Deep Learning for System Identification

Introduction

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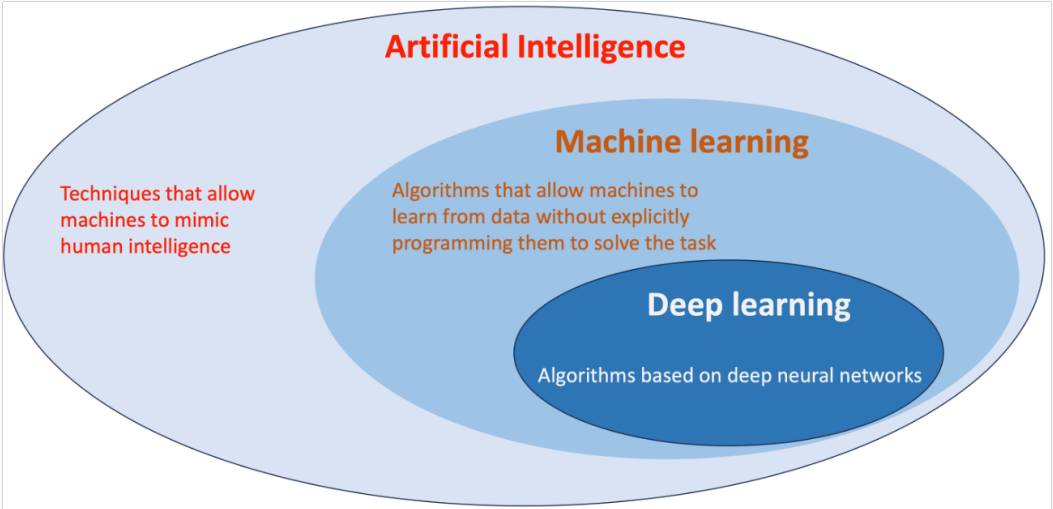
Artificial Intelligence, Machine Learning, Deep Learning

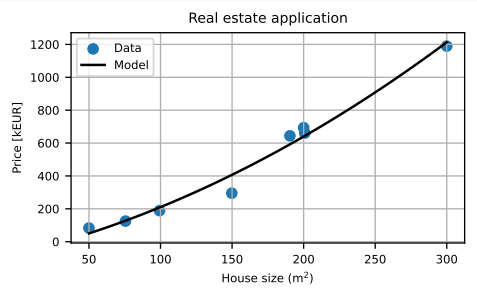


Machine learning: Regression

Nothing really special...

- Dataset: $D = \{(x_i, y_i)\}_{i=1}^{N}$
- Model structure: $\hat{y} = M(x; \theta)$
- Loss: $\mathcal{L}(\theta) = \frac{1}{N} \sum_{i=1}^{N} \|y_i - \hat{y}_i(\theta)\|^2$

$$\hat{\theta} = \arg\min_{\theta} \mathcal{L}(\theta)$$



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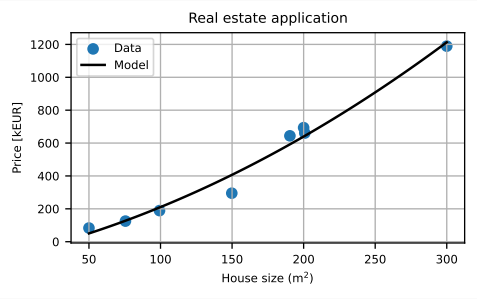
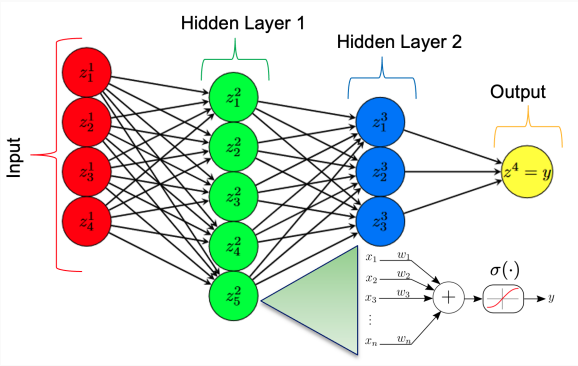


In deep learning:

- Expressive model structures like **neural networks**
- Iterative optimization, often gradient-based
- Efficient software with support for **automatic differentiation**

Is it just a form of *glorified curve fitting*?

