Model order reduction of deep structured state-space models: A system-theoretic approach

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Deep structured state-space models

Growing interest within the machine-learning community. They dominate long-range sequence learning where Transformers suffer the $\mathcal{O}(N^2)$ scaling.

- Interconnection of linear dynamical systems with static non-linearities (and Normalization layers, skip connection, ...)
- The architecture should ring a bell to sysid researchers.

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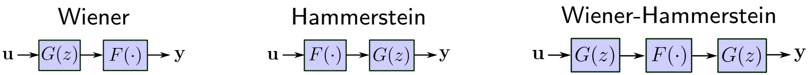
The classic block-oriented modeling framework.

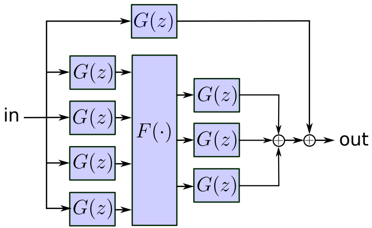


E.W. Bai, F. Giri Block-oriented nonlinear system identification. *Springer*, 2010

The dynoNet architecture

LTI blocks for deep learning also in our 2021 dynoNet architecture

dynoNet architecture



Python code

```
G1 = LinearMimo(1,  4, ...) # a SIMO tf
F = StaticNonLin(4, 3, ...) # a static NN
G2 = LinearMimo(3, 1, ...) # a MISO tf
G3 = LinearSiso(1, 1, ...) # a SISO tf

def model(in_data):
    y1 = G1(in_data)
    z1 = F(y1)
    y2 = G2(z1)
    out = y2 + G3(in_data)
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M. Forgione and D.Piga. *dynoNet*: A Neural Network architecture for learning dynamical systems. *International Journal of Adaptive Control and Signal Processing*, 2021

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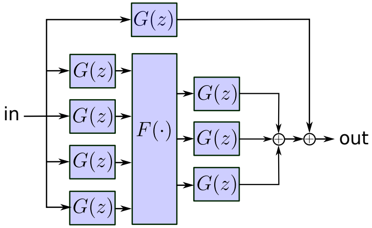

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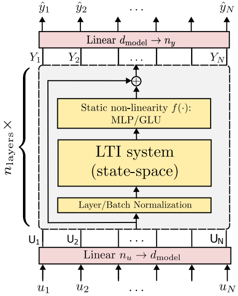
Differentiable transfer functions are now also in the `torchaudio` library.

https://pytorch.org/audio/main/generated/torchaudio.functional.lfilter.html

Deep Structured State-Space Model Architecture

- Architecture with normalization layers, skip connections, (dropout).
- State-space parameterization of the linear dynamical system



Focus on making the LTI system learning fast and well-posed.

A. Gu, K. Goel, C. Ré. Efficiently Modeling Long Sequences with Structured State Spaces. *ICLR*, 2022

Our idea: bring in model order reduction to simplify these architectures!

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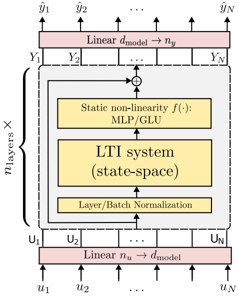

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Deep Linear Recurrent Units

We consider the deep LRU architecture recently introduced by DeepMind:

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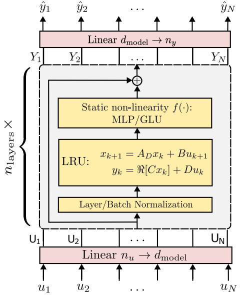

- Discrete-time, MIMO LTI system
- Complex diagonal state-transition matrix A_D:

