16 Jan Wed 9 am

Session 9: Logistic Regression

Stay Linear while

Powerful

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Based on slides by Neil D. Lawrence

Overview

Session 8 - Naive Bayes

- Naive Bayes classifier
- Numerical issues due to multiplication of probabilities: use **log**, use [Laplace smoothing (https://en.wikipedia.org/wiki/Additive_smoothing)](https://en.wikipedia.org/wiki/Additive_smoothing) (also in Session 8 notebook, search keywords).

This session

- Logistic regression: direct estimation of the probability for each class. Note: a **classification** rather than regression algorithm.
- Why stay *linear*: scalability (speed), interpretation, model complexity
- Two **default** linear classifier in Matlab: logistic regression (LR) and linear SVM
- For your possible interest only: [Watch: Conditional Logistic Regression Applied to Predicting Horse Race Winners in Hong Kong (https://www.youtube.com/watch?v=5gW0PO7g6pY)](https://www.youtube.com/watch?v=5gW0PO7g6pY)
- For your possible interest only: [Kaggle Competition: Hong Kong Horse Racing Results 2014-17 Seasons (https://www.kaggle.com/lantanacamara/hong-kong-horse-racing)](https://www.kaggle.com/lantanacamara/hong-kong-horse-racing) with notebook, code, data, discussions. Happy exploring!
- Many examples (with notebooks) at [sklearn.linear_model.LogisticRegression (https://scikit-learn.org/stable/modules/generated/sklearn.linear_model.LogisticRegression.html)](https://scikit-learn.org/stable/modules/generated/sklearn.linear_model.LogisticRegression.html)

In [8]:

```
import numpy as np
import pandas as pd
import pods
import matplotlib.pyplot as plt
%matplotlib inline
```

Logistic Regression

Log Odds

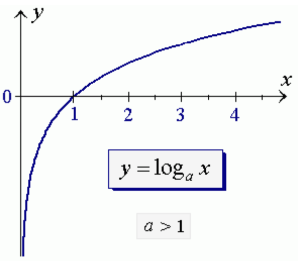
- Model the *log-odds* with the basis functions.
- [odds (http://en.wikipedia.org/wiki/Odds)](http://en.wikipedia.org/wiki/Odds) are defined as the ratio of the probability of a positive outcome, to the probability of a negative outcome.
- Probability is between zero and one, odds are:

$$\frac{\pi}{1-\pi}$$

binary

$\pi\ [0, 1]$

- Odds are between 0 and ∞.
- Logarithm of odds maps them to $-\infty$ to ∞.



Logit Link Function

- The [Logit function (http://en.wikipedia.org/wiki/Logit)](http://en.wikipedia.org/wiki/Logit),

$$g^{-1}(\pi_i) = \log \frac{\pi_i}{1-\pi_i}.$$

This function is known as a *link function*. i is the data point index.
- For a standard regression we take,

$$f(\mathbf{x}_i) = \mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x}_i),$$

- For classification we perform a logistic regression.

$$\log \frac{\pi_i}{1-\pi_i} = \mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x}_i)$$

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$\frac{\pi}{1-\pi} = e^{w^T\phi(x)}$

$\frac{1}{\frac{1}{\pi}-1} = e^{w^T\phi(x)}$

$\frac{1}{\pi} - 1 = \frac{1}{e^{w^T\phi(x)}}$

$\frac{1}{\pi} = 1 + e^{-w^T\phi(x)}$

$\pi = \frac{1}{1+e^{-w^T\phi(x)}}$

Inverse Link Function

We have defined the link function as taking the form $g^{-1}(\cdot)$ implying that the inverse link function is given by $g(\cdot)$. Since we have defined,

$$g^{-1}(\pi(\mathbf{x})) = \mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x})$$

