Model order reduction of deep structured state-space models: A system-theoretic approach

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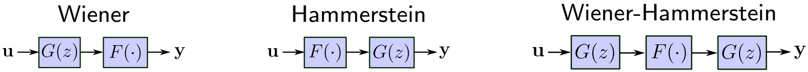
Deep structured state-space models

Growing interest within the machine-learning community. They dominate long-range sequence learning where Transformers suffer the $\mathcal{O}(N^2)$ scaling.

- Interconnection of linear dynamical systems with static non-linearities (and Normalization layers, skip connection, ...)
- The architecture should ring a bell to sysid researchers.

The classic block-oriented modeling framework.

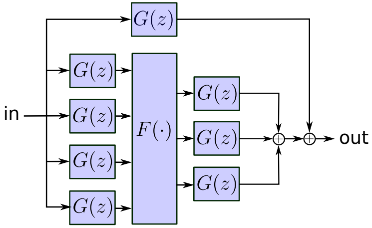


E.W. Bai, F. Giri Block-oriented nonlinear system identification. *Springer*, 2010

The dynoNet architecture

LTI blocks for deep learning in our 2021 dynoNet architecture

dynoNet architecture



Python code

```
G1 = LinearMimo(1,  4, ...) # a SIMO tf
F = StaticNonLin(4, 3, ...) # a static NN
G2 = LinearMimo(3, 1, ...) # a MISO tf
G3 = LinearSiso(1, 1, ...) # a SISO tf

def model(in_data):
    y1 = G1(in_data)
    z1 = F(y1)
    y2 = G2(z1)
    out = y2 + G3(in_data)
```

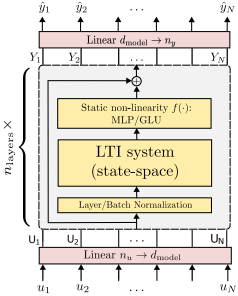

M. Forgione and D.Piga. *dynoNet*: A Neural Network architecture for learning dynamical systems. *International Journal of Adaptive Control and Signal Processing*, 2021

Differentiable transfer functions are now also in the `torchaudio` library. Our paper included in the documentation of `torchaudio.lfilter`.

https://pytorch.org/audio/main/generated/torchaudio.functional.lfilter.html

Deep Structured State-Space Model Architecture

- Architecture with normalization layers, skip connections, (dropout).
- State-space parameterization of the linear dynamical system



Several strategies to make the LTI system simulation fast and efficient.

A. Gu, K. Goel, C. Ré. Efficiently Modeling Long Sequences with Structured State Spaces. *ICLR*, 2022

Our idea: bring in model order reduction to simplify these architectures!

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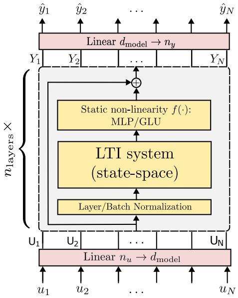

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Deep Linear Recurrent Units

We consider the deep LRU architecture recently introduced by DeepMind:

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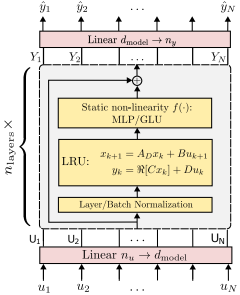

- Discrete-time, MIMO LTI system
- Complex diagonal state-transition matrix A_D:

$$A_D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_x})$$

- Stable parameterization with $|\lambda_i| < 1$
- Implementation with either:
  - $\mathcal{O}(N)$ sequential ops (standard recursion)
  - N parallel jobs, $\mathcal{O}(\log N)$ ops each (parallel scan)
- SOTA on long-range sequences

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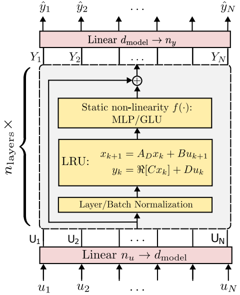

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Model Order Reduction

Consider a LTI state-space system partitioned as:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k-1) \\ x_2(k-1) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Du(k),$$

Reduction is often applied to systems in

- Modal form (A diagonal, states sorted from slowest to fastest)
- Balanced form (states sorted for decreasing Hankel singular values)

The states x_2 can be removed by:

- Truncation $\Rightarrow$ keep (A_{11}, B_1, C_1, D)
- Singular perturbation $\Rightarrow$ solve $x_2(k) = x_2(k-1)$

We tested all combinations for LRU simplification: MT, MSP, BT, BSP.

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Model Order Reduction

- Modal reduction is almost directly applicable to the LRU, which is already in modal (diagonal) form.
  1. Sort system (A_D, B, C, D) for decreasing eigenvalues magnitude
  2. Eliminate fastest states
- Balanced reduction techniques require a three-step procedure:
  1. Balance (A_D, B, C, D) to a non-diagonal (A_b, B_b, C_d, D)
  2. Reduce (A_b, B_b, C_b, D) to a non-diagonal (A_r, B_r, C_r, D)
  3. Diagonalize (A_r, B_r, C_r, D) to fit the LRU structure

Regularization

Regularization introduced to enhance the MOR:

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \frac{1}{N} \sum_{k=1}^{N} (y_k - \hat{y}_k(\theta))^2 + \gamma \mathcal{R}(\theta),$$

Modal ℓ_1

$$\mathcal{R}(\theta) = \sum_{j=1}^{n_x} |\lambda_j|$$

- Push some modes towards 0
- Tailored for modal reduction MT/MSP

Hankel nuclear norm (HNN)

$$\mathcal{R}(\theta) = \sum_{j=1}^{n_x} \sigma_j$$

- Push some HSV towards 0
- Tailored for balanced reduction BT/BSP

- Modal ℓ_1 efficient (λ_j are on the diagonal)
- HNN also efficient exploiting diagonal A_D for HSV computation

Hankel nuclear norm minimization

Let us denote as G and G_r the original and reduced system.