Feedforward Neural Networks



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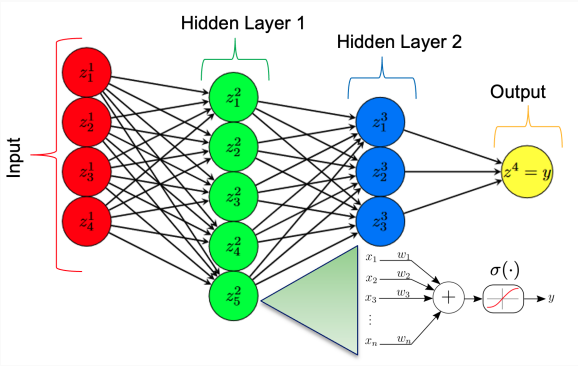

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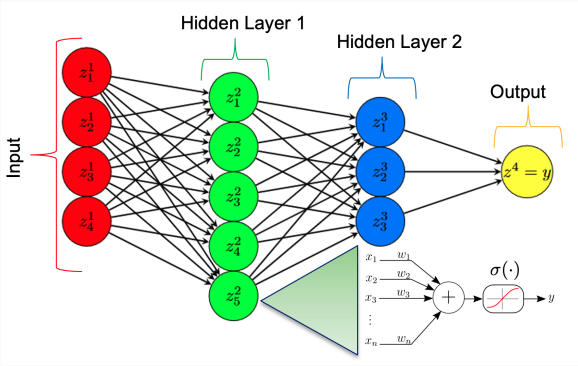

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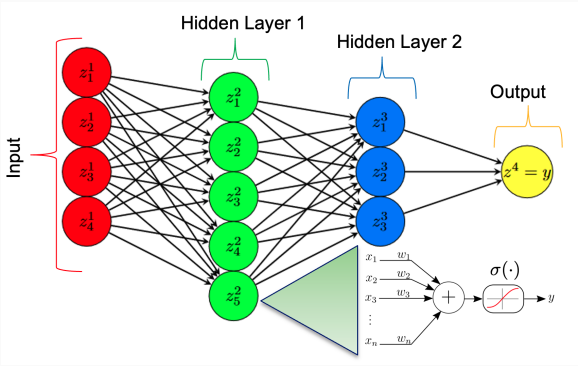

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- Non-linearities essential for **expressiveness**

Gradient Descent

1. Initialize: $\hat{\theta}$
2. **for** $k = 1, \dots, N$:
   - Compute Gradient: $g = \nabla_\theta \mathcal{L}(\theta^k)$
   - Update Parameters:

$$\hat{\theta} = \hat{\theta} - \gamma g$$

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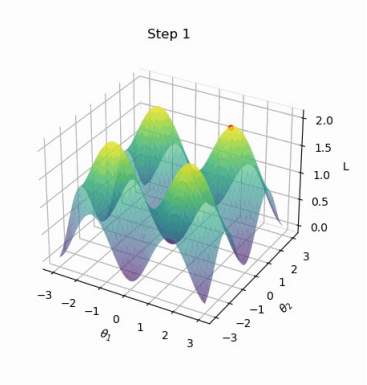

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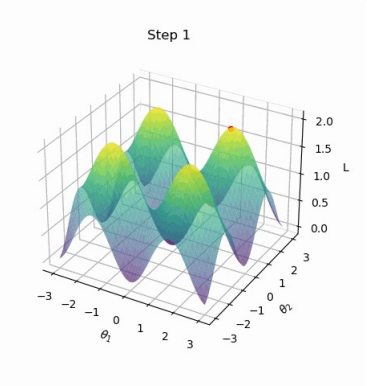


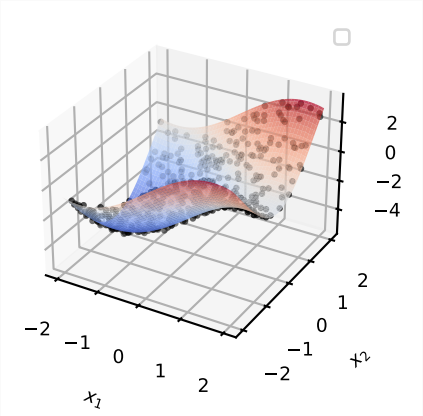
- Mini-batching: at each iteration, compute loss on a subset of the dataset
- Variants of plain gradient descent like `Adam` are more common and effective
- Second-order methods like (L)-BFGS also suitable for small/medium-scale problems

Example: synthetic toy dataset

Consider the 2D function $f : \mathbb{R}^2 \mapsto \mathbb{R}$

$$f(x) = 2\sin(x_1) - 3\cos(x_2)$$
$$x \in [-2, 2]^2 \subset \mathbb{R}^2$$

- Training and test datasets: 500 random points in the domain
- Additive noise with standard deviation 0.1