we can write π in terms of the *inverse link* function, $g(\cdot)$ as

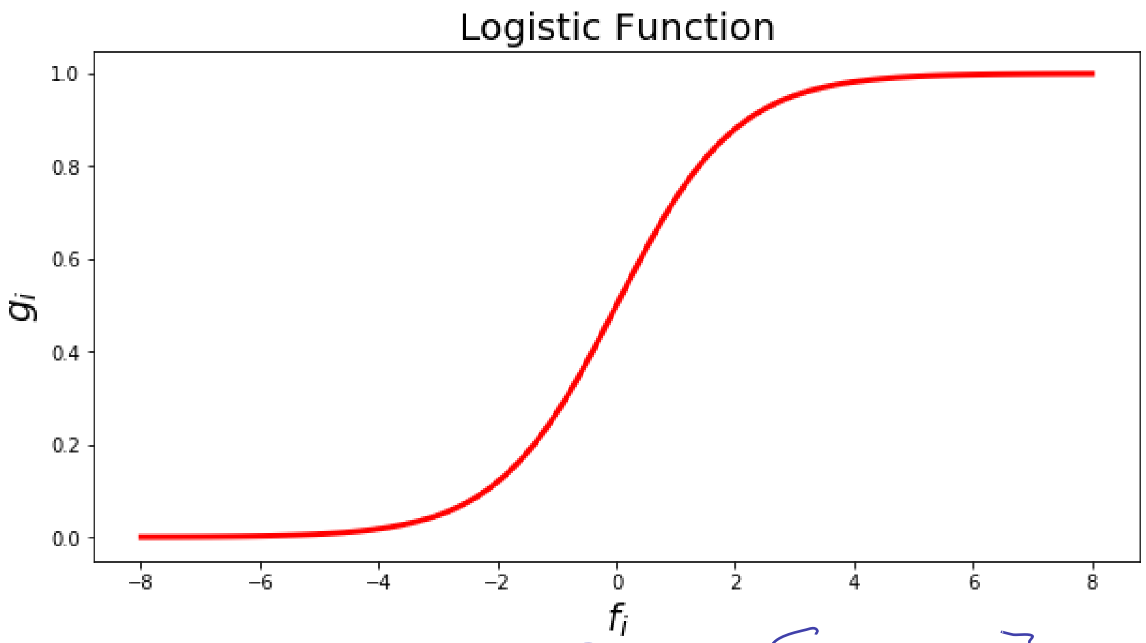
$$\pi(\mathbf{x}) = g(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x})).$$

prob

In [9]:

```
fig, ax = plt.subplots(figsize=(10,5))
f = np.linspace(-8, 8, 100)
g = 1/(1+np.exp(-f))

ax.plot(f, g, 'r-', linewidth=3)
ax.set_title('Logistic Function', fontsize=20)
ax.set_xlabel('$f_i$', fontsize=20)
ax.set_ylabel('$g_i$', fontsize=20)
plt.savefig('./diagrams/logistic.svg')
```



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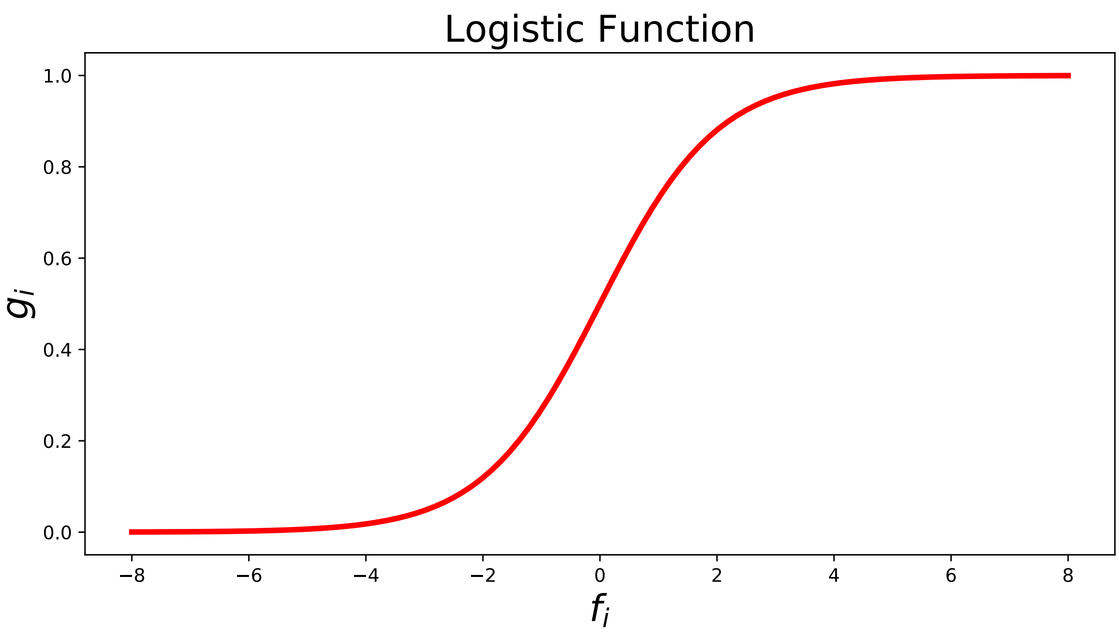
0

