## 2021/22 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Final examination (December)

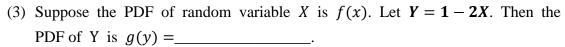
## **INSTRUCTIONS TO STUDENTS**

1) Duration of the exam: 120 minutes

- 2) This paper contains 3 pages. There are 8 questions.
- 3) You must answer all questions.
- 4) This is a closed book exam. No books or notes may be brought into the exam room.
- 5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
- 6) Some values that might be useful:

$$\Phi(0.2381) = 0.5941,$$
  $\Phi(0.9037) = 0.8169,$   $\Phi(1) = 0.8413,$   $\Phi(1.5921) = 0.9443,$   $\Phi(1.8856) = 0.97033,$   $\Phi(3.4848) = 0.999754$ 

1.	(20pt)
(1)	Suppose $P(A-B) = P(A) - P(B)$ . Then $P(B-A) = $
(2)	A bag contains 3 white balls and 7 black balls. Take 2 balls, without replacement,
	from the bag. The probability that at least one of the two balls taken is white is



- (4) Assume that  $X \sim \mathbf{B}(\mathbf{40}, \mathbf{0}, \mathbf{1})$ , then the distribution of X can be approximated by the Poisson distribution with  $\lambda = \underline{\phantom{A}}$ .
- (5) The continuous random variable X has memoryless property, then X follows the \_\_\_\_\_\_ distribution.
- (6) Let  $X \sim U(1,7)$ ,  $Y \sim N(0,4)$ , and X and Y are independent. Let Z = 3X 2Y. Then E(Z) =\_\_\_\_\_\_\_, Var(Z) =\_\_\_\_\_\_.
- (7) Given sample data (8, 4, 0, 3, 5). The sample mean is \_\_\_\_\_\_, and the sample variance is \_\_\_\_\_.
- (8) There are three criteria for assessing estimators: consistency, \_\_\_\_\_\_, and efficiency.
- 2. (**8pt**) Prove that for every events A and B, the following inequality holds:  $\max(0, P(A) + P(B) 1) \le P(AB) \le \min(P(A), P(B))$ .
- **3.** (**10pt**) Customers are used to evaluate preliminary product designs. In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products.
- (a) (**5pt**) What is the probability that a product attains a good review?
- (b) (**5pt**) If a new design attains a good review, what is the probability that it will be a highly successful product?
- **4.** (10pt) Random variables X and Y are independent,  $X \sim N(1,4)$ ,  $Y \sim N(0,1)$ . Determine P(2X Y > 10).
- 5. (14pt) Suppose that X and Y have the following joint PDF:

$$f(x,y) = \begin{cases} c(x+y), & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) (3pt) Find the constant c.
- (b) (3pt) Find the marginal PDF  $f_X(x)$ .
- (c) (3pt) Determine  $P(X \le \frac{1}{2})$ .
- (d) (3pt) Find the conditional PDF of Y given that X = x.
- (e) (**2pt**) Are *X* and *Y* independent? Why?
- 6. (16pt) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \frac{x+y}{8} & 0 < x < 2, 0 < y < 2, \\ 0 & otherwise. \end{cases}$$

Determine

- (a) (4pt) E(X) and E(Y)
- (b) (5pt) Var(X) and Var(Y)
- (c) (7pt) Cov(X,Y) and  $\rho_{X,Y}$ .
- **7.** (12pt) Let  $X_1, X_2, \dots, X_{30}$  be independent random variables each having a discrete distribution with PF

$$f(x) = \begin{cases} \frac{1}{4} & if \ x = 0 \ or \ 2, \\ \frac{1}{2} & if \ x = 1, \\ 0 & otherwise. \end{cases}$$

Use the central limit theorem and the correction for continuity to approximate the probability that  $X_1 + X_2 + \cdots + X_{30}$  is at most 33.

**8.** (10pt) Let  $X_1, X_2, \dots, X_n$  be a random sample from population  $X \sim Exp(\lambda)$ . Derive the maximum likelihood estimator of the parameter  $\lambda$ .