Lancaster University College
at Beijing Jiaotong University

2019/20 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Midterm examination (November)

INSTRUCTIONS TO STUDENTS

1) Duration of the exam: 120 minutes
2) This paper contains 3 pages. There are 8 questions.
3) You must answer all questions.
4) This is a closed book exam. No books or notes may be brought into the exam room.
5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
6) Some values of CDF of the standard Normal distribution:

$$\Phi(1) = 0.8413, \qquad \Phi(1.645) = 0.95,$$

$$\Phi(1.96) = 0.975, \qquad \Phi(2) = 0.9772, \qquad \Phi(3) = 0.9987\,.$$

1. (20pt)

Use the random events A, B, C to express the events in (1) and (2).

(1) The event that A occurs but neither B nor C occurs is $A\,\bar{B}\bar{C}$.

(2) The event that at least one of A, B, C occurs is $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$.

(3) An urn contains ten balls numbered from 1 to 10. If three balls are taken, the probability that the maximum number of them is 5 is $\frac{C_4^2}{C_{10}^3} = 0.05$.

(4) A and B are two random events. $P(A) = 0.4, P(B) = 0.3, P(A \cup B) = 0.6$, then $P(A\bar{B}) = 0.3$.

(5) A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected randomly without replacement from the batch. The probability that the second part is defective given that the first part is defective is $\frac{49}{849} = 0.0577$.

(6) If the random variable $X \sim N(1,2), Y = 2X + 3$, then $Y \sim N(5,8)$.

(7) If random variables $X \sim N(1,2), Y \sim N(3,4)$, X and Y are independent, then $2X + 3Y \sim N(11,44)$.

(8) Suppose that X follows the Poisson distribution, and $P(X = 1) = P(X = 2)$. Then, $P(X = 3) = \frac{4}{3e^2} = 0.1804$.

(9) The CDF of random variable X is $F(x) = \frac{1}{1+2^{-x}}$. Then $P(0 < X < 1) =$ $\frac{1}{6}$.

(10) Two people independently decode a message. The probabilities that they succeed are $\frac{1}{5}, \frac{1}{4}$, respectively. The probability that they can decode the message is 0.4.

2. (10pt) Let X be a continuous random variable with the following PDF:

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & otherwise. \end{cases}$$

Find the PDF of $Y = \sqrt{1 - x^2}$.

Solution: $y = r(x) = \sqrt{1-x^2}$ is one-to-one for $0 \le x \le 1$. It maps $0 \le x \le 1$ to $0 \le y \le 1$, and $\mathrm{r}^{-1}(\mathrm{y}) = \sqrt{1-y^2}$. Let $g(y)$ be the PDF of Y. Then [0,1] is the support of $g(y)$, and in this interval

$$g(y) = f\left(r^{-1}(y)\right)\left|\frac{dr^{-1}(y)}{dy}\right| = 2\sqrt{1-y^2}\left|\frac{d\sqrt{1-y^2}}{dy}\right|$$

$$=2\sqrt{1-y^2} \cdot \tfrac{1}{2}\left(1-y^2\right)^{-\frac{1}{2}} \cdot 2y = 2y.$$

Hence finally,

$$g(y) = \begin{cases} 2y, & 0 \le y \le 1, \\ 0, & otherwise. \end{cases}$$

3. (10pt) The height of a population of women has the normal distribution with mean 64 inches and standard deviation 2 inches.

(1) What is the probability that a randomly selected woman from this population is between 58 inches and 70 inches?

(2) Determine the height that defeats (i.e. is taller than) 95% of the population.

(3) What is the probability that five women selected independently at random from this population are all taller than 68 inches?

Solution: Let X be the height in inch of this population of women. We have $X \sim N(64,4)$.

(1) The desired probability is $P(58 < X < 70) = P(\frac{58-64}{2} < \frac{X-64}{2} < \frac{70-64}{2})$

$$= \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 2 * 0.9987 - 1 = 0.9974.$$

(2) The height is $64 + 2\Phi^{-1}(0.95) = 64 + 2 * 1.645 = 67.29$.

(3) $P(X > 68) = P\left(\frac{X-64}{2} > 2\right) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$. The desired probability is $(1 - \Phi(2))^5 = 6.1 * 10^{-9}$.

4. (10pt) An inspector working for a manufacturing company has a 0.99 chance of identifying a defective item if it is defective, and a 0.005 chance of incorrectly classifying a good item as defective. We have known that 0.01 of the items produced by the company are defective.

(1) What is the probability that an item selected for inspection is classified as defective?

(2) If an item selected at random is classified as good, what is the probability that it is indeed good?

Solution: Let D be the event that the item is defective. Let A be the event that the item is identified as defective. We know that $P(A|D) = 0.99$ and $P(A|\bar{D}) = 0.005$, and $P(D) = 0.01$.