$[-\infty, \infty] \rightarrow [0, 1]$

Logistic function

- Logistic (http://en.wikipedia.org/wiki/Logistic_function) (or sigmoid) squashes real line to between 0 & 1. Sometimes also called a 'squashing function'.



Prediction Function

- Can now write π as a function of the input and the parameter vector as,

$$\pi(\mathbf{x}, \mathbf{w}) = \frac{1}{1 + \exp\left(-\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x})\right)}.$$

- Compute the output of a standard linear basis function composition ($\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x})$, as we did for linear regression)
- Apply the inverse link function, $g(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x}))$.
- Use this value in a Bernoulli distribution to form the likelihood.

Bernoulli Reminder

- From last time

$$P(y_i|\mathbf{w}, \mathbf{x}) = \pi_i^{y_i}(1 - \pi_i)^{1-y_i}$$

- Trick for switching betwen probabilities

```
def bernoulli(y, pi):
  if y == 1:
      return pi
  else:
      return 1-pi
```

Maximum Likelihood

- Conditional independence of data:

$$P(\mathbf{y}|\mathbf{w}, \mathbf{X}) = \prod_{i=1}^{n} P(y_i|\mathbf{w}, \mathbf{x}_i).$$

Log Likelihood

$$\log P(\mathbf{y}|\mathbf{w}, \mathbf{X}) = \sum_{i=1}^{n} \log P(y_i|\mathbf{w}, \mathbf{x}_i) = \sum_{i=1}^{n} y_i \log \pi_i + \sum_{i=1}^{n} (1 - y_i)\log(1 - \pi_i)$$

Objective Function

- Probability of positive outcome for the ith data point

$$\pi_i = g\left(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x}_i)\right),$$

where $g(\cdot)$ is the *inverse* link function
- Objective function of the form

$$E(\mathbf{w}) = -\sum_{i=1}^{n} y_i \log g\left(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x}_i)\right) - \sum_{i=1}^{n}(1 - y_i)\log\left(1 - g\left(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x}_i)\right)\right).$$

error

Minimize Objective

- Gradient wrt $\pi(\mathbf{x}; \mathbf{w})$

$$\frac{\mathrm{d}E(\mathbf{w})}{\mathrm{d}\mathbf{w}} = -\sum_{i=1}^{n} \frac{y_i}{g\left(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x})\right)} \boxed{\frac{\mathrm{d}g(f_i)}{\mathrm{d}f_i}\boldsymbol{\phi}(\mathbf{x}_i)} + \sum_{i=1}^{n} \frac{1-y_i}{1-g\left(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x})\right)} \boxed{\frac{\mathrm{d}g(f_i)}{\mathrm{d}f_i}\boldsymbol{\phi}(\mathbf{x}_i)}$$

Link Function Gradient

- Also need gradient of inverse link function wrt parameters.

