Complex Functions and Integral Transforms
Model Answer of 2021 Final Exam Paper B
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Note: x, y, u, v denote real numbers or real functions.

1. (12pt) Compute the following:
(a) $(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})^{85}$
(b) all values of $\sqrt[3]{i}$
(c) principal value of $(-i)^{\frac{1}{\pi}}$
(d) $\mathrm{Re}[\exp(2 - i\frac{\pi}{3})]$

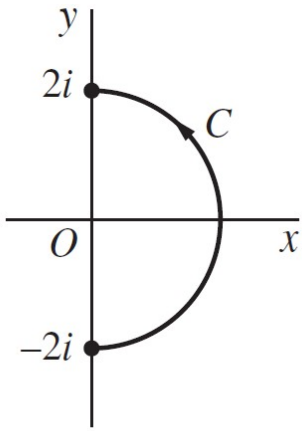
Solution: (a) $(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})^{85} = (-i)^{42}(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) = (-1)^{21}(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) = -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} + \frac{\mathbf{i}}{\sqrt{\mathbf{2}}}$.
(b) $\sqrt[3]{-1} = \sqrt[3]{e^{i\pi}} = e^{i\frac{\pi}{3}}, e^{i\frac{3\pi}{3}}, e^{i\frac{5\pi}{3}} = \frac{\mathbf{1}}{\mathbf{2}} + \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\mathbf{i}, -\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}} - \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\mathbf{i}$.
(c) P.V. $(-i)^{\frac{1}{\pi}} = \exp(\frac{1}{\pi}\mathrm{Log}(-i)) = \exp(\frac{1}{\pi}(-\frac{\pi}{2}i)) = \cos(\frac{\mathbf{1}}{\mathbf{2}}) - \mathbf{i}\sin(\frac{\mathbf{1}}{\mathbf{2}})$.
(d) $\mathrm{Re}[\exp(2 - i\frac{\pi}{3})] = \mathbf{e^2}$.

2. (10pt) Let $f(z) = x^2 + iy^2$ where $z = x + iy$ with x, y being real numbers. Find all values of z where $f'(z)$ exists using the Cauchy-Riemann equations, and give the value of $f'(z)$ where it exists.

Solution: $u_x = 2x$, $v_x = 0$, $u_y = 0$, $v_y = 2y$. For the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ to hold, we have $\mathbf{x = y}$. If $x = y$, then

$$f'(z) = u_x + iv_x = \mathbf{2x}.$$

3. (10pt) Evaluate the integral $\int_C \mathrm{Re}(z)dz$, where C is the contour shown below.



Solution: Let the parameterization of the circle be $z(\theta) = 2e^{i\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. So

$$\int_C \mathrm{Re}(z)dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta i e^{i\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{i\theta} + e^{-i\theta}) i e^{i\theta} d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i(e^{2i\theta} + 1) d\theta$$
$$= \pi \mathbf{i}.$$

4. (10pt) Evaluate the integral $\int_C \cos(\frac{z}{2})dz$ for the following two cases:
(a) C is the semi-circle from $-2i$ to $2i$ shown in the figure above.
(b) C is the line segment from $-2i$ to $2i$.

Solution: $f(z) = \cos(\frac{z}{2})$ has an anti-derivative $F(z) = 2\sin(\frac{z}{2})$ along both contours. So

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = 2\sin(\frac{z}{2})|_{-2i}^{2i} = 4\sin i$$
$$= \mathbf{2(e - e^{-1})i}.$$

5. (12pt) Let $f(z) = \frac{z}{(z+1)(z+2)}$. Find the following:
(a) Taylor series representation of $f(z)$ in $|z| < 1$
(b) Laurent series representation of $f(z)$ in $1 < |z| < 2$.

Solution: (a) $f(z) = \frac{z}{(z+1)(z+2)} = \frac{-1}{z+1} + \frac{2}{z+2} = \frac{-1}{1+z} + \frac{1}{1+\frac{1}{2}z}$

$$= \sum_{\mathbf{n=0}}^{\infty} [-(-\mathbf{1})^{\mathbf{n}} + (-\frac{\mathbf{1}}{\mathbf{2}})^{\mathbf{n}}]\mathbf{z}^{\mathbf{n}}.$$

(b) $f(z) = \frac{-1}{z+1} + \frac{2}{z+2} = \frac{1}{1+\frac{1}{2}z} - z^{-1}\frac{1}{1+z^{-1}}$

$$= \sum_{\mathbf{n=0}}^{\infty} (-\frac{\mathbf{1}}{\mathbf{2}})^{\mathbf{n}}\mathbf{z}^{\mathbf{n}} + \sum_{\mathbf{n=1}}^{\infty} (-\mathbf{1})^{\mathbf{n}}\mathbf{z}^{-\mathbf{n}}.$$