(1) $P(A) = P(D)P(A|D) + P(\bar{D})P(A|\bar{D}) = 0.01 * 0.99 + 0.99 * 0.005$
$= 0.01485.$

(2) $$P(\bar{D}|\bar{A}) = \frac{P(\bar{D}\bar{A})}{P(\bar{A})} = \frac{P(\bar{D})P(\bar{A}|\bar{D})}{P(D)P(\bar{A}|D)+P(\bar{D})P(\bar{A}|\bar{D})} =$$
$$= \frac{0.99*(1-0.005)}{0.01*(1-0.99)+0.99*(1-0.005)} = 0.9998985.$$

5. (12pt) Two cards are drawn from a deck of 52 cards. Let X and Y be the number of kings and queens drawn (so that $X + Y \leq 2$), respectively.

(1) Determine the joint PF of X and Y (in table form).

(2) Are X and Y independent?

Solution: (1)

$$P(X=0,Y=0) = \frac{C_{44}^2}{C_{52}^2} = \frac{946}{1326},\ P(X=0,Y=1) = \frac{C_4^1C_{44}^1}{C_{52}^2} = \frac{176}{1326},\ P(X=0,Y=2) = \frac{C_4^2}{C_{52}^2} = \frac{6}{1326},$$

$$P(X=1,Y=0) = \frac{176}{1326},\ P(X=1,Y=1) = \frac{C_4^1C_4^1}{C_{52}^2} = \frac{16}{1326},\ P(X=1,Y=2) = 0$$

$$P(X=2,Y=0) = \frac{C_4^2}{C_{52}^2} = \frac{6}{1326},\ P(X=2,Y=1) = 0,\ P(X=2,Y=2) = 0$$

	Y=0	Y=1	Y=2
X=0	$\frac{946}{1326}$	$\frac{176}{1326}$	$\frac{6}{1326}$
X=1	$\frac{176}{1326}$	$\frac{16}{1326}$	0
X=2	$\frac{6}{1326}$	0	0

(2) Not independent. Because $P(X=2) = P(Y=2) \neq 0$ but $P(X=2,Y=2) = 0$.

6. (12pt) The joint PDF of X and Y is given by

$$f(x,y)=\begin{cases} kx^2y, & x^2\le y\le 1,\\ 0, & otherwise.\end{cases}$$

Find (1) The constant k. (2) $P(X\ge Y)$.

Solution: (1)

$$1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dy\,dx=\int_{-1}^{1}\int_{x^2}^{1}kx^2y dy\,dx=\int_{-1}^{1}kx^2\int_{x^2}^{1}y dy\,dx$$

$$=\int_{-1}^{1}kx^2\frac{1-x^4}{2}dx=\frac{k}{2}\int_{-1}^{1}(x^2-x^6)dx=\frac{k}{2}\left(\frac{1}{3}-\frac{1}{7}\right)2=\frac{4}{21}k.$$

$$\text{So } k=\frac{21}{4}.$$

(2) $P(X\ge Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x,y)dy\,dx=\int_{0}^{1}\int_{x^2}^{x}\frac{21}{4}x^2y dy\,dx$

$$=\int_{0}^{1}\frac{21}{4}x^2\int_{x^2}^{x}y dy\,dx=\int_{0}^{1}\frac{21}{4}x^2\frac{x^2-x^4}{2}dx=\frac{21}{8}\int_{0}^{1}(x^4-x^6)dx$$

$$=\frac{21}{8}\left(\frac{1}{5}-\frac{1}{7}\right)=\frac{3}{20}.$$

7. (14pt) Suppose that X and Y have the following joint PDF:

$$f(x,y)=\begin{cases} 2(x+y), & 0<x<y<1,\\ 0, & otherwise.\end{cases}$$

Determine (1) the marginal PDF of X;
(2) the conditional PDF of Y given that $X=x$.

Solution: (1) Let $f(x)$ be the PDF of X. Then

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy.$$

1.a) If $x\le 0$ or $x\ge 1$, then $f(x)=0$.

1.b) If $0<x<1$, then $f(x)=\int_{x}^{1}2(x+y)dy=[2xy+y^2]|_{y=x}^{y=1}$

$$=[2xy+y^2]|_{y=x}^{y=1}=1+2x-3x^2.$$

Therefore,

$$f(x)=\begin{cases} 1+2x-3x^2, & 0<x<1,\\ 0, & \text{otherwise.}\end{cases}$$

(2) $f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)}$. Thus for $0 < x < 1$,

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)} = \begin{cases} \dfrac{2(x+y)}{1+2x-3x^2}, & 0 < x < y < 1, \\ 0 \quad, & \text{otherwise.} \end{cases}$$

8. (12pt) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{1}{2}(x+y)e^{-(x+y)}, & x \geq 0, y \geq 0, \\ 0, & otherwise. \end{cases}$$

Find the PDF of $Z = X + Y$.

Solution: Let $h(z)$ be the PDF of Z. Then

$$h(z) = \int_{-\infty}^{\infty} f(x, z-x)dx$$

1) If $z < 0$, then $h(z) = 0$.

2) If $z \geq 0$, then $h(z) = \int_0^z \frac{1}{2}(x+z-x)e^{-(x+z-x)}dx = \int_0^z \frac{1}{2}ze^{-z}dx$

$$= \frac{1}{2}z^2e^{-z}.$$

$$h(z) = \begin{cases} \dfrac{1}{2}z^2e^{-z}, & z \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$