$$\begin{aligned} g(f_i) &= \frac{1}{1+\exp(-f_i)} \\ &= (1+\exp(-f_i))^{-1} \end{aligned}$$

and the gradient can be computed as

$$\begin{aligned} \frac{\mathrm{d}g(f_i)}{\mathrm{d}f_i} &= \exp(-f_i)(1+\exp(-f_i))^{-2} \\ &= \frac{1}{1+\exp(-f_i)}\frac{\exp(-f_i)}{1+\exp(-f_i)} \\ &= g(f_i)(1-g(f_i)) \end{aligned}$$

Objective Gradient

$$\frac{\mathrm{d}E(\mathbf{w})}{\mathrm{d}\mathbf{w}} = -\sum_{i=1}^{n} y_i \left(1 - g\left(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x})\right)\right) \boldsymbol{\phi}(\mathbf{x}_i) + \sum_{i=1}^{n}(1-y_i)\left(g\left(\mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x})\right)\right) \boldsymbol{\phi}(\mathbf{x}_i).$$

LR 1
LR 2
:
10

Optimization of the Function

- Can't find a stationary point of the objective function analytically.
- Optimization has to proceed by *numerical methods.*
  - Newton's method (http://en.wikipedia.org/wiki/Newton%27s_method) or
  - gradient based optimization methods (http://en.wikipedia.org/wiki/Gradient_method)
- Similarly to matrix factorization, for large data *stochastic gradient descent* (Robbins Munroe optimization procedure) works well.

→ Seed (100)

Ad Matching for Facebook

- This approach used in many internet companies, making lots of money for them.
- Example: ad matching for Facebook.
  - Millions of advertisers
  - Billions of users
  - How do you choose who to show what?
- Logistic regression used in combination with decision trees ()
- Practical Lessons from Predicting Clicks on Ads at Facebook (http://www.herbrich.me/papers/adclicksfacebook.pdf)

posterior $P(y|x)$

#6 Bayesian Regression

Discriminative vs Generative Models

- Naive Bayes: A **generative** model, modelling the joint density $p(y, \mathbf{x})$.
  - *Strong* assumptions about data generation distribution -- feature independence.

  - Higher bias but lower variance
  - Not a linear model in general (linear in some special cases only): discussion at stackexchange (https://stats.stackexchange.com/questions/142215/how-is-naive-bayes-a-linear-classifier)

Quora

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- Logistic regression: A **discriminative** model, modelling the posterior probability $p(y|\mathbf{x})$ directly.
  - Learn a mapping from the input to the label directly.
- For your possible interest only, a paper by Andrew Ng and Michael Jordan: On Discriminative vs. Generative Classifiers: A comparison of logistic regression and naive Bayes (http://ai.stanford.edu/~ang/papers/nips01-discriminativegenerative.pdf)

2001

Other GLMs (*optional*)

- Logistic regression is part of a family known as *generalized linear models*
- They all take the form

$$g^{-1}(f_i(x)) = \mathbf{w}^\top \boldsymbol{\phi}(\mathbf{x}_i)$$

- As another example let's look at *Poisson regression*.

Poisson Distribution (*optional*)

- Poisson distribution (https://en.wikipedia.org/wiki/Poisson_distribution) is used for 'count data'. For non-negative integers, y,

