A) = P(A|B)$. Then $P(B^c|A^c) =$ ________.

2. Of the following six families of distributions: binomial, Poisson, negative binomial, uniform, normal, and Gamma, there are two families that a random variable having distribution in them may take negative values with positive probability. These two families of distributions are the ________ distributions and the ________ distributions.

3. Suppose that there is a binomial distribution with parameter n and p, where n is large. 1) If np is moderate, then it can be approximated by a ________ distribution. 2) If np is large, then according to the ________ theorem, it can be approximated by a ________ distribution.

4. The lower and upper limits of correlation coefficients are ________ and ________, respectively.

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2, then an unbiased and consistent estimator of μ is ________, and an unbiased and consistent estimator of σ^2 is ________.

2

(8 points) Suppose that a box contains r red balls and w white balls. Suppose also that balls are drawn from the box one at a time, at random, without replacement.

(a) What is the probability that all red balls will be obtained before any white balls are obtained?

(b) What is the probability that the first two balls drawn will be of the same color?

3

(12 points) Two students, Adam and Brian, are going to take a test. They take the test independently of each other. The probability for Adam to pass the test is a, and the probability for Brian to pass the test is b. Let X be the number of people of these two students who pass the test. Prove that $\mathrm{Var}(X) = a - a^2 + b - b^2$.

4

(12 points) Suppose that random variables X and Y are **independent**. X has the uniform distribution on the interval $[0,1]$, and Y has the exponential distribution with parameter β. Suppose also that $E(X) = E(Y)$.

(a) Determine the value of β.

(b) Find $P(X > Y)$.

5

(8 points) Suppose that X and Y are independent random variables that both have the Poisson distribution and are such that $\text{Var}(X) + \text{Var}(Y) = 5$. Evaluate $P(X + Y < 2)$.

6

(14 points) A random sample $X_1, \ldots, X_n$ is to be taken from a distribution with mean μ and standard deviation σ^2.

(a) Use the Chebyshev inequality to determine the smallest n such that the following relation will be satisfied:

$$P(|\overline{X}_n - \mu| < \frac{\sigma}{4}) > 0.99$$

(b) Suppose additionally that the random sample $X_1, \ldots, X_n$ has normal distribution. Determine the smallest n such that the above relation will be satisfied.

7

(14 points) Suppose that X and Y have the **bivariate normal distribution** with covariance matrix $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. The real number a is such that

$$aX + Y \text{ and } aX - Y \text{ are independent, and}$$

$$\text{Var}(aX + Y) = 1.$$

(a) Determine the value of a.

(b) Compute $\text{Var}(aX - Y)$.

8

(12 points) Suppose that $X_1, \ldots, X_n$ form a random sample from a distribution for which the p.d.f. is as follows:

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $\theta > 1$ is the parameter to be estimated. Derive the M.L.E. of θ.

Table of the Standard Normal Distribution Function

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du$$

z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998