6. (10pt) Determine the order and residue of the pole $z = 0$ of $f(z)$:
(a) $f(z) = \frac{e^z - 1}{z^3}$
(b) $f(z) = \frac{1}{z \sin z}$

Solution: (a) $f(z) = \frac{1}{z^2}\frac{e^z-1}{z} = \frac{1}{z^2}\frac{z+\frac{z^2}{2}+\cdots}{z} = \frac{1}{z^2}(1+\frac{z}{2}+\cdots)$. **2-order pole**. $\mathrm{Res}(\mathbf{f}, \mathbf{0}) = \mathbf{\frac{1}{2}}$.
(b) $f(z) = \frac{1}{z \sin z} = \frac{1}{z^2}\frac{z}{\sin z}$. **2-order pole**. $\mathrm{Res}(\mathbf{f}, \mathbf{0}) = \mathbf{0}$.

7. (14pt) Use the residue theorem to evaluate the following integrals. All the contours are in the counterclockwise direction.
(a) $\int_{|z|=2} \frac{e^{2z}}{(z-1)^2} dz$
(b) $\int_{|z|=2} \frac{z}{z^2-1} dz$**.**

Solution: (a) $\frac{e^{2z}}{(z-1)^2}$ has one isolated singular point in $|z| < 2$: 1, which is a 2nd-order pole. $\mathrm{Res}(\frac{e^{2z}}{(z-1)^2}, 1) = [e^{2z}]'_{z=1} = 2e^2$. Therefore,

$$\int_{|z|=2} \frac{e^{2z}}{(z-1)^2} dz = 2\pi i \mathrm{Res}(\frac{e^{2z}}{(z-1)^2}, 1) = \mathbf{4\pi e^2 i}.$$

(b) $\frac{z}{z^2-1}$ has two isolated singular points in $|z| < 3$: -1, 1, which are both simple poles. $\mathrm{Res}(\frac{z}{z^2-1}, -1) = [\frac{z}{2z}]_{z=-1} = \frac{1}{2}$. Likewise, $\mathrm{Res}(\frac{z}{z^2-1}, 1) = \frac{1}{2}$. Therefore,

$$\int_{|z|=3} \frac{z}{z^2-1} dz = 2\pi i[\mathrm{Res}(\frac{z}{z^2-1}, -1) + \mathrm{Res}(\frac{z}{z^2-1}, 1)] = \mathbf{2\pi i}.$$

8. (12pt) Use the residue theorem to evaluate the real integral $\int_0^{2\pi} \frac{1}{2-\sin\theta} d\theta$.

Solution: Let C be the contour $z = e^{i\theta}, \quad \theta \in [0, 2\pi]$. Then $dz = ie^{i\theta}d\theta = izd\theta$, and $d\theta = \frac{dz}{iz}$. $\cos\theta = \frac{z+z^{-1}}{2}$.

$$\int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \int_C \frac{1}{2 - \frac{z-z^{-1}}{2i}} \frac{dz}{iz}$$

$$= \int_C \frac{-2}{z^2 - 4iz - 1} dz = \int_C \frac{-2}{(z-(2+\sqrt{3})i)(z-(2-\sqrt{3})i)} dz$$

The integrand function has a pole inside the contour, which is $(2-\sqrt{3})i$, the residue there is

$$[\frac{-2}{2z-4i}]_{z=(2-\sqrt{3})i} = \frac{1}{\sqrt{3}i}.$$

So by Cauchy's residue theorem, the desired integral is

$$2\pi i \cdot \frac{1}{\sqrt{3}i} = \mathbf{\frac{2\pi}{\sqrt{3}}}.$$

9. (10pt) Find the Fourier transform of the following signal

$$f(t) = \begin{cases} -1, & \textbf{if } -1 < t < 0, \\ 1, & \textbf{if } 0 \le t < 1, \\ 0, & \textbf{otherwise.} \end{cases}$$

Solution: Let $F(\omega)$ denote the Fourier transform of $f(t)$. Then

$$F(\omega) = \int_{-1}^{0} -e^{-i\omega t} dt + \int_{0}^{1} e^{-i\omega t} dt = -\int_{0}^{1} e^{i\omega u} du + \int_{0}^{1} e^{-i\omega t} dt$$

$$= -\int_{0}^{1} e^{i\omega t} dt + \int_{0}^{1} e^{-i\omega t} dt = \int_{0}^{1} -2i \sin(\omega t) dt = [\frac{2i\cos(\omega t)}{\omega}]|_0^1$$

$$= \mathbf{2i} \cdot \frac{\cos\omega - \mathbf{1}}{\omega}.$$