$$A_D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_x})$$

- Stable parameterization with $|\lambda_i| < 1$
- Implementation with either:
  - $\mathcal{O}(N)$ sequential ops (standard recursion)
  - N parallel jobs, $\mathcal{O}(\log N)$ ops each (parallel scan)
- SOTA on long-range sequences

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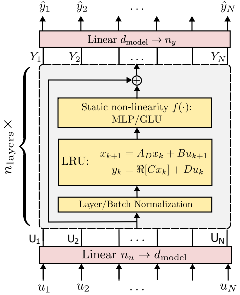

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Model Order Reduction

Consider a LTI state-space system partitioned as:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k-1) \\ x_2(k-1) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Du(k),$$

Reduction is often applied to systems in

- **M**odal form (A diagonal, states sorted from slowest to fastest)
- **B**alanced form (states sorted for decreasing Hankel singular values)

The states x_2 can be removed by:

- **T**runcation $\Rightarrow$ keep (A_{11}, B_1, C_1, D)
- **S**ingular perturbation $\Rightarrow$ solve for $x_2(k) = x_2(k-1)$

We tested all combinations: MT, MSP, BT, BSP.

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Model Order Reduction

- Modal reduction is almost directly applicable to the LRU, which is already in modal (diagonal) form.
  1. Sort system (A_D, B, C, D) for decreasing eigenvalues magnitude
  2. Eliminate fastest modes

- Balanced reduction techniques require a three-step procedure:
  1. Balance (A_D, B, C, D) to a non-diagonal (A_b, B_b, C_d, D)
  2. Reduce (A_b, B_b, C_b, D) to a non-diagonal (A_r, B_r, C_r, D)
  3. Diagonalize (A_r, B_r, C_r, D) to fit the LRU structure

Regularization

Regularization introduced to enhance the MOR:

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \frac{1}{N} \sum_{k=1}^{N} \left(y_k - \hat{y}_k(\theta)\right)^2 + \gamma \mathcal{R}(\theta),$$

Modal ℓ_1

$$\mathcal{R}(\theta) = \sum_{j=1}^{n_x} |\lambda_j|$$

- Push some modes λ_j towards 0
- Tailored for modal reduction MT/MSP

Hankel nuclear norm (HNN)

$$\mathcal{R}(\theta) = \sum_{j=1}^{n_x} \sigma_j$$

- Push some HSV σ_j towards 0
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- Modal ℓ_1 efficient (λ_j are on the diagonal of A_D)
- HNN also efficient exploiting diagonal A_D structure...

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Hankel nuclear norm minimization

- The HNN $\sum_{j=1}^{n_x} \sigma_j$ is a convex approximation to the McMillan degree of a linear system G (minimium realization order).
- The choice of HNN regularization is further motivated by the bound:

$$\|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n_x} \sigma_j,$$

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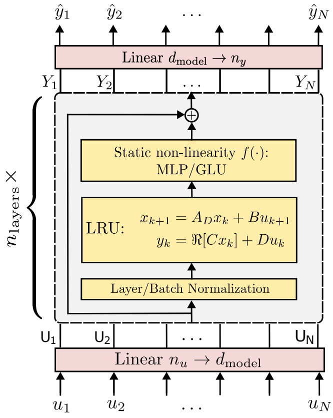
