Lancaster University College
at Beijing Jiaotong University

2021/22 Examinations

Course code: WB73L004Q

Course name: Probability Theory and Mathematical Statistics (B)

Final examination (December)

INSTRUCTIONS TO STUDENTS

1) Duration of the exam: 120 minutes
2) This paper contains 3 pages. There are 8 questions.
3) You must answer all questions.
4) This is a closed book exam. No books or notes may be brought into the exam room.
5) A scientific calculator is allowed in the examination. Other electronic devices are not allowed in the exam room.
6) Some values that might be useful:

$$\Phi(0.2381) = 0.5941, \qquad \Phi(0.9037) = 0.8169,$$

$$\Phi(1) = 0.8413, \qquad \Phi(1.5921) = 0.9443,$$

$$\Phi(1.8856) = 0.97033, \qquad \Phi(3.4848) = 0.999754$$

1. (20pt)

(1) Suppose $\boldsymbol{P(A-B)=P(A)-P(B)}$. Then $\boldsymbol{P(B-A)=}$ ____________.

(2) A bag contains 3 white balls and 7 black balls. Take 2 balls, without replacement, from the bag. The probability that at least one of the two balls taken is white is ______________.

(3) Suppose the PDF of random variable X is $f(x)$. Let $\boldsymbol{Y=1-2X}$. Then the PDF of Y is $g(y)=$__________________.

(4) Assume that $\boldsymbol{X \sim \mathrm{B}(40, 0.1)}$, then the distribution of $\boldsymbol{X}$ can be approximated by the Poisson distribution with $\boldsymbol{\lambda=}$ _______________.

(5) The continuous random variable $\boldsymbol{X}$ has memoryless property, then $\boldsymbol{X}$ follows the ___________________________ distribution.

(6) Let $\boldsymbol{X \sim U(1,7)}$, $\mathbf{Y} \sim \boldsymbol{N(0,4)}$, and $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent. Let $\boldsymbol{Z=3X-2Y}$. Then $\mathbf{E(Z)=}$ _____________, $\mathbf{Var(Z)=}$ _____________.

(7) Given sample data (8, 4, 0, 3, 5). The sample mean is _____________, and the sample variance is _______________.

(8) There are three criteria for assessing estimators: consistency, _______________, and efficiency.

2. (**8pt**) Prove that for every events A and B, the following inequality holds:

$$\max(0, P(A)+P(B)-1) \le P(AB) \le \min(P(A), P(B)).$$

3. (**10pt**) Customers are used to evaluate preliminary product designs. In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products.

(a) (**5pt**) What is the probability that a product attains a good review?

(b) (**5pt**) If a new design attains a good review, what is the probability that it will be a highly successful product?

4. (**10pt**) Random variables X and Y are independent, $X \sim N(1,4)$, $Y \sim N(0,1)$. Determine $P(2X-Y>10)$.

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5. (**14pt**) Suppose that X and Y have the following joint PDF:

$$f(x,y) = \begin{cases} c(x+y), & 0 < x < y < 1, \\ 0 \quad , & \text{otherwise.} \end{cases}$$

(a) (**3pt**) Find the constant c.
(b) (**3pt**) Find the marginal PDF $f_X(x)$.
(c) (**3pt**) Determine $P(X \le \frac{1}{2})$.
(d) (**3pt**) Find the conditional PDF of Y given that $X = x$.
(e) (**2pt**) Are X and Y independent? Why?

6. (**16pt**) The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{x+y}{8} & 0 < x < 2, 0 < y < 2, \\ 0 & otherwise. \end{cases}$$

Determine
(a) (**4pt**) $E(X)$ and $E(Y)$
(b) (**5pt**) $Var(X)$ and $Var(Y)$
(c) (**7pt**) $Cov(X,Y)$ and $\rho_{X,Y}$.

7. (12pt) Let $X_1, X_2, \cdots, X_{30}$ be independent random variables each having a discrete distribution with PF

$$f(x) = \begin{cases} \dfrac{1}{4} & if\ x = 0\ or\ 2, \\ \dfrac{1}{2} & if\ x = 1, \\ 0 & otherwise. \end{cases}$$

Use the central limit theorem and the correction for continuity to approximate the probability that $X_1 + X_2 + \cdots + X_{30}$ is at most 33.

8. (**10pt**) Let $X_1, X_2, \cdots, X_n$ be a random sample from population $X \sim Exp(\lambda)$. Derive the maximum likelihood estimator of the parameter λ.