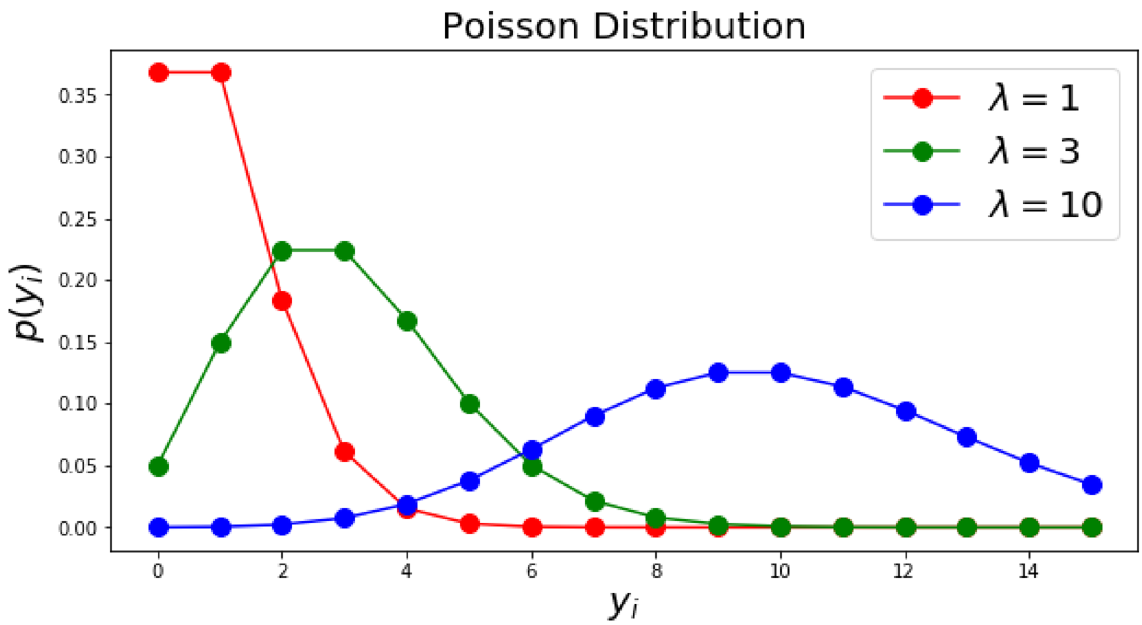
$$P(y) = \frac{\lambda^y}{y!}\exp(-y)$$

- Here λ is a *rate* parameter that can be thought of as the number of arrivals per unit time (customers at restaurant, cars at traffic lights, packets on network, etc).
- Poisson distributions can be used for disease count data. E.g. number of incidence of malaria in a district.

In [10]:

```
from scipy.stats import poisson
fig, ax = plt.subplots(figsize=(10,5))
y = np.asarray(range(0, 16))
p1 = poisson.pmf(y, mu=1.)
p3 = poisson.pmf(y, mu=3.)
p10 = poisson.pmf(y, mu=10.)

ax.plot(y, p1, 'r.-', markersize=20, label='$\lambda=1$')
ax.plot(y, p3, 'g.-', markersize=20, label='$\lambda=3$')
ax.plot(y, p10, 'b.-', markersize=20, label='$\lambda=10$')
ax.set_title('Poisson Distribution', fontsize=20)
ax.set_xlabel('$y_i$', fontsize=20)
ax.set_ylabel('$p(y_i)$', fontsize=20)
ax.legend(fontsize=20)
plt.savefig('./diagrams/poisson.svg')
```



Poisson Regression (*optional*)

- Poisson regression make rate a function of space/time.

$$\log \lambda(\mathbf{x}, t) = \mathbf{w}_x^\top \boldsymbol{\phi}_x(\mathbf{x}) + \mathbf{w}_t^\top (\phi_t(t))$$

- This is known as a *log linear* or *log additive* model.
- The link function is a logarithm.
- We can rewrite such a function as

$$\log \lambda(\mathbf{x}, t) = f_x(\mathbf{x}) + f_t(t)$$

Multiplicative Model (*optional*)

- Be careful though ... a log additive model is really multiplicative.

$$\log \lambda(\mathbf{x}, t) = f_x(\mathbf{x}) + f_t(t)$$

- Becomes

$$\lambda(\mathbf{x}, t) = \exp(f_x(\mathbf{x}) + f_t(t))$$

- Which is equivalent to

$$\lambda(\mathbf{x}, t) = \exp(f_x(\mathbf{x})) \exp(f_t(t))$$

- Link functions can be deceptive in this way.

Reading

- Section 5.2.2 of @Rogers:book11 up to pg 182.