Example

Experiments on the F16 ground vibration benchmark.

Deep LRU with:

- $n_{\text{layers}} = 6$ layers
- $n_x = 100$ states per layer
- $d_{\text{model}} = 50$ units per layer
- Layer Normalization
- MLP non-linearity

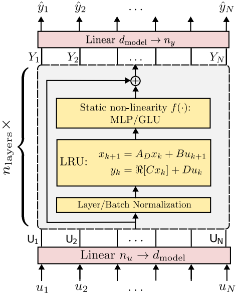


https://github.com/forgi86/lru-reduction

Nominal results

Training repeated with:

- No regularization
- Modal ℓ_1 regularization
- Hankel norm regularization.



Results on test set `FullMSine_Level6_Validation`:

Regularization	Channel 1			Channel 2			Channel 3		
	fit	RMSE	NRMSE	*fit*	RMSE	NRMSE	*fit*	RMSE	NRMSE
No reg.	86.5	0.180	0.134	90.0	0.167	0.099	76.2	0.368	0.237
Modal ℓ_1	85.4	0.195	0.145	89.8	0.171	0.102	74.5	0.395	0.254
Hankel norm	85.8	0.190	0.142	89.0	0.185	0.110	75.5	0.379	0.245

In line with literature. Regularization has a large effect on the LTI blocks!

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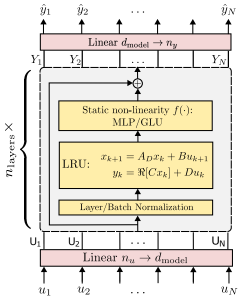

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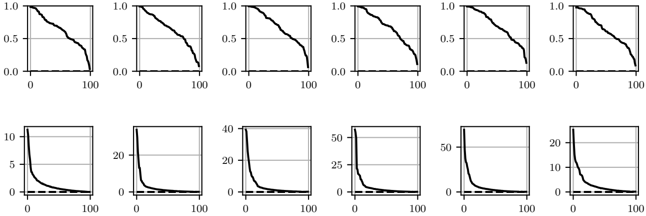
No regularization: eigenvalues magnitude (top) and HSV (bottom)



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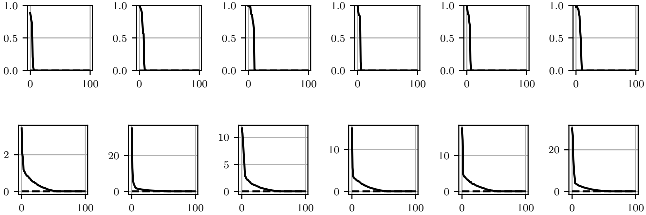

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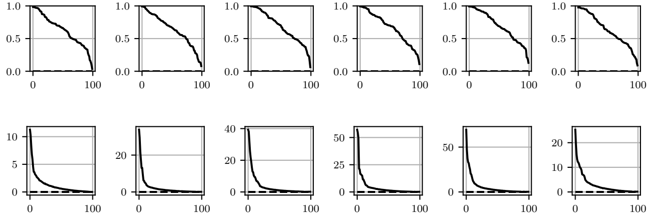
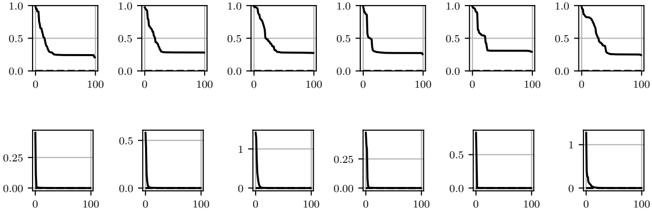


HNN regularization: eigenvalues magnitude (top) and HSV (bottom)

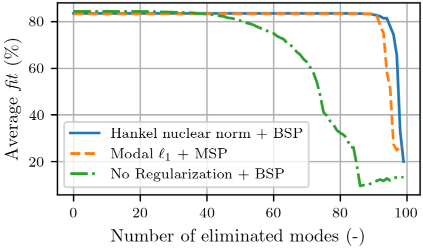


Model Order Reduction

Performance of all regularizers/model order reduction techniques assessed.

	Truncation Method			
Regularization Method	BT	BSP	MT	MSP
No Regularization	28	43	3	35
Modal ℓ_1	56	73	0	**91**
Hankel nuclear norm	89	**91**	18	76

Table: Number of states eliminated s.t. performance decrease is less than 1%



- Best results with Hankel nuclear norm + balanced singular perturbation and modal ℓ_1 + modal singular perturbation
- Without regularization, MOR is significantly less effective

Conclusions & future research

Preliminary efforts to improve deep SSMs with System Theoretic tools.
MOR + tailor-made regularization to reduce state dimensionality.

Future research:

- More extensive simulations (e.g., effect of the regularization strength)
- Other model order reduction (e.g., Kyrilov-based) and regularizers
- Reduction also of layers and channels
- Application to other models where LTI blocks are key, e.g.
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  - dynoNet
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Thank you.
Questions?

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