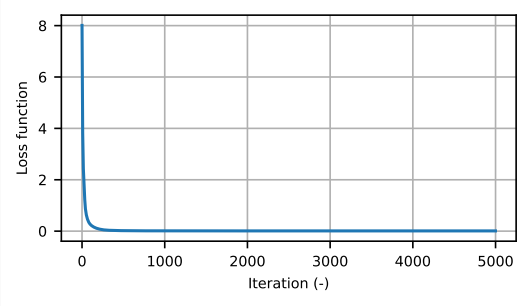
Feed-forward neural network

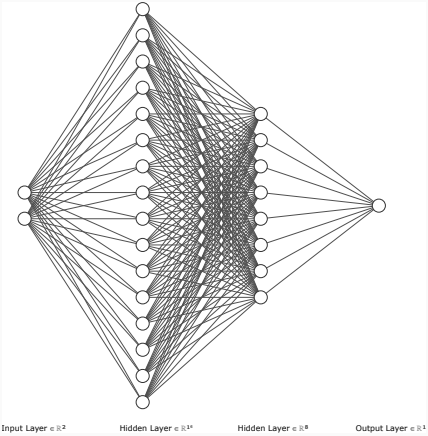
Model with:

- 2 inputs, 1 output - just like $f(x)$
- 2 hidden layers with $[16, 8]$ neurons
- $\tanh(\cdot)$ activation functions

$$\hat{y} = W_3 \tanh\left(W_2 \tanh(W_1 x + b_1) + b_2\right) + b_3$$

Training: 5000 iterations of `Adam`...



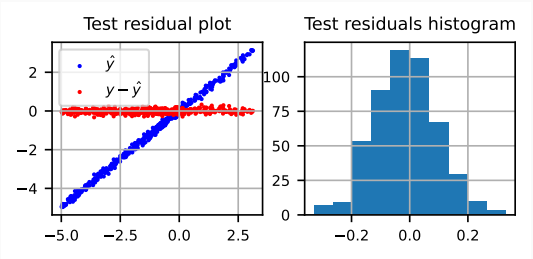


Model evaluation

Common to inspect on the test:

- Predictions and residuals vs measured output (left)
- Histogram of residuals (right)

Keep on following the good practices!



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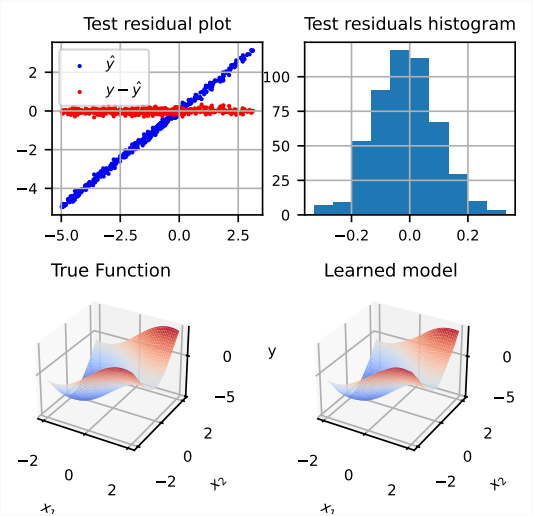
2D toy example. We can also visualize:

- The known true function (right)
- The learned model over a grid (left)



Deep Learning For System Identification

- Feedforward nets fed by lagged input/outputs for NARX identification:

$$\hat{y}(k) = \mathrm{FF}(\overbrace{y(k-1), \ldots, y(k-n_a), u(k), u(k-1), \ldots, u(k-n_b-1)}^{=x(k)}; \theta)$$

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- State-space models with neural-network updata/output maps:
  - Also known as Recurrent Neural Networks

$$x(k+1) = \mathrm{FF}_x(x(k), u(k); \theta)$$
$$y(k) = \mathrm{FF}_y(x(k); \theta)$$

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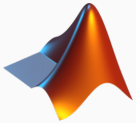
- Theoretical aspects
  - learning theory
  - uncertainty quantification
  - ...

Software for Deep Learning

An essential aspect is the software implementation. A few options:

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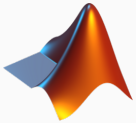

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We will try out Google's **JAX**!

Some Literature

Machine learning reference books

- Murphy, Kevin P. *Probabilistic machine learning: An introduction.* MIT press, 2022.
- Murphy, Kevin P. *Probabilistic machine learning: Advanced topics.* MIT press, 2023

Deep learning in system identification (my biased perspective)

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