- The HNN $\sum_{j=1}^{n_x} \sigma_j$ is a convex approximation to the McMillan degree of G (minimium realization order)
- The choice of HNN regularization is justified by the bound:

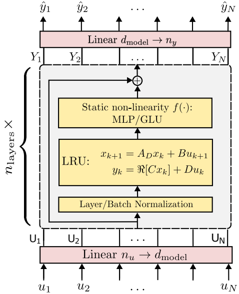
$$\|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n_x} \sigma_j,$$

valid when G_r is obtained with balanced reduction methods BT/BSP

Example

Experiments on the F16 ground vibration benchmark. Deep LRU with

- $n_{\text{layers}} = 6$ layers
- $n_x = 100$ states per layer
- $d_{\text{model}} = 50$ units per layer
- Layer Normalization
- MLP non-linearity



Results on test set `FullMSine_Level6_Validation`

Regularization	Channel 1			Channel 2			Channel 3		
	fit	RMSE	NRMSE	*fit*	RMSE	NRMSE	*fit*	RMSE	NRMSE
No reg.	86.5	0.180	0.134	90.0	0.167	0.099	76.2	0.368	0.237
Modal ℓ_1	85.4	0.195	0.145	89.8	0.171	0.102	74.5	0.395	0.254
Hankel norm	85.8	0.190	0.142	89.0	0.185	0.110	75.5	0.379	0.245

In line with literature. Regularization has a large effect on the LTI blocks!

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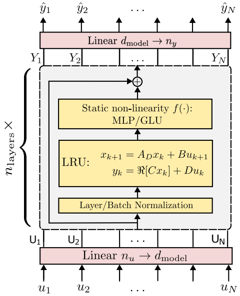

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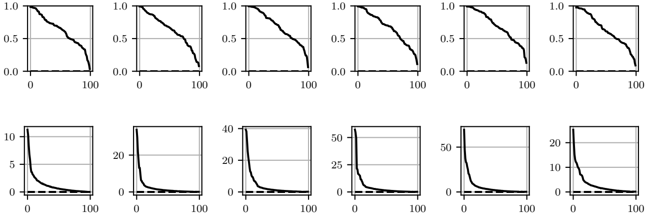
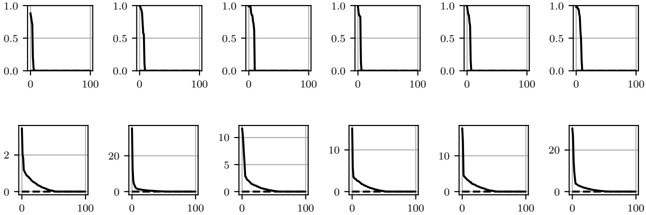
No regularization: eigenvalues magnitude (top) and HSV (bottom)



Modal ℓ_1 regularization: eigenvalues magnitude (top) and HSV (bottom)



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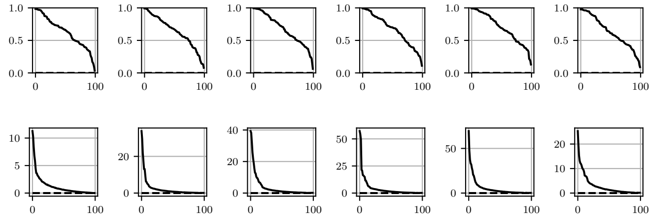
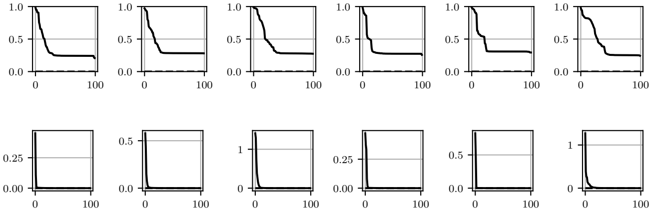


HNN regularization: eigenvalues magnitude (top) and HSV (bottom)

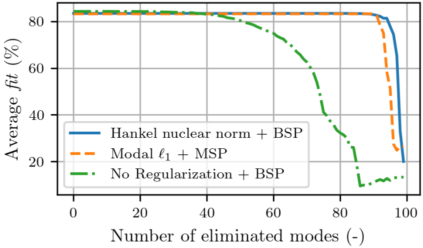


Model Order Reduction

Performance of all regularizers/model order reduction techniques assessed.

	Truncation Method			
Regularization Method	BT	BSP	MT	MSP
No Regularization	28	43	3	35
Modal ℓ_1	56	73	0	**91**
Hankel nuclear norm	89	**91**	18	76

Table: Number of states eliminated s.t. performance decrease is less than 1%



- Best results with Hankel nuclear norm + balanced singular perturbation and modal ℓ_1 + modal singular perturbation
- Without regularization, MOR is significantly less effective

Thank you.
Questions?

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